Analysis of a circular Couette flow of incompressible fluid between rotating coaxial cylinder

A thesis Submitted to Department of Mathematics, Jimma University in Partial Fulfillment of the Requirements for the Degree of Masters of Sciences in Mathematics

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DECLARATION

I, the under signed declare that the research report entitled as **"Analysis of a circular Couette flow of incompressible fluid between rotating coaxial cylinder."** is original and it has not been submitted to any institution elsewhere for the award of an academic degree or like.

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The work has been with supervision and approval of advisors

Supervisor: Dr. Mitiku Daba

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ABSTRACT

This study aims to analyze analytical solutions of the Couette flow of incompressible fluid between two coaxial cylinders, generated due to constant density and viscosity using no-slip boundary conditions. Two distinct cases have been identified in Couette flow of incompressible fluid between rotating coaxial cylinders. Those are when inner and outer cylinder rotate with different angular velocity in the same direction and the outer cylinder rotate with angular velocity Ω_o and inner cylinder removed. In each case the steady state velocity and pressure distribution in the field are determined. The velocity and pressure profiles of the flows are presented and the effect of r on velocity and pressure is discussed.

ACKNOWLEDGMENT

Frist of all I would like to thank my advisor Dr. Mitiku Daba for his guidance and advising throughout my study and for unreserved support, unlimited advice, constructive comments and immediate responses that helped me.

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CHAPTER ONE

1. INTRODUCTION

1.1. Background of the study

The geometric simplicity of the flow of a fluid between rotating concentric cylinders has attracted the interest of scientists for centuries. Incompressible flow is an approximation of flow where flow speed is insignificant everywhere compared to the speed of sound of the medium. In fluid mechanics incompressible flow refers to a flow in which the material density is constant within a fluid parcel. If incompressible flow is defined in this way, majority of the fluid and associated flow we encounter in our daily life belong to the incompressible category. One of the earliest mathematical models of incompressible flow is the famous equation by Bernoulli, who in 1730 developed the model equation while investigating blood flow by (Quarteroni, 2000). It is not surprising that scientists have been investigating incompressible flow analytically, experimentally and computationally.

Computational study of compressible flow problems in both basic research and engineering application has been performed for several decades. Numerical solutions for such basic fluid dynamics problems are: flow through a circular cylinder, flow through channels, ducts and pipes and flow over a backward facing step were present early as the 1930s **(**Thorn, 1933) for a circular cylinder. Liquid contained in a narrow annulus between two coaxial cylinders, one or both of which rotate, experiences a nearly uniform shear rate. In the simplest situation one cylinder is stationary and the other is set in motion with either a constant velocity or constant torque. Torque measurements and flow visualization are both performed to determine the flow characteristics.

The flow between two concentric cylinders is well known and studied in fluid dynamics problem prominent in the development of rotating machinery. The classical Couette flow problem consists of infinitely long concentric cylinders and an incompressible Newtonian fluid between them. The Navier-Stokes equation method is used to compute the fluid of pressure and velocity field. This kind of methods with applications in various fields of continuum mechanics is given by (Nem´enyi, 1951).

Moreover, a number of reviews on the exact solutions for Navier-Stokes equations have been published by (Dochan et al., 2000; Hron et al., 2008). Due to its vast applications in engineering and industry, the Couette flow has attracted attention of various researchers. The Couette flow comprising concentric rotating cylinder is widely used in many industrial and research process found in chemical, mechanical, civil engineering and nuclear engineering. The device can be only one cylinder rotating and the other at rest or two cylinders rotating in the same or counter direction. Ataide et al. (2003) analyzed the fluid flow in annular regions is an area of great interest in the petroleum industry both in the drilling and in the artificial rising of the petroleum. Carrasco et al. (2009) analyzed the effects of rotation and axial motion of the inner cylinder during the displacement flow between two Newtonian fluids of differing density and viscosity. Pressure fluctuations are in some way responsible because large density differences at low speeds have very little effect. There is considerable interest in the wakes of axisymmetric bodies moving at high speeds, with reference to the detection of reentering missiles. In this case, the most important variables are the temperature and the electron density in the partly ionized gas (Demetriades, 1976). There are a large number of experimental and theoretical studies of flow between concentric rotating cylinders (circular Couette flow) in the century since the earliest studies were conducted by (Mallock, 1888). This motivates the researcher to construct analytical solutions of the Couette flow of incompressible fluid between two coaxial cylinders by applying appropriate assumptions. The momentum equation is a vector equation obtained by applying Newton's law of motion to a fluid element which is given by (John, 1996).

$$\rho \frac{Du}{Dt} = \rho g - \nabla p + \mu \nabla^2 u$$

The equation of continuity for cylindrical coordinates (r,θ,z) is given by (Bird et al., 2007).

$$\frac{\partial \rho}{\partial t}\frac{1}{r}\frac{\partial(\rho r \boldsymbol{v}_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho \boldsymbol{v}_\theta)}{\partial \theta} + \frac{\partial(\boldsymbol{\rho v}_z)}{\partial z} = 0$$

The continuity equation which expresses conservation of mass is given by:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho u) = 0$$

For incompressible flow this equation is reduced to $\nabla . \mathrm{u} = 0$, where u is velocity component of a fluid which is a function of (t, x, y, z) and ρ is fluid density. According to Temesgen Degu et al. (2016) analytical solutions of steady fully developed flows of incompressible fluid between two concentric cylinders generated due to constant density and viscosity using no-slip boundary conditions, they used three cases:

- ❖ The outer cylinder rotating with constant angular velocity Ω_o and the inner cylinder at rest.
- ❖ The inner cylinder rotating with constant angular velocity Ω_I and the outer cylinder at rest.
- ❖ Both inner and outer cylinder rotating in the same direction with the same constant angular velocity Ω_I and Ω_O respectively. But in this study, we consider another two cases that are different from the above cases. Those are:
- ➢ When inner and outer cylinder rotate with different angular velocity in the same direction
- ➢ The outer cylinder rotate with angular velocity Ω_o and inner cylinder removed and additionally torque which is needed to rotate cylinder and free surface shape for rotating liquid in cylinder will be studied. So the aim of the study is to construct an analytical solution of a circular Couette flow of incompressible fluid between rotating coaxial cylinders.

1.2. Statement of the problem

According to other authors, analytical solution of Couette flow problem of rotating coaxial cylinders depend on both inner and outer cylinders rotating in the same direction with the same angular velocity, the outer cylinder rotating with constant angular velocity Ω_o and the inner cylinder at rest and the inner cylinder rotating with constant angular velocity Ω_I and the outer cylinder at rest. But in this study, we consider both inner and outer cylinders rotate with different angular velocity in the same direction and the outer cylinder rotate with angular velocity Ω_o and inner cylinder removed. The aim of study is to construct an analytical solution of a circular Couette flow of incompressible fluid between rotating coaxial cylinder.

The study attempts to:

- Provide the relationship between radius & the velocity of fluid.
- Find the velocity and pressure of fluid flow of rotating cylinder.
- Determine the shape of free surface of liquid in rotating cylinder.

1.3. Objective of the study

1.3.1. General objective

The general objective of the study is to develop an analytical solution to Couette flow problem between rotating coaxial cylinders.

1.3.2. Specific objective

The specific objectives of the study:

- To provide the relationship between radius & the velocity of fluid.
- To find velocity and pressure of the fluid flow between rotating coaxial cylinder.
- To determine the shape of the free surface of liquid in rotating cylinder.

1.4. Significance of the study

This study may have the following advantages.

- ❖ It may provide further understanding on the characteristics of fluid between rotating coaxial cylinder.
- ❖ It may familiarize the researcher with scientific communications in applied mathematics.
- ❖ It may serve as background information for researcher who works around this area.

1.5. Delimitation of the study

The study is delimited to the governing partial differential conservation equations for laminar free convection and focuses only on the appropriate assumptions to construct analytical solutions for circular Couette flow of incompressible fluid between rotating coaxial cylinder.

1.6. Definition of terms

Steady flow is in which the conditions of velocity, pressure and cross-section may differ from point to point but do not change with time.

Newtonian fluids are a real fluid in which shear stress is directly proportional to the rate of shear strain (velocity of gradient).

Couette flow is the flow of a viscous fluid in the space between two surfaces.

Boundary condition is the condition for the velocity components of a fluid when it makes contact with a solid surface.

Compressible fluid is a fluid in which the fluid density changes when it is subjected to high pressure gradients.

Shear stress($\boldsymbol{\tau}$) is the product of viscosity and the transverse velocity gradient ($\boldsymbol{\tau} = \boldsymbol{\mu}\frac{d\boldsymbol{u}}{dy}$).

Navier-Stokes equations are the fundamental partial differentials equations that describe the flow of incompressible fluid.

Torque is an action that causes objects to rotate.

Fully developed: a flow is said to be fully developed if the velocity of the flow does not change any more as a function of space in the direction of the flow.

CHAPTER TWO

2. LITERATURE REVIEW

2.1. Incompressible flow

An incompressible fluid is a fluid whose density does not change when the pressure changes or incompressible flow is an approximation of flow where flow speed is insignificant everywhere compared to the speed of sound of the medium. According to Dochan et al. (2000) the difference between incompressible and compressible Navier-Stokes formulation is in the continuity equation. The incompressible formulation can be viewed as a singular limit of the compressible one satisfying the mass conservation equation. Therefore, incompressible flow is characterized by elliptic behavior of the pressure waves, the speed of which in a truly incompressible flow is infinite.

According to Victor (1962), mathematically the incompressible flow formulation poses unique issues not present in compressible equations because of the incompressibility requirement. Physically, information travels at infinite speed in an incompressible medium, which imposes stringent requirements on computational algorithms for satisfying incompressibility as well as difficulties in designing downstream boundary conditions. The differences in various methods of solving incompressible flow equations originate from strategies of satisfying incompressibility. No convincing explanation of the compressibility effects exists clearly pressure fluctuations are in some way responsible because large density differences at low speeds have very little effect. There is considerable interest in the wakes of axisymmetric bodies moving at high speeds, with reference to the detection of reentering missiles. In this case the most important variables are the temperature and the electron density in the partly ionized gas (Demetriades, 1976).

2.2. Couette flow between two parallel plates

Couette flow between parallel plates is a classical problem that has important applications in power generators, pumps and etc. Couette flow is a classical problem of primary importance in the history of fluid mechanics, which is a typical example of exact solutions for Navier-Stokes equation. Couette flow is perhaps the simplest of all viscous flows, while at the same time retaining much of the same physical characteristics of a more complicated boundary-layer flow (Panton, 1996). Since the gap between the barrel and the screw of extruder is small, assuming a fluid flowing between parallel plates leads to representative results. Etemad et al. (1994) solved the simultaneously developed case of the motion and energy equation for power law fluid between parallel stationary plates when the variation of viscosity with temperature and viscous dissipation could not be neglected. They solved the problem numerically using finite element method and as a special case, calculated the flow and heat transfer characteristics for fully developed conditions. The steady flow of a viscous incompressible fluid between two parallel flat plates caused by the motion of one of the plates, under constant pressure is quite well known as plane Couette flow. Pia (1956), expressed the velocity distribution and temperature distribution between two parallel plates.

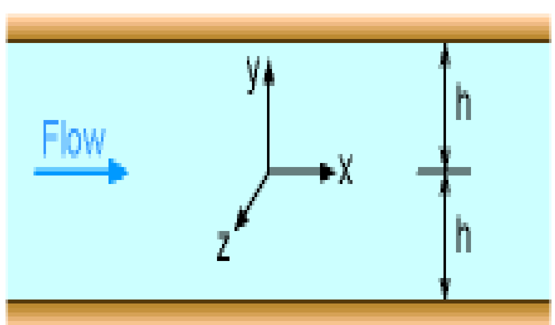


Figure 1. Flow between two parallel plates.

Consider steady incompressible flow between two infinite parallel horizontal plates as shown in the figure 1, above the flow is in the x direction, hence there is no velocity component in either the y or z direction (i.e. v = 0 and w = 0). The steady-state continuity equation becomes:

$$\frac{\partial u}{\partial x} = 0 \qquad (1)$$

From Eqn.1, we can conclude that the velocity u is a function of both y and z only. Since the plates are infinitely wide, it can be argued that the velocity u should not be a function of z, i.e., it must be a function of y only, u = u(y). Applying Navier-Stokes equations using the assumptions that v = 0, w = 0 and u = u(y), yields

$$\frac{\partial p}{\partial x} = \mu \frac{d^2 u}{dy^2} \qquad (2)$$

$$\frac{\partial p}{\partial y} = -\rho g \qquad (3)$$

$$\frac{\partial p}{\partial z} = 0 \qquad (4)$$

Eqn. 4, indicates that the pressure is a function of x and y. Integrating Eqn. 3, to yield

$$p = -\rho g y + g_1(x)$$

Hence it can be concluded that $\frac{\partial p}{\partial x}$ is a function of x only. Now integrate Eqn. 2, twice with respect to y and treat $\frac{\partial p}{\partial x}$ as a constant (with respect to y) to give:

$$U = \frac{1}{2\mu}\frac{\partial p}{\partial x} y^2 + C_1 y + C_2$$

Applying the no-slip conditions (i.e., the fluid is "stuck" to the plates or u = 0 at y = ±h) to determine the coefficients as follows:

$$C_1 = 0 \quad \text{and } C_2 = -\frac{h^2}{2\mu}\frac{\partial p}{\partial x}$$

The velocity profile becomes:

$$U = \frac{1}{2\mu}\frac{\partial p}{\partial x}(y^2 - h^2)$$

2.3. Circular Couette flow

Couette flow problem consists of infinitely long concentric cylinders and an incompressible Newtonian fluid between them. These flows are named in honor of M. Couette (1890), who performed experiments on the flow between fixed and moving concentric cylinders. For a system with a rotating inner cylinder and stationery outer cylinder, the fluid flow will pass on stable circular Couette flow. Circular Couette flow occurs in the gap between two rotating coaxial cylinders. The inner cylinder of radius R_I has the angular velocity Ω_I while the outer cylinder of radius R_o spins at Ω_o. The apparatus has a height L which is much larger than the radius of cylinder so that the apparatus height is supposed infinite. Rotating Couette flow is a flow between two concentric circular cylinders rotating with different velocities. Due to its simple and common geometry it has numerous industrial prototypes. Some examples from most common to rather exotic include flows in bearings, flows in particle separators, flows in rotational viscosity meters, flow between the drill string that is the inner cylinder to which the drill bit is attached and which rotates rapidly in the drilled hole in the drilling of oil wells. The cylindrical coordinates system (r, θ, z) in the steady state velocity field is such that:

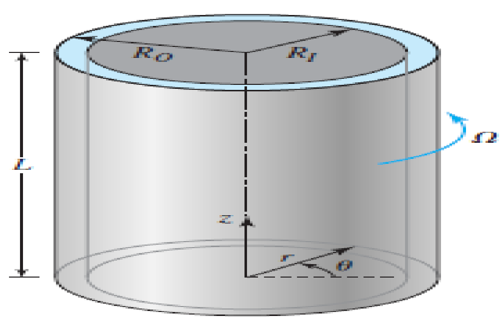


Fig 2. Circular Couette flow - I

$$v_r = 0, \qquad v_\theta = v_\theta(r), \;\; v_z = 0$$

This v_θ(velocity field) is then determined from the integration of the momentum equation $\rho\frac{Du}{Dt} = \rho g - \nabla p + \mu\nabla^2 u$. Subjected to the boundary condition and appropriate assumptions where, g is gravity, ρ is density, p is fluid pressure and μ is viscosity.

According to Laglois et al. (2014) the motion of fluid contained between two concentric circular pipes of constant radii R_O and R_I with $R_I < R_o$, the pipes rotate about their common axis with constant angular velocities Ω_o and Ω_I respectively. Since the cylindrical coordinate system is not accelerated, the fluid motion is governed by the continuity equation and momentum equation. The no-slip condition requires that:

$$v_r = v_z = 0 \qquad v_\theta = R_I \Omega_I \qquad \text{at r} = R_I$$

$$v_r = 0, v_z = U \qquad v_\theta = R_o \Omega_o, \qquad \text{at r} = R_o$$

Flow over rotating cylinders is important in a wide number of applications from shafts and axles to spinning projectiles. Also consider the flow in an annulus formed between two concentric cylinders where one or both of the cylindrical surface are rotating. As the rotating flow associated with discs, the proximity of a surface can significantly alter the flow structure. A boundary layer will form on a rotating body of revolution due to the no-slip condition at the body surface. The flow about a body of revolution rotating about its axis and simultaneously subjected to a flow in the direction of the axis of rotation is relevant to a number of applications, including certain rotating machinery and the ballistics of projectiles with spin. Various parameters such as the drag, moment coefficient and the critical Reynolds number are dependent on the ratio of the circumferential to free-stream velocity. A linear stability analysis was carried out for axial flow between a rotating porous inner cylinder and a concentric stationary, porous outer cylinder when radial flow is present for several radius ratios. Murakami and Kikuyama (1980), measured the velocity profile and hydraulic loss in a hydro dynamically fully-developed flow region of a rotating pipe. The boundary layer on a rotating body of revolution in an axial flow consists of the axial component of velocity and the circumferential component due to the no-slip condition at the body surface.

2.4. Torque

Torque is required to rotate an object, just as a force is required to move an object in a line. Torque is created by force, but it also depends on where the force is applied and the point about which the object rotates. Force is the action that creates changes in linear motion. Torque is defined as the tendency to produce a change in rotational motion. The torque imparted by the fluid acting on the inner cylinder is defined as the product of the total force acting on the surface of the inner cylinder and the lever arm. The total force is determined by evaluating the inward pointing momentum flux ($-\tau_{r\theta}$) at the surface of the cylinder, and then multiplying this result by the total external surface area of the cylinder with the lever arm.

CHAPTER THREE

3. METHODOLOGY

3.1. Study design

The research is conducted by using both analytical and experimental approaches.

3.2. Study site and period

The study is conducted in Jimma University under the College of Natural Science in Mathematics Department from October 2016 to September 2017.

3.3. Sources of information

The sources of our data are different books, journals, internet, articles and etc.

3.4. Mathematical Procedure of the study

The general procedure for solving each problem involves the following steps:

1. Defining the problem and making reasonable assumptions.
2. Write down the continuity equation of momentum and simplify them according to the assumptions.
3. Integrate the simplified equations in order to obtain the expressions for velocity & pressure.
4. Constructing analytical solutions for velocity, pressure and torque.
5. Finally a graph is produced using MATLAB.

3.5. Ethical consideration

For this study, I need journals, books, information and other related materials. To collect all the above materials the researcher takes permission letter from Jimma University College of Natural Sciences Department of Mathematics before collecting the materials.

CHAPTER FOUR

4. RESULT AND DISCUSSION

4.1. Discussion

A fluid is incompressible if and only if one of the following conditions are satisfied

i. $\nabla u = 0$

ii. $\frac{D\rho}{Dt} = 0$

i.e. if the fluid is incompressible, ρ = constant and independent of space and time, The continuity equation $\frac{\partial \rho}{\partial t} + \nabla.(\rho u) = 0$, is simplified to $\frac{D\rho}{D\,t} = 0$. Therefore the continuity equation is reduces to $\nabla \cdot u = 0$. Circular Couette flow occurs in the gap between two rotating concentric cylinders. The inner cylinder of radius R_I has the angular velocity Ω_I while the outer cylinder of radius R_O has angular velocity $\Omega_{O.}$ The apparatus has a height L which is much larger than the radius of either cylinder so that the apparatus height is supposed infinite.

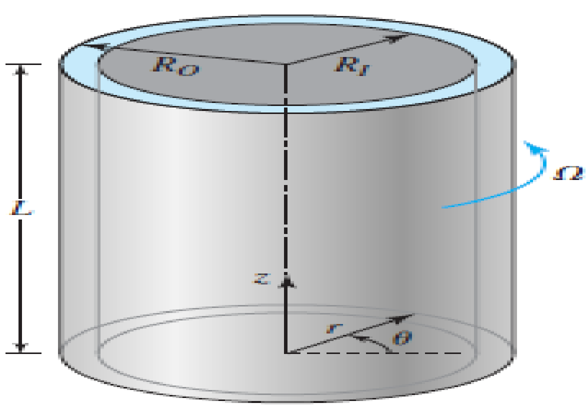


Fig.3. Circular Couette flow -II

The fluid flow analysis aims to determine the relationship between pressure and velocity of fluid flow by solving the Navier- Stokes equation which is subjected to a geometric boundary condition, which is the interface surface at which a fluid contacts a solid object. In order to solve a circular Couette flow of incompressible fluid over a circular cylinder some assumptions are necessary, since mathematical expressions become simpler and the solutions are still close to real cases.

Assumptions

1. The flow is fully developed.
2. The flow is axisymmetric flow: no change in tangential direction or in θ direction.
3. The fluid is Newtonian fluid.
4. For θ momentum , $\frac{d}{dr}\left(\frac{1d}{r}\frac{(rv_\theta)}{dr}\right) = 0$

For z momentum, $\frac{\partial p}{\partial z} + \rho g = 0$

For r momentum $\frac{\rho v_\theta{}^2}{r} = \frac{\partial p}{\partial r}$

Governing Equations

The flow between two rotating cylinders are computed by solving the Navier-Stokes equations for incompressible fluid in a three dimensional geometry. The governing equations are equations of viscos, incompressible fluid flow, known as the Navier-Stokes (N-S) equation.

The momentum equation is given by;

$$\rho\frac{Du}{Dt} = \rho g - \nabla p + \mu\nabla^2 u$$

The continuity equation expresses conservation of mass which is given by;

$$\frac{\partial\rho}{\partial t} + \nabla.(\rho u) = 0$$

For incompressible flow this equation reduces to $\nabla. u = 0$, by (Chacon and Lewandowski, 2014).

As Bird et al. (1987) equation of motion for incompressible Newtonian fluid (Navier-Stokes equation) has three components in Cartesian coordinates given by,

$$\frac{\partial \boldsymbol{u}}{\partial x} + \frac{\partial \boldsymbol{v}}{\partial y} + \frac{\partial \boldsymbol{w}}{\partial z} = 0,$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \rho g_x,$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \rho g_y,$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + \rho g_z ,$$

Where u, v and w are velocity components in the directions of x, y and z respectively, P is the pressure, μ is the coefficient of viscosity, ρ is the density of the fluid. Equation of motion for incompressible, Newtonian fluid (Navier-Stokes equation), three components in cylindrical coordinate,

$$\frac{1}{r}\frac{\partial(r\boldsymbol{v}_r)}{\partial r}+\frac{1}{r}\frac{\partial \boldsymbol{v}_\theta}{\partial \theta}+\frac{\partial \boldsymbol{v}_z}{\partial z}=0, \tag{1}$$

Navier-Stokes equations of cylindrical coordinates for incompressible fluid are given by:

$$\rho\left(\frac{\partial(v_r)}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+v_z\frac{\partial v_r}{\partial z}-\frac{{v_\theta}^2}{r}\right)=$$

$$\rho g_r-\frac{\partial p}{\partial r}+\mu\left(\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}+\frac{\partial {v_r}^2}{\partial z^2}-\frac{v_r}{r^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right), \tag{2}$$

$$\rho\left(\frac{\partial(v_\theta)}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+v_z\frac{\partial v_\theta}{\partial z}+\frac{v_r v_\theta}{r}\right)=$$

$$\rho g_\theta-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left(\frac{\partial^2 v_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial v_\theta}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{\partial {v_\theta}^2}{\partial z^2}-\frac{v_\theta}{r^2}-\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right), \tag{3}$$

$$\rho\left(\frac{\partial(v_z)}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)= \rho g_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 v_z}{\partial r^2}+\frac{1}{r}\frac{\partial v_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial {v_z}^2}{\partial z^2}\right), \tag{4}$$

by (Tritton, 1998)

4.2. Main Results

4.2.1. Analysis of a circular Couette flow of incompressible fluid between rotating coaxial cylinders

Consider rotating coaxial cylinder with the inner cylinder of radius R_I has the angular velocity Ω_I while the outer cylinder of radius R_O has angular velocity Ω_O and which is filled with an incompressible fluid that can be modeled as a Newtonian fluid. We made the following determinations about the fluid flow field:

1. The velocity distribution of the fluid in the cylinders.
2. The pressure distribution of fluid in the cylinders.
3. The relation between pressure and velocity with radius of cylinder.

In laminar flow the fluid travels in a circular motion. To verify this we are going to use the following.

In r direction $\mathrm{v_r} = \text{function of } (\mathrm{t, r, \theta, z})$

In z direction $\mathrm{v_z} = \text{function of } (\mathrm{t, r, \theta, z})$

In θ direction $\mathrm{v_\theta} = \text{function of } (\mathrm{t, r, \theta, z})$

Flow conditions are no change in the axial or in z direction is assumed. Thus

In r direction $\mathrm{v_r} = \text{function of } (\mathrm{t, r, \theta, z})$

In z direction $\mathrm{v_z} = 0$

In θ direction $\mathrm{v_\theta} = \text{function of } (\mathrm{t, r, \theta, z})$

In cylindrical coordinates $\mathrm{r, \theta}$ and z continuity equations for incompressible fluids are given by:

$$\frac{1}{\mathrm{r}}\frac{\partial(\mathrm{rv_r})}{\partial \mathrm{r}} + \frac{1}{\mathrm{r}}\frac{\partial \mathrm{v_\theta}}{\partial \theta} + \frac{\partial \mathrm{v_z}}{\partial \mathrm{z}} = 0$$

Implies that $\frac{1}{\mathrm{r}}\frac{\partial(\mathrm{rv_r})}{\partial \mathrm{r}} = 0$, since $\frac{\partial \mathrm{v_\theta}}{\partial \theta}$ and $\frac{\partial \mathrm{v_z}}{\partial \mathrm{z}}$ is zero

The velocity component in r direction is not constant in order to satisfy flow through the walls and therefore we need to choose $\mathrm{v_r} = 0$. From this we get;

In r direction $\mathrm{v_r} = 0$

In z direction $\mathrm{v_z} = 0$

In θ direction $\mathrm{v_\theta} = \text{function of } (\mathrm{t, r, \theta, z})$

Now we are going to use the momentum equation which is a vector equation obtained by applying Newton's law of motion to a fluid element. The r, θ and z component of the Navier Stokes equation in cylindrical coordinates for the incompressible fluid with constant viscosity are given by,

Radial component: $\rho\left(\frac{\partial(v_r)}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{{v_\theta}^2}{r}\right) =$

$$-\frac{\partial p}{\partial r} + \mu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right). \qquad (5)$$

Tangential component: $\rho\left(\frac{\partial(v_\theta)}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right) =$

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_\theta}{r^2} - \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right). \qquad (6)$$

Axial component: $\rho\left(\frac{\partial(v_z)}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial z} + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right)$ (7)

From radial component of the momentum equation we have:

$$\rho\left(\frac{{v_\theta}^2}{r}\right) = \frac{\partial p}{\partial r}\,. \qquad (8)$$

From axial component of the momentum equation we have:

$$\frac{\partial p}{\partial z} = 0 \qquad (9)$$

From tangential component of the momentum we have:

$$\frac{d}{dr}\left(\frac{1d}{r}\frac{(rv_\theta)}{dr}\right) = 0 \qquad (10)$$

To get tangential velocity component (v_θ) integrate equation (10).

$$\int \frac{d}{dr}\left(\frac{1d}{r}\frac{(rv_\theta)}{dr}\right) = 0$$

$$\frac{1}{r}\frac{d(rv_\theta)}{dr} = C_1$$

$$\frac{d(rv_\theta)}{dr} = C_1 r$$

And again integrate with respect to r

$$\int \frac{d(rv_\theta)}{dr}dr = \int C_1 r\, dr$$

$$rv_\theta = \frac{C_1 r^2}{2} + C_2 \qquad \text{(Dividing both side by r)}$$

$$v_\theta = \frac{C_1 r}{2} + \frac{C_2}{r} \tag{11}$$

Where C_1 and C_2 are unknown constants, then equation (11) is the velocity distribution.

To get pressure distribution substitute equation (11) in equation (8)

$$\rho\left(\frac{{v_\theta}^2}{r}\right) = \frac{\partial p}{\partial r}$$

$$\rho\left[\frac{{C_1}^2 r}{4} + \frac{{C_2}^2}{r^3} + \frac{C_1 C_2}{r}\right] = \frac{\partial p}{\partial r}$$

$$\rho\left[\frac{{C_1}^2 r}{4} + \frac{{C_2}^2}{r^3} + \frac{C_1 C_2}{r}\right] dr = dp \tag{12}$$

Integrating equation (12) we can get:

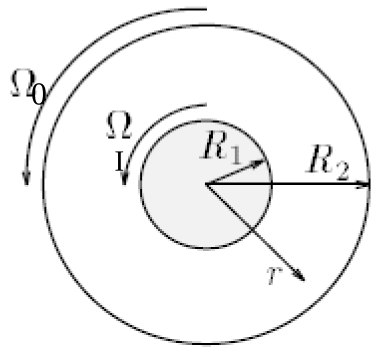
$$\rho \int \left[\frac{{C_1}^2 r}{4} + \frac{{C_2}^2}{r^3} + \frac{C_1 C_2}{r}\right] dr = \int dp$$

$$p = \rho\left[\frac{{C_1}^2 r^2}{8} - \frac{{C_2}^2}{2r^2} + C_1 C_2 \ln r\right] + C^* \tag{13}$$

Where C* is the constant of integration. Now equation (13) is the pressure distribution of fluid in the cylinder.

Case I

Consider two cylinders with the inner and the outer cylinder rotating in the same direction with constant angular velocity Ω_I and Ω_O respectively. Determine the steady state velocity distribution and the pressure distribution in the field. (In the case of $\Omega_I \neq \Omega_O$)



The boundary conditions are that the fluid no slip at the surface of the two cylindrical surfaces. When the inner and outer cylinders are rotating, then the BCs are:

B.C.1. at $r = R_O \implies v_\theta = \Omega_O R_O$

B.C.2. at $r = R_I \implies v_\theta = \Omega_I R_I$

Depending on this boundary condition we can find v_θ

$$\text{At} \quad r = R_O \implies v_\theta = \frac{C_1 R_O}{2} + \frac{C_2}{R_O} = \Omega_O R_O \tag{14}$$

$$\text{At} \quad r = R_I \implies v_\theta = \frac{C_1 R_I}{2} + \frac{C_2}{R_I} = \Omega_I R_I \tag{15}$$

From equation (14) we have: $C_1 {R_O}^2 + 2C_2 = 2\Omega_O {R_O}^2$

From equation (15) we have: $C_1 {R_I}^2 + 2C_2 = 2\Omega_I {R_I}^2$

Solving simultaneously we have can get C_1 and C_2 then:

$$C_1 = \frac{2(\Omega_0 R^2{}_O - \Omega_I R^2{}_I)}{R^2{}_O - R^2{}_I}$$

$$C_2 = -\frac{R^2{}_O R^2{}_I}{R^2{}_O - R^2{}_I}(\Omega_O - \Omega_I)$$

Substituting these values for constants of integration into equation (11) we get the final expression for the velocity profile.

$$v_\theta = \frac{C_1 r}{2} + \frac{C_2}{r}$$

$$v_\theta = \frac{\left(\frac{2(\Omega_0 R^2{}_O - \Omega_I R^2{}_I)}{R^2{}_O - R^2{}_I}\right) r}{2} + \frac{-\frac{R^2{}_O R^2{}_I}{R^2{}_O - R^2{}_I}(\Omega_O - \Omega_I)}{r}$$

$$v_\theta = \frac{1}{R^2{}_0 - R^2{}_I}\left[(\Omega_0 R^2{}_0 - \Omega_I R^2{}_I) r - R^2{}_O R^2{}_I (\Omega_O - \Omega_I)\frac{1}{r}\right] \tag{16}$$

Equation (16) is velocity profile.

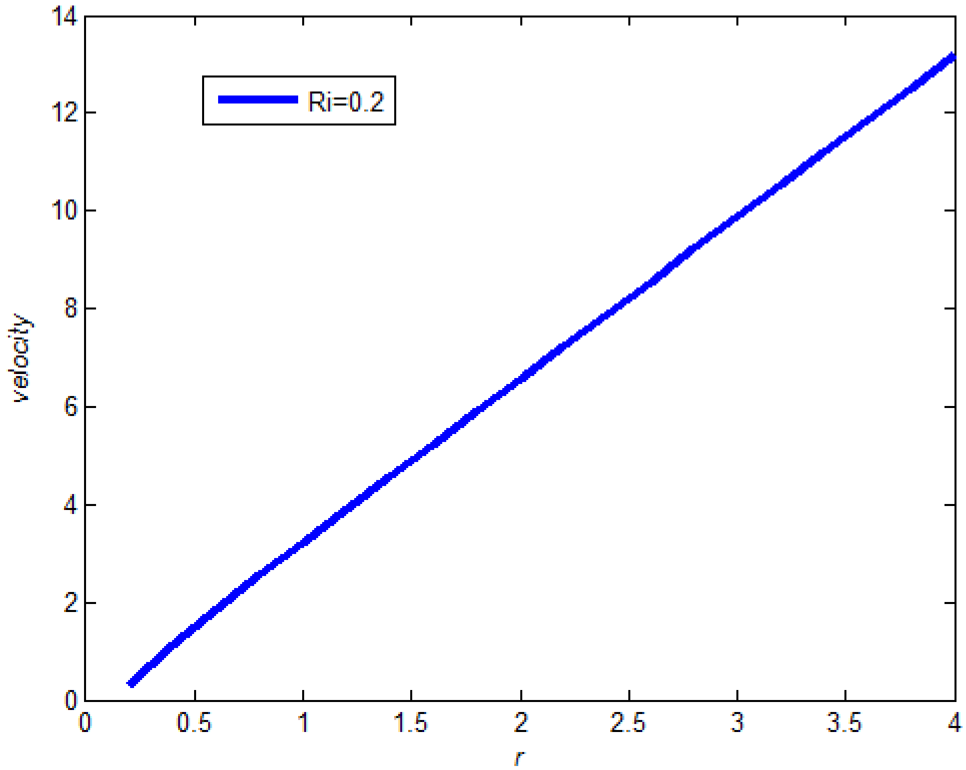


Figure 4.a. Velocity profile, $r \in [0.2, 4]$

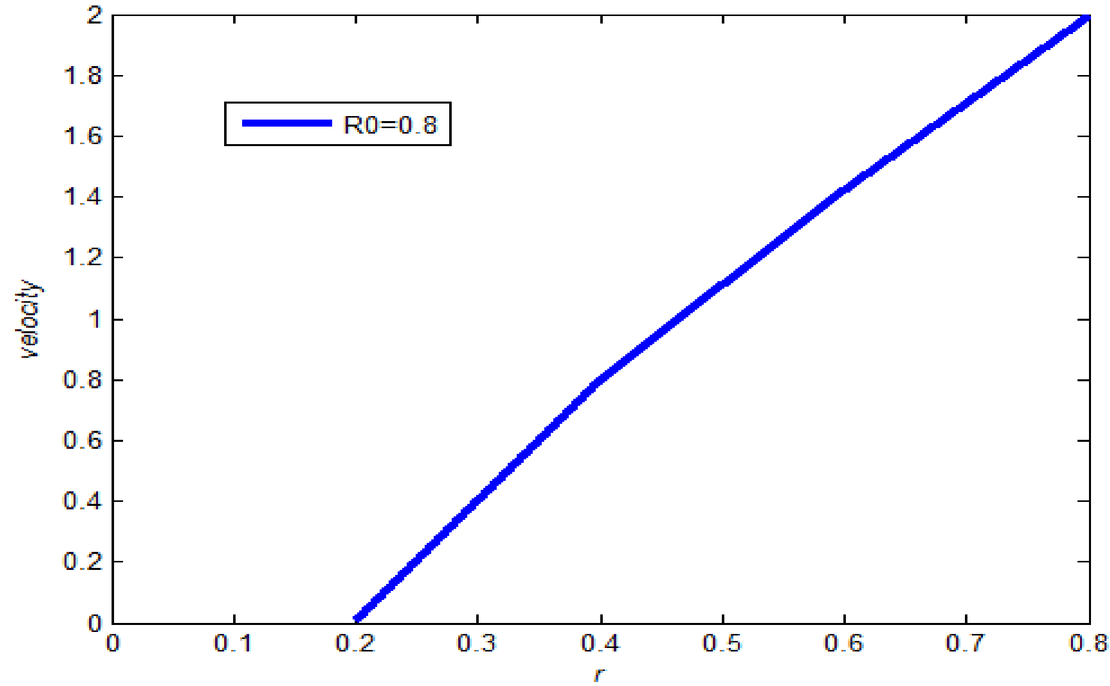


Figure 4.b. Velocity profile, $r \in [0.2, 0.8]$

From Figure 4.a and 4.b, we observe that as the value of radius increases the velocity of fluid flow increases and also when the values of radius decreases the velocity of fluid flow in cylinder decreases. So velocity is affected by motion of the fluid.

To find the pressure distribution using equation (13)

$$p = \rho\left[\frac{{C_1}^2 r}{8} - \frac{{C_2}^2}{2r^2} + C_1 C_2 \ln r\right] + C^*$$

$$p = \rho\left[\frac{\left(\frac{2(\Omega_O {R^2}_O - \Omega_I {R^2}_I)}{{R^2}_O - {R^2}_I}\right)^2 r}{8} - \frac{\left(-\frac{{R^2}_O {R^2}_I}{{R^2}_O - {R^2}_I}(\Omega_O - \Omega_I)\right)^2}{2r^2} + \left(\frac{2(\Omega_O {R^2}_O - \Omega_I {R^2}_I)}{{R^2}_O - {R^2}_I}\right)\left(-\frac{{R^2}_O {R^2}_I}{{R^2}_O - {R^2}_I}(\Omega_O - \Omega_I)\right)\ln r\right] + C^*$$

$$p = \rho\frac{1}{({R^2}_o - {R^2}_I)^2}\left[\left(\frac{1}{2}(\Omega_o {R^2}_0 - \Omega_I {R^2}_I)^2 r\right) - 2{R^2}_O {R^2}_I(\Omega_o {R^2}_0 - \Omega_I {R^2}_I)(\Omega_O - \Omega_I)\ln r - \frac{1}{2}\left(\frac{{R_O}^4 {R_I}^4(\Omega_O - \Omega_I)^2}{r^2}\right)\right] + C^*$$

Where C* is constant integration. (17)

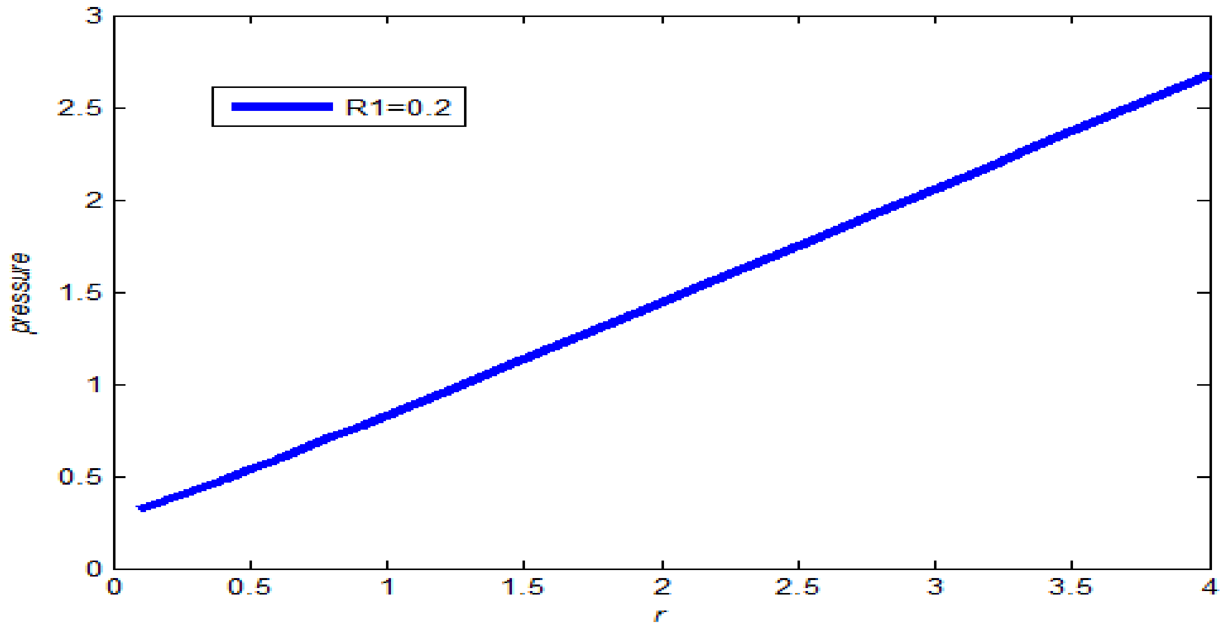


Figure.5.a. Pressure profile, $r \in [0.2, 4]$

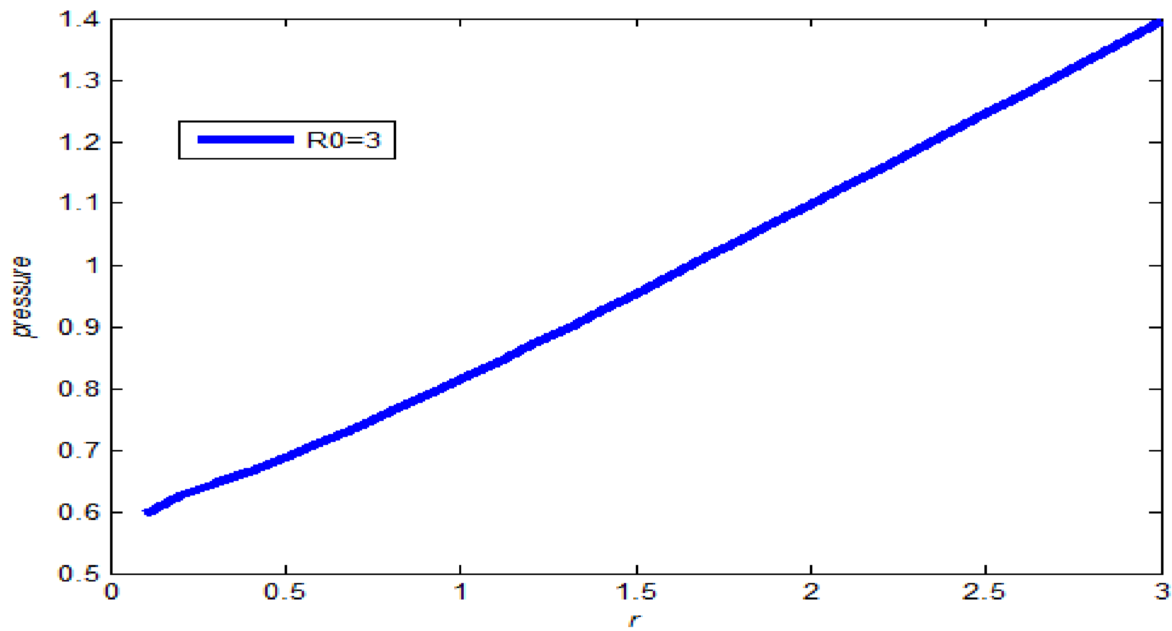
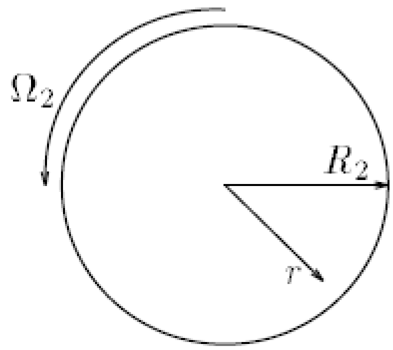


Figure.5.b. Pressure profile, $r \in [0.1, 3]$

From Figure 5.a and 5.b we observe that the increasing values of radius between two cylinders have an increasing effect on the pressure and also the decreasing value of radius between two cylinders have a decreasing pressure of fluid flow.

Case II

The inner cylinder is removed and the outer cylinder rotates with constant angular velocity Ω_2. Determine the steady state velocity distribution and the pressure distribution in the field.



The boundary conditions are that the fluid does not slip at the cylindrical surfaces. Then the BCs are:

B.C.1. at r = R_2, $v_\theta = \Omega_2 R_2$ 18

B.C.2. at r = 0, v_θ = finite 19

From eq. (18) and (19) we can get:

$$v_\theta = \frac{C_1 R_2}{2} + \frac{C_2}{R_2} = \Omega_2 R_2 \quad 20$$

$$v_\theta = \text{Finite at } r = 0 \quad 21$$

From eq. (21) we can get, C_2 = 0 and using equation (20),

$$\frac{C_1 R_2}{2} + \frac{0}{R_2} = \Omega_2 R_2$$

$$C_1 = 2\Omega_2 \quad 22$$

To get velocity profile in this case using equation (11),

$$v_\theta = \frac{C_1 r}{2} + \frac{C_2}{r}$$

$$v_\theta = \frac{2\Omega_2 r}{2} + \frac{0}{r}$$

$$v_\theta = \Omega_2 r \quad (23)$$

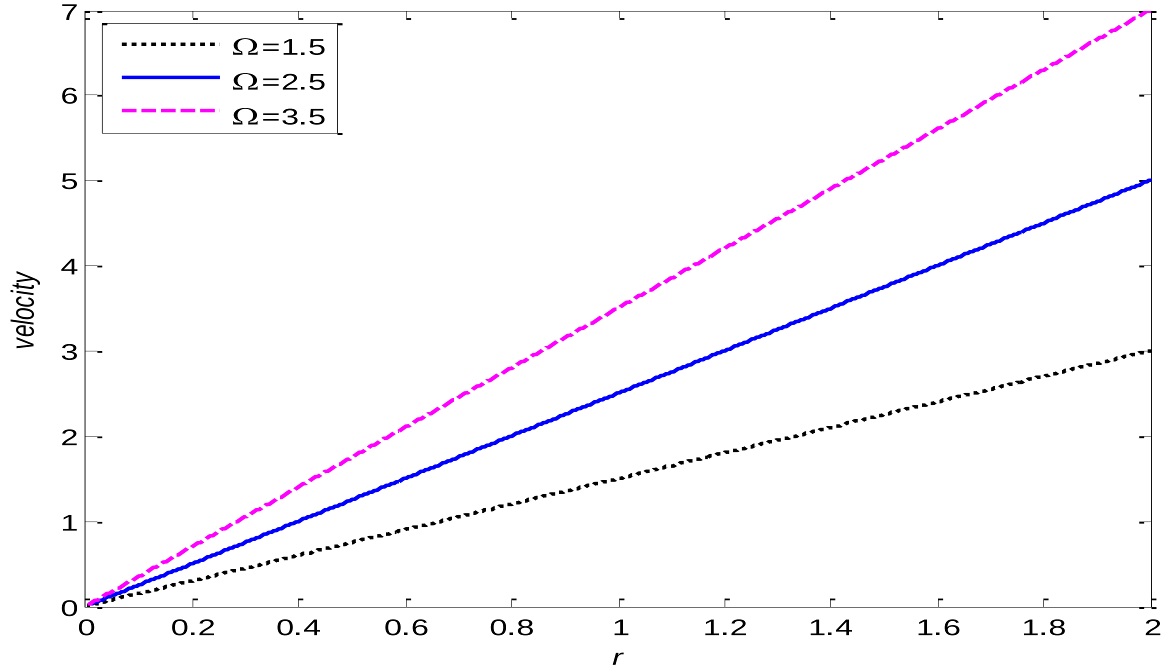


Figure 6.Velocity profile, $r \in [0, 2]$

From figure 6, we observe that as the value of radius increases then velocity of the fluid increases and also when the values of radius decreases the velocity fluid flow also decreases, so the relation between radius and velocity of fluid flow is direct proportionality.

To get pressure profile under these case using equation (8),

$$\frac{\partial p}{\partial r} = \rho\left(\frac{{v_\theta}^2}{r}\right)$$

$$\frac{\partial p}{\partial r} = \rho\left(\frac{(\Omega_O r)^2}{r}\right) \quad = \rho\left(\frac{{\Omega_o}^2 r^2}{r}\right)$$

$\frac{\partial p}{\partial r} = \rho{\Omega_o}^2$ r and integrate both sides with respect to r

$$\int dp = \int \rho{\Omega_o}^2 \, r \, dr$$

$$p = \frac{1}{2}\rho{\Omega_o}^2 r^2 + C^* \tag{24}$$

Where C* can be evaluated by specifying the value of the pressure,

$$P = P_{R_O} \text{ at } r = R_O \text{ that is,}$$

$$C^* = P_{R_O} - \frac{1}{2}\rho\Omega_o{}^2 r^2$$

Equation (24) is called pressure profile when the inner cylinder is removed.

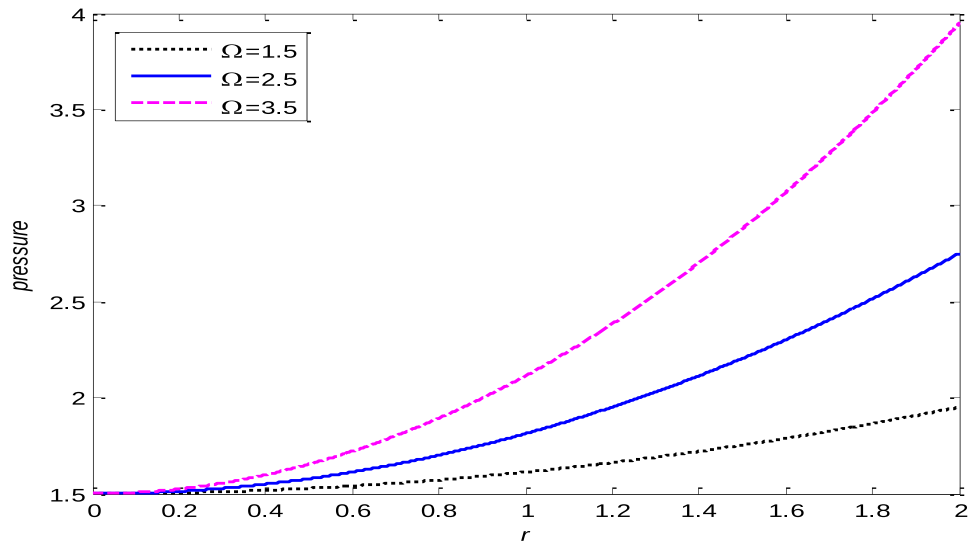


Figure.7. Pressure profile, $r \in [0, 2]$

As we observe from figure 7, when the value of radius is increased the pressure of fluid flow increased and if the value of radius decreased the pressure of fluid flow also decreased.

4.2.2. Analysis of torque which is needed to rotate cylinder

I. Consider two long concentric cylinders with the outer cylinder rotating with constant angular velocity Ω_o and the inner cylinder at rest. Determine the torque of fluid which is exerted to the outer cylinder. The boundary conditions (BCs) are that the fluid does not slip (no-slip) at the surface of the two cylindrical surfaces. When the outer cylinder is rotating and the inner cylinder is fixed, then the BCs are:

at $r = R_o$, $\boldsymbol{v}_\theta = R_o \Omega_o,$

at $r = R_I$, $\boldsymbol{v}_\theta = 0.$

From this we can find $\boldsymbol{v}_\theta$ at $r = R_I$

$$\boldsymbol{v}_\theta = \frac{C_1 R_I}{2} + \frac{C_2}{R_I} = 0,$$

then implies that $C_2 = -\frac{C_1 R^2{}_I}{2}$

Similarly at $r = R_o$, $\boldsymbol{v}_\theta = R_o \Omega_o,$

$$\frac{C_1 R_o}{2} + \frac{C_2}{R_o} = R_o \Omega_o,$$

and then $2{R_o}^2 \Omega_o = C_1 {R_o}^2 + 2C_2$ (25)

After substituting C_2 in to equation (25) we obtain

$$C_1 = \frac{2{R_o}^2}{{R_o}^2 - {R_I}^2}\Omega_o \text{ and } C_2 = \frac{-{R_o}^2 {R_I}^2}{{R_o}^2 - {R_I}^2}\Omega_o$$

Substituting those values for constants of integration into $v_\theta = \frac{C_1 r}{2} + \frac{C_2}{r}$ yields the final expression for the velocity profile given by:

$$v_\theta = \frac{{R_o}^2}{({R_o}^2 - {R_I}^2)}\frac{1}{r}\left(r^2 - {R_I}^2\right)\Omega_o \,. \tag{26}$$

Shear stresses of the fluid are obtained by:

$\tau_{r\theta} = \eta r \frac{d}{dr}\left(\frac{v_\theta}{r}\right)$. Replacing v_θ from equation (26) we can get:

$$\tau_{r\,\theta} = \eta r \frac{d}{dr}\left(\frac{\frac{R_o{}^2}{(R_o{}^2 - R_I{}^2)r}\frac{1}{r}(r^2 - R_I{}^2)\Omega_o}{r} \right) \;=\; \eta r \frac{d}{dr}\left(\frac{\frac{R_o{}^2}{(R_o{}^2 - R_I{}^2)}(r^2 - R_I{}^2)\Omega_o}{r^2} \right)$$

The derivative of these with respect to r gives:

$$\tau_{r\theta} = 2\eta \frac{R_o{}^2\, R_I{}^2}{R_o{}^2 - R_I{}^2}\Omega_o \frac{1}{r^2} \tag{27}$$

The shear stress exerted by the liquid to the outer cylinder is

$$\tau_w = -\,\tau_{r\theta}|\; r = R_0 = \;-\,2\eta \frac{R_o{}^2\, R_I{}^2}{R_o{}^2 - R_I{}^2}\Omega_o \frac{1}{R_o{}^2}$$

$$\tau_w = \;-\,2\eta \frac{R_I{}^2}{R_o{}^2 - R_I{}^2}\Omega_o \tag{28}$$

The torque T per unit height L at the outer Cylinder is:

$$\frac{T}{L} = 2\pi R_0{}^2\,(-\tau_w)$$

$$\frac{T}{L} = 2\pi R_0{}^2\left(-\,2\eta \frac{R_I{}^2}{R_o{}^2 - R_I{}^2}\Omega_o\right)$$

$$\frac{T}{L} = 4\eta\pi \frac{R_o{}^2\, R_I{}^2}{R_o{}^2 - R_I{}^2}\Omega_o$$

$$T = 4\eta\pi \frac{R_o{}^2\, R_I{}^2}{R_o{}^2 - R_I{}^2}\Omega_o L \tag{29}$$

Equation (29) is the torque which is exerted by the liquid to the outer cylinder.

II. Consider two concentric cylinders with the inner cylinder rotating with constant angular velocity Ω_I and the outer cylinder is fixed. Determine the torque of fluid which is exerted to inner cylinder. The boundary conditions are that the fluid does not slip at the surface of the two cylindrical surfaces. When the inner cylinder is rotating and the outer cylinder is fixed, then the BCs are: at $r = R_o$, $v_\theta = 0$,

at $r = R_I$, $v_\theta = \Omega_I R_I$.

From this we can find v_θ

at $r = R_O$ $\quad v_\theta = \frac{C_1 R_O}{2} + \frac{C_2}{R_O} = 0$, and we have $\quad C_2 = -\frac{C_1 R^2{}_O}{2}$

at r $= R_I$, $\quad v_\theta = R_I \Omega_I$

Implies that $\frac{C_1 R_I}{2} + \frac{C_2}{R_I} = R_I \Omega_I$ and then $2{R_I}^2 \Omega_I = C_1 {R_I}^2 + 2C_2$ (30)

After substituting C_2 in to equation (30), we obtain

$$C_1 = \frac{2{R_I}^2}{{R_o}^2 - {R_I}^2} \Omega_I \text{ and } C_2 = \frac{-{R_o}^2 {R_I}^2}{{R_o}^2 - {R_I}^2} \Omega_I.$$

Substituting of those values for constants of integration into $v_\theta = \frac{C_1 r}{2} + \frac{C_2}{r}$, then velocity profile is given by;

$$v_\theta = \left(\frac{2R^2{}_I \Omega_I}{R^2{}_I - R^2{}_O}\right)\left(\frac{r}{2}\right) - \left(\frac{{R_o}^2 R^2{}_I \Omega_I}{R^2{}_I - R^2{}_O}\right)\left(\frac{1}{r}\right),$$

$$v_\theta = \frac{{R_I}^2}{({R_I}^2 - {R_O}^2)} \frac{1}{r} \left(r^2 - {R_O}^2\right) \Omega_I \tag{31}$$

Shear stresses of the fluid are obtained by:

$$\tau_{r\theta} = \eta r \frac{d}{dr}\left(\frac{v_\theta}{r}\right)$$

$$\tau_{r\,\theta} = \eta r \frac{d}{dr}\left(\frac{\frac{{R_I}^2}{({R_I}^2 - {R_O}^2)} \frac{1}{r}(r^2 - {R_O}^2)\Omega_I}{r}\right) = \eta r \frac{d}{dr}\left(\frac{\frac{{R_I}^2}{({R_I}^2 - {R_O}^2)}(r^2 - {R_O}^2)\Omega_I}{r^2}\right)$$

$$\tau_{r\theta} = \eta r \left(2 \frac{{R_o}^2 {R_I}^2}{{R_I}^2 - {R_o}^2} \Omega_I \frac{1}{r^3}\right)$$

$$\tau_{r\theta} = 2\eta \frac{{R_o}^2 {R_I}^2}{{R_I}^2 - {R_o}^2} \Omega_I \frac{1}{r^2} \tag{32}$$

The shear stress exerted by the liquid to the inner cylinder is:

$$\tau_w = -\tau_{r\theta}|\, r = R_I \;=\; -2\eta \frac{{R_o}^2 {R_I}^2}{{R_I}^2 - {R_O}^2} \Omega_I \frac{1}{{R_I}^2}$$

$$\tau_w = -2\eta \frac{R_o^2}{R_I^2 - R_O^2} \Omega_I \tag{33}$$

The torque T per unit height L to the inner cylinder:

$$\frac{T}{L} = 2\pi R_I^2 (-\tau_w)$$

$$\frac{T}{L} = 2\pi R_I^2 \left(-2\eta \frac{R_o^2}{R_I^2 - R_O^2} \Omega_I\right)$$

$$\frac{T}{L} = 4\eta\pi \frac{R_o^2 R_I^2}{R_I^2 - R_O^2} \Omega_I$$

$$T = 4\eta\pi \frac{R_o^2 R_I^2}{R_I^2 - R_O^2} \Omega_I L \tag{34}$$

Equation (34), is the torque which is exerted by the liquid to the inner cylinder.

4.2.3. Free surface shape for cylindrical vessel rotating about its own axis

I) let us consider the inner cylinder and outer cylinder rotates with equal angular velocity when the inner and outer cylinders are rotating, then the BCs are:

At $r = R_o$, $\quad \boldsymbol{v}_\theta = \Omega_O R_O$

At $r = R_I$, $\quad \boldsymbol{v}_\theta = \Omega_I R_I$

From this we can find v_θ

$$\text{At } r = R_O \qquad \boldsymbol{v}_\theta = \frac{C_1 R_O}{2} + \frac{C_2}{R_O} = \Omega_O R_O \tag{35}$$

$$\text{At } r = R_I \qquad \boldsymbol{v}_\theta = \frac{C_1 R_I}{2} + \frac{C_2}{R_I} = \Omega_I R_I \tag{36}$$

From equation (35), we have: $C_1 {R_O}^2 + 2C_2 = 2\Omega_O {R_O}^2$

From equation (36) we have: $C_1 R_I^2 + 2C_2 = 2\Omega_I R_I^2$

Solving equation (35) and equation (36), simultaneously and we get

$C_1 = 2\Omega_O$ & $C_2 = 0$ Since $\Omega_I = \Omega_O = \Omega$

Then the velocity distribution is given by:

$$v_\theta = \frac{C_1 r}{2} + \frac{C_2}{r}$$

$$v_\theta = \frac{2\Omega_O r}{2} + \frac{0}{r}$$

$$v_\theta = \Omega r \qquad (37)$$

The pressure is given by $\frac{\partial p}{\partial r} = \rho\left(\frac{v^2{}_\theta}{r}\right) = \rho\frac{\Omega^2 r^2}{r} = \rho\Omega^2 \mathrm{r}$

$$\frac{\partial p}{\partial r} = \rho\Omega^2 \mathrm{r} \qquad \text{integrating with respect to r}$$

$$\int \frac{\partial p}{\partial r} = \int \rho\Omega^2 \mathrm{r}$$

$$\int dp = \int \rho\Omega^2 \mathrm{r}\, \mathrm{dr}$$

$$\mathrm{P(r, z)} = \frac{1}{2}\rho\Omega^2 r^2 + \mathrm{f}_1\,(\mathrm{z}) \qquad (38)$$

Substitute in z component $\frac{\partial \mathrm{p}}{\partial \mathrm{z}} = -\ \rho \mathrm{g}$

$$\frac{df_1}{dz} = -\ \rho \mathrm{gz} + \mathrm{c} \qquad (39)$$

Substituting equation (39), in equation (38)

$$\mathrm{P(r, z)} = \frac{1}{2}\rho\Omega^2 r^2 - \rho \mathrm{gz} + \mathrm{c} \qquad (40)$$

Let z_0 be height of liquid in cylinder at r = 0 and $p = p_0$ where p_0 is atmospheric pressure, we can get the constant c which is given by:

$$\mathrm{c} = p_0 + \rho g z_0 \qquad (41)$$

Substituting equation (41) in equation (40)

$$p = \frac{1}{2}\rho\Omega^2 r^2 - \rho \mathrm{gz} + p_0 + \rho g z_0$$

$$p = \frac{1}{2}\rho\Omega^2 r^2 - \rho \mathrm{g}\,(\mathrm{z} - z_0) + p_0$$

Then $\mathrm{p} - p_0 = 0 = \frac{1}{2}\rho\Omega^2 r^2 - \rho \mathrm{g}\,(\mathrm{z} - z_o)$

$$\rho g\,(z - z_o) = \frac{1}{2}\rho\Omega^2 r^2$$

$$z = z_o + \frac{1}{2g}\,\Omega^2 r^2 \quad \text{where } z_o \text{ is height of cylinder.} \tag{42}$$

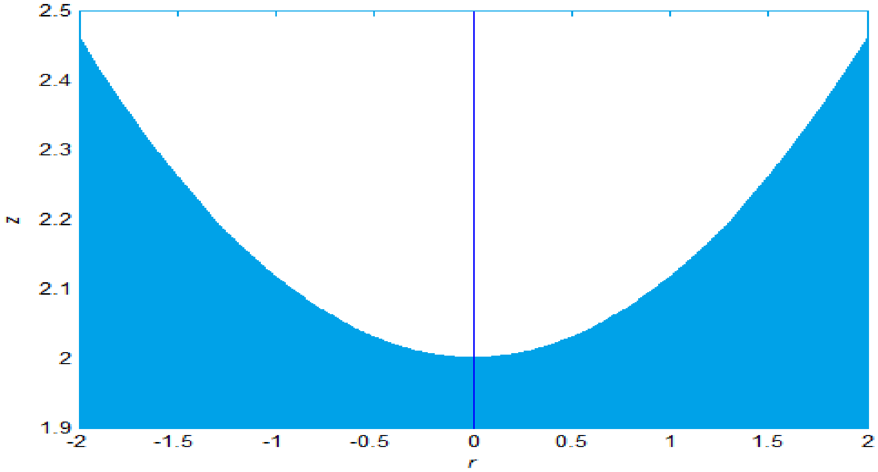


Figure 8. Free surface shape of fluid in cylinder -I

II. Let us consider the inner cylinder is removed and outer cylinder rotate with angular velocity of Ω_0. From equation (23) we have $v_\theta = \Omega_O r$ and using r- component the pressure is given by

$$\frac{\partial p}{\partial r} = \rho\left(\frac{{v_\theta}^2}{r}\right)$$

$$\frac{\partial p}{\partial r} = \rho(\Omega_0)^2 r \tag{43}$$

Integrating equation (43) with respect to r

$$\int dp = \int \rho(\Omega_0)^2 r dr$$

$$P\,(r, z) = \frac{1}{2g}\,\rho(\Omega_0)^2 r^2 + f_1(z) \tag{44}$$

Substitute in z momentum we obtain

$$\frac{\partial p}{\partial z} = -\rho g$$

$$\frac{d_{f1}}{dz} = -\rho g z + c$$

$f_1\,(z) = -\rho g z + c$ substituting in equation (44) we obtain:

$$P = \frac{1}{2g}\,\rho(\Omega_0)^2 r^2 - \rho g z + c \tag{45}$$

Let z_0 be height of liquid in cylinder at r = 0 and p = p_0, where p_0 is atmospheric pressure, we can get the constant c which is given by c = p_0 + $\rho g z_0$ (46)

Substitute equation (46) in equation (45) we obtain

$$P = \frac{1}{2g}\, \rho(\Omega_0)^2 r^2 - \rho g\,(z - z_0) + p_0$$

$$0 = p - p_0 = \frac{1}{2}\, \rho(\Omega_0)^2 r^2 - \rho g\,(z - z_0)$$

$$\rho g\,(z - z_0) = \frac{1}{2}\, \rho(\Omega_0)^2 r^2$$

$$z = z_0 + \frac{(\Omega_0)^2 r^2}{2g} \qquad (47)$$

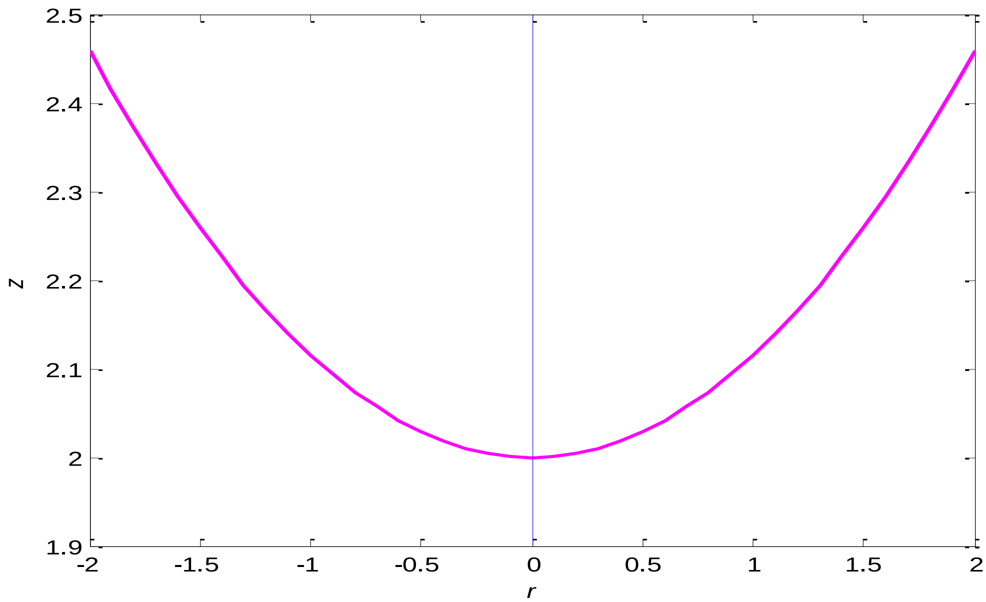


Figure 9. Free surface shape of fluid in cylinder -II

From figure 8 & 9 we conclude that the shape of free surface of fluid in cylinder is parabola. So the shape of the free surface of liquid in rotating cylinder at steady state has a parabolized shape.

CHAPTER FIVE

CONCLUSIONS AND FUTURE SCOPE

5.1. Conclusion

In this study two cases are presented for a flow of an incompressible fluid between two rotating coaxial cylinder. Namely, when inner and outer cylinder rotate with different angular velocity Ω_I & Ω_o respectively in the same direction and the outer cylinder rotate with angular velocity Ω_o and inner cylinder removed. In a circular Couette flow of incompressible fluid, the velocity field involves only v_θ and the magnitude is a function of the radial coordinate.

In a circular Couette flow of incompressible fluid, the pressure distribution is a function of r. From Figure 4.a and 4.b, we observed that as the value of radius increases the velocity of fluid flow increases and also when the values of radius decreases the velocity of fluid flow in cylinder decreases. That is, the more close the fluid no-slips at the boundary, the less its velocity is affected by the motion of the boundary. From Figure 5.a. and 5.b, we can observe that increasing the radius between two cylinders has an increasing effect of pressure; also the decreasing value of radius between two cylinders, pressure of fluid flow is decreasing.

From figure 6, we observe that as the value of radius increases the velocity of the fluid increases and also when the value of radius decrease the velocity fluid flow also decreases, so the relation between radius and velocity of fluid flow is direct proportionality. As we observe from figure 7, when the value of radius increases the pressure of fluid flow increases and if the value of radius decreases the pressure of fluid flow also decreases.

A free surface is a surface between two homogeneous fluids, for example between liquid and the air in the Earth's atmosphere. If a liquid is contained in a cylinder rotating around a vertical axis coinciding with the axis of the cylinder, as we see from figure 8 & 9 the free surface is parabolic. This parabolic shape is used to create liquid mirror telescope. Liquid mirror telescope is a telescope made from a reflective liquid. Liquid mirrors rotate in the shape of a parabola, this shape needed for the liquid mirror to be able to collect and focus incoming starlight.

5.2. Future Scope

In fluid mechanics, incompressible flow refers to a flow in which the material density is constant within a fluid parcel. Incompressible fluid flow between rotating coaxial cylinders has received notable attention in fluid mechanics, applied mathematics and chemical engineering. Based on this, it can be recommended that the upcoming post graduate student and other researchers who are interested in this area to use the result of this as platform and make further investigation in Couette flow when three coaxial cylinders rotating in different angular velocity.

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