SOME TOPOLOGICAL PROPERTIES OF GENERALIZED INTEGRATION OPERATORS ON FOCK SPACES

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS

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Declaration

I, Aklilu Defar Lambore, with student ID RM 0197/14-0, declare that this thesis entitled Some topological properties of generalized integration operators on Fock spaces is my own original work and it has not been submitted to any institution or University elsewhere for the award of any academic degree, and sources of information that I have been used or quoted are indicated and acknowledged.

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Abstract

On the past several years, different topological properties of Volterra-type integral operators is among an operator on several functional spaces, which was studied widely. In particular, on the Fock spaces (Constantine 2012, and Mengestie, 2014) have been studied about boundedness and compactness of the operators. Recently (Mengestie, 2018), studied path-connected components of the properties of V_g on the spaces $V(\mathcal{F}_p, \mathcal{F}_q)$. However, path-connected components and isolated points of the operator were not studied for the generalized integral operator on Fock spaces. So, the purpose of this thesis is to study path-connected and isolated points of properties of the operator on Fock spaces $\mathcal{F}_p$. The result of this thesis was generalized the works of (Mengestie, 2018).

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Chapter 1

Introduction

1.1 Background of the study

In Mathematics, Analytic Functions is defined as a function that is locally given by the convergent power series. The analytic function is classified in to two different types, such as real and complex analytic functions. Functions of each type are infinitely differentiable, but complex analytic functions exhibit properties that do not generally hold for real analytic functions. So that, a function f defined on some subsets of the real line is said to be real analytic at a point x if there is anehborhood D of x on which f is real analytic. And, a function f defined on some subsets of complex plane is said to be complex analytic at a point x if there is anehborhood D of x on which f is complex analytic. In generally, Analytic Function is defined as, a function $f : U \to \mathbb{C}$ is analytic on U if it is continuously differentiable. That is, f is differentiable and the derivative is continuous on U.
For a given space $H(U)$ of analytic functions on a domain $U \subseteq \mathbb{C}$. The Volterra type integral operator on $H(U)$ induced by analytic symbol function g is given by,

$$V_g f(z) = \int_0^z f(w)g'(w)dw,$$

The study of these operators acting on various analytic functions of different linear operators on the spaces analytic functions defined over domain $U \subseteq \mathbb{C}$ is a rich history, where many authors are participated any many papers and books are written on it. This operator is first introduced by (Pommerenke, 1977), where he studied its boundedness on the Hilbert Hardy spaces H^2. In particular, integral operators including the Volterra-type are among widely studied linear operators. This is due to their applicability in solving real world problems.

Let X and Y be Banach spaces and $L : X \to Y$ is a linear operator. If there is a constant $k > 0$ such that $||Lx|| \leq k||x||, x \in X$, then we can say that L is bounded operator. Moreover, if $||Lx_n|| \to 0$, whenever $x_n \to 0$ over a compact subset of X, then we say that L is compact. Thus, a linear operator defined as: A linear operator L between real (or complex) linear spaces X, Y is a function $L : X \to Y$ such that for all $x, y \in X$ and for all $\alpha, \beta \in R$ (or $\mathbb{C}$) satisfy:

$$L(\alpha x + \beta y) = \alpha(Lx) + \beta(Ly) = \alpha Lx + \beta Ly.$$

The operators like, $V_g f(z) = \int_0^z f(w)g'(w)dw$, were introduced a lot by other authors with the aim to explore the connection between their operator theoretic behaviors with the function theoretic properties of the symbol g. (Pommereke, 1977) studied boundedness of the operator on the Hilbert space of Hardy space H^2 and this result extended to $H^p, 0 < p \leq \infty$, in general by (Aleman and Siskakis, 1995) and further more they studied compactness property also. Later, (Aleman and Siskakis, 1997) gave the similar characterization on the Hardy and Bergman spaces. But, those studies are considered on a spaces of analytic functions defined over a disk. (Constantine, 2012) and (Mengestie, 2018) considered the problem over a space defined over the whole complex plane, namely the Classical-Fock spaces F_p. later, (Mengestie and Worku, 2019) studied over the generalized Fock spaces F_∞. So, main goal of this study is to consider the spaces of all bounded generalized Fock spaces, and characterize different topological properties, like path-connected components and connectedness in the space F_p. (Chalmoukis, 2020) introduced a new generalization of Volterra-type integral operator, which is defined below. For a nonnegative integers m and n with $0 \leq m < n$, and an entire function g, the generalized integration operator, $V_g^{n,m}$, is defined by

$$V_g^{n,m} f = I^n(f^{(m)} g^{(n-m)}),$$

where I^n is the n^{th} iterate of integration operator $I(f)(z) = \int_0^z f(w)dw$. Chalmoukis studied the operator on the Hardy spaces and very recently other researchers namely (Du, J., Li, Qu, D. and Qian, R., Zhu, X.) considered the operator on other spaces. In particular, for the values of $n = 1$ and $m = 0$, it gives the Volterra-type integral operator V_g.The aim of this thesis taking further study on the some topological properties of generalized integration operators on Fock spaces. Thus, we plan to define some topological spaces, like; path-connected, compact and boundedness as follow:

1.1.1 Some Topological Properties

Now, we shall define, some topological properties like, path-connected,isolated points, compactness and boundedness of the spaces including Fock spaces.

Definition 1.1.1. Let X be a topological space,
I. A path in X is continuous map $f : [0,1] \to X$, if $f(0) = x$ and $f(1) = y$, we said that f is a path from x to y.
II. A space X is path-connected if it is nonempty and for all,$x, y \in X$, there exists a path from x to y in X.
III. A point is called an isolated point of a subset X,if $x \in X$ and there exists a neibghborhood of x that does not contain any other points of U.
Iv. A space X is compact if every open cover of X has a finite sub cover.
V. A space X is said to be bounded if and only if the size of that space set is finite.

1.1.2 Fock Spaces

Let $\mathbb{C}$ be the complex plane and, $0 < p \leq \infty$.Then Fock Space is a space of analytic functions on $\mathbb{C}$, for which,

$$||f||_p = \begin{cases} [\frac{p}{2\pi} \int_{\mathbb{C}} |f(z)|^p e^{-\frac{p|z|^2}{2}} dA(z)]^{\frac{1}{p}}, & \text{if } p < \infty \\ \sup\limits_{z\in\mathbb{C}} |f(z)| e^{-\frac{|z|^2}{2}}, & \text{if } p = \infty, \end{cases}$$

where dA usually the Lebesgue area on $\mathbb{C}$. We note that, the space $\mathcal{F}_p$ for $1 \leq p \leq \infty$ is Banach space. The space is named after the Soviet Physicist V. Aleksandrovich Fock (1898-1974) and has a wide application in quantum physics, harmonic analysis on the Heisenberg group and partial differential equations. In particular, the space F_2 is a reproducing kernel Hilbert space with kernel and normalized kernel functions, which is given by $k_w(z) = e^{\frac{z\overline{w}}{2}}$ and $k_w(z) = e^{z\overline{w} - \frac{|w|^2}{2}}$. More over,the inner product is given by
$\langle f, g\rangle = \frac{1}{\pi} \int_{\mathbb{C}} f(z) g(\bar{z}) e^{-|z|^2} dA(z)$.

Furthermore, the kernel function K_w belongs to all Fock spaces $\mathcal{F}_p$ with norms, $||K_w||_p = e^{\frac{|w|^2}{2}}$, for all $w \in \mathbb{C}$ and $0 < p \leq \infty$. The space $\mathcal{F}_p$ is called Fock space.

Example 1.1.2. The function $f(z) = z^n$ is an entire function in $\mathcal{F}_p$, since

$$\begin{aligned} ||z^n|| &= \frac{p}{2\pi}\int_{\mathbb{C}} |z^n|^p e^{-\frac{p|z|^2}{2}} dA(z) \\ &= p\int_0^{\infty} \Gamma^{np} e^{-\frac{p\Gamma^2}{2}} \Gamma d\Gamma \\ &= (\frac{1}{p})^{\frac{np}{2}} \Gamma(\frac{np}{2}+1) < \infty, \end{aligned}$$

where $\Gamma(z)$ is gamma function.

1.2 Statement of the problem

On the past several years different topological properties of Volterra-type integral operators is among an operator on several functional spaces, which was studied widely. In particular, on the Fock spaces (Constantine 2012, and Mengestie, 2014) have been studied about boundedness and compactness of the operators. Recently, (Mengestie, 2018) studied path-connected components of the properties of V_g on the spaces $V(\mathcal{F}_p, \mathcal{F}_q)$. However, path-connected components and isolated points of the operator were not studied for the generalized integral operator on Fock spaces. So, the purpose of this thesis is to study path-connected properties of $V_g^{n,m}$ of the operators on Fock spaces $\mathcal{F}_p$. The result of this thesis will be generalized the works of (Mengestie, 2018).

1.3 Objectives of the study

1.3.1 General objectives

The general objective of this thesis is to study some topological properties of generalized integration operators on Fock spaces.

1.3.2 Specific objectives

The specific objectives of this thesis were to;

- describe the path-connected components and isolated points of the operators.
- give sufficient and necessary condition for path-connectedness and isolated points.

- describe essentially path-connected components of generalized integration on Fock spaces.
- find isolated points in the spaces of bounded generalized integration operators.

1.4 Significance of the study

The result of this study have the following importance:

- It generalizes the study of the operators in to more general spaces on Fock spaces.
- It may be used as a base for any researcher who is interested to study other properties of the path-connected and connected components of the integration on Fock spaces.
- Help the graduate students to acquire research skills and scientific procedures.

1.5 Delimitation of the study

This study was delimited to study on establishing some topological properties of generalized integration operators on Fock spaces.

Chapter 2

Review of Related Literature

(Pommereke, 1977), introduced and studied boundedeness property of Voltera-type integral operator V_g on the Hardy space H^2. Then (Aleman and Siskakis, 1995), studied on bounded and compact properties of the operator on the Hardy and Bergman spaces. Later, (Mengestie, 2018) studied path-connected components and other properties of V_g on the spaces $V(\mathcal{F}_p, \mathcal{F}_q)$. Following that, a number of researchers are motivated to study different topological properties of the integration operators on different spaces. (Mengestie and Worku, 2019) studied on the isolated and essentially isolated Volterra type integral operators on generalized Fock spaces $\mathcal{F}_\infty$. We state it by the following theorems: Then the study was continued by (Mengestie, 2013), on the growth type Fock space $\mathcal{F}_\infty$, which is stated by the following theorem.

Theorem 2.0.1 (Mengestie, 2013)**.**
1. Let $0 < p \leq q \leq \infty$. Then $V_g : \mathcal{F}_q \to \mathcal{F}_\infty$ is:
(i) bounded if and only if $g(z) = az^2 + bz + c$, for some a,b,c∈ C
(ii)compactifandonlyif g(z)=az+b, forsome a,b∈ C.

Then the study was continued by (Constantine, 2012 and Mengestie, 2018) on the classical Fock spaces $\mathcal{F}_\infty$, which is stated by the following theorem:

Theorem 2.0.2. *(Constantin, 2013 and Mengesti, 2018)*
Let $0 < p \leq q \leq \infty$. Then the following are equivalent
(i) $V_g : \mathcal{F}_q \to \mathcal{F}_p$ is bounded.
(ii) $V_g : \mathcal{F}_p \to \mathcal{F}_q$ is compact.
(iii) $q > \begin{cases} \frac{2p}{p+2}, & p < \infty \\ 2, & p = \infty \end{cases}$ and $g(z) = az + b$ for some $a, b \in \mathbb{C}$.

Then the study was continued by (Mengestie, 2018) he studied on generalized Volterra- type integral operator V_g on classical Fock spaces $\mathcal{F}_p$.

Theorem 2.0.3 (Mengestie, 2018).
Let $0 < p \leq q \leq \infty$, then the set of all compact operators $V_g : \mathcal{F}_p \to \mathcal{F}_q$ is a path-connected component of the space $V(\mathcal{F}_p, \mathcal{F}_q)$.

Theorem 2.0.4 (Mengestie, 2018).
Let $0 < p \leq q \leq \infty$, then the set of all compact operators $V_g : \mathcal{F}_{m,p} \to \mathcal{F}_{m,q}$ is a path-connected component of the space $V(\mathcal{F}_{m,p}, \mathcal{F}_{m,q})$, where $V(\mathcal{F}_{m,p}, \mathcal{F}_{m,q})$ is space of all bounded Volterra-type integral operators $V_g : \mathcal{F}_{m,p} \to \mathcal{F}_{m,q}$ is space of entire functions for which the following is finite.

$$||f|| = \begin{cases} [\frac{p}{2\pi} \int_{\mathbb{C}} |f(z)|^p e^{-\frac{p|z|^m}{2}} dA(z)]^{\frac{1}{p}}, & \text{if } p < \infty \\ \sup_{z \in \mathbb{C}} |f(z)| e^{-\frac{|z|^m}{m}}, & \text{if } p = \infty. \end{cases}$$

Theorem 2.0.5 (Mengestie and Worku, 2019).
Let $0 < p \leq q \leq \infty$ and $V_g : \mathcal{F}_p \to \mathcal{F}_q$, where $m > 2$ be a bounded operator. The following are equivalent:
I. V_g is isolated in $V(\mathcal{F}_m^p, \mathcal{F}_m^q)$.
II. V_g is not compact, and, hence $g(z) = ax^2 + bz + c$, with $a \neq 0$.
III. The space $V(\mathcal{F}_m^p, \mathcal{F}_m^q)$ has the same connected and path-connected components which is only the set of all compact operators in $V(\mathcal{F}_m^p, \mathcal{F}_m^q)$.

Following the line of research in this research, we study establishing some topological properties of generalized integration operators on Fock spaces.

Chapter 3

Methodology of the study

3.1 Study area and Period

The study was conducted in Jimma University department of mathematics under the functional analysis stream from January, 2023 G.C. to June, 2023 G.C.

3.2 Study design

In this research work we employed analytical method of design.

3.3 Source of information

The relevant sources of information for this study were journals, books, published articles and related studies from Internet.

3.4 Mathematical Procedure of the study

The mathematical procedure that the researcher follows for this research work is the following:

- Giving definitions.
- Establishing theorems.
- Proving theorems.

- Providing sufficient and necessary condition for path-connected components and isolated points of the operators.
- Characterize isolated points and path-connectedness.
- Giving conclusions or results based on the main findings.

Chapter 4

Main Results and Discussion

We begin the section with some preliminaries, regarding our working space $\mathcal{F}_p$.

Lemma 4.0.1. *(Hu,2013 and Ueke,2016). Let $0 < p \leq \infty$. Then for each $f \in \mathcal{F}_p$, it holds that*

$$||f||_p \simeq \begin{cases} (\int_{\mathbb{C}} \frac{|f^{(n)}(z)|^P}{(1+|Z|)^{np}} e^{-\frac{p|z|^2}{2}} dA(z))^{\frac{1}{p}}, & \text{for } 0 < p < \infty \\ \sup\limits_{z\in\mathbb{C}} \frac{|f^{(n)}(z)|}{(1+|Z|)^n} e^{-\frac{|z|^2}{2}}, & \text{for } p = \infty. \end{cases}$$

The above Lemma characterizes the space $\mathcal{F}_p$ in terms of the nth derivative and it plays important role in the study of different properties of integral operators on the space. Our next proposition simplifies the conditions for boundedness and compactness of $V_g^{n,m} : \mathcal{F}_p \to \mathcal{F}_q$, where $0 < p \leq q \leq \infty$, which are stated in Theorem 4.0.3 (Hafz, 2022), follows from (Mangiste, 2014), we now stated it as follows:

Theorem 4.0.2. *Let $0 < p \leq q \leq \infty$, m, n nonnegative integers with $0 \leq m < n$ and $V_g^{n,m}$ maps from $\mathcal{F}_p$ into $\mathcal{F}_q$.*

(I) If $p \leq q$, then $V_g^{n,m}$ is bounded (respectively, compact) if and only if the function $\frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n}$ is bounded (respectively, $\lim_{|z|\to\infty} \frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n} = 0$).

(II) If $q < p$, then $V_g^{n,m}$ is bounded or compact if and only if

$$\begin{cases} \int_{\mathbb{C}} \left(\frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n}\right)^{\frac{pq}{p-q}} dm(z) < \infty, & \text{for } p < \infty \\ \int_{\mathbb{C}} \left(\frac{|g^{(n-m)}(z)||z|^m}{(1+|z|)^n}\right)^{q} dm(z) < \infty, & \text{for } p = \infty. \end{cases}$$

Proposition 4.0.3. *Let $0 < p \le q \le \infty$. Then $V_g^{n,m} : \mathcal{F}_p \to \mathcal{F}_q$ is:*
(I) bounded if and only if g is a polynomial of degree at most $2(n-m)$.
(II) Compact if and only if g is a polynomial of degree at most $2(n-m)-1$.

Proof. (I) From Theorem 4.0.3 (Hafz, 2022), $V_g^{n,m}$ is bounded if and only if the function $\frac{|g^{(n-m)}(z)|}{(1+|z|)^n}|z|^m$ is bounded. First, suppose $\frac{|g^{(n-m)}(z)|}{(1+|z|)^n}|z|^m$ is bounded, then there exists a number $M > 0$ such that,

$$\frac{|g^{(n-m)}(z)|}{(1+|z|)^n}|z|^m \le M.$$

This implies

$$|g^{(n-m)}(z)| \le M(\frac{1}{|z|^m} + |z|^{n-m}), \text{ for } z \neq 0.$$

By Lienville's Theorem $g^{(n-m)}$ is a polynomial of degree at most n-m. This implies that, g is a polynomial of degree at most $2(n-m)$. On the other hand, if g is a polynomial of degree at most $2(n-m)$, then clearly the function $\frac{|g^{(n-m)}(z)|}{(1+|z|)^n}|z|^m$ is bounded.

(II) Since $V_g^{n,m}$ is compact if and only if $\frac{|g^{(n-m)}(z)|}{(1+|z|)^n}|z|^m$ goes to zero as $|z| \to \infty$, from (I) above, this holds if and only if g is a polynomial of degree at most $2(n-m)-1$.
We denote by $V(\mathcal{F}_p, \mathcal{F}_q)$ the space of all bounded generalized integration operators $V_g^{n,m}$: $\mathcal{F}_p \to \mathcal{F}_q$.

□

We now state our main result.

Theorem 4.0.4. *Let $0 < p \le q \le \infty$ and $V_g^{n,m} : \mathcal{F}_p \to \mathcal{F}_q$ be a compact operator. Then $V_g^{n,m}$ and $V_{g_{(0)}}^{n,m}$ belongs to the same path-connected components of the space $V(\mathcal{F}_p, \mathcal{F}_q)$.*

Proof. Since $V_g^{n,m}$ is compact, by proposition 4.0.3 g is a polynomial of degree at most $2(n-m)-1$. Thus, g has the form

$$g(z) = a_l z^l + a_{l-1} z^{l-1} + ... + a_0,$$

where $l = 2(n-m)-1$. If $a_k = 0$ for all k, $\frac{l+1}{2} \le k \le l$, then $V_g^{n,m} = V_{g_{(0)}}^{n,m}$ are zero operators and the result holds trivially. We assume, $a_k \neq 0$ for some k, $\frac{l+1}{2} \le k \le l$. And define scaling functions $g_s : [0,1] \to \mathbb{C}$ by $g_s(z) = g(sz)$. Then, $V_{g_{(s)}}^{n,m} : \mathcal{F}_p \to \mathcal{F}_q$ is compact for all s and satisfies $V_{g_1}^{n,m} = V_g^{n,m}$ and $V_{g_0}^{n,m} = V_{g_{(0)}}^{n,m}$. We want to show that the map $s \longmapsto V_{g_s}^{n,m}$ continuous. It is enough to show that for every $s \in [0,1]$,

$$\lim_{t \to s} ||V_{g_t}^{n,m} - V_{g_s}^{n,m}|| = 0.$$

First, for $q < \infty$ and $f \in \mathcal{F}_p$ with $||f||_p \leq 1$, an application of Lemma 4.0.1 gives,

$$
\begin{aligned}
||V_{g_t}^{n,m} f - V_{g_s}^{n,m} f||_q^q &\simeq \int_{\mathbb{C}} \frac{|f^{(m)}(z)|^q |g_t^{(n-m)}(z) - g_s^{(n-m)}(z)|^q}{(1+|z|)^{nq}} e^{-\frac{q|z|^2}{2}} dA(z) \\
&= |t-s|^q \int_{\mathbb{C}} \frac{|f^{(m)}(z)|^q |g_{(t,s)}^{(n-m)}(z)|^q}{(1+|z|)^{nq}} e^{-\frac{q|z|^2}{2}} dA(z),
\end{aligned}
$$

where

$$
g_{(t,s)}^{n-m}(z) = \sum_{i=(n-m)+1}^{l} (a_i \frac{i!}{(i-(n-m))!} z^{i-(n-m)} \sum_{j=n-m}^{i-1} s^{j-(n-m)} t^{i-1-j}).
$$

Since $\deg(g_{(t,s)}) \leq l = 2(n-m) - 1$, by proposition 4.0.3,

$$
V_{g_{(t,s)}}^{n,m)} : \mathcal{F}_p \to \mathcal{F}_q
$$

is uniformly bounded independent of s and t. Thus, an application of Lemma 4.0.1, gives

$$
\begin{aligned}
||V_{g_t}^{n,m} f - V_{g_s}^{n,m} f||_q^q &\simeq |t-s|^q \int_{\mathbb{C}} \frac{|f^{(m)}(z)|^q |g_{(t,s)}^{n-m}(z)|^q}{(1+|z|)^{nq}} e^{-\frac{q|z|^2}{2}} dA(z) \\
&\simeq |t-s|^q ||V_{g_{(t,s)}}^{n,m} f||_q^q \\
&\leq |t-s|^q ||V_{g_{(t,s)}}^{n,m}||^q ||f||_p^q.
\end{aligned}
$$

Therefore,

$$
\lim_{t \to s} ||V_{g_t}^{n,m} - V_{g_s}^{n,m}|| = 0.
$$

For $q = \infty$, using Lemma 4.0.1, similarly we get

$$
\begin{aligned}
||V_{g_t}^{n,m} f - V_{g_s}^{n,m} f||_\infty &\simeq \sup_{z \in \mathbb{C}} |t-s| \frac{|f^{(m)}(z)| |g_{(t,s)}^{n,m}(z)|}{(1+|z|)^n} e^{-\frac{|z|^2}{2}} \\
&\lesssim |t-s| ||V_{g_{t,s}}^{n,m} f||_\infty \\
&\lesssim |t-s| ||f||_p,
\end{aligned}
$$

which implies that

$$
\lim_{t \to s} ||V_{g_t}^{n,m} - V_{g_s}^{n,m}|| = 0.
$$

□

Corollary 4.0.5. *Let $0 < p \leq q \leq \infty$. Then the set of all compact operators $V_g^{n,m} : \mathcal{F}_p \to \mathcal{F}_q$ is a path-connected component of the space $V(\mathcal{F}_p, \mathcal{F}_q)$.*

Proof. Suppose $V_{g_1}^{n,m}$ and $V_{g_2}^{n,m}$ are compact operators,then by proposition 4.0.3, g_1 and g_2 are polynomial of degree at most $2(n-m)-1$. By Theorem 4.0.4 above $V_{g_1}^{n,m}$ and $V_{g_{1(0)}}^{n,m}$ belongs to the same path-connected component. Similarly, $V_{g_2}^{n,m}$ and $V_{g_{2(0)}}^{n,m}$ belongs the same path-connected component. But $V_{g_{1(0)}}^{n,m}$ and $V_{g_{2(0)}}^{n,m}$ are zero operators and hence $V_{g_1}^{n,m}$ and $V_{g_2}^{n,m}$ are in the same path-connected component. □

Our next result characterizes isolated generalized integration operators in $V(\mathcal{F}_p, \mathcal{F}_q)$.

Theorem 4.0.6. *Let $0 < p \leq q \leq \infty$ and $V_g^{n,m} : \mathcal{F}_p \to \mathcal{F}_q$ be a bounded operator, then the following statements are equivalent:*
(I) $V_g^{n,m}$ is isolated in $V(\mathcal{F}_p, \mathcal{F}_q)$,
(II) $V_g^{n,m}$ is not compact, and hence g has the form $g(z) = a_l z^l + a_{l-1} z^{l-1} + ... + a_0$, where $l = 2(n-m)$ and $a_l \neq 0$.

Proof. $(I \Rightarrow II)$ follows from Corolary 4.0.5 above, we will prove $(II \Rightarrow I)$. If (II) holds, then it suffices to show that there exists a positive number α such that

$$||V_g^{n,m} - V_{g_1}^{n,m}|| \geq \alpha,$$

for all polynomials of the form $g_1(z) = b_l z^l + b_{l-1} z^{l-1} + ... + b_0$ with $b_l \neq a_l$. For $p \leq q < \infty$,

$$\begin{aligned} ||V_g^{n,m} - V_{g_1}^{n,m}||^q &\geq ||V_g^{n,m} k_w - V_{g_1}^{n,m} k_w||_q^q \\ &\geq C_1 \int_{\mathbb{C}} \frac{|a_l - b_l|^q |z|^{(n-m)q} |w|^{mq} |k_w(z)|^q}{(1+|z|)^{nq}} e^{-\frac{q|z|^2}{2}} dA(z) \\ &\geq C_2 \int_{D(w,1)} \frac{|z|^{(n-m)q} |w|^{mq} |k_w(z)|^q}{(1+|z|)^{nq}} e^{-\frac{q|z|^2}{2}} dA(z) \\ &\simeq C_3 \int_{D(w,1)} |k_w(z)|^q e^{-\frac{q|z|^2}{2}} dA(z). \end{aligned}$$

For some positive constants C_1, C_2 and C_3. By sub harmonicity of $|k_w|^q$ it follows that

$$C_3 \int_{D(w,1)} |k_w(z)|^q e^{-\frac{q|z|^2}{2}} dA(z) \geq C_4 |k_w(w)|^q e^{-\frac{q|w|^2}{2}} = C_4 > 0.$$

Therefore, there exists $\alpha = C_4 > 0$ such that

$$||V_g^{n,m} - V_{g_1}^{n,m}|| \geq \alpha.$$

Similarly, for $p \leq q = \infty$, applying Lemma 4.0.1, we obtain

$$\begin{aligned} ||V_g^{n,m} - V_{g_1}^{n,m}|| &\geq ||V_g^{n,m} k_w - V_{g_1}^{n,m} k_w||_\infty \\ &\geq C_1 \sup_{z \in \mathbb{C}} \frac{|a_l - b_l||z|^{(n-m)}|w|^m |k_w(z)|)}{(1+|z|)^n} e^{-\frac{|z|^2}{2}} \\ &\geq C_2 |k_w(z)|) e^{-\frac{|z|^2}{2}} = C_2 > 0. \end{aligned}$$

Therefore, there exists $\alpha = C_2 > 0$ such that

$$||V_g^{n,m} - V_{g_1}^{n,m}|| \geq \alpha.$$

□

Chapter 5

Conclusion

This thesis includes a number of results, which deals some topological properties of the generalized integration operators $V_g^{n,m}$ when they act between Fock spaces $V(\mathcal{F}_p, \mathcal{F}_q)$ when $0 < p \leq q \leq \infty$. Our results in chapter four, which deals about bounded, compact, path-connected components and connectedness on the space of generalized integration operators are new and it is simple to apply to study other properties defined whenever the operator is bounded (res, compact). In addition, our result improve and generalize some of results that have been obtained for this important class of linear operators. In generally, this thesis results generalizes the results of (Mengestie, 2018) from integration operators to generalized integration operators on Fock spaces $\mathcal{F}_p$.

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