Schatten Class Generallized Integration Operators on Fock Spaces

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS

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Declaration

I, Kamil Biya, with student ID number RM 0189 /14, declare that this thesis entitled "Schatten Class Generallized Integration Operators on Fock Spaces" is my own original work and it has not been submitted to any institution or University elsewhere for the award of any academic degree, and sources of information that I have been used or quoted are indicated and acknowledged.

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Abstract

Different properties of Volterra-type integral operator have been studied in the past two decades on several functional spaces. In particular, on Fock spaces Volterra type integral operator studied by (Constantin, 2012 and Mengestie, 2013). The schatten class membership of Volterra type integral operators on Fock spaces were studied also by (Constantin, 2012 and Mengestie, 2013). However, the schatten class membership of generalized integration operator on Fock spaces is not studied. So the purpose of this thesis is to fill this gap and study schatten class membership of generalized integration operator on Fock spaces. The result of this thesis generalize the works of (Constantin, 2012 and Mengestie, 2013).

Contents

Chapter 1

Introduction

1.1 Background of the study

For analytic functions f and g on the complex plane $\mathbb{C}$, we consider the Volterra type integral operator V_g is defined by

$$V_g f(z) = \int_0^z f(w) g'(w) dw,$$

The operator is first introduced by (Pommerenke, 1977). Pommerenke studied boundedness of the operator on the Hilbert Hardy space H^2. The study was continued by (Aleman and Siskakis, 1995) on the whole Hardy space, extending the result of Pommerenke. The study aims at finding the form of g for which the operator is bounded and compact acting on different spaces. (Aleman and Siskakis, 1997) have also characterized those properties of the operator acting on the Bergman space. The study of the operator on Fock space were initiated by (Constantin, 2012) and continued by (Mengestie, 2013). We recall that Fock space $\mathcal{F}_p$, for $0 < p \leq \infty$ is space of analytic function f for which the following are finite:

$$||f|| = \begin{cases} [\frac{p}{2\pi} \int_{\mathbb{C}} |f(z)|^p e^{-\frac{p|z|^2}{2}} dA(z)]^{\frac{1}{p}}, & \text{if } p < \infty \\ \sup_{z \in \mathbb{C}} |f(z)| e^{-\frac{|z|^2}{2}}, & \text{if } p = \infty. \end{cases}$$

The space $\mathcal{F}_p$ for $1 \leq p \leq \infty$ is a Banach space. Fock spaces play important role in quantum physics, harmonic analysis on the Heisenberg group and partial differential equations. In particular, the normalized reproducing kernels in the Fock space are exactly the so-called coherent states in quantum physics. The space $\mathcal{F}_2$ is a reproducing kernel Hilbert space with $k_w(z) = e^{z\overline{w}}$. In 2008, (Li and Stevic, 2008) raised an idea to extend the Volterra-type integral operator V_g by considering its product with composition operator $C_\varphi f(z) = f(\varphi(z))$ and they studied their operator theoretic properties in terms of the inducing pair of symbols on some spaces of analytic functions on the unit disk. They eventually considered the following operators; For analytic functions g and ψ, the generalized Volterra type integral and composition operator are defined as;

$$V_g^\psi f(z) = \int_0^z f(\psi((w))g'(w)dw$$

and

$$C_g^\psi f(z) = \int_0^z f'(\psi((w))g'(w)dw$$

Since a particular choice of $\varphi(z) = z$, reduce V_g^ψ to the Volterra-type integral operator V_g, the operator V_g^ψ is called the generalized Volterra-type integral operator. Up to constant C_g^ψ reduces to the composition operator and hence called generalized composition operator. Boundedness and compactness of this operator have been studied on different spaces and the characterization of these properties on the classical Fock space have been given by (Mengestie, 2014) and later by (Mengestie and Worku, 2018). (Chalmoukis, 2020) introduced a new generalization of Volterra-type integral operator, which is defined as, for a nonnegative integers m and n with $0 \leq m < n$, and an entire function g, the generalized integration operator $V_g^{n,m}$, is defined by,

$$V_g^{n,m} f = I^n(f^{(m)} g^{(n-m)}),$$

where I^n is the n^{th} iterate of integration operator $I(f)(z) = \int_0^z f(w)dw$. Chalmoukis studied the operator on the hardy space and very recently other researcher considered the operator on other space in particular for the values of $m = 0$ and $n = 1$ it gives the Volterra type integral operator V_g. The purpose of this research is to study the schatten class membership of generalized integration operators on Fock spaces.

Definition 1.1.1. A positive operator T on $\mathcal{F}_2$ belongs to the trace class if

$$\sum_{k=0}^{\infty} \langle Te_n, e_n \rangle) < \infty.$$

For some orthonormal basis $\{e_n\}$ of $\mathcal{F}_2$. If $0 < p < \infty$, a bounded operator T on $\mathcal{F}_2$ belongs to the schatten class S_p if the operator $(T^*T)^{\frac{p}{2}}$ is in the trace class. In particular for $p = 2$, S_p is called Hilbert Schmidt class.

1.2 Statement of the problem

Different properties of Volterra-type integral operator have been studied in the past two decades on several functional spaces. In particular, on Fock spaces Volterra type integral operator studied by (Constantin, 2012 and Mengestie, 2013). The schatten class membership of Volterra type integral operators on Fock spaces were studied also by (Constantin, 2012 and Mengestie, 2013). However, the schatten class membership of generalized integration operator on Fock spaces is not studied. So the purpose of this thesis is to fill this gap and study schatten class membership of generalize integration operator on Fock spaces. The result of this thesis was generalized the works of Constantin, (2012) and Mengestie, (2013).

1.3 Objectives of the study

1.3.1 General objectives

The general objective of this thesis was to study the Schatten class of properties of generalized integration operator on Fock spaces.

1.3.2 Specific objectives

The specific objectives of this thesis are;

- Describing Schatten class membership of generalized integration operator on Fock spaces by giving sufficient and necessary condition for it.
- Finding a condition on which generalized integration operators belong to the trace class and Hilbert Schmidt class.
- Describing Schatten class membership of Volterra type integral operator on Fock spaces in particular.

1.4 Significance of the study

The result of this study have the following importance:

- It generalizes study of Volterra-type integral operators into more general operators.

- It can be used as a base for any researcher who is interested to study other properties of generalized integration operators on Fock spaces.
- Help the graduate students to acquire research skills and scientific procedures.

1.5 Delimitation of the study

This study focused only on characterizing the schatten class membership of generalized integration operators acting between Fock spaces.

Chapter 2

Review of Related Literature

(Pommerenke, 1977) introduced and studied Volterra type integral operator Vg on the Hilbert Hardy spaces and then (Aleman and Siskakis, 1995 and 1997) characterized bounded and compact Volterra type integral operator properties on Hardy and Bergman spaces. Following that, a number of researchers are motivated to study different properties of the Volterra-type integral operator on different spaces. (Constantin, 2012), studied bounded Volterra type integral operator and other properties of V_g on the Fock spaces. We state it by the following theorem.

Theorem 2.0.1. *(Constantin, 2012)*
1. Let $0 < p \leq q < \infty$*. Then* $V_g : \mathcal{F}_p \to \mathcal{F}_q$ *is*
(i) bounded if and only if $g(z) = az^2 + bz + c$*.*
(ii) compact if and only if $g(z) = az + b$*.*
2. Let $0 < p \leq q < \infty$*. Then the following are equivalent*
(i) $V_g : \mathcal{F}_p \to \mathcal{F}_q$ *is bounded.*
(ii) $V_g : \mathcal{F}_p \to \mathcal{F}_q$ *is compact.*
(iii) $q > \frac{2p}{p+1}$ *and* $g(z) = az + b$*.*

Then the study was continued by (Mengestie, 2013), on the growth type Fock space $\mathcal{F}_\infty$, which is stated by the following theorem.

Theorem 2.0.2 (Mengestie, 2013)**.**
1. Let $0 < q \leq \infty$*. Then* $V_g : \mathcal{F}_q \to \mathcal{F}_\infty$ *is:*
(i) bounded if and only if $g(z) = az^2 + bz + c$*.*
(ii) compact if and only if $g(z) = az + b$*.*
2. Let $0 < q < \infty$*. Then the following are equivalent*

(i) $V_g : \mathcal{F}_q \to \mathcal{F}_\infty$ *is bounded*
(ii) $V_g : \mathcal{F}_q \to \mathcal{F}_\infty$ *is compact*
(iii) $q > 2$ *and* $g(z) = az + b$*.*

The schatten class membership of V_g on Fock spaces were characterized in (Constantin, 2012 and Mengistie, 2013), which we state it as follows:

Theorem 2.0.3 (Mengestie and Worku, 2018)**.**
Suppose V_g *is compact. Then* V_g *belongs to the Schatten class* S_p *for all* $p > 2$*, but it fails to be Hilbert-Schmidt unless* g *is constant.*

Following the line of research in this research, we study schatten class membership of generalized integration operators on Fock spaces.

Chapter 3

Methodology of the study

3.1 Study area and Period

The study was conducted in Jimma University department of mathematics under the functional analysis stream from January, 2023 G.C. to June, 2023 G.C.

3.2 Study design

In this research work we employed analytical method of design.

3.3 Source of information

The relevant sources of information for this study were journals, books, published articles and related studies from Internet.

3.4 Mathematical Procedure of the study

The mathematical procedure that the researcher follows for this research work is the following:

- Providing a sufficient and necessary condition for generalized integration operators to be in the shatten class membership.
- Characterizing for Volterra type integral operator on Fock spaces.
- Giving conclusion based on the main findings.

Chapter 4

Preliminaries and Main Results

4.0.1 Preliminaries

We begin the section with the following lemmas. The first lemma characterizes Fock spaces in terms of the n^{th} derivative. That, is it characterizes the norm of function f in $\mathcal{F}_p$ in terms n^{th} derivative of f. This characterization, for $n = 1$, is called Littlewood-paley type estimate.

Lemma 4.0.1. *(Hu, 2013 and Ueke, 2016) For a positive integer $n \geq 1$, we have*

$$||f||_p \simeq \begin{cases} [\frac{p}{2\pi}\int_{\mathbb{C}} \frac{|f^{(n)}(z)|^P}{(1+|Z|)^{np}} e^{-\frac{p|z|^2}{2}} dA(z)]^{\frac{1}{p}}, & for\ 0 < p < \infty \\ \sup\limits_{z\in\mathbb{C}} \frac{|f^{(n)}(z)|}{(1+|Z|)^n} e^{-\frac{|z|^2}{2}}, & for\ p = \infty. \end{cases}$$

Lemma 4.0.2. *(Zhu, 2007) For $0 < p \leq q \leq \infty$. Then $\mathcal{F}_p \subseteq \mathcal{F}_q$.*

Proof. 1. For $q < \infty$, assume $f \in \mathcal{F}_p$. Then,

$$\begin{aligned} ||f||_q^q &= \frac{q}{2\pi}\int_{D(z,1)} |f(z)|^q e^{\frac{-q|z|^2}{2}} dA(z) \\ &= \frac{q}{2\pi}\int_{D(z,1)} |f(z)|^p |f(z)|^{q-p} e^{\frac{-q|z|^2}{2}} dA(z) \end{aligned}$$

Applying the point wise estimate, we have the right hand side is

$$\begin{aligned} ||f||_q^q &\leq ||f||_p^{q-p} \frac{qp}{2\pi p}\int_{D(z,1)} |f(z)|^p e^{-\frac{q-p|z|^2}{2}} dA(z) \\ &= \frac{q}{p}||f||_p \end{aligned}$$

Thus,

$$||f||_q \leq \frac{q}{p}^{\frac{1}{q}} ||f||_p.$$

And, 2. If $q = \infty$, then applying the above Lemma 4.0.1, we have

$$\begin{aligned} |f(z)|e^{-\frac{|z|^2}{2}} &\leq (k \int_{D(z,1)} |f(w)|^p e^{-\frac{|z|^2}{2}} dA)^{\frac{1}{p}} \\ &= (\frac{2\pi}{p})^{\frac{1}{p}} \int_{D(z,1)} |f(w)|^p e^{-\frac{|w|^2}{2}} dA)^{\frac{1}{p}} \\ &\leq (\frac{2\pi}{p})^{\frac{1}{p}} \int_{D(z,1)} |f(w)|^p e^{-\frac{|w|^2}{2}} dA)^{\frac{1}{p}} \\ &= (\frac{2\pi}{p})^{\frac{1}{p}} ||f||_p \leq \infty, \end{aligned}$$

which implies

$$||f||_\infty = \sup_{z\in\mathbb{C}} |f(z)|e^{-\frac{|z|^2}{2}} \leq (\frac{k2\pi}{p})^{\frac{1}{p}} ||f||_p < \infty.$$

Therefore, $\mathcal{F}_p \subseteq \mathcal{F}_q$. □

We next recall some preliminary results on the schatten class of an operator.If T is compact operator on a separable Hilbert space H, then there exist orthonormal sets α_k and β_k in H such that $\sum_{k=1}^{\infty} \lambda_k \langle x, \alpha_k \rangle \beta_k, x \in H$, where λ_k is the k^{th} singular value of T. We note also that, an operator T is in S_p if and only if T^*T is in $S_{\frac{p}{2}}$, where T^* is adjoint of T, and there exists a sequence of orthonormal basis $\{f_k\}$ in H such that

$$(T^*T)_f = \sum_{k=1}^{\infty} \beta_k \langle f, f_k \rangle f_k,$$

where β_k are singular values of T^*T. For more details on the subject, we refer to the book (Zhu, 2007).

Proposition 4.0.3. *(Mengestie, 2014) Let H be any Hilbert space and T be a bounded operator from $\mathcal{F}_2 \to \mathcal{F}_2$.*
i. If $p \geq 2$ and $T \in S_p$, then $\int_{\mathbb{C}} ||T_{k_z}||_2^p dA(z) < \infty$;
ii. If $0 < p \leq 2$ and $\int_{\mathbb{C}} ||T_{k_z}||_2^p dA(z) < \infty$, then $T \in S_p$.

Proposition 4.0.4. *(Zhu, 2007) Let $0 < p < \infty$. Then a bounded operator T on a Hilbert space H belongs to the schatten class $S_p(H)$ if and only if $\sum_{k=1}^{\infty} ||T_{e_k}||^p < \infty$, for any orthonormal basis $\{e_k\}$ of H.*

4.0.2 Main Results

Before we state our main results, we give some norm estimates of kernel $K_w(z)$ and normalized kernel $k_w(z)$ of $\mathcal{F}_2$. For any orthonormal basis $\{e_k\}$ of $\mathcal{F}_2$, we have

$$K_w(z) = \sum_{k=1}^{\infty} e_k(z)\overline{e_k(w)},$$

which implies that

$$||K_w(z)||_2{}^2 = \sum_{k=1}^{\infty} |e_k(w)|^2.$$

From this we further have,

$$\frac{d^n}{d\overline{w}^n} K_w(z) = \sum_{k=1}^{\infty} e_k(z)\overline{e_k{}^n(w)} \tag{4.0.1}$$

and

$$||\frac{d^n}{d\overline{w}^n} K_w(z)||_2{}^2 = \sum_{k=1}^{\infty} |e_k^n(z)|^2. \tag{4.0.2}$$

By completing the square and making change of variables, we can get the following estimates:

$$\begin{aligned} \int_{\mathbb{C}} |w|^{pn}|k_w(z)|^p dA(w) &= e^{\frac{p|z|^2}{2}} \int_{\mathbb{C}} |w|^{pn} e^{\frac{-p|z-w|^2}{2}} dA(w) \\ &\simeq |z|^{pn} e^{\frac{p|z|^2}{2}}. \end{aligned} \tag{4.0.3}$$

and

$$\begin{aligned} ||\frac{d^n}{d\overline{w}^n} K_w(z)||_2^2 &= \int_{\mathbb{C}} |z|^{2n} |K_w(z)|^2 e^{-|z|^2} dA(z) \\ &\simeq |w|^{2n} e^{|w|^2}. \end{aligned} \tag{4.0.4}$$

Now, we state our first main result.

Theorem 4.0.5. *Let $0 < p < \infty$ and $V_g^{n,m} : \mathcal{F}_2 \to \mathcal{F}_2$ be bounded. Then $V_g^{n,m}$ belongs to the schatten S_p class if and only if $\int_{\mathbb{C}} ||V_g^{n,m} k_z||_2{}^p dA(z) < \infty$.*

Proof. first we assume $V_g^{n,m}$ is in S_p. By Proposition 4.0.3, the assertion holds for $p \geq 2$. Thus, we proceed to show for the case $0 < p < 2$. Since, $V_g^{n,m} \in S_p$, $(V_g^{n,m})^* V_g^{n,m}$ is in $S_{\frac{p}{2}}$ and there exists a sequence of orthonormal basis $\{f_k\}$ in $\mathcal{F}_2$ such that

$$(V_g^{n,m})^* V_g^{n,m} f = \sum_{k=1}^{\infty} \beta_k \langle f, f_k \rangle f_k,$$

where β_k are singular values of $(V_g^{n,m})^* V_g^{n,m}$. Using the derivative characterization in Lemma 4.0.1 and the inclusion property $\mathcal{F}_P \subseteq \mathcal{F}_2$ for $0 < p < 2$, we obtain

$$\begin{aligned} \int_{\mathbb{C}} ||V_g^{n,m} k_z||_2{}^p dA(z) &\simeq \int_{\mathbb{C}} (\int_{\mathbb{C}} \frac{|k_z^m(w)|^2 |g^{(n-m)}(w)|^2}{(1+|w|)^{2n}} e^{-|w|^2} dA(w))^{\frac{p}{2}} dA(z) \\ &\lesssim \int_{\mathbb{C}} (\int_{\mathbb{C}} \frac{|z|^{pm} |k_z(w)|^p |g^{(n-m)}(w)|^p}{(1+|w|)^{pn}} e^{-\frac{p|w|^2}{2}} dA(w)) dA(z). \end{aligned}$$

Applying Fubini's theorem, the above integral is equal to

$$\int_{\mathbb{C}} \frac{|g^{(n-m)}(w)|^p}{(1+|w|)^{pn}} e^{-\frac{p|w|^2}{2}} \int_{\mathbb{C}} |z|^{pm} |k_z(w)|^p dA(z) dA(w).$$

Using the estimates in (4.0.3) and (4.0.4), the above integral is estimated as:

$$\int_{\mathbb{C}} \frac{|g^{(n-m)}(w)|^p}{(1+|w|)^{pn}} e^{-\frac{p|w|^2}{2}} . |w|^{pm} e^{\frac{p|w|^2}{2}} dA(w) \simeq \int_{\mathbb{C}} \frac{|g^{(n-m)}(w)|^p}{(1+|w|)^{pn}} |w|^{pm} \times (\frac{||\frac{d^m}{d\overline{w}^m} k_w||_2^2}{(1+|w|)^{2m} e^{|w|^2}}) dA(w),$$

which by (4.0.2) is equal to:

$$\sum_{k=1}^{\infty} \int_{\mathbb{C}} \frac{|g(n-m)(w)|^p |w|^{pm}}{(1+|w|)^{pn}} (\frac{|f_k^m(w)|^2}{(1+|w|)^{2m} e^{|w|^2}}) dA(w).$$

Applying Holder's Inequality and Lemma 4.0.1, the above sum is bounded by

$$\begin{aligned}
&\textstyle\sum_{k=1}^{\infty} \quad \left(\int_{\mathbb{C}} \frac{|g^{(n-m)}(w)|^2}{(1+|w|)^{2n}} |f_k^{(m)}(w)|^2 e^{-|w|^2} dA(w)\right)^{\frac{p}{2}} \times \left(\int_{\mathbb{C}} \frac{|f_k^{(m)}(w)|^2}{(1+|w|)^{2m}} e^{-|w|^2} dA(w)\right)^{\frac{2-p}{2}} \\
\simeq \quad & \sum_{k=1}^{\infty} \left(\int_{\mathbb{C}} \frac{|g^{(n-m)}(w)|^2}{(1+|w|)^{2n}} |f_k^{(m)}(w)|^2 e^{-|w|^2} dA(w)\right)^{\frac{p}{2}} \times ||f_k||_2{}^{2-p} \\
\lesssim \quad & \sum_{k=1}^{\infty} \langle (V_g^{n,m})^* V_g^{n,m} f_k, f_k \rangle^{\frac{p}{2}} \\
= \quad & \sum_{k=1}^{\infty} \beta_k^{\frac{p}{2}} \\
= \quad & ||(V_g^{n,m})^* V_g^{n,m}||_{s_{\frac{p}{2}}}^{\frac{p}{2}} \\
= \quad & ||V_g^{n,m}||_{s_p}^p < \infty.
\end{aligned}$$

Therefore,

$$\int_{\mathbb{C}} ||V_g^{n,m} k(z)||_2^p dA(z) < \infty.$$

The proof for the case $0 < p < 2$ of the reverse side again follows from Proposition 4.0.3. Thus, we will show for the case $p \geq 2$. If $\int_{\mathbb{C}} ||V_g^{n,m} k(z)||_2^p dA(z) < \infty$, then the operator $V_g^{n,m}$ by Theorem 4.0.3 (Hafiz) is compact. Thus, $V_g^{n,m}$ belongs to S_p if and only if $\sum_{k=1}^{\infty} ||V_g^{n,m} e_k||_2^p < \infty$, for any orthonormal set $\{e_k\}$ of $\mathcal{F}_2$. We proceed to show the above sum is finite. By Lemma 4.0.1,

$$\begin{aligned}
\sum_{k=1}^{\infty} ||V_g^{n,m} e_k||_2^p \quad \simeq \quad & \sum_{k=1}^{\infty} \left(\int_{\mathbb{C}} \frac{|g^{(n-m)}(z)|^2}{(1+|z|)^{2n}} |e_k^{(m)}(z)|^2 e^{-|z|^2} dA(z)\right)^{\frac{p}{2}} \\
\lesssim \quad & \sum_{k=1}^{\infty} \left(\int_{\mathbb{C}} \frac{|g^{(n-m)}(z)|^2 |z|^{2m}}{(1+|z|)^{2n}} e^{-|z|^2} \left(\frac{|e_k^{(m)}(z)|^2}{(1+|z|)^{2m}}\right) dA(z)\right)^{\frac{p}{2}}.
\end{aligned}$$

Applying Holder's inequality, the above sum is bounded from above by,

$$\begin{aligned}
&\textstyle\sum_{k=1}^{\infty} \quad \left(\int_{\mathbb{C}} \frac{|e_k^{(m)}(z)|^2}{(1+|z|)^{2m}} e^{-|z|^2} dA(z)\right)^{\frac{p-2}{2}} \times \left(\int_{\mathbb{C}} \frac{|e_k^{(m)}(z)|^2}{(1+|z|)^{pn}} |g^{(n-m)}(z)|^p |z|^{pm} e^{-|z|^2} dA(z)\right) \\
\simeq \quad & \sum_{k=1}^{\infty} \left(\int_{\mathbb{C}} \frac{|e_k^{(m)}(z)|^2 |g^{(n-m)}(z)|^p}{(1+|z|)^{pn}} |z|^{pm} e^{-|z|^2} dA(z)\right)
\end{aligned}$$

Which by estimate (4.0.2), gives

$$\begin{aligned}
\int_{\mathbb{C}} & \frac{\|\frac{d^m}{d\bar{z}^m}k_z\|_2^2|g^{(n-m)}(z)|^p}{(1+|z|)^{pn}}|z|^{pm}e^{-|z|^2}dA(z) \\
= & \int_{\mathbb{C}} \frac{|g^{(n-m)}(z)|^p}{(1+|z|)^{pn}}(\int_{\mathbb{C}}|w|^{pm}|k_w(z)|^p dA(w))dA(z) \\
= & \int_{\mathbb{C}} \|V_g^{n,m}k_w\|_2^p dA(w) < \infty.
\end{aligned}$$

Therefore, $V_g^{n,m} \in S_p$. □

Theorem 4.0.6. *Let* $0 < p < \infty$. *Then* $V_g^{n,m} : \mathcal{F}_2 \to \mathcal{F}_2$ *belongs to the schatten class* S_p *if and only if the function*

$$\frac{|g^{(n-m)}(z)|}{(1+|z|)^n}|z|^m \in L^p(\mathbb{C}, dA).$$

Proof. Applying Lemma 4.0.1, and subharmonicity of $|z|^{2m}|g^{(n-m)}(z)|^2$, we have

$$\begin{aligned}
\|V_g^{n,m}k_w\|_2^2 & \simeq \int_{\mathbb{C}} \frac{|w|^{2m}||k_w(z)|^2|g^{(n-m)}(z)|^2}{(1+|z|)^{2n}}e^{-|z|^2}dA(z) \\
& \gtrsim \int_{D(z,1)} \frac{|w|^{2m}||k_w(z)|^2|g^{(n-m)}(z)|^2}{(1+|z|)^{2n}}e^{-|z|^2}dA(z) \\
& \simeq \int_{D(z,1)} \frac{|w|^{2m}|g^{(n-m)}(z)|^2}{(1+|z|)^{2n}}dA(z) \\
& \gtrsim \frac{|z|^{2m}|g^{(n-m)}(z)|^2}{(1+|z|)^{2n}},
\end{aligned}$$

where in the last estimate we used the fact that $|z| \simeq |w|$ for $w \in D(w,1)$. Therefore,

$$\|V_g^{n,m}k_w\|_2 \gtrsim \frac{|z|^m|g^{(n-m)}(z)|}{(1+|z|)^n}$$

and hence the conclusion follows from Theorem 4.0.5. On the other hand, for $0 < p \leq 2$, using the inclusion $\mathcal{F}_p \subseteq \mathcal{F}_2$, Fubini's theorem and the estimate in the Lemma 4.0.1, we have

$$\begin{aligned}
\int_{\mathbb{C}} ||V_g^{n,m} k_z||_2^p dA(z) &\simeq \int_{\mathbb{C}} (\int_{\mathbb{C}} \frac{|z|^{2m}||k_z(w)|^2|g^{(n-m)}(w)|^2}{(1+|w|)^{2n}} e^{-|w|^2} dA(w))^{\frac{p}{2}} dA(z) \\
&\lesssim \int_{\mathbb{C}} (\int_{\mathbb{C}} \frac{|z|^{pm}||k_z(w)|^p|g^{(n-m)}(w)|^p}{(1+|w|)^{pn}} e^{-\frac{p}{2}|w|^2} dA(z)) dA(w) \\
&= \int_{\mathbb{C}} \frac{|g^{(n-m)}(w)|^p}{(1+|w|)^{pn}} e^{-\frac{p}{2}|w|^2} (\int_{\mathbb{C}} |z|^{pm}|k_w(w)|^p dA(z)) dA(w) \\
&\simeq \int_{\mathbb{C}} \frac{|g^{(n-m)}(w)|^p}{(1+|w|)^{pn}} dA(w).
\end{aligned}$$

Hence, the conclusion follows from Theorem 4.0.5 above. If $p > 2$, applying Holder's inequality.

$$\begin{aligned}
||V_g^{n,m} k_z||_2^p &\simeq (\int_{\mathbb{C}} \frac{|g^{(n-m)}(w)|^2|z|^{2m}||k_z(w)|^2}{(1+|w|)^{2n}} e^{-|w|^2} dA(w))^{\frac{p}{2}} \\
&\leq (\int_{\mathbb{C}} \frac{|g^{(n-m)}(w)|^p|z|^{pm}||k_z(w)|^2}{(1+|w|)^{pn}} e^{-|w|^2} dA(w)) \times (\int_{\mathbb{C}} |k_z(w)|^2 e^{-|w|^2} dA(w))^{\frac{p-2}{2}} \\
&\simeq \int_{\mathbb{C}} \frac{|g^{(n-m)}(w)|^p|z|^{pm}||k_z(w)|^2}{(1+|w|)^{pn}} e^{-|w|^2} dA(w).
\end{aligned}$$

Thus, using Fubini's theorem, we obtain

$$\begin{aligned}
\int_{\mathbb{C}} ||V_g^{n,m} k_z||_2^p dA(z) &\simeq \int_{\mathbb{C}} \int_{\mathbb{C}} \frac{|g^{(n-m)}(w)|^p|z|^{pm}||k_z(w)|^2}{(1+|w|)^{pn}} e^{-|w|^2} dA(z) dA(w) \\
&= \int_{\mathbb{C}} \frac{|g^{(n-m)}(w)|^p}{(1+|w|)^{pn}} e^{-|w|^2} \int_{\mathbb{C}} |k_z(w)|^2|z|^{pm} dA(z) dA(w) \\
&\simeq \int_{\mathbb{C}} \frac{|g^{(n-m)}(w)|^p|w|)^{pm}}{(1+|w|)^{pn}} dA(w)
\end{aligned}$$

and the conclusion follows from Theorem 4.0.5. □

Corollary 4.0.7. *Suppose V_g is compact on $\mathcal{F}_2$. Then*
(i) V_g belongs to $S_p(\mathcal{F}_2)$, for all $p > 2$.
(ii) V_g belongs to $S_p(\mathcal{F}_2)$, class for $p \leq 2$ only if g is constant function.

Proof. Putting $n = 1$ and $m = 0$ in the above theorem, it follows that $V_g \in S_p(\mathcal{F}_2)$ if and only if

$$\int_{\mathbb{C}} \frac{|g^{'}(z)|^p}{(1+|z|)^p} dA(z) < \infty.$$

Since V_g is compact, $g(z) = az + b, a, b \in \mathbb{C}$.

$$\Rightarrow \int_{\mathbb{C}} \frac{|g^{'}(z)|^p}{(1+|z|)^p} dA(z) = \int_{\mathbb{C}} \frac{|a|^p}{(1+|z|)^p} dA(z),$$

which is clearly finite for $p > 2$. And, for $p \leq 2$ the above integral is finite only if $a = 0$, i.e., g is constant function. $\square$

Chapter 5

Conclusion

This thesis includes a number of results, which characterize generalized integration operators acting between Fock spaces. Our results in chapter 4, which is about the schatten class membership of generalized integration operator on Fock spaces are new and may be applied to study other properties defined whenever the operator is schatten class membership of generalized integration operator on Fock spaces. In addition, our results generalizes some of the results that have been obtained for the Volterra-type integral operators. In particular, it generalizes the result of the works of (Constantin, 2012 and Mengestie, 2013) from Volterra-type integral to the schatten class membership of generalized integration operator on Fock spaces.

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