Cubic Non-Polynomial Spline on Piecewise Mesh for Robin Type Singularly Perturbed Reaction Diffusion Problem

A Research Submitted to the Department of Mathematics, Jimma University in Partial Fulfillment for the Requirements of the Degree of Masters of Science in Mathematics (Numerical Analysis)

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June, 2024

Jimma, Ethiopia

DECLARATION

I undersigned declare that, this research entitled "**Cubic Non-Polynomial Spline on Piecewise Mesh for Robin Type Singularly Perturbed Reaction Diffusion Problem"**is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

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Acknowledgement

Foremost, I express my gratitude to the Almighty for guiding me through every step and enabling me to complete this endeavor. I extend heartfelt thanks to my family, particularly my mother and father, for their unwavering moral support throughout this journey. Special appreciation goes to my mentors, Dr. Tesfaye Aga, and Prof. Gemechis File, for their invaluable support and guidance. Lastly, I am grateful to Jimma University for providing me with the opportunity to pursue this research and for their financial assistance.

Table of Contents Page

List of Tables

List of Figures

Abstract

This thesis introduces a novel approach to solving Robin type singularly perturbed reaction-diffusion problems through cubic non-polynomial spline approximation on a piecewise mesh. The methodology involves discretizing the solution domain using a piecewise mesh size, followed by defining a cubic non-polynomial spline function and obtaining its derivatives. Subsequently, the derivatives in the differential equations are transformed into difference approximations, leading to a linear system of algebraic equations expressed in a three-term recurrence relation, solvable through an elimination algorithm. The stability and consistency of the method are thoroughly investigated to ensure its convergence. Furthermore, numerical model examples are employed to validate the proposed method, with results compared against other methods cited in the literature. The maximum absolute error and order of convergence for each model example are presented to demonstrate the significant advancements offered by the present method.

Chapter One
Introduction

1.1. Background of the Study

Mathematically modeled problems refer to a set of consistent equations intended to describe the particular features or behavior of a physical system that we seek to understand. Physical phenomena modeled by differential equations include the prescribed defined conditions [1]. Any differential equation in which the highest order derivative is multiplied by a small positive parameter is called singularly perturbed differential equation, and the parameter is known as the perturbation parameter. The quantities of interest in many areas of applied mathematics are often to be found as the solution of singularly perturbed problems. Thus, singularly perturbed problems arise very frequently in varied fields of applied mathematics and engineering. For instance, in fluid mechanics, elasticity, hydrodynamics, quantum mechanics, chemical-reactor theory, aerodynamics, plasma dynamics, rarefied-gas dynamics, oceanography, meteorology, modeling of semiconductor devices, diffraction theory and reaction-diffusion processes and many others allied areas, [2, 3, 4, 5].

Owing to the importance of these problems in real life situations, the requisite to develop numerical methods for approximation its solution is valuable. Numerical methods are techniques that are used to approximate mathematical modeled problems. We need approximations since we either cannot solve analytically or the analytical method is intractable, [2]. Mostly, finite-difference methods are a class of numerical techniques for solving differential equations with approximating derivatives by finite difference approximations. This methods obtained by converting differential equations into a system of equation that can be solved by appropriate iterative techniques.

Singularly perturbed problems broadly categorized into reaction-diffusion and convection-diffusion types, accordingly the order of differential equation reduced by two or one when the perturbation parameter approaching to zero, [6, 7, 8, 9]. Also, the problems may be classified in different types regarding the type's defined conditions like initial or boundary (Dirichlet, Robin or mixed conditions). Particularly, the linear second order singularly perturbed reaction-diffusion problems are categorized into with Dirichlet or robin boundary conditions exhibiting twin (both left and right) boundary layers. Thus, in this work, the study will focus on numerical solution of the linear second order singularly perturbed reaction diffusion ordinary differential equation with robin type boundary conditions.

Numerical solutions for the singularly perturbed problems depend on perturbation parameter that causes the solution varies rapidly in some parts and slowly in some other parts of the domain, [7, 8]. Due to this, if one body tries to solve these types of problems using standard numerical methods, then less accurate solutions are obtained. As a result, it should be important to develop appropriate numerical methods for such problems, whose convergence does not affected by the perturbation parameter. Thus, it is necessary to formulate a uniformly convergent numerical method for robin type singularly perturbed reaction diffusion problems.

There are several numerical methods presented for solving the different types of the singularly perturbation reaction-diffusion problems mentioned in the literature, [2, 5, 10, 11, 12, 13, 14]. Most authors in these literatures developed higher order finite difference methods for the singularly perturbation reaction-diffusion problems with Dirichlet boundary conditions.

Mainly, different numerical methods developed for solving the robin type singularly perturbed reaction diffusion problems, [15, 16, 17, 18]. Hence, several numerical approximation techniques were presented to solve singularly perturbed reaction diffusion differential equation with robin-type boundary conditions. Further, the error estimates indicate that the proposed techniques are parameter-uniform second-order convergent and are numerically stable. Furthermore, numerical experiments performed and presented to validate the theoretical results. Henceforth, different finite difference methods have been adopted for solving the robin type singularly perturbed reaction-diffusion problems. However, yet, the obtained accuracy of the numerical solution and the rate of convergence are takes attention to be work more. Therefore, the main objective of this study is to present a higher order uniformly convergent numerical method for solving the robin type singularly perturbed reaction diffusion problems.

1.2 Statement of the Problem

There are various numerical methods presented for solving the linear second order singularly perturbed reaction diffusion differential equations with the robin boundary conditions. From the literatures, we observe that the maximum rate or order of convergence of the developed methods is order two. As far as we observe from the literature review, in robin type singularly perturbed reaction diffusion problems, the existence of small positive parameter in the differential equation and the robin type boundary conditions make it difficult to produce accurate solutions for numerical methods. Thus, providing an alternative numerical approach to these types of problems is remarkable. Therefore, it is necessary to provide uniformly convergent numerical method for the numerical approach to solve the problems under consideration.

The study attempt to answer the following questions:

- How to formulate a numerical scheme using cubic non-polynomial spline for solving singularly perturbed reaction diffusion problems with robin-boundary conditions?
- How to provide the convergence analysis for the formulated scheme?
- To what extent the accuracy and rate of convergence will be yields by the proposed scheme, are more advantageous than the existing methods?

1.3 Objectives of the Study

1.3.1 General objective

The general objective of this study is to develop a cubic non-polynomial spline over piecewise mesh for robin type singularly perturbed reaction diffusion problems.

1.3.1 Specific objectives

The specific objectives of the study are:

- To formulate a numerical scheme using cubic non-polynomial spline for solving singularly perturbed reaction diffusion problems with robin-boundary conditions
- To provide the convergence analysis for the formulated numerical scheme.
- To investigate the more accuracies' and higher rate of convergence of the formulated scheme.

1.4 Delimitation of the Study

This study is delimited to study a cubic non-polynomial spline over piecewise mesh for robin type singularly perturbed reaction diffusion problem.

$$-\varepsilon y''(x) + b(x)y(x) = f(x), \qquad 0 < x < 1, \tag{1.1}$$

subject to the robin boundary conditions:

$$\begin{cases} a_0 y(0) - a_1\sqrt{\varepsilon}\, y'(0) = \eta_1, \\ \\ d_0 y(1) + d_1\sqrt{\varepsilon}\, y'(1) = \eta_2, \end{cases} \tag{1.2}$$

where $0 < \varepsilon << 1$ is the singular perturbation parameter, and $a_0, a_1, d_0, d_1, \eta_0, \eta_1$ with $a_1, d_1 \neq 0$ are given constants. Assume that the functions $b(x)$ and $f(x)$ are sufficiently smooth such that $b_1 \geq b(x) \geq b_0, \ \forall x \in [0, 1]$. The constant terms in robin boundary conditions satisfy $a_j, d_j \geq 0$ and $a_j + d_j > 0, \ j = 0,1$ with $a_0 \neq a_1\sqrt{b(0)}$. Then, the problem in Eqn. (1.1) and (1.2) has a unique solution $y(x) \in C^2(0, 1) \bigcap C^1[0, 1]$ that typically exhibits exponential boundary layers at $x = 0$ and $x = 1$ when the perturbation parameter ε tends to zero.

1.5 Significance of the Study

The outcomes of this study:

- Produce a more accurate uniformly convergent numerical solution for the considered problem.
- Provide a reference material for scholars to develop higher order numerical methods for solving robin type singularly perturbed reaction diffusion problems.
- Help the graduate students to acquire research skills and scientific procedures.

Chapter Two

Review of Related Literature

2.1 Singularly Perturbed Problems

Singular perturbation problem was first introduced by Prandtl [19] during his report on fluid motion with small friction in a seven page report presented at the third international congress of mathematics in Heidelberg in 1904 in which he demonstrated that fluid flow past over a body. Prandtl [19] originally introduces the term ‘boundary layer’ but this term comes in to more general following the work of Wasow [20]. The study of many theoretical and applied problems in science and technology leads to boundary value problems for singularly perturbed differential equations. The second order differential equations in which its second order derivative term is multiplied by a small positive parameter (say ε) is called second-order singularly perturbed differential equations. These equations subject to the defined conditions (initial and or boundary or interval boundary) is known as second order singularly perturbed problems.

Kadalbajoo and Reddy [21] gave survey of various asymptotic and numerical methods developed from 1908−1986 for the determination of approximate solution of singular perturbation problems of various kinds. Kadalbajoo and Patidar extended the work done by Kadalbajoo and Reddy and surveyed the work done by various researchers in the area of singular perturbation from 1984 − 2000. In this work they considered one dimensional problem only and discussed the work done on linear, non-linear, semi-linear and quasi-linear singularly perturbed problems [22].

The area of singular perturbations is a field of increasing interest to applied mathematicians. During the last few years much progress has been made in the theory and in the computer implementation of the numerical treatment of singular perturbation problems. The existence of boundary layer theory in various fields of applied mathematics, for instance, fluid mechanics, fluid dynamics, quantum mechanics, plasticity, chemical-reactor theory, aerodynamics, plasma dynamics, rarefied-gas dynamics, meteorology, diffraction theory, reaction-diffusion processes and radiating flows and other domains of the great world of fluid motion [23].

Generally, second-order singularly perturbed problems broadly can be categorized in terms of linear or nonlinear, convection-diffusion or reaction-diffusion, Dirichlet or robin boundary condition types, [7, 8]. Particularly, linear second order singularly perturbed reaction-diffusion problems have different types depending on boundary conditions (Dirichlet, robin or mixed). Hence, second order singularly perturbed reaction-diffusion boundary value problems exhibits boundary and/ or interior layers, [7, 8]. In this review section, we investigate the solution methods for the class of second-order singularly perturbed differential equations with robin boundary conditions.

2.2 Methods for linear singularly perturbed convection-diffusion problems

In this subsection, we give a brief description of the developed numerical methods for the linear second-order singularly perturbed convection-diffusion problem of the form:

$$\begin{cases} \varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \quad 0 < x < 1, \\ y(0) = A, \\ y(1) = B, \end{cases} \tag{2.1}$$

where $0 < \varepsilon << 1$ is a small positive perturbation parameter, and A and B are given constants. Assumed that the functions $a(x), b(x), f(x)$ are sufficiently smooth with the properties of:

$$a(x) \neq 0, \quad \begin{cases} a(x) \leq \alpha < 0, & \forall x \in [0,1] \\ a(x) \geq \alpha > 0, & \forall x \in [0,1] \end{cases} \tag{2.2}$$

Contained by these conditions, the singularly perturbed convection-diffusion problem in Eq. (2.1) possesses a unique smooth solution with boundary layer respectively on the right or left side of the solution domain regarding Eq. (2.2).

Numerical methods involve the B-spline collocation method and different finite difference schemes with appropriate fitted operator or fitted meshes were concerned in [7, 8, 24]. The solution of these type problem exhibits the boundary layer on the right or left-hand side of the domain depending on the sign of the coefficient in the convection term. The bounds of most methods are established for the derivative of the analytical solution. Moreover, most developed methods are boundary layer resolving as well as second-order uniformly convergent in the maximum norm.

2.3 Methods for singularly perturbed reaction-diffusion with Dirichlet boundary conditions

Also, in this subsection, we give a quick description of the developed numerical methods for the linear second-order singularly perturbed reaction-diffusion problem of the form:

$$\begin{cases} -\varepsilon y''(x) + b(x)y(x) = f(x), \quad 0 < x < 1, \\ y(0) = \alpha, \\ y(1) = \beta, \end{cases} \tag{2.3}$$

where, $0 < \varepsilon < 1$ is a singular perturbation parameter, α and β are given constant numbers. For the existence and uniqueness of the solution, the functions $b(x)$ and $f(x)$ are assumed to be sufficiently smooth and bounded $b(x) \geq \min_{\forall x \in [0,1]} b(x) = b_0 > 0$ for small positive constant b_0.

Most methods like in [5, 11, 12, 13, 14] and others on reaction-diffusion problems presented to solve the problems in Eq. (2.3) were developed on uniform mesh and the methods are finite difference based. Principally, the higher order (fourth, sixth, eighth and tenth) order compact finite difference method is presented for solving linear singularly perturbed reaction–diffusion problems. These methods described by replacing the derivative in the differential equation by finite difference approximations that transformed to linear systems of algebraic equations in the form of a three-term recurrence relation, which can be solved using Thomas algorithm. Both the theoretical error bounds and the numerical rate of convergence have been established for the methods.

Further, most of the solution methodologies presented in [5, 11, 12, 13] to solve Eq. (2.3) were works for the constant coefficient reaction-term. Furthermore, the methods were gives accurate solution when the perturbation and uniform mesh-size parameters are nearly comparable. Hence, standard numerical methods give valuable accurate solution when mesh-size parameter is too smaller than perturbation parameter which is costly to solve algebraic equations. Thus, this implies to think of alternative method to solve Eq. (2.3) for the variable coefficient of the reaction term and when the perturbation parameter is very small compared to non-uniform mesh sizes of the solution domain.

2.4 Methods for nonlinear or semi-linear singularly perturbed reaction-diffusion problems

In this subsection, we give a brief description of the developed numerical methods for the semi-linear second-order singularly perturbed reaction-diffusion problem of the form:

$$\begin{cases} -\varepsilon y''(x) + q(x)y(x) + g(x, y(x)) = 0, \quad 0 < x < 1 \\ y(0) = y_0, \\ y(1) = y_1, \end{cases} \tag{2.4}$$

Where y_0 and y_1 are given constants. Assume that the functions $q(x)$ and the nonlinear term $g(x, y(x))$ are given sufficiently smooth. Also, assume $q(x) + \frac{\partial g(x, y(x))}{\partial y} \geq \beta > 0, \quad \forall x \in [0,1]$ for some constant β to ensure the existence and unique solution with dual boundary layers near the two end points of the solution domain.

Numerical methods like in [2, 24, 25, 26] presented to solve the problems in Eq. (2.4) were developed on uniform and non-uniform meshes. Most of these numerical methods based on finite difference on non-uniform meshes, orthogonal spline and Sine collocation methods on uniform meshes. Also, many of these methods are presented after the considered problem converted in to its linearization form by using quasilinearization technique. Uniform convergence and stability of the difference methods are discussed in the discrete maximum norm. During each presentation of the numerical methods, the numerical results support the theoretical results and illustrate the efficiency and accuracy of the methods compared with the results in the previously existing methods, were provided. However, most of the presented methods except in [2], others are limited to the second-order convergence of the method which gives satisfactory solution. Besides, very few methods have been developed to solve the problem under consideration in Eq. (2.4). Thus, the developed methods for the problem in Eq. (2.4) are at its kid stage special in terms of higher order rate convergence.

2.5 Methods for robin type singularly perturbed reaction-diffusion problems

In this subsection, we try to see the developed numerical methods for the linear second-order robin type singularly perturbed reaction-diffusion problem of the form:

$$
\begin{cases}
-\varepsilon y''(x)+b(x)y(x)=f(x), \qquad 0<x<1, \\
\\
a_0 y(0)-a_1\sqrt{\varepsilon}\,y'(0)=\eta_1, \\
d_0 y(1)+d_1\sqrt{\varepsilon}\,y'(1)=\eta_2,
\end{cases}
\tag{2.5}
$$

where $0<\varepsilon<<1$ is the singular perturbation parameter, and $a_0, a_1, d_0, d_1, \eta_0, \eta_1$ with $a_1,\ d_1 \neq 0$ are given constants. Assume that the functions $b(x)$ and $f(x)$ are sufficiently smooth such that $b_1 \geq b(x) \geq b_0,\ \forall x \in [0, 1]$. The constant terms in robin boundary conditions satisfy $a_j,\ d_j \geq 0$ and $a_j + d_j > 0,\ \ j=0,1$ with $a_0 \neq d_0\sqrt{b_0}$. Then, the problem in Eqs. (1.1) and (1.2) has a unique solution $y(x) \in C^2(0, 1) \cap C^1[0, 1]$ that typically exhibits exponential boundary layers at $x=0$ and $x=1$ when the perturbation parameter ε tends to zero.

Several numerical methods which involve both the cubic spline and classical finite difference schemes are presented to solve the problem in Eq. (2.5) in [15, 16, 17, 18]. The error estimates indicate that the provided methods are parameter-uniform second-order convergent and numerically stable. For each of the formulated methods, numerical experiments performed and presented to validate the theoretical results.

As observed from the more recent existing literatures in [15, 16, 17,] regarding the robin type singularly perturbed reaction diffusion problems described by the governing equation in Eq. (2.5) is solved by using various numerical methods like robust and the family of spline methods that were developed based on uniform and non-uniform or adaptive mesh discretization techniques. It is also observable from these literatures that the maximum rate or order of convergence of the developed methods is order two. However, yet, the obtained accuracy of the numerical solution and the rate of convergence are takes attention to be work more. Therefore, this implies takes attention to think of an alternative method to solve Eq. (2.5) that produces more accurately solution and higher order convergence than the existing mentioned methods.

Chapter Three
Research Methodology

3.1 Study Site and Period

The study was conducted at Jimma University under college of natural science department of mathematics in numerical analysis stream from February 2024-June 2024 G.C.

3.2 Study Design

The study employed both documentary review and experimental design on uniformly convergent numerical method for robin type singularly perturbed reaction-diffusion problems.

3.3 Sources of Information

The relevant source of information for this study were journals, books, published articles, related articles from internets and the experimental results obtained by writing MATLAB code for the proposed numerical scheme.

3.4 Mathematical procedure

In order to achieve the stated specific objectives, the study followed the following procedures:

- Defining the robin type singularly perturbed reaction-diffusion problem.
- Describing the properties of continuous solution.
- Discretizing the solution domain or interval by defining non-uniform mesh widths.
- Formulating a uniformly convergent numerical scheme.
- Establishing the convergence of the formulated scheme.
- Writing MATLAB code for the formulated scheme.
- Provide numerical illustrations to validate the applicability scheme.

Chapter Four

Description of the Method, Result and Discussion

4.1 Description of the problem

In this section, we consider the Robin type singularly perturbed reaction diffusion problem of the form:

$$\begin{cases} Ly(x) \equiv -\varepsilon y''(x) + b(x)y(x) = f(x), \quad 0 < x < 1, \\ B_L y(0) \equiv a_0 y(0) - a_1\sqrt{\varepsilon}y'(0) = \eta_1, \\ B_R y(1) \equiv d_0 y(1) + d_1\sqrt{\varepsilon}y'(1) = \eta_2, \end{cases} \tag{4.1}$$

where $0 < \varepsilon << 1$ is the singular perturbation parameter, and $a_0, a_1, d_0, d_1, \eta_0, \eta_1$ with $a_1, d_1 \neq 0$ are given constants. The differential operators are denoted by L, B_L, B_R.

The functions $b(x)$ and the source $f(x)$ function are assume to be sufficiently smooth functions such that $b_1 \geq b(x) \geq b_0 > 0, \ \forall x \in [0, 1]$. Also, assume that the given constant values given in robin type boundary conditions satisfy $a_j, d_j \geq 0$ and $a_j + d_j > 0, \ j = 0,1$.Then, the problem in Eq. (4.1) has a unique solution $y(x) \in C^2(0, 1) \cap C^1[0, 1]$ that typically exhibits boundary layers both at $x = 0$ and $x = 1$ when the perturbation parameter sufficiently small. The existence and uniqueness of the continuous solution for this defined problem is provided in [15].

4.2 Properties of continuous solution

First, let begin with an initial awareness of the analytical behavior of the exact solution. Due to the boundary layer behavior of the continuous solution, it can be divided into a smooth and a layer parts. The derivative bounds of the solution and its components are used to analyze the discretization techniques on piecewise meshes of the solution domain. The ordinary decomposition is applied to obtain the bounds of the exact solution and its higher-order derivatives. These bounds are used to determine the error estimates of the numerical discretization.

Lemma 1: **[Bounds for the derivatives]** assume that $b(x) \geq b_0 > 0$. Then, for $k = 0,1, \dots, r$, the solution $y(x)$ to the problem of Eq. (4.1) satisfies:

$$\left|y^{(k)}(x)\right| \le C\left(1+\varepsilon^{-\frac{k}{2}}\left(e^{-x\sqrt{\frac{b_0}{\varepsilon}}}+e^{-(1-x)\sqrt{\frac{b_0}{\varepsilon}}}\right)\right),$$

where the order r depends on the smoothness of the functions $b(x)$ and $f(x)$.

The next lemma provides the derivative bounds for the smooth component $v(x)$, and layer component $w(x)$. This lemma will be used to estimate the local truncation error related with formulated difference methods [15].

Lemma 2: **[Bounds for the solution component]** the solution $y(x)$ is decomposed into:

$$y(x) = v(x) + w(x),$$

Where $v(x)$ is the smooth part (regular component), and $w(x)$ is the layer part (interior layer component). The smooth part $v(x)$ satisfies:

$$\begin{aligned}
&Lv(x) = \ f(x), \qquad 0 < x < 1, \\
&B_L v(0) = -B_L w(0), \\
&B_R v(1) = -B_R w(1),
\end{aligned}$$

and the layer part satisfies

$$\begin{aligned}
&Lw(x) = \ 0, \quad 0 < x < 1, \\
&B_L w(0) = \eta_1 - B_L v(0), \\
&B_R w(1) = \eta_2 - B_R v(1).
\end{aligned}$$

The derivative bounds of these components are

$$\begin{aligned}
&\left|v^{(k)}(x)\right| \le C, \\
&\left|w^{(k)}(x)\right| \le C\varepsilon^{-\frac{k}{2}}\left(e^{-x\sqrt{\frac{b_0}{\varepsilon}}}+e^{-(1-x)\sqrt{\frac{b_1}{\varepsilon}}}\right),
\end{aligned}$$

Where $k = 0,1,\ ...\ ,\ 2p+1$, for arbitrary p.

Proof: An asymptotic expansion of the smooth part $v(x)$ produces

$$v(x) = \sum_{j=0}^{2p+1} \varepsilon^{\frac{j}{2}} v_j(x) + \varepsilon^{p+1} v^*_{2p+2}(x).$$

Substitute $v(x)$ in the problem of Eq. (4.1) and then by equating the coefficients of like powers of ε, we get:

$$
\begin{aligned}
&L_0 v_0(x) = f(x),\\
&L_0 v_{2i}(x) = L_1 v_{2i-2}(x), \quad i = 1, \dots, p,\\
&L v_{2p+2}(x) = L_1 v_{2p}(x),\\
&B_L v_{2p+2}(0) = 0 = B_R v_{2p+2}(1),
\end{aligned}
$$

Where $L_0 = b(x)I$ and $L_1 = \frac{d^2}{dx^2}$ with $v_{2i+1}(x) = 0, \ \forall i.$

Hence, following the procedures in [16], and it follows from Lemma 1, we have

$$\left| v^{(k)}(x) \right| \le C, \quad \text{for } k = 0, \dots, 2p+1.$$

Subsequently, the smooth part $v(x)$ fails to satisfy the boundary conditions, a layer part $w(x)$ is familiarized so that the boundary conditions are satisfied. The layer part is additional decomposed as its left hand layer part $w^L(x)$, and right hand layer part $w^R(x)$, such that

$$w(x) = w^L(x) + w^R(x),$$

where

$$w^L(x) = \sum_{j=0}^{2p+1} \varepsilon^{\frac{j}{2}} w_j^L(x) + \varepsilon^{p+1} w_{2p+2}^{*L}(x),$$

$$w^R(x) = \sum_{j=0}^{2p+1} \varepsilon^{\frac{j}{2}} w_j^R(x) + \varepsilon^{p+1} w_{2p+2}^{*R}(x).$$

Let us consider the left segment of the boundary layer and substitute $\xi = \frac{x}{\sqrt{\varepsilon}}$, Then in terms of the new variable ξ, the Taylor's expansion of $b(\xi\sqrt{\varepsilon})$ that following the detail procedures in [16, 15] concludes the step at

$$
\begin{aligned}
&\left| w^L(0) \right| \le C,\\
&\left| w^L(1) \right| \le C\, e^{-\sqrt{\frac{b(0)}{\varepsilon}}}, \qquad \text{and}\\
&\left| w^{L(k)}(1) \right| \le C\, \varepsilon^{\frac{-k}{2}} e^{-x\sqrt{\frac{b(0)}{\varepsilon}}}
\end{aligned}
$$

Similarly, $w^R(x)$ satisfies the boundary conditions for the right segment of the boundary layer. Henceforth the proof concluded.

4.3 Piecewise Mesh Generation

We construct a piecewise mesh that contains more number of nodal points in the layer regions than non-layer region. The domain $[0,1]^N$, $N \geq 4$ is divided into three subintervals, $[0,\tau], [\tau, 1-\tau], [1-\tau, 1]$ where the chosen transition parameter,

$$\tau = \min\left\{\frac{1}{4},\ 2\sqrt{\varepsilon}\ln(N)\right\}, \tag{4.2}$$

denotes the width of the boundary layers. The domain $[0,1]^N$ is obtained by putting a non-uniform mesh with $\frac{N}{4}$ mesh elements in both the layer intervals and a uniform mesh with $\frac{N}{2}$ mesh elements in the outer layer region. A general piecewise mesh

$$[0,1]^N = \{0 = x_0, x_1, x_2, ..., x_N = 1\}$$

with step size will be defined:

$$h_i = x_i - x_{i-1} = \begin{cases} \dfrac{4\tau}{N},\ i = 1, 2, ..., \dfrac{N}{4} \\ \dfrac{2(1-2\tau)}{N},\ i = \dfrac{N}{4}+1, ..., \dfrac{3N}{4}, \\ \dfrac{4\tau}{N},\ i = \dfrac{3N}{4}+1, ..., N. \end{cases} \tag{4.3}$$

4.4 Formulation of the Method (Cubic Non-Polynomial Spline Approximation)

Now, in order to develop the non-polynomial cubic spline approximation for the problems in Eq. (4.1), the interval $[0, 1]$ is divided into N sub-intervals. For this, we introduce the set of grid points from Eq. (4.3) as

$$x_i = x_{i-1} + h_i,\quad \begin{cases} i = 1, 2,\ ...\ , N-1, \\ h_i = x_i - x_{i-1}, \\ 0 = x_0 < x_1 < ... < x_N = 1. \end{cases} \tag{4.4}$$

Let $y(x_i)$ be the exact solution to the problem in Eq. (4.1) , and y_i be an approximation to $y(x_i)$, obtained by the segment $S(x)$ of the spline function passing through the points $(x_i,\ y_i)$ and $(x_{i+1},\ y_{i+1})$. Also, let us consider the non-polynomial cubic spline function $S(x)$ in subinterval $[x_i,\ x_{i+1}],\ i=1,\ ...\ ,N-1$, of the form:

$$S(x)=c_i+d_i(x-x_i)+u_i(e^{\omega(x-x_i)}-e^{-\omega(x-x_i)})+v_i(e^{\omega(x-x_i)}+e^{-\omega(x-x_i)}) \tag{4.5}$$

where, c_i, d_i, u_i, and v_i are unknown coefficients to be determined, and ω is a free parameter, which is used to raise the accuracy of the method.

To determine the unknown coefficient in Eq. (4.5), let us denote:

$$\begin{aligned} &S(x_i)=y_i, && S'(x_i)=m_i, && S''(x_i)=M_i, \\ &S(x_{i+1})=y_{i+1}, && S'(x_{i+1})=m_{i+1}, && S''(x_{i+1})=M_{i+1}, \end{aligned} \tag{4.6}$$

Differentiating Eq. (4.5) successively, we get:

$$S'(x)=d_i+\omega u_i(e^{\omega(x-x_i)}+e^{-\omega(x-x_i)})+\omega v_i(e^{\omega(x-x_i)}-e^{-\omega(x-x_i)}) \tag{4.7}$$

$$S''(x)=\omega^2 u_i(e^{\omega(x-x_i)}-e^{-\omega(x-x_i)})+\omega^2 v_i(e^{\omega(x-x_i)}+e^{-\omega(x-x_i)}) \tag{4.8}$$

Using relations in Eqs. (4.6) and (4.8), we have:

$$\begin{aligned} S''(x_i) &= \omega^2 u_i(e^{\omega(x_i-x_i)}-e^{-\omega(x_i-x_i)})+\omega^2 v_i(e^{\omega(x_i-x_i)}+e^{-\omega(x_i-x_i)})=M_i \\ &= \omega^2 u_i(1-1)+\omega^2 v_i(1+1)=M_i \\ &= 2\omega^2 v_i = M_i \end{aligned}$$

Which results

$$v_i=\frac{M_i}{2\omega^2} \tag{4.9}$$

Again, using the relation in Eq. (4.6), Eq. (4.9) into Eq. (4.5) at the point x_i, yields:

$$
\begin{aligned}
S(x_i) &= c_i + d_i(x_i - x_i) + u_i(e^{\omega(x_i - x_i)} - e^{-\omega(x_i - x_i)}) + v_i(e^{\omega(x_i - x_i)} + e^{-\omega(x_i - x_i)}) = y_i \\
&= c_i + u_i(1-1) + v_i(1+1) = y_i \\
& c_i + 2v_i = y_i
\end{aligned}
$$

and it gives

$$
c_i = y_i - 2v_i = y_i - \frac{M_i}{\omega^2} \tag{4.10}
$$

Using the relation in Eq. (4.6), Eq. (4.8) at the point x_{i+1} and letting $\theta_2 = \omega h_{i+1}$, we have:

$$
S''(x_{i+1}) = \omega^2 u_i(e^{\omega(x_{i+1} - x_i)} - e^{-\omega(x_{i+1} - x_i)}) + \omega^2 v_i(e^{\omega(x_{i+1} - x_i)} + e^{-\omega(x_{i+1} - x_i)}) = M_{i+1}
$$

$$
\begin{aligned}
&\omega^2 u_i(e^{\omega h_{i+1}} - e^{-\omega h_{i+1}}) + \omega^2 v_i(e^{\omega h_{i+1}} + e^{-\omega h_{i+1}}) = M_{i+1} \\
&u_i(e^{\theta_2} - e^{-\theta_2}) + v_i(e^{\theta_2} + e^{-\theta_2}) = \frac{M_{i+1}}{\omega^2} \\
&u_i(e^{\theta_2} - e^{-\theta_2}) = \frac{M_{i+1}}{\omega^2} - v_i(e^{\theta_2} + e^{-\theta_2}), \qquad But,\ v_i = \frac{M_i}{2\omega^2}
\end{aligned}
$$

It gives

$$
u_i = \frac{M_{i+1}}{\omega^2(e^{\theta_2} - e^{-\theta_2})} - \frac{M_i}{2\omega^2}\frac{(e^{\theta_2} + e^{-\theta_2})}{(e^{\theta_2} - e^{-\theta_2})} \tag{4.11}
$$

Also, using the relation in Eq. (4.6), Eq. (4.5) at the point x_{i+1} and for $\theta_2 = \omega h_{i+1}$, we have:

$$
S(x_{i+1}) = c_i + d_i(x_{i+1} - x_i) + u_i(e^{\omega(x_{i+1} - x_i)} - e^{-\omega(x_{i+1} - x_i)}) + v_i(e^{\omega(x_{i+1} - x_i)} + e^{-\omega(x_{i+1} - x_i)}) = y_{i+1}
$$

$$
\begin{aligned}
y_{i+1} &= c_i + d_i h_{i+1} + u_i(e^{\omega h_{i+1}} - e^{-\omega h_{i+1}}) + v_i(e^{\omega h_{i+1}} + e^{-\omega h_{i+1}}) \\
y_{i+1} &= c_i + d_i h_{i+1} + u_i(e^{\theta_2} - e^{-\theta_2}) + v_i(e^{\theta_2} + e^{-\theta_2})
\end{aligned}
$$

$$
\begin{aligned}
y_{i+1} &= y_i - \frac{M_i}{\omega^2} + d_i h_{i+1} + \frac{M_{i+1}}{\omega^2} - \frac{M_i}{2\omega^2}(e^{\theta_2} + e^{-\theta_2}) + \frac{M_i}{2\omega^2}(e^{\theta_2} + e^{-\theta_2}) \\
y_{i+1} - y_i &= d_i h_{i+1} + \frac{M_{i+1} - M_i}{\omega^2}
\end{aligned}
$$

Which results

$$d_i = \frac{y_{i+1} - y_i}{h_{i+1}} - \frac{M_{i+1} - M_i}{h_{i+1}\omega^2} \tag{4.12}$$

Using the continuity condition of the first derivative at x_i, $S'_{\Delta-1}(x_i) = S'_{\Delta}(x_i)$, we obtain:

$$\begin{aligned} &d_{i-1} + \omega u_{i-1}(e^{\omega(x_{i-1}-x_i)} + e^{-\omega(x_{-1}-x_i)}) + \omega v_{i-1}(e^{\omega(x_{i-1}-x_i)} - e^{-\omega(x_{i-1}-x_i)}) \\ &\quad = d_i + \omega u_i(e^{\omega(x_i-x_i)} + e^{-\omega(x_i-x_i)}) + \omega v_i(e^{\omega(x_i-x_i)} - e^{-\omega(x_i-x_i)}), \quad \theta_1 = \omega h_i \end{aligned}$$

$$d_{i-1} + \omega u_{i-1}(e^{-\theta_1} + e^{\theta_1}) + \omega v_{i-1}(e^{-\theta_1} - e^{\theta_1}) = d_i + \omega u_i(1+1) + \omega v_i(1-1)$$

$$d_{i-1} + \omega u_{i-1}(e^{\theta_1} + e^{-\theta_1}) - \omega v_{i-1}(e^{\theta_1} - e^{-\theta_1}) = d_i + 2\omega u_i \tag{4.13}$$

Reducing indices of Eqs. (4.11) and (4.12) by one and substituting into Eq. (4.13) gives:

$$d_{i-1} + \omega u_{i-1}(e^{\theta_1} + e^{-\theta_1}) - \omega v_{i-1}(e^{\theta_1} - e^{-\theta_1}) = d_i + 2\omega u_i$$

$$\begin{aligned} &\frac{y_i - y_{i-1}}{h_i} - \frac{M_i - M_{i-1}}{h_i\omega^2} + \frac{M_i(e^{\theta_1} + e^{-\theta_1})}{\omega(e^{\theta_1} - e^{-\theta_1})} - \frac{M_{i-1}}{2\omega}\frac{(e^{\theta_1} + e^{-\theta_1})^2}{(e^{\theta_1} - e^{-\theta_1})} - \frac{M_{i-1}}{2\omega}(e^{\theta_1} - e^{-\theta_1}) \\ &\quad = \frac{y_{i+1} - y_i}{h_{i+1}} - \frac{M_{i+1} - M_i}{h_{i+1}\omega^2} + \frac{2M_{i+1}}{\omega(e^{\theta_2} - e^{-\theta_2})} - \frac{M_i}{\omega}\frac{(e^{\theta_2} + e^{-\theta_2})}{(e^{\theta_2} - e^{-\theta_2})} \end{aligned}$$

$$\begin{aligned} &\left[\frac{1}{h_i\omega^2} - \frac{(e^{\theta_1} + e^{-\theta_1})^2 + (e^{\theta_1} - e^{-\theta_1})^2}{2\omega(e^{\theta_1} - e^{-\theta_1})}\right]M_{i-1} + \left[\frac{(e^{\theta_1} + e^{-\theta_1})}{\omega(e^{\theta_1} - e^{-\theta_1})} + \frac{(e^{\theta_2} + e^{-\theta_2})}{\omega(e^{\theta_2} - e^{-\theta_2})} - \frac{h_i + h_{i+1}}{h_i h_{i+1}\omega^2}\right]M_i \\ &\quad + \left[\frac{1}{h_{i+1}\omega^2} - \frac{2}{\omega(e^{\theta_2} - e^{-\theta_2})}\right]M_{i+1} = \frac{y_{i+1} - y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i} \end{aligned}$$

After multiplying both sides by $\dfrac{2}{h_i + h_{i+1}}$, and due to the parameters $\theta_1 = \omega h_i$, and $\theta_2 = \omega h_{i+1}$, we have the parameter $\omega = \dfrac{\theta_1 + \theta_2}{h_i + h_{i+1}}$, so that the above equation re-written as:

$$\frac{2}{h_i+h_{i+1}}\left[\frac{y_{i+1}-y_i}{h_{i+1}}-\frac{y_i-y_{i-1}}{h_i}\right]=\alpha M_{i-1}+2\beta M_i+\alpha M_{i+1} \tag{4.14}$$

Where the coefficients are denoted by:

$$\alpha=\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_i\omega^2}-\frac{e^{2\theta_1}+e^{-2\theta_1}}{\omega(e^{\theta_1}-e^{-\theta_1})}\right)=\frac{2}{h_i+h_{i+1}}\left(\frac{1}{h_{i+1}\omega^2}-\frac{2}{\omega(e^{\theta_2}-e^{-\theta_2})}\right),$$
$$\beta=\frac{1}{h_i+h_{i+1}}\left(\frac{(e^{\theta_1}+e^{-\theta_1})}{\omega(e^{\theta_1}-e^{-\theta_1})}+\frac{(e^{\theta_2}+e^{-\theta_2})}{\omega(e^{\theta_2}-e^{-\theta_2})}-\frac{h_i+h_{i+1}}{h_ih_{i+1}\omega^2}\right).$$

As $\theta_1\to 0,\ \theta_2\to 0$ in Eq. (4.14), we get the sum of the two constants, $\alpha+\beta=\frac{1}{2}$ and their values will be determined from local truncation error.

Considering the second order differential equation in Eq. (4.1) at the nodal point x_i and using the relation in Eq. (4.6), corresponding to $M_i=y_i''=S''(x)$ which yields:

$$\begin{aligned}M_i&=\frac{b_iy_i-f_i}{\varepsilon},\\ M_{i-1}&=\frac{b_{i-1}y_{i-1}-f_{i-1}}{\varepsilon},\\ M_{i+1}&=\frac{b_{i+1}y_{i+1}-f_{i+1}}{\varepsilon}.\end{aligned} \tag{4.15}$$

Finally, substituting Eq. (4.15) into Eq. (4.14) and also rearranging yields the finite difference scheme that re-written in three term recurrence relations:

$$\frac{2}{h_i+h_{i+1}}\left[\frac{y_{i+1}-y_i}{h_{i+1}}-\frac{y_i-y_{i-1}}{h_i}\right]=\alpha\left[\frac{b_{i-1}y_{i-1}-f_{i-1}}{\varepsilon}\right]+2\beta\left[\frac{b_iy_i-f_i}{\varepsilon}\right]+\alpha\left[\frac{b_{i+1}y_{i+1}-f_{i+1}}{\varepsilon}\right]$$
$$\frac{2}{h_i+h_{i+1}}\left[\frac{y_{i+1}}{h_{i+1}}-\frac{y_i}{h_{i+1}}-\frac{y_i}{h_i}+\frac{y_{i-1}}{h_i}\right]=\frac{\alpha b_{i-1}y_{i-1}}{\varepsilon}-\frac{\alpha f_{i-1}}{\varepsilon}+\frac{2\beta b_iy_i}{\varepsilon}-\frac{2\beta f_i}{\varepsilon}+\frac{\alpha b_{i+1}y_{i+1}}{\varepsilon}-\frac{\alpha f_{i+1}}{\varepsilon}$$
$$\frac{2y_{i+1}}{h_{i+1}\left(h_i+h_{i+1}\right)}-\frac{2y_i}{h_{i+1}\left(h_i+h_{i+1}\right)}-\frac{2y_i}{h_i\left(h_i+h_{i+1}\right)}+\frac{2y_{i-1}}{h_i\left(h_i+h_{i+1}\right)}-\frac{\alpha b_{i-1}y_{i-1}}{\varepsilon}-\frac{2\beta b_iy_i}{\varepsilon}-\frac{\alpha b_{i+1}y_{i+1}}{\varepsilon}$$
$$=-\frac{\alpha f_{i-1}}{\varepsilon}-\frac{2\beta f_i}{\varepsilon}-\frac{\alpha f_{i+1}}{\varepsilon}$$

$$y_{i-1}\left[\frac{2}{h_i\left(h_i+h_{i+1}\right)}-\frac{\alpha b_{i-1}}{\varepsilon}\right]+y_i\left[\frac{-2}{h_{i+1}\left(h_i+h_{i+1}\right)}-\frac{2}{h_i\left(h_i+h_{i+1}\right)}-\frac{2\beta b_i}{\varepsilon}\right]+y_{i+1}\left[\frac{2}{h_{i+1}\left(h_i+h_{i+1}\right)}-\frac{\alpha b_{i+1}}{\varepsilon}\right]$$
$$=-\frac{\alpha f_{i-1}}{\varepsilon}-\frac{2\beta f_i}{\varepsilon}-\frac{\alpha f_{i+1}}{\varepsilon}$$

Multiplying both sides by $-\varepsilon$ gives:

$$y_{i-1}\left[\frac{-2\varepsilon}{h_i\left(h_i+h_{i+1}\right)}+\alpha b_{i-1}\right]+y_i\left[\frac{2\varepsilon}{h_{i+1}\left(h_i+h_{i+1}\right)}+\frac{2\varepsilon}{h_i\left(h_i+h_{i+1}\right)}+2\beta b_i\right]+y_{i+1}\left[\frac{-2\varepsilon}{h_{i+1}\left(h_i+h_{i+1}\right)}+\alpha b_{i+1}\right]$$
$$=\alpha f_{i-1}+2\beta f_i+\alpha f_{i+1}$$

where,

$$\left[\frac{2\varepsilon}{h_{i+1}\left(h_i+h_{i+1}\right)}+\frac{2\varepsilon}{h_i\left(h_i+h_{i+1}\right)}\right]=\frac{2\varepsilon h_i+2\varepsilon h_{i+1}}{h_i h_{i+1}\left(h_i+h_{i+1}\right)}=\frac{2\varepsilon\left(h_i+h_{i+1}\right)}{h_i h_{i+1}\left(h_i+h_{i+1}\right)}=\frac{2\varepsilon}{h_i h_{i+1}}$$

$$y_{i-1}\left[\frac{-2\varepsilon}{h_i\left(h_i+h_{i+1}\right)}+\alpha b_{i-1}\right]+y_i\left[\frac{2\varepsilon}{h_i h_{i+1}}+2\beta b_i\right]+y_{i+1}\left[\frac{-2\varepsilon}{h_{i+1}\left(h_i+h_{i+1}\right)}+\alpha b_{i+1}\right]=\alpha f_{i-1}+2\beta f_i+\alpha f_{i+1}$$

$$E_i y_{i-1}+F_i y_i+G_i y_{i+1}=H_i,\qquad i=1,2,\ldots,N-1 \tag{4.16}$$

where,

$$E_i=\frac{-2\varepsilon}{h_i\left(h_i+h_{i+1}\right)}+\alpha b_{i-1},\qquad F_i=\frac{2\varepsilon}{h_i h_{i+1}}+2\beta b_i,$$
$$G_i=\frac{-2\varepsilon}{h_{i+1}\left(h_i+h_{i+1}\right)}+\alpha b_{i+1},\qquad H_i=\alpha f_{i-1}+2\beta f_i+\alpha f_{i+1}.$$

Truncation error (To determine the values of α and β)

For the formulated cubic spline method, the truncation error is obtained from Eq. (4.14) as:

$$T_i = \frac{2}{h_i + h_{i+1}}\left[\frac{y_{i+1} - y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}\right] - (\alpha M_{i-1} + 2\beta M_i + \alpha M_{i+1}) \tag{4.17}$$

From the relation in Eq. (4.6) and by expanding Eq. (4.17) in Taylor's series about x_i, we have the truncation error term:

$$\begin{aligned} T_i = \frac{2}{h_i + h_{i+1}}\Bigg[& \frac{1}{h_{i+1}}\left\{ y_i + h_{i+1}y_i' + \frac{h_{i+1}^2}{2}y_i'' + \frac{h_{i+1}^3}{3!}y_i''' + \frac{h_{i+1}^4}{4!}y_i^{(4)} + \frac{h_{i+1}^5}{5!}y_i^{(5)} + \frac{h_{i+1}^6}{6!}y_i^{(6)} + \frac{h_{i+1}^7}{7!}y_i^{(7)} + \frac{h_{i+1}^8}{8!}y_i^{(8)} + \ldots - y_i \right\} \\ & - \frac{1}{h_i}\left\{ y_i - \left[y_i - h_i y_i' + \frac{h_i^2}{2}y_i'' - \frac{h_i^3}{3!}y_i''' + \frac{h_i^4}{4!}y_i^{(4)} - \frac{h_i^5}{5!}y_i^{(5)} + \frac{h_i^6}{6!}y_i^{(6)} - \frac{h_i^7}{7!}y_i^{(7)} + \frac{h_i^8}{8!}y_i^{(8)} + \ldots \right] \right\} \Bigg] \\ & - \alpha\left\{ y_i'' - h_i y_i''' + \frac{h_i^2}{2}y_i^{(4)} - \frac{h_i^3}{3!}y_i^{(5)} + \frac{h_i^4}{4!}y_i^{(6)} - \frac{h_i^5}{5!}y_i^{(7)} + \ldots \right\} - 2\beta y_i'' \\ & - \alpha\left\{ y_i'' + h_{i+1}y_i''' + \frac{h_{i+1}^2}{2}y_i^{(4)} + \frac{h_{i+1}^3}{3!}y_i^{(5)} + \frac{h_{i+1}^4}{4!}y_i^{(6)} + \frac{h_{i+1}^5}{5!}y_i^{(7)} + \ldots \right\}. \end{aligned}$$

This can be simplified in the form of:

$$\begin{aligned} T_i = & \left(1 - 2(\alpha + \beta)\right)y_i'' + \left(h_{i+1} - h_i\right)\left\{\frac{1}{3} - \alpha\right\}y_i''' + \left\{\frac{2}{h_i + h_{i+1}}\left(\frac{h_{i+1}^3 + h_i^3}{4!}\right) - \frac{\alpha}{2}\left(h_{i+1}^2 + h_i^2\right)\right\}y_i^{(4)} + \\ & \left\{\frac{2}{h_i + h_{i+1}}\left(\frac{h_{i+1}^4 - h_i^4}{5!}\right) - \frac{\alpha}{3!}\left(h_{i+1}^3 - h_i^3\right)\right\}y_i^{(5)} + \left\{\frac{2}{h_i + h_{i+1}}\left(\frac{h_{i+1}^6 + h_i^6}{6!}\right) - \frac{\alpha}{4!}\left(h_{i+1}^4 + h_i^4\right)\right\}y_i^{(6)} + \ldots \end{aligned} \tag{4.18}$$

In order to obtain the higher order method, and for sufficiently cases of the mesh lengths, we choose the value of α and β as:

$$\begin{cases} \alpha+\beta=\dfrac{1}{2}, \\ \dfrac{1}{3}-\alpha=0. \end{cases}, \alpha=\frac{1}{3}, \beta=\frac{1}{6},$$

and

$$\begin{cases} \alpha+\beta=\dfrac{1}{2}, \\ \dfrac{1}{12}-\alpha=0. \end{cases}, \alpha=\frac{1}{12}, \beta=\frac{5}{12},$$

This yields the second order and fourth order convergent respectively.

Boundary Conditions (End Conditions)

For $i=0,$ at $x_0=0$, we have the discretized form of the first boundary condition, $B_L y(0)$ as

$$a_0 y_0 - a_1\sqrt{\varepsilon}\, y_0' = \eta_1 \tag{4.19}$$

In order to formulate the finite difference approximation to Eq. (4.19), adapting the Taylor series expansion,

$$y_{i+1} = y_i + h_{i+1}y_i' + \frac{h_{i+1}^2}{2}y_i'' + \frac{h_{i+1}^3}{3!}y_i''' + \frac{h_{i+1}^4}{4!}y_i^{(4)} + \frac{h_{i+1}^5}{5!}y_i^{(5)} + \frac{h_{i+1}^6}{6!}y_i^{(6)} + \ldots \tag{4.20}$$

Considering Eq. (4.20) at the nodal point x_0, to approximation the first derivative in boundary condition, we obtain

$$\begin{aligned} y_1 &= y_0 + h_1 y_0' + \frac{h_1^2}{2}y_0'' + \frac{h_1^3}{3!}y_0''' + \frac{h_1^4}{4!}y_0^{(4)} + \frac{h_1^5}{5!}y_0^{(5)} + \frac{h_1^6}{6!}y_0^{(6)} + \ldots \\ y_0' &= \frac{y_1-y_0}{h_1} - \frac{h_1}{2}y_0'' - \frac{h_1^2}{3!}y_0''' - \frac{h_1^3}{4!}y_0^{(4)} - \frac{h_1^4}{5!}y_0^{(5)} - \frac{h_1^5}{6!}y_0^{(6)} + \ldots \\ y_0' &= \frac{y_1-y_0}{h_1} - \frac{h_1}{2}y_0'' - \frac{h_1^2}{3!}y_0''' - \frac{h_1^3}{4!}y_0^{(4)} + TB_L, \end{aligned} \tag{4.21}$$

where the local truncation error for the left boundary condition is

$$TB_L = -\frac{h_1^4}{5!} y_0^{(5)} - \frac{h_1^5}{6!} y_0^{(6)} + \dots \quad .$$

Differentiating the second order differential equation Eq. (4.1) twice with respect to the independent variable and considering it at the discretized points, we obtain

$$\begin{aligned} y_i'' &= \frac{b_i y_i - f_i}{\varepsilon} \\ y_i''' &= \frac{b_i y_i' + b_i' y_i - f_i'}{\varepsilon}, \\ y_i^{(4)} &= \frac{b_i y_i'' + 2b_i' y_i' + b_i'' y_i - f_i''}{\varepsilon}. \end{aligned} \tag{4.22}$$

Hence, from Eq. (4.22) the values of y_0'', y_0''', $y_0^{(4)}$, obtained at the nodal point x_0 as:

$$\begin{aligned} y_0'' &= \frac{b_0 y_0 - f_0}{\varepsilon}, \\ y_0''' &= \frac{b_0 y_0' + b_0' y_0 - f_0'}{\varepsilon}, \\ y_0^{(4)} &= \frac{b_0 y_0'' + 2b_0' y_0' + b_0'' y_0 - f_0''}{\varepsilon}. \end{aligned} \tag{4.23}$$

Substituting Eq. (4.23) into Eq. (4.21) and rearranging yields

$$y_0' = \frac{y_1 - y_0}{h_1} - \frac{h_1}{2} y_0'' - \frac{h_1^2}{3!}\left\{\frac{b_0 y_0' + b_0' y_0 - f_0'}{\varepsilon}\right\} - \frac{h_1^3}{4!}\left\{\frac{b_0 y_0'' + 2b_0' y_0' + b_0'' y_0 - f_0''}{\varepsilon}\right\},$$

$$y_0' = \frac{y_1 - y_0}{h_1} - \left\{\frac{h_1^2 b_0'}{3!\varepsilon} + \frac{h_1^3 b_0''}{4!\varepsilon}\right\} y_0 - \left\{\frac{h_1^2 b_0}{3!\varepsilon} + \frac{2h_1^3 b_0'}{4!\varepsilon}\right\} y_0' - \left\{\frac{h_1^3 b_0}{4!\varepsilon} + \frac{h_1}{2}\right\} y_0'' + \frac{h_1^2}{3!\varepsilon} f_0' + \frac{h_1^3}{4!\varepsilon} f_0'', \tag{4.24}$$

$$y_0' = \frac{\frac{1}{h_1} y_1 - \left\{\frac{h_1^2 b_0'}{3!\varepsilon} + \frac{h_1^3 b_0''}{4!\varepsilon} + \frac{b_0}{\varepsilon}\left(\frac{h_1^3 b_0}{4!\varepsilon} + \frac{h_1}{2}\right) + \frac{1}{h_1}\right\} y_0 + \frac{1}{\varepsilon}\left\{\frac{h_1^3 b_0}{4!\varepsilon} + \frac{h_1}{2}\right\} f_0 + \frac{h_1^2}{3!\varepsilon} f_0' + \frac{h_1^3}{4!\varepsilon} f_0''}{1 + \frac{h_1^2 b_0}{3!\varepsilon} + \frac{2h_1^3 b_0'}{4!\varepsilon}}.$$

Hence, for $i = 0$, substituting Eq. (4.24) into Eq. (4.19), we get the scheme:

$$\left[a_0(1+\frac{h_1^2b_0}{3!\varepsilon}+\frac{2h_1^3b_0'}{4!\varepsilon})+a_1\sqrt{\varepsilon}\left\{\frac{h_1^2b_0'}{3!\varepsilon}+\frac{h_1^3b_0''}{4!\varepsilon}+\frac{b_0}{\varepsilon}(\frac{h_1^3b_0}{4!\varepsilon}+\frac{h_1}{2})+\frac{1}{h_1}\right\}\right]y_0-\frac{a_1\sqrt{\varepsilon}}{h_1}y_1$$
$$=\eta_1(1+\frac{h_1^2b_0}{3!\varepsilon}+\frac{2h_1^3b_0'}{4!\varepsilon})+a_1\sqrt{\varepsilon}\left\{\frac{1}{\varepsilon}\left\{\frac{h_1^3b_0}{4!\varepsilon}+\frac{h_1}{2}\right\}f_0+\frac{h_1^2}{3!\varepsilon}f_0'+\frac{h_1^3}{4!\varepsilon}f_0''\right\}. \tag{4.25}$$

Similarly, for $i=N,$ at x_N, we have the discretized form of the second boundary condition, $B_R y(1)$ as

$$d_0y_N+d_1\sqrt{\varepsilon}y_N'=\eta_2. \tag{4.26}$$

At any nodal point x_i, the Taylors' series expansion written as

$$y_{i-1}=y_i-h_iy_i'+\frac{h_i^2}{2}y_i''-\frac{h_i^3}{3!}y_i'''+\frac{h_i^4}{4!}y_i^{(4)}-\frac{h_i^5}{5!}y_i^{(4)}+\frac{h_i^6}{6!}y_i^{(6)}+\ldots \tag{4.27}$$

Then, adapting this expansion for the nodal point x_N, to approximation the first derivative in the second boundary condition, we get:

$$\begin{aligned}
y_{N-1}&=y_N-h_Ny_N'+\frac{h_N^2}{2}y_N''-\frac{h_N^3}{3!}y_N'''+\frac{h_N^4}{4!}y_N^{(4)}-\frac{h_N^5}{5!}y_N^{(4)}+\frac{h_N^6}{6!}y_N^{(6)}+\ldots\\
y_N'&=\frac{y_N-y_{N-1}}{h_N}+\frac{h_N}{2}y_N''-\frac{h_N^2}{3!}y_N'''+\frac{h_N^3}{4!}y_N^{(4)}-\frac{h_N^4}{5!}y_N^{(5)}+\frac{h_N^5}{6!}y_N^{(6)}+\ldots\\
y_N'&=\frac{y_N-y_{N-1}}{h_N}+\frac{h_N}{2}y_N''-\frac{h_N^2}{3!}y_N'''+\frac{h_N^3}{4!}y_N^{(4)}+TB_R,
\end{aligned} \tag{4.28}$$

where the local truncation error for the second boundary condition is

$$TB_R=-\frac{h_N^4}{5!}y_N^{(5)}+\frac{h_N^5}{6!}y_N^{(6)}+\ldots\ .$$

Similar to Eq. (4.23), from Eq. (4.22) at the nodal point x_N, considering the values of y_N'', y_N''', $y_N^{(4)}$, and also substituting it into Eq. (4.28), we get

$$y'_N = \frac{y_N - y_{N-1}}{h_N} + \frac{h_N}{2} y''_N - \frac{h_N^2}{3!} y'''_N + \frac{h_N^3}{4!} y_N^{(4)},$$

$$y'_N = \frac{y_N - y_{N-1}}{h_N} + \frac{h_N}{2} y''_N - \frac{h_N^2}{3!} \left\{ \frac{b_N y'_N + b'_N y_N - f'_N}{\varepsilon} \right\} + \frac{h_N^3}{4!} \left\{ \frac{b_N y''_N + 2b'_N y'_N + b''_N y_N - f''_N}{\varepsilon} \right\},$$

$$y'_N = \frac{\frac{-y_{N-1}}{h_N} + \left\{ \frac{1}{h_N} - \frac{h_N^2 b'_N}{3!\varepsilon} + \frac{h_N^3 b''_N}{4!\varepsilon} + \frac{h_N b_N}{2\varepsilon} + \frac{h_N^3 b_N^2}{4!\varepsilon^2} \right\} y_N - (\frac{h_N}{2\varepsilon} + \frac{h_N^3 b_N}{4!\varepsilon^2}) f_N + \frac{h_N^2 f'_N}{3!\varepsilon} - \frac{h_N^3 f''_N}{4!\varepsilon}}{1 + \frac{h_N^2 b_N}{3!\varepsilon} - \frac{2h_N^3 b'_N}{4!\varepsilon}}.$$

Now, substituting this value of y'_N in Eq. (4.29) into Eq. (4.26) gives the scheme

$$\begin{aligned} &\frac{-d_1\sqrt{\varepsilon}}{h_N} y_{N-1} + \left[d_0 (1 + \frac{h_N^2 b_N}{3!\varepsilon} - \frac{2h_N^3 b'_N}{4!\varepsilon}) + d_1\sqrt{\varepsilon} (\frac{1}{h_N} - \frac{h_N^2 b'_N}{3!\varepsilon} + \frac{h_N^3 b''_N}{4!\varepsilon} + \frac{h_N b_N}{2\varepsilon} + \frac{h_N^3 b_N^2}{4!\varepsilon^2}) \right] y_N \\ &= \eta_2 (1 + \frac{h_N^2 b_N}{3!\varepsilon} - \frac{2h_N^3 b'_N}{4!\varepsilon}) + d_1\sqrt{\varepsilon} (\frac{h_N}{2\varepsilon} + \frac{h_N^3 b_N}{4!\varepsilon^2}) f_N - d_1\sqrt{\varepsilon} \frac{h_N^2 f'_N}{3!\varepsilon} + d_1\sqrt{\varepsilon} \frac{h_N^3 f''_N}{4!\varepsilon}. \end{aligned} \tag{4.30}$$

To conclude formulation of the method, the three equations in Eq. (4.16), Eq. (4.25) and Eq. (4.30) constitute the system of $(N+1)\times(N+1)$ linear algebraic equations, that give the approximations $y_0, y_1, \dots, y_N$ of the solution $y(x)$ at the nodal points, $x_0, x_1, \dots, x_N$, with piecewise mesh length h_i. This can be re-written in matrix form

$$MY = R, \tag{4.31}$$

where the three matrices are defined

$$M = (m_{ij})_{(N+1)\times(N+1)} = \begin{bmatrix} B_0 & C_0 & 0 & \cdots & 0 \\ A_1 & B_1 & C_1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & A_{N-1} & B_{N-1} & C_{N-1} \\ 0 & \cdots & 0 & A_N & B_N \end{bmatrix}, \quad Y = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \\ y_N \end{bmatrix}, \quad R = \begin{bmatrix} H_0 \\ H_1 \\ \vdots \\ H_{N-1} \\ H_N \end{bmatrix},$$

with the specific entries are denoting:

For $i=0$, $B_0 = a_0(1+\frac{h_1^2 b_0}{3!\varepsilon}+\frac{2h_1^3 b_0'}{4!\varepsilon})+a_1\sqrt{\varepsilon}(\frac{h_1^2 b_0'}{3!\varepsilon}+\frac{h_1^3 b_0''}{4!\varepsilon}+\frac{b_0}{\varepsilon}(\frac{h_1^3 b_0}{4!\varepsilon}+\frac{h_1}{2})+\frac{1}{h_1})$,

$$C_0 = -\frac{a_1\sqrt{\varepsilon}}{h_1},$$

$$H_0 = \eta_1(1+\frac{h_1^2 b_0}{3!\varepsilon}+\frac{2h_1^3 b_0'}{4!\varepsilon})+a_1\sqrt{\varepsilon}\left\{\frac{1}{\varepsilon}(\frac{h_1^3 b_0}{4!\varepsilon}+\frac{h_1}{2})f_0+\frac{h_1^2}{3!\varepsilon}f_0'+\frac{h_1^3}{4!\varepsilon}f_0''\right\},$$

For $i=1, \dots, N-1$: $A_i = E_i$, $B_i = F_i$, $C_i = G_i$, and $H_i = \alpha f_{i-1}+2\beta f_i+\alpha f_{i+1}$.

For $i=N$: $A_N = \frac{-d_1\sqrt{\varepsilon}}{h_N}$,

$$B_N = d_0(1+\frac{h_N^2 b_N}{3!\varepsilon}-\frac{2h_N^3 b_N'}{4!\varepsilon})+d_1\sqrt{\varepsilon}(\frac{1}{h_N}-\frac{h_N^2 b_N'}{3!\varepsilon}+\frac{h_N^3 b_N''}{4!\varepsilon}+\frac{h_N b_N}{2\varepsilon}+\frac{h_N^3 b_N^2}{4!\varepsilon^2}),$$

$$H_N = \eta_2(1+\frac{h_N^2 b_N}{3!\varepsilon}-\frac{2h_N^3 b_N'}{4!\varepsilon})+d_1\sqrt{\varepsilon}(\frac{h_N}{2\varepsilon}+\frac{h_N^3 b_N}{4!\varepsilon^2})f_N - d_1\sqrt{\varepsilon}\frac{h_N^2 f_N'}{3!\varepsilon}+d_1\sqrt{\varepsilon}\frac{h_N^3 f_N''}{4!\varepsilon}.$$

Algorithm and its Stability

The obtained finite difference approximation yields an $(N+1)\times(N+1)$ algebraic tri-diagonal system that can be solved by Gaussian elimination method. Hence, in using this method, we have the algorithm:

$$\text{Forward Elimination} \rightarrow \begin{cases} W_0 = \frac{C_0}{B_0}, \\ T_0 = \frac{R_0}{B_0}, \\ W_i = \frac{C_i}{B_i - A_i W_{i-1}}, \quad i=1,2, \dots, N+1, \\ T_i = \frac{R_i - A_i T_{i-1}}{B_i - A_i W_{i-1}}, \quad i=1,2, \dots, N+1. \end{cases}$$

$$\text{Backward Elimination} \rightarrow \begin{cases} \text{Setting } C_N = 0, \\ y_{N+1} = T_{N+1}, \\ y_i = T_i - W_i y_{i+1}, \quad i=N, N-1, \dots 2, 1. \end{cases}$$

Theorem 1: (Stability)

Let M be a coefficient matrix of the tri-diagonal system, in Eq. (4.31). Then, the matrix M is an irreducible and diagonally dominant matrix and hence the scheme is stable. [27]

Proof: To estimate the stability of the formulated method, we show that the coefficient matrix of the method satisfy the diagonal dominance. First, to check this, consider that $b(x) \geq b_0 > 0,\ \forall x \in [0, 1]$ and for $i = 1, \ldots, N-1$, from Eq. (4.16), we have:

$$
\begin{aligned}
&B_i \geq A_i + C_i, \quad (F_i \geq E_i + G_i), \quad i = 1, 2, \ldots, N-1, \\
&\frac{2\varepsilon}{h_i h_{i+1}} + 2\beta b_i \geq \frac{-2\varepsilon}{h_i\left(h_i + h_{i+1}\right)} + \alpha b_{i-1} + \frac{-2\varepsilon}{h_{i+1}\left(h_i + h_{i+1}\right)} + \alpha b_{i+1}, \\
&\frac{2\varepsilon}{h_i h_{i+1}} + 2\beta b_0 \geq \frac{-2\varepsilon}{h_i\left(h_i + h_{i+1}\right)} + \alpha b_0 + \frac{-2\varepsilon}{h_{i+1}\left(h_i + h_{i+1}\right)} + \alpha b_0, \\
&\frac{2\varepsilon}{h_i h_{i+1}} + \frac{5b_0}{6} > \frac{-2\varepsilon}{h_i h_{i+1}} + \frac{b_0}{12}.
\end{aligned}
$$

Second, we consider the Robin boundary conditions at the two extremities $i = 0$ and $i = N$.

For $i = 0$, $B_0 > C_0$,

$$
B_0 = a_0\left(1 + \frac{h_1^2 b_0}{3!\varepsilon} + \frac{2h_1^3 b_0'}{4!\varepsilon}\right) + a_1\sqrt{\varepsilon}\left(\frac{h_1^2 b_0'}{3!\varepsilon} + \frac{h_1^3 b_0''}{4!\varepsilon} + \frac{b_0}{\varepsilon}\left(\frac{h_1^3 b_0}{4!\varepsilon} + \frac{h_1}{2}\right) + \frac{1}{h_1}\right), \quad C_0 = -\frac{a_1\sqrt{\varepsilon}}{h_1}.
$$

For $i = N$: $B_N > A_N$,

$$
A_N = \frac{-d_1\sqrt{\varepsilon}}{h_N}, \quad B_N = d_0\left(1 + \frac{h_N^2 b_N}{3!\varepsilon} - \frac{2h_N^3 b_N'}{4!\varepsilon}\right) + d_1\sqrt{\varepsilon}\left(\frac{1}{h_N} - \frac{h_N^2 b_N'}{3!\varepsilon} + \frac{h_N^3 b_N''}{4!\varepsilon} + \frac{h_N b_N}{2\varepsilon} + \frac{h_N^3 b_N^2}{4!\varepsilon^2}\right).
$$

Thus, this implies that for each row of M, the sum of the two off-diagonal elements is less than or equal to the modulus of the diagonal element. Therefore, M is diagonally dominant. Hence, M is irreducible matrix. Henceforth, the Gaussian elimination algorithm is used for solving the formulated scheme in Eq. (4.31) is stable for the described scheme.

Error Analysis

In order to analyze the convergence behavior of the numerical approximation approach made by the proposed technique, we need to investigate its order of consistency. For this determination, similar to the decomposition of the exact solution $y(x)$, the corresponding numerical approximation is also decomposed into smooth and layer parts as

$$y_i = V_i + W_i, \quad i = 0,1, \ \dots \ , N.$$

The smooth part satisfies

$$\begin{cases} L^N V_i = f(x_i), \\ B_L V_0 = -B_L w(0), \\ B_R V_N = -B_R w(1), \end{cases} \tag{4.32}$$

while the layer part satisfies

$$\begin{cases} L^N W_i = 0, \\ B_L W_0 = \eta_1 - B_L v(0), \\ B_R W_N = \eta_2 - B_R v(1). \end{cases} \tag{4.33}$$

An application of the triangular inequality to the error at the nodes yields

$$\left| y(x_i) - y_i \right| \le \left| v(x_i) - V_i \right| + \left| w(x_i) - W_i \right|. \tag{4.34}$$

Therefore, the consistency error of the numerical approximation can be obtained from the error bounds of the smooth and layer parts. The basics for the error analysis are labeled in the following Lemmas. Hence, to estimate the consistency of the method, we now evaluate the truncation error.

Lemma 3: The error due to smooth part V_i is given by

$$\left| TV_i \right| \le CN^{-4},$$

for arbitrary constant $\boldsymbol{C}$ independent of perturbation parameter ε.

Proof: For $i = 1, 2, \ \dots \ , N-1,$ and for the constants $\alpha = \dfrac{1}{12}, \beta = \dfrac{5}{12},$ from Taylor's series expansion in Eq. (4.18), the smooth part of the truncation error TV_i at any node x_i is estimated as

Proof: For $i=1,2,\ \dots\ ,N-1$, and for the constants $\alpha=\dfrac{1}{12},\ \ \beta=\dfrac{5}{12}$, from the Taylor's series expansion in Eq. (4.18), the smooth part of the truncation error TV_i at any node x_i is estimated as

$$TV_i=\left\{\frac{2}{h_i+h_{i+1}}\left(\frac{h_{i+1}^6+h_i^6}{6!}\right)-\frac{\alpha}{4!}\left(h_{i+1}^4+h_i^4\right)\right\}y_i^{(6)}+\dots$$

$$\left|TV_i\right|\le\frac{1}{288}\left(h_{i+1}^4+h_i^4\right)\left|v_i^{(6)}\right|,\qquad \text{But } h_i=\frac{4\tau}{N},\ h_{i+1}=\frac{2-4\tau}{N},\ (\text{Eq.}(4.3)),$$

$$\text{and } \left|v^{(k)}(x)\right|\le C,\ (\text{Lemma } 2),$$

$$\left|TV_i\right|\le\frac{1}{288}\left[\frac{\left(2-4\tau\right)^4}{N^4}+\frac{\left(4\tau\right)^4}{N^4}\right]C=\frac{1}{288}\left[\frac{16-128\tau+384\tau^2-512\tau^3+512\tau^4}{N^4}\right]C,$$

$$\left|TV_i\right|\le CN^{-4}.$$

For $i=0$, and Eq. (4.21), the smooth local truncation error for the left boundary condition is given by

$$TB_L=-\frac{h_1^4}{5!}y_0^{(5)}-\frac{h_1^5}{6!}y_0^{(6)}+\dots$$

$$\left|TV_0\right|\le\frac{1}{120}h_1^4\left|v_0^{(5)}\right|,\qquad h_1=\frac{4\tau}{N},\ \text{ and } \left|v^{(k)}(x)\right|=\left|v_0^{(5)}\right|\le C_0,\quad .$$

$$\left|TV_0\right|\le\frac{1}{120}\left(\frac{4\tau}{N}\right)^4C_0=CN^{-4}.$$

Similarly, for $i=N$, and Eq. (4.28), the smooth local truncation error for the right boundary condition is given by

$$TB_R=-\frac{h_N^4}{5!}y_N^{(5)}+\frac{h_N^5}{6!}y_N^{(6)}+\dots$$

$$\left|TV_N\right|\le\frac{1}{120}h_N^4\left|v_0^{(5)}\right|,\qquad h_N=\frac{4\tau}{N},\ \text{ and } \left|v^{(k)}(x)\right|=\left|v_N^{(5)}\right|\le C_N,\quad .$$

$$\left|TV_N\right|\le\frac{1}{120}\left(\frac{4\tau}{N}\right)^4C_N=CN^{-4}.$$

Hence the proof.

Lemma 4: the error due to layer part W_i is given by

$|TW_i| \le CN^{-4}$, For arbitrary constant C.

Proof: For $i = 1, 2, ..., N-1$, and for the constants $\alpha = \dfrac{1}{12}, \beta = \dfrac{5}{12}$, from the Taylors series expansion in Eq. (4.18), the layer part of the truncation error TW_i at any node x_i is estimated as

$$TW_i = \left\{ \frac{2}{h_i + h_{i+1}} (\frac{h_{i+1}^6 + h_i^6}{6!}) - \frac{\alpha}{4!}\left(h_{i+1}^4 + h_i^4\right) \right\} y_i^{(6)} + ...$$

$$|TW_i| \le \frac{1}{288}\left(h_{i+1}^4 + h_i^4\right)\left|w_i^{(6)}\right|, \qquad \text{But } h_i = \frac{4\tau}{N},\ h_{i+1} = \frac{2-4\tau}{N}, \text{ (Eq.(4.3))},$$

$$\text{and } \left|w^{(k)}(x)\right| \le C\varepsilon^{-\frac{k}{2}}\left(e^{-x\sqrt{\frac{b_0}{\varepsilon}}} + e^{-(1-x)\sqrt{\frac{b_1}{\varepsilon}}}\right), \text{ (Lemma 2)},$$

$$|TW_i| \le \frac{1}{288}[\frac{16 - 128\tau + 384\tau^2 - 512\tau^3 + 512\tau^4}{N^4}]C\varepsilon^{-3}\left(e^{-x_{i-1}\sqrt{\frac{b_0}{\varepsilon}}} + e^{-(1-x_{i+1})\sqrt{\frac{b_1}{\varepsilon}}}\right),$$

$$|TW_i| \le CN^{-4}.$$

For $i = 0$, and Eq. (4.21), the layer truncation error for the left boundary condition is given by

$$TB_L = -\frac{h_1^4}{5!} y_0^{(5)} - \frac{h_1^5}{6!} y_0^{(6)} + ...$$

$$|TW_0| \le \frac{1}{120} h_1^4 \left|w_0^{(5)}\right|, \quad h_1 = \frac{4\tau}{N}, \text{ and } \left|w^{(k)}(x)\right| \le C\varepsilon^{-\frac{k}{2}}\left(e^{-x\sqrt{\frac{b_0}{\varepsilon}}} + e^{-(1-x)\sqrt{\frac{b_1}{\varepsilon}}}\right), \quad .$$

$$|TW_0| \le \frac{1}{120}\left(\frac{4\tau}{N}\right)^4 C_0\varepsilon^{-\frac{5}{2}} e^{-x_0\sqrt{\frac{b_0}{\varepsilon}}} = \frac{1}{120}\left(\frac{4\tau}{N}\right)^4 C_0\varepsilon^{-\frac{5}{2}} \le CN^{-4}.$$

Similarly, for $i = N$, and Eq. (4.28), the layer truncation error for the right boundary condition is given by

$$TB_R = -\frac{h_N^4}{5!} y_N^{(5)} + \frac{h_N^5}{6!} y_N^{(6)} + ...$$

$$|TW_N| \le \frac{1}{120} h_N^4 \left|w_0^{(5)}\right|, \quad h_N = \frac{4\tau}{N}, \text{ and } \left|w^{(k)}(x)\right| \le C\varepsilon^{-\frac{k}{2}}\left(e^{-x\sqrt{\frac{b_0}{\varepsilon}}} + e^{-(1-x)\sqrt{\frac{b_1}{\varepsilon}}}\right),$$

$$|TW_N| \le \frac{1}{120}\left(\frac{4\tau}{N}\right)^4 C\varepsilon^{-\frac{5}{2}} e^{-(1-1)\sqrt{\frac{b_1}{\varepsilon}}} = \frac{1}{120}\left(\frac{4\tau}{N}\right)^4 C\varepsilon^{-\frac{5}{2}} \le CN^{-4}.$$

Hence the proof.

4.5 Numerical Examples and Results

In order to test the validity of the proposed method and to demonstrate their convergence computationally, we have taken different model examples. The maximum absolute error is evaluated by:

$$E_\varepsilon^N = \max_{x_i \in [0,1]^N} | y(x_i) - y_i |,$$

Where $y(x_i)$ and y_i respectively, denotes the exact and approximate solutions. Further, the order of convergence calculated by the formula:

$$R = \frac{\log(E_\varepsilon^N) - \log(E_\varepsilon^{2N})}{\log(2)}.$$

Example 1: Consider the singularly perturbed problem [15]

$$\begin{cases} -\varepsilon y''(x) + (1 + x(1-x))y(x) = f(x), \quad 0 < x < 1, \\ y(0) - \sqrt{\varepsilon}y'(0) = -1 - \sqrt{\varepsilon} + e^{\frac{-1}{\sqrt{\varepsilon}}}, \\ y(1) + \sqrt{\varepsilon}y'(1) = -1 - \sqrt{\varepsilon} + e^{\frac{-1}{\sqrt{\varepsilon}}}. \end{cases}$$

.

The source function is given by

$$f(x) = 1 + x(1-x) + [2\sqrt{\varepsilon} - x(1-x)^2]e^{\frac{-x}{\sqrt{\varepsilon}}} + [2\sqrt{\varepsilon} - x^2(1-x)]e^{\frac{-(1-x)}{\sqrt{\varepsilon}}}.$$

The exact solution to the problem that satisfies the above boundary conditions is given by

$$y(x) = 1 + (x-1)e^{\frac{-x}{\sqrt{\varepsilon}}} - xe^{\frac{-(1-x)}{\sqrt{\varepsilon}}}.$$

Example 2: Consider the singularly perturbed problem [16]

$$\begin{cases} -\varepsilon y''(x)+\dfrac{4(1+\sqrt{\varepsilon}(1+x))}{(1+x)^4}y(x)=f(x), \quad 0<x<1, \\ y(0)-\sqrt{\varepsilon}y'(0)=2+\dfrac{6}{1-e^{\frac{-1}{\sqrt{\varepsilon}}}}, \\ y(1)+\sqrt{\varepsilon}y'(1)=\dfrac{-2-e^{\frac{-1}{\sqrt{\varepsilon}}}}{2(1-e^{\frac{-1}{\sqrt{\varepsilon}}})}, \end{cases}$$

.
with the source function is

$$f(x)=\frac{-4}{(1+x)^4}\left[(1+\sqrt{\varepsilon}(1+x)+4\pi^2\varepsilon)\cos(\frac{4\pi x}{1+x})-2\pi\varepsilon(1+x)\sin(\frac{4\pi x}{1+x})+3(1+\sqrt{\varepsilon}(1+x))\frac{e^{\frac{-1}{\sqrt{\varepsilon}}}}{1-e^{\frac{-1}{\sqrt{\varepsilon}}}}\right].$$

The mixed boundary conditions are chosen in a way, such that the exact solution for this problem becomes

$$y(x)=-\cos(\frac{4\pi x}{1+x})+3\frac{e^{\frac{-2x}{(1+x)\sqrt{\varepsilon}}}-e^{\frac{-1}{\sqrt{\varepsilon}}}}{1-e^{\frac{-1}{\sqrt{\varepsilon}}}}.$$

Example 3: Consider the singularly perturbed problem [15]

$$\begin{cases} -\varepsilon y''(x)+y(x)=-\cos^2(\pi x)=2\varepsilon\pi^2\cos(2\pi x), \quad 0<x<1, \\ y(0)-2\sqrt{\varepsilon}y'(0)=2-\dfrac{4e^{\frac{-1}{\sqrt{\varepsilon}}}}{1+e^{\frac{-1}{\sqrt{\varepsilon}}}}, \\ y(1)+3\sqrt{\varepsilon}y'(1)=\dfrac{3(1-e^{\frac{-1}{\sqrt{\varepsilon}}})}{1+e^{\frac{-1}{\sqrt{\varepsilon}}}}. \end{cases}$$

.

The exact solution that satisfies the given boundary conditions is given by

$$y(x) = \frac{e^{\frac{-(1-x)}{\sqrt{\varepsilon}}} + e^{\frac{-x}{\sqrt{\varepsilon}}}}{1 + e^{\frac{-1}{\sqrt{\varepsilon}}}} - \cos^2(\pi x)$$

Example 4: Consider singularly perturbed problem [16]

$$\begin{cases} -\varepsilon y''(x) + y(x) = -\cos^2(\pi x) - 2\varepsilon\pi^2 \cos(2\pi x), \quad x \in (0,1) \\ y(0) - 2\sqrt{\varepsilon}\, y'(0) = 1, \\ y(1) + 3\sqrt{\varepsilon}\, y'(1) = 0. \end{cases}$$

As the exact solution for the Example 4 is not available, so the accuracy of its numerical solution was computed using double mesh principle.

Table 1: Computed maximum absolute errors for Example 1 by the proposed method on uniform mesh when $\alpha = \frac{1}{12}$, $\beta = \frac{5}{12}$

$\varepsilon \downarrow N \rightarrow$	64	128	256	512	1024	2048
10^{-3}	2.9485e-04	2.0460e-05	1.3407e-06	8.5692e-08	5.4143e-09	3.4022e-10
10^{-4}	1.5204e-02	1.4654e-03	1.0970e-04	7.4149e-06	4.8043e-07	3.0548e-08
10^{-6}	6.0163e-01	3.6419e-01	1.4058e-01	2.8215e-02	3.1066e-03	2.4622e-04
10^{-8}	9.5028e-01	9.0287e-01	8.1506e-01	6.6455e-01	4.4341e-01	2.0299e-01
10^{-10}	9.9491e-01	9.8983e-01	9.7975e-01	9.5989e-01	9.2137e-01	8.4894e-01
10^{-12}	9.9949e-01	9.9898e-01	9.9796e-01	9.9591e-01	9.9184e-01	9.8375e-01
10^{-16}	9.9999e-01	9.9999e-01	9.9998e-01	9.9996e-01	9.9992e-01	9.9984e-01

Table 2: Computed maximum absolute errors for Example 1 by the proposed method on piecewise mesh

$\varepsilon \downarrow N \rightarrow$	64	128	256	512	1024	2048
$\alpha = \frac{1}{12}$, $\beta = \frac{5}{12}$						
10^{-3}	2.9485e-04	2.0460e-05	1.3407e-06	8.5692e-08	5.4143e-09	3.4022e-10
10^{-4}	3.2535e-04	4.1312e-05	4.6331e-06	4.7689e-07	4.6128e-08	4.2567e-09
10^{-6}	3.1203e-04	4.7604e-05	4.6331e-06	1.2766e-06	1.3469e-07	1.0771e-08
10^{-8}	3.1070e-04	6.2649e-05	8.7563e-06	3.4607e-06	7.6676e-07	1.5275e-07
10^{-10}	3.1057e-04	6.4374e-05	1.4857e-05	3.8392e-06	9.4147e-07	2.2909e-07

10^{-12}	3.1055e-04	6.4549e-05	1.5662e-05	3.8792e-06	9.6101e-07	2.3864e-07
10^{-16}	3.1055e-04	6.4568e-05	1.5754e-05	3.8836e-06	9.6318e-07	2.3971e-07
$\alpha=\frac{1}{3},\ \beta=\frac{1}{6}$						
10^{-3}	1.6639e-02	4.5091e-03	1.1885e-03	3.0595e-04	7.7666e-05	1.9568e-05
10^{-4}	1.7402e-02	6.3386e-03	2.1634e-03	7.0684e-04	2.2270e-04	6.8177e-05
10^{-6}	1.6980e-02	6.1878e-03	2.1102e-03	6.8962e-04	2.1718e-04	6.6467e-05
10^{-8}	1.6941e-02	6.1728e-03	2.1049e-03	6.8790e-04	2.1663e-04	6.6296e-05
10^{-10}	1.6936e-02	6.1713e-03	2.1043e-03	6.8772e-04	2.1658e-04	6.6279e-05
10^{-12}	1.6936e-02	6.1711e-03	2.1043e-03	6.8771e-04	2.1657e-04	6.6277e-05
10^{-16}	1.6936e-02	6.1711e-03	2.1043e-03	6.8771e-04	2.1657e-04	6.6277e-05

Table 3: Computed maximum absolute errors for Example 2 by the proposed method

$\varepsilon\downarrow N\rightarrow$	64	128	256	512	1024	2048
$\alpha=\frac{1}{12},\ \beta=\frac{5}{12}$						
10^{-1}	2.2525e-05	1.9012e-06	1.8025e-07	1.8949e-08	2.1447e-09	2.5399e-10
10^{-3}	1.6510e-03	2.1045e-04	2.3558e-05	2.4019e-06	2.2735e-07	2.0083e-08
10^{-6}	1.2543e-03	1.7671e-04	3.8181e-05	7.3433e-06	1.1485e-06	1.4224e-07
10^{-9}	1.2425e-03	1.9312e-04	4.6931e-05	1.1473e-05	2.7969e-06	6.7217e-07
10^{-12}	1.2421e-03	1.9369e-04	4.7250e-05	1.1645e-05	2.8866e-06	7.1764e-07
$\alpha=\frac{1}{3},\ \beta=\frac{1}{6}$						
10^{-1}	1.1943e-02	2.9936e-03	7.4887e-04	1.8727e-04	4.6820e-05	1.1705e-05
10^{-3}	6.3931e-02	2.3480e-02	8.2158e-03	2.7294e-03	8.6616e-04	2.6643e-04
10^{-6}	5.6933e-02	2.0894e-02	8.4326e-03	2.4112e-03	7.6410e-04	2.3476e-04
10^{-9}	5.6713e-02	2.0813e-02	7.2498e-03	2.4013e-03	7.6092e-04	2.3378e-04
10^{-12}	5.6706e-02	2.0811e-02	7.2489e-03	2.4010e-03	7.6082e-04	2.3375e-04

Table 4: Computed maximum order of convergence for Example 2 by the proposed method

$\varepsilon\downarrow N\rightarrow$	64	128	256	512	1024
$\alpha=\frac{1}{12},\ \beta=\frac{5}{12}$					
10^{-1}	3.5665	3.3988	3.2498	3.1433	3.0779
10^{-3}	2.9718	3.1592	3.2940	3.4012	3.5009
10^{-6}	2.8274	2.2105	2.3784	2.6767	3.0134

10^{-9}	2.6857	2.0409	2.0323	2.0363	2.0569
10^{-12}	2.6810	2.0354	2.0206	2.0123	2.0080
$\alpha=\frac{1}{3},\ \beta=\frac{1}{6}$					
10^{-1}	1.9962	1.9991	1.9996	1.9999	2.0000
10^{-3}	1.4451	1.5150	1.5898	1.6559	1.7009
10^{-6}	1.4462	1.3090	1.8062	1.6579	1.7026
10^{-9}	1.4462	1.5215	1.5941	1.6580	1.7026
10^{-12}	1.4462	1.5215	1.5941	1.6580	1.7026

Table 5: Comparison of maximum absolute errors and orders of convergence for Example 3

$\varepsilon \downarrow N \rightarrow$	64	128	256	512	1024	2048
Present Method						
10^{-1}	1.5804e-07	9.5736e-09	6.0110e-10	3.8343e-11	2.4281e-12	1.7421e-13
	4.0101	3.9934	3.9706	3.9811	3.8009	
10^{-2}	4.4426e-06	2.8574e-07	1.8107e-08	1.1394e-09	7.1458e-11	4.4797e-12
	3.9586	3.9801	3.9902	3.9950	3.9956	
10^{-3}	3.8440e-04	2.6622e-05	1.7427e-06	1.1132e-07	7.0321e-09	4.4198e-10
	3.8519	3.9332	3.9685	3.9846	3.9919	
10^{-4}	4.6569e-04	5.9110e-05	6.6270e-06	6.8198e-07	1.0174e-07	2.3644e-08
	2.9779	3.1570	3.2806	2.7448	2.1053	
Result in [15]						
10^{-1}	3.208e-04	7.957e-05	2.005e-05	5.007e-06	1.252e-06	3.131e-07
	2.01	1.99	2.00	2.00	2.00	
10^{-2}	4.066e-04	1.017e-04	2.544e-05	6.359e-06	1.590e-06	3.975e-07
	2.00	2.00	2.00	2.00	2.00	
10^{-3}	3.998e-04	1.003e-04	2.509e-05	6.274e-06	1.569e-06	3.922e-07
	1.99	2.00	2.00	2.00	2.00	
10^{-4}	3.859e-04	9.938e-05	2.504e-05	6.271e-06	1.568e-06	3.922e-07
	1.96	1.99	2.00	2.00	2.00	

Table 6: Comparison of maximum absolute errors and orders of convergence for Example 4

$N \rightarrow$	64	128	256	512	1024	2048	4096
Present Method							
E^N	2.5673e-04	3.2707e-05	3.6738e-06	3.7844e-07	3.6621e-08	3.3801e-09	7.2549e-10
R^N	2.9726	3.1543	3.2791	3.3693	3.4375	2.2200	----
Result in [16]							
E^N	8.3652e-04	1.9349e-04	4.6383e-05	1.1343e−05	2.8032e-06	6.9671e−07	1.7369e-07
R^N	2.1122	2.0606	2.0319	2.0166	2.0085	2.0040	----

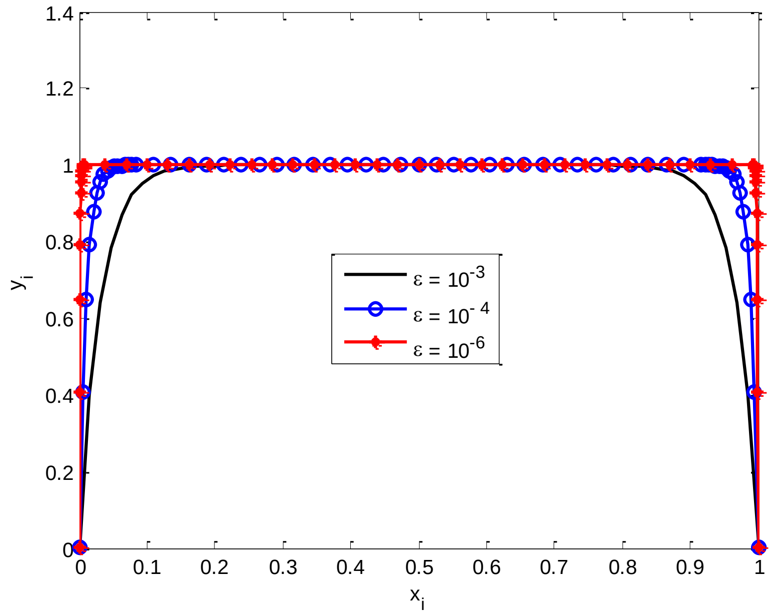

Figure 1: Effects of perturbation parameter on the solution for Example 1 when, $N = 64$.

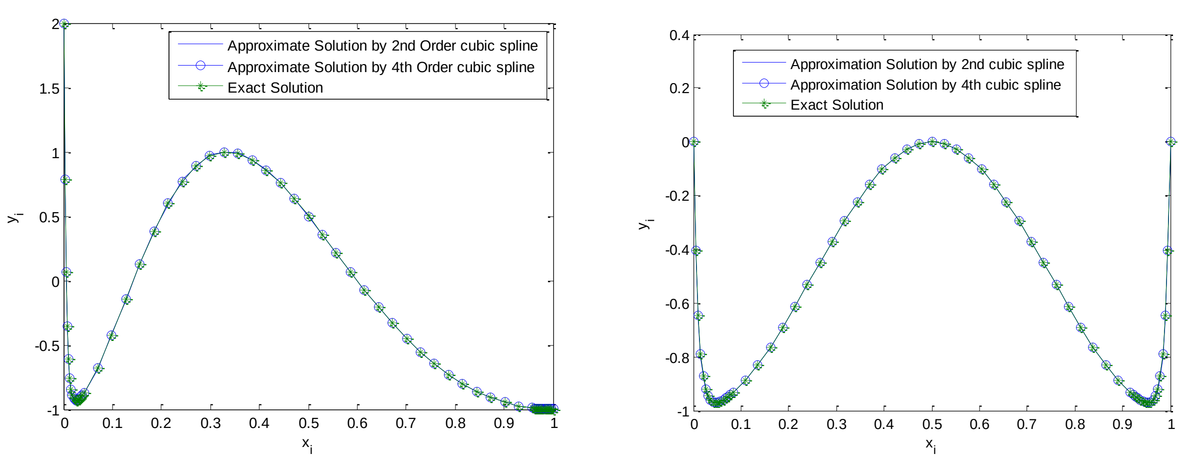

Figure 2: Numerical solution versus exact solution for Examples 2 and 3 respectively,

when $N = 64,\ \varepsilon = 10^{-4}$.

4.6 Discussion

In this study, the family of non-polynomial cubic spline approximation method has been presented for solving a Robin type singularly perturbed reaction-diffusion problems. First, the solution domain or interval is discretized depending on piecewise mesh lengths. Then, the second order derivative is replaced by non-polynomial cubic spline approximations. This approximation yields the second or fourth order convergent numerical scheme that forms the finite difference can be reduced into tri-diagonal system. The two given Robin type boundary conditions replaced by fourth order finite difference approximation from the Taylor's series expansions around the two end points of the independent variable. Collectively, as one can observe from the system in Eq. (4.31), the formulated method forms $(N+1)\times(N+1)$ linear algebraic equations.

Since, the sum of the two off-diagonal elements is less than or equal to the diagonal element, the coefficient matrix of the system is diagonally dominant. Hence, the system is can be solved by using Gaussian elimination method and the formulated scheme in Eq. (4.31) is stable for the described method. Further, the consistency and stability analysis of the obtained tri-diagonal scheme is investigated to guarantee the convergence of the method. To validate the applicability of the proposed method, different modeled examples have been provided and their numerical results are illustrated in Tables 1 – 6 as well as in Figures 1 - 2. Particularly, Figure 1 used to visualize the effect of perturbation parameter and the boundary layer behaviors. As observed from the obtained results in Table 1, the proposed method cannot produces accurate solution to the defined problem on the uniform mesh due to the existence of small positive perturbation parameter. To attack this challenge the method is considered on piecewise mesh lengths that is fine mesh in the layer region and cores in outer layer region.

To confirm the contribution of using piecewise mesh lengths, the numerical results in Table 2 and 3 are provided. Furthermore, the results in this Tables indicates that the effect of choosing the two parameter constants α and β. Also, the obtained maximum absolute errors and convergence rate of convergence in Tables 4 – 6, indicates that the comparison that confirms the superiority of the proposed method. Likewise, the proposed method produces more accurate solution than some existing methods in the literature. Besides, from the results in tables, we observe that as the number mesh increases the maximum absolute error decreases in each row implying that the proposed method is convergent.

Chapter Five

Conclusion and the Future Work

5.1. Conclusion

Cubic non-polynomial spline approximation method is established on piecewise mesh length of the solution domain for solving the Robin type singularly perturbed reaction diffusion problems. The stability and consistency analysis are investigated and shows that the present method is convergent. Different modeled numerical examples are considered for numerical illustrations. As a numerical illustration provided, one can see that the maximum absolute error and order of convergences validate the efficiencies of the method. Furthermore, the result of the present method gives more accurate solution than some existing methods in the literature. Hence, we have computationally shown from supporting numerical experiments that the proposed method provides better convergence with reasonably fewer errors, which supports the theoretical result. Thus, the present method approximates the exact solution very well.

5.2. Future Work

In this study, a family cubic non-polynomial spline approximation approach has been presented for solving the Robin type singularly perturbed reaction diffusion problems. Hence, the method presented in this work can also be extended to higher order spline method to produce a more accurate solution for solving the problem under consideration.

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