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ANALYSIS OF A FINITE BUFFER GENERAL INPUT QUEUE WITH MARKOVIAN SERVICE PROCESS AND ACCESSIBLE AND NON-ACCESSIBLE BATCH SERVICE

Pikkala Vijaya Laxmi

Department of Applied Mathematics, Andhra University
Visakhapatnam - 530 003, India

Obsie Mussa Yesuf

Department of Mathematics, Andhra University
Visakhapatnam - 530 003, India

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Abstract. Queues with Markovian service process (MSP) are mainly useful in modeling and performance analysis of telecommunication networks based on asynchronous transfer mode (ATM) environment. This paper analyzes a finite buffer single server batch service (a, b) queue with general input and Markovian service process (MSP). The server accesses new arrivals even after service has started on any batch of initial number a. This operation continues till the service time of the ongoing batch is completed or the maximum accessible capacity d $(a \leq d < b)$ of the batch being served is attained whichever occurs first. Using the embedded Markov chain technique and the supplementary variable technique we obtain the steady state queue length distributions at pre-arrival and arbitrary epochs. The primary focus is on the various performance measures of the steady state distribution of the batch service, special cases and also on numerical illustrations.

1. Introduction.

Batch service queues have been extensively studied by many researchers over the last three decades. These queues have wide applications in many areas, for example, in loading and unloading of cargoes at a seaport, in manufacturing systems, in semiconductor foundries, in transportation systems, in lift operations, in traffic signal systems, in distribution logistics, etc. Recent applications include computer networks where jobs are processed in batches with a limit on the number of jobs taken at a time for processing. For extensive studies related to batch service queues, see Medhi [20], Chaudhry and Templeton [7], Dshalalow [8], Gold and Tran-Gia [9], Hébuterne and Rosenberg [16], Chakravarthy [5, 6] and Gupta and Vijaya Laxmi [15], etc.

In typical batch service queueing models, once the service is started, arriving customers cannot enter the service station though enough space is available to accommodate them. But in many practical situations the arriving customers are considered for service with current batch in service with some limitation, for example, cinema hall, elevator etc. That is, in the general batch service (a, b) rule, if

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a batch being served does not employ its full capacity of service, late arrivals may join the ongoing service as long as the number in that service batch is less than a pre-defined threshold d ($a \leq d < b$). The service time of the batch is not changed by inclusion of such arriving customers in course of ongoing service. Such a batch is said to be an accessible batch (AB). However, if the number in the service batch exceeds d, the batch becomes non-accessible for late arriving customers and such a batch is called non-accessible batch (NAB). This has been considered by Gross et al. [13] and Kleinrock [17]. The infinite buffer queue with accessible and non-accessible batch service rule has been studied by Sivasamy [24]. In discrete-time systems, the same type of model has been studied by Goswami et al. [10] with finite and infinite buffers. Recently, the infinite buffer discrete time batch service queue with accessible batch and geometric arrivals and negative Binomial distributed service times has been analyzed by Sivasamy and Pukazhenthi [25]. Goswami and Sikdar [11] have discussed the discrete time batch service $GI/Geo^{(a,b)}/1/N$ queue with accessible and non-accessible batches using recursive method.

Queueing models with non-renewal arrivals and Markovian services are often used to model networks of complex computer and telecommunication systems. In such systems both the arrival and service processes may exhibit correlations which have significant impact on queueing performance. Markovian arrival process (MAP) is used to capture the correlation among the inter-arrival times. Similarly, Batch Markovian arrival process ($BMAP$) is used to capture the correlations among the inter-batch arrival times. $BMAP$ is a versatile Markovian point process (N-process) which was introduced by Neuts [21] and later formalized by Lucantoni [18]. Like these non-renewal arrival processes, Markovian service process (MSP) is a versatile service process which can capture the correlation among successive service times. Several other service processes, for example, Poisson process, Markov modulated Poisson process ($MMPP$), PH-type renewal process, Interrupted Poisson process (IPP), etc, are the special cases of MSP. For details of MSP, readers are referred to Bocharov [4], Albores and Tajonar [1], Gupta and Banik [14], etc.

Recently, Banik et al. [3] have analyzed the batch service $GI/MSP^{(a,b)}/1/N$ queue using the methods of supplementary variable and embedded Markov chain and obtained the queue length distributions at pre-arrival and arbitrary epochs. It may be noted that the accessible batch service has more economic utilizations in providing better service to the queue. For example, in many shuttle transportation systems, we observe units being transported according to accessibility rule with some limitation. One can view this as priority services where the late arrival (priority) job gets service without affecting the service time of the ongoing batch. Therefore, this paper gives an extension of the work of Banik et al. [3]. To be more specific, we present the analysis of a finite buffer general input queue with Markovian service process and with accessible and non-accessible batch service i.e., $GI/MSP^{(a,d,b)}/1/N$ queue. Using the supplementary variable and embedded Markov chain techniques we have obtained the steady state distributions of the number in the system (queue) at pre-arrival and arbitrary epochs. Some numerical results have been presented in the form of tables and graphs. As a special case, when the accessibility limit equals the minimum batch size, i.e., $a = d$, the present model reduces to the general batch service $GI/MSP^{(a,b)}/1/N$ queue, Banik et al. [3]. The model presented in this paper may be useful in polling systems, cinema theatres, communication routers where the trade-off between batch services and arrival times is adopted to capture processing times accurately and to control the access to the communication media.

The rest of this paper is organized as follows: Section 2 presents the necessary notations and description of the model. Section 3 gives the analytic analysis of the model. Sections 4 and 5 deal with performance measures and numerical results, respectively and Section 6 concludes the paper.

2. **Notations and description of the model.** Let us consider a finite buffer queue wherein the customers inter-arrival times are independent, identically distributed (i.i.d.) random variables with probability distribution function $A(u)$, probability density function $a(u), u \geq 0$, Laplace-Stieltjes transform (LST) $A^*(\theta), Re(\theta) \geq 0$ and mean inter-arrival time $1/\lambda = -A^{*(1)}(0)$ where $h^{(1)}(0)$ is the first derivative of $h(\theta)$ evaluated at $\theta = 0$. The customers are served by a single server in batches of maximum size b with a minimum threshold value a. However, if the number of customers in the queue is less than a, the server remains idle until the queue size reaches a. If b or more customers are present in the queue at service initiate epoch then only b of them are taken into service and the rest of the customers will wait in the queue whose size is taken as finite N. It is further assumed that the late entries can join a batch in course of ongoing service as long as the number of customers in that batch is strictly less than d (called accessible limit). At every departure epoch, that is, before initiating service of the next batch, the server may find the system in any one of the following three cases: (i) $0 \leq n \leq a-1$, (ii) $a \leq n \leq d-1$ and (iii) $n \geq d$. In case (i), the server cannot initiate service, it remains idle. In case (ii), the server takes the entire queue for batch service and admits the subsequent arrivals in the batch while the service is on, till the accessible limit d is reached, and such a batch is called an accessible batch (AB). In case (iii), it takes $min(n, b)$ customers for the service and does not allow further arrivals into the batch being served even if the current batch size is not b, that is, when the batch size is greater than or equal to d, the batch becomes non-accessible (NAB) for late arriving customers.

The Markovian service process is a generalization of the Poisson process where the services are governed by an underlying m-state Markov chain. With transition rate $L_{ij},\ 1 \leq i,j \leq m,\ i \neq j$ there is a transition from state i to state j in the underlying Markov chain without a service completion, and with transition rate $M_{ij},\ 1 \leq i,j \leq m$, there is a transition from state i to state j in the underlying Markov chain with a service completion. The matrix $\mathbf{L} = [L_{ij}]$ has non-negative off-diagonal and negative diagonal elements, and the matrix $\mathbf{M} = [M_{ij}]$ has non-negative elements and both have at least one positive entry. Let $N(t)$ denotes the number of customers served in $(0, t]$ and $J(t)$ be the state of the underlying Markov chain at time t with state space $\{i : 1 \leq i \leq m\}$. Then $\{N(t), J(t)\}$ is a two dimensional Markov process with state space $\{(n, i) : n \geq 0, 1 \leq i \leq m\}$. The infinitesimal generator of the above Markov process is given by

$$\mathbf{Q} = \begin{pmatrix} \mathbf{L} & \mathbf{M} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ 0 & \mathbf{L} & \mathbf{M} & \mathbf{0} & \mathbf{0} & \cdots \\ 0 & 0 & \mathbf{L} & \mathbf{M} & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and $\{N(t), J(t)\}$ is called the Markovian service process (MSP). Since $\mathbf{Q}$ is the infinitesimal generator of the MSP, we have $(\mathbf{L} + \mathbf{M})\mathbf{e} = \mathbf{0}$ where $\mathbf{e}$ is a $m \times 1$ vector with all its elements equal to 1. Further, $\mathbf{L}+\mathbf{M}$ is the infinitesimal generator of the underlying Markov chain $\{J(t)\}$, there exists a stationary probability vector $\overline{\mathbf{\Pi}}$ such that $\overline{\mathbf{\Pi}}(\mathbf{L}+\mathbf{M}) = \mathbf{0}$, $\overline{\mathbf{\Pi}}\mathbf{e} = 1$. The fundamental service rate of the above Markov process is given by $\mu^* = \overline{\mathbf{\Pi}}\mathbf{M}\mathbf{e}$ and the lag k coefficient of correlation is computed by, see [14],

$$r[k] = \frac{\mu^*\overline{\mathbf{\Pi}}[(-\mathbf{L})^{-1}\mathbf{M}]^k(-\mathbf{L})^{-1}\mathbf{e}-1}{2\mu^*\overline{\mathbf{\Pi}}(-\mathbf{L})^{-1}\mathbf{e}-1},\ k > 1.$$

The case when the server remains idle for certain time interval and then a customer enters, the service process begins with the initial service phase distribution given by $f_j,\ j = 1, 2, \dots, m,\ \sum_{j=1}^{m} f_j = 1$. This phase process is independent of the path followed by the previous service period. This is called idle-restart service phase distribution suggested by Neuts [21] and later by Albores and Tajonar [1]. Thus, a MSP is characterized by the matrices $\mathbf{L}$, $\mathbf{M}$ and the phase distribution vector $\mathbf{f} = (f_1, f_2, \dots, f_m)$.

The customers are served according to the Markovian service process and accessibility to batches with stationary mean service rate equal to μ^*. The traffic intensity ρ is given by $\rho = \lambda/b\mu^*$. The state of the system at time t is described by the following random variables, namely

- $N_s(t)\big(N_q(t)\big)$ = number of customers present in the system (queue),
- $U(t)$ = remaining inter-arrival time for the next arrival,
- $\zeta(t) = \begin{cases} 0, & \text{if the server is idle or busy with an accessible batch,} \\ 1, & \text{if the server is busy with a non-accessible batch.} \end{cases}$

Let us define the joint probabilities by

$$\begin{aligned} P^j_{n,0}(u,t)du &= Pr\left[N_s(t) = n,\ J(t) = j,\ u < U(t) \le u + du,\ \zeta(t) = 0\right],\ u \ge 0, \\ &\qquad\qquad 0 \le n \le d-1, \\ P^j_{n,1}(u,t)du &= Pr\left[N_q(t) = n,\ J(t) = j,\ u < U(t) \le u + du,\ \zeta(t) = 1\right],\ u \ge 0, \\ &\qquad\qquad 0 \le n \le N. \end{aligned}$$

In the limiting case, as $t \to \infty$, the above probabilities will be denoted by $P^j_{n,0}(u)$ and $P^j_{n,1}(u)$, respectively. Further, let $\mathbf{P}_{n,0}(u), (0 \le n \le d-1)$ and $\mathbf{P}_{n,1}(u), (0 \le n \le N)$ denote the row vectors of order $1 \times m$ whose j-th components are $P^j_{n,0}(u)$ and $P^j_{n,1}(u)$, respectively.

3. Analysis of the model.

In this section, we shall carry out the analytic analysis of the model and obtain the queue length distribution at various epochs.

3.1. Steady state distribution at pre-arrival epoch.

Consider the system just before an arrival of a customer which are taken as embedded points. Let $t_0,\ t_1,$, ... be the time epochs at which successive arrivals occur and t_n^- the time epochs just before the arrival instant t_n. The inter-arrival times $T_{n+1} = t_{n+1} - t_n,\ n = 0, 1,$ ... are i.i.d. random variables with common distribution function $A(u)$. The state of the system at t_i^- is defined as $\{N_s(t_i^-),\ \zeta(t_i^-)\}$, where $N_s(t_i^-)$ is the number of customers in the system and $\zeta(t_i^-) = 0$ represents whether the server is idle/busy with accessible batch and $\zeta(t_i^-) = 1$ represents that the server is busy with a non-accessible batch. In the limiting case, let us assume

$$\begin{aligned} P^{j-}_{n,0} &= \lim_{i\to\infty} Pr\left[N_s(t_i^-) = n,\ \zeta(t_i^-) = 0, J(t_i^-) = j\right],\ 0 \le n \le d-1, 1 \le j \le m, \\ P^{j-}_{n,1} &= \lim_{i\to\infty} Pr\left[N_q(t_i^-) = n,\ \zeta(t_i^-) = 1, J(t_i^-) = j\right],\ 0 \le n \le N, 1 \le j \le m, \end{aligned}$$

where $P_{n,0}^{j-}$ be the probability that the server is idle or busy with n $(0 \leq n \leq d-1)$ customers at pre-arrival epoch and $P_{n,1}^{j-}$ denotes the probability that the server is busy with non-accessible batch and n $(0 \leq n \leq N)$ customers waiting in the queue at pre-arrival epoch and the phase of the service process is j. Further, let $\mathbf{P}_{n,k}^{-}$ be the row vector of order $1 \times m$ of pre-arrival epoch probability whose j-th component is $P_{n,k}^{j-},\ k = 0,\ 1$.

Let $\mathbf{S}_k,\ k \geq 0$, denotes an $m \times m$ matrix whose (i,j)-th element represents the conditional probability that k batches of customers have been served during an inter-arrival time and the underlying Markov chain of the service process is in phase j just before the arrival given that the underlying Markov chain was in phase i at the previous pre-arrival epoch.

Now observing the state of the system at two consecutive embedded points, we have an embedded Markov chain whose finite state space is equivalent to $\Omega = \{(i,j), 0 \leq i \leq d-1,\ 1 \leq j \leq m\} \bigcup \{(i,j), 0 \leq i \leq N,\ 1 \leq j \leq m\}$. The one step transition probability matrix (TPM) $\bar{\mathbf{P}}$ of the above Markov chain has dimension $(N+d+1) \times (N+d+1)$ with four block matrices as given below:

$$\bar{\mathbf{P}} = \begin{pmatrix} \mathbf{A}_{d\times d} & \mathbf{B}_{d\times(N+1)} \\ \mathbf{C}_{(N+1)\times d} & \mathbf{D}_{(N+1)\times(N+1)} \end{pmatrix}$$

The pre-arrival epoch probabilities $\mathbf{P}_{n,0}^{-},\ 0 \leq n \leq d-1$ and $\mathbf{P}_{n,1}^{-},\ 0 \leq n \leq N$ can be determined by solving the system of equations $\mathbf{\Pi} = \mathbf{\Pi}\bar{\mathbf{P}}$, $\mathbf{\Pi}\bar{\mathbf{e}} = 1$, using GTH algorithm (Grassmann et al. [12]), where $\mathbf{\Pi} = (\mathbf{P}_{0,0}^{-}, \mathbf{P}_{1,0}^{-}, \cdots, \mathbf{P}_{a,0}^{-}, \mathbf{P}_{a+1,0}^{-}, \cdots, \mathbf{P}_{d-1,0}^{-}, \mathbf{P}_{0,1}^{-}, \ldots, \mathbf{P}_{N,1}^{-})$ and $\bar{\mathbf{e}}$ is a $(N+d+1)$ dimensional column vector with all its components being unity.

Blocks $\mathbf{A}$ and $\mathbf{B}$ represent the probability of transition from idle/accessible batch service state to idle/ accessible batch service state and non-accessible batch service state, respectively. Their elements have the following expressions:

$$\mathbf{A}_{i,j} = \begin{cases} \mathbf{I}_m & : \ 0 \leq i \leq a-2,\ 0 \leq j \leq a-1,\ i+1 = j, \\ \mathbf{S}_o & : \ a-1 \leq i \leq d-1,\ a \leq j \leq d-1,\ i+1 = j, \\ \psi_1(i) & : \ a-1 \leq i \leq d-1,\ j = 0, \\ \mathbf{0} & : \ \text{otherwise.} \end{cases}$$

$$\mathbf{B}_{i,j} = \begin{cases} \mathbf{S}_o & : \ i = d-1,\ j = 0, \\ \mathbf{0} & : \ \text{otherwise.} \end{cases}$$

Similarly, blocks $\mathbf{C}$ and $\mathbf{D}$ represent the probability of transitions from non-accessible batch service state to idle/accessible batch service and non-accessible batch service state, respectively. Their elements are listed below:

$$\mathbf{C}_{i,j} = \begin{cases} \mathbf{S}_{\lfloor \frac{i}{b} \rfloor + 1} & : \ a-1 \leq i \leq N-1,\ a \leq j \leq d-1,\ \lfloor \frac{i}{b} \rfloor + 1 = j, \\ \psi_2(i) & : \ a-1 \leq i \leq N-1,\ j = 0,\ \lfloor \frac{i}{b} \rfloor = \text{ either } a-1 \text{ or } a \text{ or } \ldots \text{ or } b-1, \\ \psi_3(i) & : \ 0 \leq i \leq N-1,\ 1 \leq j \leq a-1,\ i+1 \geq j,\ \frac{i+1-j}{b}, \text{ is an integer}, \\ \mathbf{C}_{i-1,j} & : \ i = N,\ a \leq j \leq d-1, \\ \mathbf{0} & : \ \text{otherwise.} \end{cases}$$

$$\mathbf{D}_{i,j} = \begin{cases} \mathbf{S}_o & : \ 0 \le i \le N-1, \ 1 \le j \le N, \ i+1 = j, \\ \mathbf{S}_{\lfloor \frac{i}{b} \rfloor + 1} & : \ d-1 \le i \le N-1, \ j = 0, \ \lfloor \frac{i}{b} \rfloor = \text{ either } \ d-1 \text{ or } \ d \text{ or } \ldots \\ & \qquad \text{or } \ b-1, \\ \mathbf{S}_{\frac{i+1-j}{b}} & : \ b \le i \le N-1, \ 1 \le j \le N-b, \ i+1 \ge j, \frac{i+1-j}{b}, \text{ is an} \\ & \qquad \text{integer}, \\ \mathbf{D}_{i-1,j} & : \ i = N, \ 0 \le j \le N, \\ \mathbf{0} & : \ \text{otherwise.} \end{cases}$$

where

$$\psi_1(i) = \Big(\mathbf{I}_m - \Big(\sum_{j=1}^{d-1} \mathbf{A}_{i,j} + \sum_{j=0}^{N} \mathbf{B}_{i,j}\Big)\Big) \, \mathbf{ef}, \ a-1 \le i \le d-1,$$

$$\psi_2(i) = \Big(\mathbf{I}_m - \Big(\sum_{j=1}^{d-1} \mathbf{C}_{i,j} + \sum_{j=0}^{N} \mathbf{D}_{i,j}\Big)\Big) \, \mathbf{ef}, \ 0 \le i \le N-1, \ \lfloor i/b \rfloor = \text{either } \ a-1 \text{ or } \ a \text{ or } \ldots \text{or } \ b-1.$$

$$\psi_3(i) = \Big(\mathbf{I}_m - \Big(\sum_{j=\lfloor \frac{i}{b} \rfloor + 2}^{d-1} \mathbf{C}_{i,j} + \sum_{j=0}^{N} \mathbf{D}_{i,j}\Big)\Big) \, \mathbf{ef}, \ 0 \le i \le N-1,$$

and $[x]$ and $\lfloor x/y \rfloor$ represent the greatest integer contained in x and the remainder obtained after dividing integer x by integer y, respectively. Also, $\mathbf{ef}$ is a stochastic matrix with invariant vector $\mathbf{f}$ and $\mathbf{I}_m$ is an identity matrix of order given in the suffix.

Remark. It may be remarked here that instead of assuming idle-restart service phase distribution, if one considers that a new busy period starts with the same phase where the previous busy period ended, then in this case, the TPM remains the same as above, but the expressions of $\psi_1(i)$, $\psi_2(i)$ and $\psi_3(i)$ will have the following changes.

$$\psi_1(i) = diag\Big(\mathbf{I}_m - \Big(\sum_{j=1}^{d-1} \mathbf{A}_{i,j}\mathbf{e}\mathbf{e}^{'} + \sum_{j=0}^{N} \mathbf{B}_{i,j}\mathbf{e}\mathbf{e}^{'}\Big)\Big), \ a-1 \le i \le d-1,$$

$$\psi_2(i) = diag\Big(\mathbf{I}_m - \Big(\sum_{j=1}^{d-1} \mathbf{C}_{i,j}\mathbf{e}\mathbf{e}^{'} + \sum_{j=0}^{N} \mathbf{D}_{i,j}\mathbf{e}\mathbf{e}^{'}\Big)\Big), \ 0 \le i \le N-1, \ \lfloor i/b \rfloor = \text{either}$$

$$\psi_3(i) = diag\Big(\mathbf{I}_m - \Big(\sum_{j=\lfloor \frac{i}{b} \rfloor + 2}^{d-1} \mathbf{C}_{i,j}\mathbf{e}\mathbf{e}^{'} + \sum_{j=0}^{N} \mathbf{D}_{i,j}\mathbf{e}\mathbf{e}^{'}\Big)\Big), \ 0 \le i \le N-1,$$

where $\mathbf{e}^{'}$ is the $1 \times m$ row vector with all its elements equal to 1, and $diag(\mathbf{W})$ is the diagonal matrix.

Further, the matrices $\mathbf{S}_n$ which occur in the TPM require numerical integration for arbitrary inter-arrival times. One can compute these matrices along the lines proposed by Neuts [23] or by Lucantoni and Ramaswami [19]. However, when the arrival time distribution is of phase type (PH-distribution), $\mathbf{S}_n$ matrices can be efficiently evaluated without any numerical integration, see Neuts [22]. In the following theorem we list some results which are needed for the computation of $\mathbf{S}_n$.

Theorem 3.1. *Let the inter-arrival times $A(x)$ follow PH-distribution with irreducible representation $(\alpha, \mathbf{T})$, where α and $\mathbf{T}$ are of dimension γ, then the matrices $\mathbf{S}_n$ which occur in TPM are given by*

$$\begin{aligned}
\mathbf{S}_n &= \mathbf{U}_n(\mathbf{I}_m \otimes \mathbf{T}^0), \quad 0 \le n \le N, \\
\mathbf{U}_0 &= -(\mathbf{I}_m \otimes \alpha)[\mathbf{L} \otimes \mathbf{I}_\gamma + \mathbf{I}_m \otimes \mathbf{T}]^{-1}, \\
\mathbf{U}_n &= -\mathbf{U}_{n-1}(\mathbf{M} \otimes \mathbf{I}_\gamma)[\mathbf{L} \otimes \mathbf{I}_\gamma + \mathbf{I}_m \otimes \mathbf{T}]^{-1}, \quad 1 \le n \le N,
\end{aligned}$$

and $\otimes$ denotes the Kronecker product of two matrices.

Proof. Following the steps given in Theorem 3.1 in Neuts [22], the matrices $\mathbf{U}_n$ and $\mathbf{S}_n$ can be derived, see Gupta and Vijaya Laxmi [15]. $\square$

3.2. **Steady state distribution at arbitrary epoch.** To obtain the queue length distribution at arbitrary epoch, we develop relations between distributions of number of customers in the system (queue) at pre-arrival and arbitrary epochs. For this we make use of the supplementary variable technique. Now relating the states of the system at two consecutive time epochs t and $t + dt$ and using probabilistic arguments, in steady state, we obtain the following system of differential-difference equations:

$$\begin{aligned}
-\frac{d}{du}\mathbf{P}_{0,0}(u) &= \sum_{k=a}^{d-1} \mathbf{P}_{k,0}(u)\mathbf{M} + \mathbf{P}_{0,1}(u)\mathbf{M}, && (1) \\
-\frac{d}{du}\mathbf{P}_{n,0}(u) &= \mathbf{P}_{n,1}(u)\mathbf{M} + a(u)\mathbf{P}_{n-1,0}(0),\ 1 \le n \le a-1, && (2) \\
-\frac{d}{du}\mathbf{P}_{n,0}(u) &= \mathbf{P}_{n,0}(u)\mathbf{L} + \mathbf{P}_{n,1}(u)\mathbf{M} + a(u)\mathbf{P}_{n-1,0}(0),\ a \le n \le d-1, && (3) \\
-\frac{d}{du}\mathbf{P}_{0,1}(u) &= \mathbf{P}_{0,1}(u)\mathbf{L} + \sum_{k=d}^{b} \mathbf{P}_{k,1}(u)\mathbf{M} + a(u)\mathbf{P}_{d-1,0}(0), && (4) \\
-\frac{d}{du}\mathbf{P}_{n,1}(u) &= \mathbf{P}_{n,1}(u)\mathbf{L} + \mathbf{P}_{n+b,1}(u)\mathbf{M} + a(u)\mathbf{P}_{n-1,1}(0),\ 1 \le n \le N-b, && (5) \\
-\frac{d}{du}\mathbf{P}_{n,1}(u) &= \mathbf{P}_{n,1}(u)\mathbf{L} + a(u)\mathbf{P}_{n-1,1}(0), \quad N-b+1 \le n \le N-1, && (6) \\
-\frac{d}{du}\mathbf{P}_{N,1}(u) &= \mathbf{P}_{N,1}(u)\mathbf{L} + a(u)[\mathbf{P}_{N-1,1}(0) + \mathbf{P}_{N,1}(0)]. && (7)
\end{aligned}$$

where $\mathbf{P}_{n,0}(0)$ and $\mathbf{P}_{n,1}(0)$ are the respective rates of arrivals. Let us define

$$\mathbf{P}^*_{n,0}(\theta) = \int_0^\infty e^{-\theta u}\mathbf{P}_{n,0}(u)du \quad \text{and} \quad \mathbf{P}^*_{n,1}(\theta) = \int_0^\infty e^{-\theta u}\mathbf{P}_{n,1}(u)du,\ Re(\theta) \ge 0.$$

Here $\mathbf{P}_{n,0} \equiv \mathbf{P}^*_{n,0}(0),\ \mathbf{P}_{n,1} \equiv \mathbf{P}^*_{n,1}(0)$ are the arbitrary epoch probabilities. Multiplying (1) to (7) by $e^{-\theta u}$ and integrating with respect to u from 0 to ∞, yields

$$
\begin{aligned}
-\theta\mathbf{P}^*_{0,0}(\theta) &= \sum_{k=a}^{d-1}\mathbf{P}^*_{k,0}(\theta)\mathbf{M} + \mathbf{P}^*_{0,1}(\theta)\mathbf{M} - \mathbf{P}_{0,0}(0), && (8)\\
-\theta\mathbf{P}^*_{n,0}(\theta) &= \mathbf{P}^*_{n,1}(\theta)\mathbf{M} + A^*(\theta)\mathbf{P}_{n-1,0}(0) - \mathbf{P}_{n,0}(0),\ 1 \le n \le a-1, && (9)\\
-\theta\mathbf{P}^*_{n,0}(\theta) &= \mathbf{P}^*_{n,0}(\theta)\mathbf{L} + \mathbf{P}^*_{n,1}(\theta)\mathbf{M} + A^*(\theta)\mathbf{P}_{n-1,0}(0) - \mathbf{P}_{n,0}(0), \\
&\qquad a \le n \le d-1, && (10)\\
-\theta\mathbf{P}^*_{0,1}(\theta) &= \mathbf{P}^*_{0,1}(\theta)\mathbf{L} + \sum_{k=d}^{b}\mathbf{P}^*_{k,1}(\theta)\mathbf{M} + A^*(\theta)\mathbf{P}_{d-1,0}(0) - \mathbf{P}_{0,1}(0), && (11)
\end{aligned}
$$

$$
\begin{aligned}
-\theta\mathbf{P}^*_{n,1}(\theta) &= \mathbf{P}^*_{n,1}(\theta)\mathbf{L} + \mathbf{P}^*_{n+b,1}(\theta)\mathbf{M} + A^*(\theta)\mathbf{P}_{n-1,1}(0) - \mathbf{P}_{n,1}(0), \\
&\qquad 1 \le n \le N-b, && (12)\\
-\theta\mathbf{P}^*_{n,1}(\theta) &= \mathbf{P}^*_{n,1}(\theta)\mathbf{L} + A^*(\theta)\mathbf{P}_{n-1,1}(0) - \mathbf{P}_{n,1}(0), \\
&\qquad N-b+1 \le n \le N-1, && (13)\\
-\theta\mathbf{P}^*_{N,1}(\theta) &= \mathbf{P}^*_{N,1}(\theta)\mathbf{L} + A^*(\theta)[\mathbf{P}_{N-1,1}(0) + \mathbf{P}_{N,1}(0)] - \mathbf{P}_{N,1}(0). && (14)
\end{aligned}
$$

Post-multiplying (8) - (14) by the vector $\mathbf{e}$, adding them and using $(\mathbf{L}+\mathbf{M})\mathbf{e} = \mathbf{0}$, we get

$$\sum_{n=0}^{d-1}\mathbf{P}^*_{n,0}(\theta)\mathbf{e} + \sum_{n=0}^{N}\mathbf{P}^*_{n,1}(\theta)\mathbf{e} = \frac{1-A^*(\theta)}{\theta}\left\{\sum_{n=0}^{d-1}\mathbf{P}_{n,0}(0) + \sum_{n=0}^{N}\mathbf{P}_{n,1}(0)\right\}\mathbf{e}.$$

Taking the limit as $\theta \to 0$, using the normalization condition, $\sum\limits_{n=0}^{d-1}\mathbf{P}_{n,0} + \sum\limits_{n=0}^{N}\mathbf{P}_{n,1} = \overline{\mathbf{\Pi}}$ and after simplification we get

$$\sum_{n=0}^{d-1}\mathbf{P}_{n,0}(0)\mathbf{e} + \sum_{n=0}^{N}\mathbf{P}_{n,1}(0)\mathbf{e} = \lambda. \tag{15}$$

The left hand side of (15) represents the probability that an arrival is about to occur, which is equal to the arrival rate of customers.

3.2.1. *Relation between distributions at arbitrary and pre-arrival epochs.* Relating the pre-arrival epoch probabilities $\mathbf{P}^-_{n,0},\ 0 \le n \le d-1,\ \mathbf{P}^-_{n,1},\ 0 \le n \le N$, with their rates $\mathbf{P}_{n,0}(0),\ 0 \le n \le d-1,\ \mathbf{P}_{n,1}(0),\ 0 \le n \le N$, applying Baye's theorem and using (15), we obtain

$$\mathbf{P}^-_{n,0} = \mathbf{P}_{n,0}(0)/\lambda,\ 0 \le n \le d-1;\ \mathbf{P}^-_{n,1} = \mathbf{P}_{n,1}(0)/\lambda,\ 0 \le n \le N. \tag{16}$$

Our main objective is to obtain the distribution of number of customers in the system (queue) at arbitrary epoch. This is discussed in the following theorems.

Theorem 3.2. *The arbitrary epoch probabilities are given by*

$$\begin{aligned}
\mathbf{P}_{N,1} &= \lambda \mathbf{P}^{-}_{N-1,1}(-\mathbf{L})^{-1},\\
\mathbf{P}_{n,1} &= \lambda\Big(\mathbf{P}^{-}_{n-1,1} - \mathbf{P}^{-}_{n,1}\Big)(-\mathbf{L})^{-1}, N-b+1 \le n \le N-1,\\
\mathbf{P}_{n,1} &= \Big(\mathbf{P}_{n+b,1}\mathbf{M} + \lambda\big(\mathbf{P}^{-}_{n-1,1} - \mathbf{P}^{-}_{n,1}\big)\Big)(-\mathbf{L})^{-1},\ n = N-b, N-b-1, \cdots, 1,\\
\mathbf{P}_{0,1} &= \Big(\sum_{k=d}^{b}\mathbf{P}_{k,1}\mathbf{M} + \lambda\big(\mathbf{P}^{-}_{d-1,0} - \mathbf{P}^{-}_{0,1}\big)\Big)(-\mathbf{L})^{-1},\\
\mathbf{P}_{n,0} &= \Big(\mathbf{P}_{n,1}\mathbf{M} + \lambda\big(\mathbf{P}^{-}_{n-1,0} - \mathbf{P}^{-}_{n,0}\big)\Big)(-\mathbf{L})^{-1},\ a \le n \le d-1.
\end{aligned}$$

Proof. Setting $\theta = 0$ in (10) - (14) and using (16), we obtain the result of the theorem. □

Here, one may note that from Theorem 3.2, we cannot get $\{\mathbf{P}_{n,0}\}_0^{a-1}$. However, these can be obtained using the following theorem.

Theorem 3.3. *The arbitrary epoch probabilities $\{\mathbf{P}_{n,0}\}_0^{a-1}$ are given by*

$$\mathbf{P}_{n,0} = \mathbf{P}^{-}_{n-1,0} - \mathbf{P}^{*(1)}_{n,1}(0)\mathbf{M}, \quad 1 \le n \le a-1,$$

where $\mathbf{P}^{*(1)}_{n,1}(0)$, $(1 \le n \le a-1)$ can be obtained from

$$\mathbf{P}^{*(1)}_{N,1}(0) = \Big(\mathbf{P}_{N,1} - \mathbf{P}^{-}_{N-1,1} - \mathbf{P}^{-}_{N,1}\Big)(-\mathbf{L})^{-1}, \tag{17}$$

$$\mathbf{P}^{*(1)}_{n,1}(0) = \Big(\mathbf{P}_{n,1} - \mathbf{P}^{-}_{n-1,1}\Big)(-\mathbf{L})^{-1}, \quad N-b+1 \le n \le N-1, \tag{18}$$

$$\mathbf{P}^{*(1)}_{n,1}(0) = \Big(\mathbf{P}^{*(1)}_{n+b,1}(0)\mathbf{M} + \big(\mathbf{P}_{n,1} - \mathbf{P}^{-}_{n-1,1}\big)\Big)(-\mathbf{L})^{-1}, \quad n = N-b, N-b-1, \cdots, 1, \tag{19}$$

$$\mathbf{P}^{*(1)}_{0,1}(0) = \Big(\sum_{k=d}^{b}\mathbf{P}^{*(1)}_{k,1}(0)\mathbf{M} + \big(\mathbf{P}_{0,1} - \mathbf{P}^{-}_{d-1,0}\big)\Big)(-\mathbf{L})^{-1}, \tag{20}$$

$$\mathbf{P}^{*(1)}_{n,0}(0) = \Big(\mathbf{P}^{*(1)}_{n,1}(0)\mathbf{M} + \big(\mathbf{P}_{n,0} - \mathbf{P}^{-}_{n-1,0}\big)\Big)(-\mathbf{L})^{-1},\ a \le n \le d-1. \tag{21}$$

Finally, the only unknown quantity $\mathbf{P}_{0,0}$ is obtained by using the normalization condition, i.e., $\mathbf{P}_{0,0} = \overline{\mathbf{\Pi}} - \Big(\sum\limits_{n=1}^{d-1}\mathbf{P}_{n,0} + \sum\limits_{n=0}^{N}\mathbf{P}_{n,1}\Big)$.

Proof. Differentiating (9) - (14) with respect to θ and using (16), the result of the theorem follows. □

Thus, once we know the pre-arrival epoch probability distributions from subsection 3.1 we can obtain the arbitrary epoch probabilities from Theorem 3.2 and Theorem 3.3.

4. **Performance measures.** Performance measures are one of the important features of queueing systems as they reveal the efficiency of the queueing system under consideration. Once the state probabilities at pre-arrival and arbitrary epochs are known, we can evaluate the various performance measures such as:

- the average queue length (L_q) is given by $L_q = \sum_{n=0}^{a-1} n\mathbf{P}_{n,0}\mathbf{e} + \sum_{n=0}^{N} n\mathbf{P}_{n,1}\mathbf{e}$,
- the probability of loss or blocking (P_{loss}) is given by $P_{loss} = \mathbf{P}^{-}_{N,1}\mathbf{e}$,

- the average waiting time in the queue (W_q) of a customer using Little's rule is given by $W_q = L_q/\lambda'$, where $\lambda' = \lambda(1 - P_{loss})$ is the effective arrival rate.

4.1. **Waiting time analysis.** Here we present the waiting time distribution in the queue of an admitted customer under the First-Come First-Served (FCFS) service discipline for $GI/MSP^{(a,d,b)}/1/N$ queueing model for the special case $a = 1$. If a customer upon arrival finds the server idle or busy with an accessible batch, then he will be served immediately, so that his waiting time in the queue is zero. Now assume that the server is busy with a non-accessible batch and n $(0 \le n \le N)$ customers waiting in the queue. Let $\phi_k(\theta)$ be the LST of the probability function that k batches of customers have been served within a time u and the service process upon completion of service passes to phase j and the service process has been in phase i at the beginning of service. Since the probability that the service of a batch of customers is completed in the interval $[u, u+du]$ is given by the matrix $e^{\mathbf{L}u}\mathbf{M}du$, see Gupta and Banik [14] and the total service time of k batches of customers is the sum of their service times, we have

$$\begin{aligned} \phi_k(\theta) &= \phi_1^k(\theta),\ k \ge 2, \\ \text{where} \quad \phi_1(\theta) &= \int_0^\infty e^{-\theta u} e^{\mathbf{L}u}\mathbf{M}du = (\theta\mathbf{I}_m - \mathbf{L})^{-1}\mathbf{M}. \end{aligned}$$

Therefore, the LST of the actual waiting time distribution is given by

$$W_A^*(\theta) = \frac{1}{1 - P_{loss}} \sum_{n=0}^{N-1} \mathbf{P}_{n,1}^{-} \phi_1^{\left[\frac{n}{b}\right]+1}(\theta).$$

The mean waiting time in the queue is given by

$$W_A = -W_A^{*(1)}(0)\mathbf{e} = \frac{1}{1 - P_{loss}} \sum_{n=0}^{N-1} \mathbf{P}_{n,1}^{-} \sum_{k=0}^{\left[\frac{n}{b}\right]} (-\mathbf{L}^{-1}\mathbf{M})^k (-\mathbf{L})^{-1}\mathbf{e}.$$

It may be noted here that the numerical value of the average waiting time in the queue obtained through waiting time analysis matches exactly with the one obtained earlier using Little's rule, as it should be.

5. **Numerical results.** Extensive computational work has been carried out to demonstrate the applicability of the analytical results obtained in previous sections. It also gives some insight into the behavior and application of the model. All the computations have been done in double precision in Mathematica software and the results are reported here up to six decimal places. Some numerical results are presented here in the form of tables and graphs.

Table 1 presents the queue length distributions of $PH/MSP^{(3,5,7)}/1/10$ queue at pre-arrival and arbitrary epochs. Various performance measures are also listed at the bottom of the table. The PH representation is taken as α=[0.35 0.65], $\mathbf{T}=\begin{bmatrix} -4.812 & 1.543 \\ 2.673 & -6.941 \end{bmatrix}$ with mean $1/\lambda = 0.267617$ and MSP is represented by $\mathbf{L}=\begin{bmatrix} -5.79 & 0.79 \\ 0.60 & -0.837 \end{bmatrix}$, $\mathbf{M}=\begin{bmatrix} 5.00 & 0.00 \\ 0.024 & 0.213 \end{bmatrix}$ with stationary service rate $\mu^* = 2.33892$ and phase distribution vector $\mathbf{f} = [0.6\ 0.4]$. Therefore, the stationary probability vector is $\overline{\mathbf{\Pi}} = [0.441301\ 0.558699]$ and traffic intensity is $\rho = 0.228231$.

TABLE 1. Distribution of number of customers in the $PH/MSP^{(3,5,7)}/1/10$ queue

pre-arrival ($P^{j-}_{n,k}$)				arbitrary ($P^{j}_{n,k}$)			
n	$j=1$	$j=2$	$\sum_{j=1}^{m=2}$	n	$j=1$	$j=2$	$\sum_{j=1}^{m=2}$
$P^{j-}_{0,0}$	0.073650	0.049100	0.122751	$P^{j}_{0,0}$	0.115562	0.009366	0.124928
$P^{j-}_{1,0}$	0.083520	0.055680	0.139200	$P^{j}_{1,0}$	0.087934	0.051580	0.139514
$P^{j-}_{2,0}$	0.090636	0.060424	0.151059	$P^{j}_{2,0}$	0.093487	0.057811	0.151298
$P^{j-}_{3,0}$	0.044339	0.058531	0.102870	$P^{j}_{3,0}$	0.043720	0.058434	0.102154
$P^{j-}_{4,0}$	0.023754	0.053240	0.076995	$P^{j}_{4,0}$	0.023454	0.053099	0.076553
$P^{j-}_{0,1}$	0.017057	0.049301	0.066357	$P^{j}_{0,1}$	0.016943	0.049202	0.066144
$P^{j-}_{1,1}$	0.010373	0.042739	0.053112	$P^{j}_{1,1}$	0.010275	0.042591	0.052865
$P^{j-}_{2,1}$	0.007179	0.036722	0.043902	$P^{j}_{2,1}$	0.007129	0.036590	0.043719
$P^{j-}_{3,1}$	0.008844	0.034317	0.043162	$P^{j}_{3,1}$	0.008852	0.034262	0.043114
$P^{j-}_{4,1}$	0.005318	0.028968	0.034286	$P^{j}_{4,1}$	0.005266	0.028851	0.034117
$P^{j-}_{5,1}$	0.003627	0.024307	0.027934	$P^{j}_{5,1}$	0.003600	0.024207	0.027807
$P^{j-}_{6,1}$	0.002708	0.020339	0.023047	$P^{j}_{6,1}$	0.002692	0.020255	0.022947
$P^{j-}_{7,1}$	0.002135	0.016999	0.019132	$P^{j}_{7,1}$	0.002124	0.016926	0.019050
$P^{j-}_{8,1}$	0.001732	0.014195	0.015928	$P^{j}_{8,1}$	0.001724	0.014136	0.015860
$P^{j-}_{9,1}$	0.001427	0.011852	0.013278	$P^{j}_{9,1}$	0.001420	0.011802	0.013223
$P^{j-}_{10,1}$	0.007128	0.059861	0.066989	$P^{j}_{10,1}$	0.007098	0.059610	0.066710
Sum	0.383427	0.616573	1.000000		0.431280	0.568720	1.000000
$L_q = 1.463290,\ W_q = 0.404160,\ P_{loss} = 0.031076$							

In Table 2, we have presented the sensitivity analysis of $E_2/MSP^{(a,d,16)}/1/20$ queue for the average queue length and blocking probability. This has been done by varying a and d, and fixing other parameters as $\lambda = 3.2,\ \rho = 0.4,\ b = 16$ and $N = 20$. We have considered particularly three different service time distributions:

- Poisson with MSP representation $\mathbf{L} = -0.5$, $\mathbf{M} = 0.5$,
- Set 1 MSP with $\mathbf{L}=\begin{bmatrix} -0.430 & 0.006252 \\ 0.500 & -8.6252 \end{bmatrix}$, $\mathbf{M}=\begin{bmatrix} 0.400 & 0.023748 \\ 2.500 & 5.6252 \end{bmatrix}$,
- Set 2 MSP with

$$\mathbf{L} = \begin{bmatrix} -0.5724 & 0.0424 & 0.00 \\ 0.0260 & -0.128098 & 0.020 \\ 0.00 & 0.030 & -2.090 \end{bmatrix},\ \mathbf{M} = \begin{bmatrix} 0.030 & 0.00 & 0.50 \\ 0.00 & 0.078098 & 0.004 \\ 2.015 & 0.005 & 0.04 \end{bmatrix}.$$

The above three service time distributions have the same service rate $\mu^* = 0.5$ and $\mathbf{f}$ is taken as 1.0, [1.0 0.0] and [1.0 0.0 0.0], respectively. Set 1 and Set 2 $MSPs$ have lag 2 correlation coefficient 0.041 and 0.144, respectively. For E_2 inter-arrival time, the PH-type representation is taken as α=[1.0 0.0], $\mathbf{T}=\begin{bmatrix} -\gamma & \gamma \\ 0.0 & -\gamma \end{bmatrix}$ with $\lambda = \gamma/2$ and by suitably varying γ one can get various values of ρ. One can observe from this table that the performance measures L_q and P_{loss} increase with a but decrease as the accessibility limit d increases for all service time distributions. This shows that with accessible batch service the performance of the system has improved. Thus, our model has more economic background than the regular batch service queue. Further, L_q and P_{loss} increase as the correlation coefficient of the service time distribution increases.

Figures 1 and 2 show the effect of traffic intensity (ρ) on the average waiting time in the queue (W_q) for different values of a and d, respectively. In Figure 3,

TABLE 2. Sensitivity analysis of $E_2/MSP^{(a,d,16)}/1/20$ queue for the average queue length and blocking probability.

d	(a,b)	L_q		
		$Poisson$	Set 1 MSP	Set 2 MSP
9	$(2,16)$	2.642640	3.542920	6.394280
	$(5,16)$	3.521270	4.339710	6.455860
	$(8,16)$	5.085560	5.797840	7.388260
12	$(2,16)$	1.611280	2.327170	5.498380
	$(5,16)$	2.466090	3.113520	5.498920
	$(8,16)$	3.847760	4.433610	6.205540
15	$(2,16)$	0.996278	1.514890	4.737900
	$(5,16)$	1.824370	2.282120	4.755580
	$(8,16)$	3.087470	3.500220	5.388320
d	(a,b)	P_{loss}		
		$Poisson$	Set 1 MSP	Set 2 MSP
9	$(2,16)$	0.021015	0.037948	0.191489
	$(5,16)$	0.023409	0.041215	0.169154
	$(8,16)$	0.028531	0.048097	0.158408
12	$(2,16)$	0.012414	0.024515	0.168886
	$(5,16)$	0.014000	0.026862	0.146448
	$(8,16)$	0.017168	0.031518	0.133238
15	$(2,16)$	0.007286	0.015540	0.147935
	$(5,16)$	0.008278	0.017130	0.126924
	$(8,16)$	0.010189	0.020174	0.113691

we have considered the effect of the arrival rate (λ) on W_q. We have considered $E_2/MSP^{(a,d,b)}/1/20$ queue with $d = 10,\ b = 15$ for Figure 1, $a = 3,\ b = 11$ for Figure 2 and $a = 3,\ d = 6$ for Figure 3. The arrival time distribution is E_2 and the service time is Set 1 MSP as discussed in Table 2. From Figures 1 and 2 we observe that as ρ increases, W_q initially decreases and then increases steadily.

From Figure 1, as a increases, the deviations are more for smaller ρ and converge to the same value for lager values of ρ. Further, as a increases, W_q also increases for a fixed ρ. In Figures 2 and 3, we observe that as d and b increase W_q decreases. Also, as ρ or λ increases, W_q initially decreases and then increases steadily. Thus, the suitable choice of threshold batch sizes a and b and accessible limit d make the system more utilizable from the design point of view.

Figures 4 and 5 show the effect of ρ on the average queue length (L_q) and blocking probability (P_{loss}), respectively for $E_2/MSP^{(3,6,9)}/1/30$ queue with the arrival time distribution E_2 and the service time distributions Poisson, Set 1 MSP and Set 2 MSP as discussed in Table 2. In Figure 4, we observe that as ρ increases, L_q shows a steady increase for all service time distributions. We may also note that up to certain level say $\rho = 1.0$, L_q increases as the correlation coefficient increases. But with further increase of ρ, the effect is reversed. In particular for $\rho > 1.3$, L_q corresponding to the highly correlated MSP, i.e., for Set 2 MSP will be the least. High correlation among service times affect the system upto certain level of traffic intensity, and thereafter the effect is quite steady.

In Figure 5, it can be seen that as ρ increases, P_{loss} increases steadily. Further, for fixed ρ, P_{loss} corresponding to highly correlated MSP is the highest. As ρ increases the deviations become smaller.

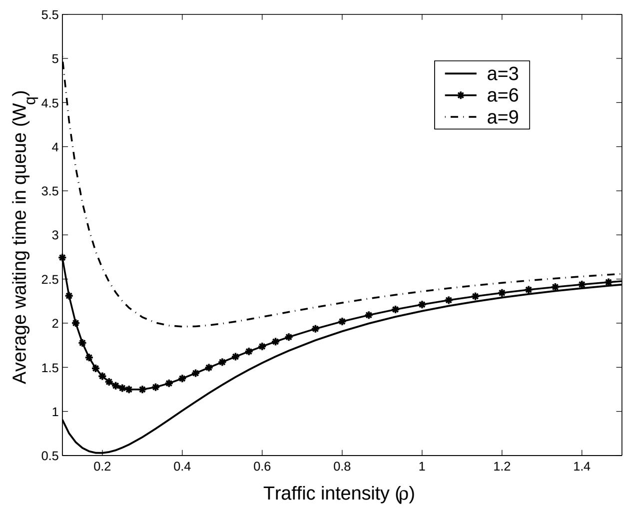

FIGURE 1. Effect of ρ on W_q.

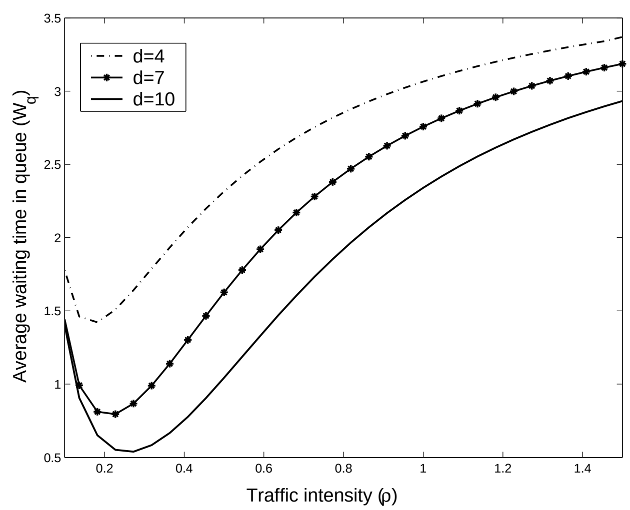

FIGURE 2. Effect of ρ on W_q.

Figure 6 depicts the effect of buffer size (N) on blocking probability (P_{loss}) for $E_2/MSP^{(3,6,9)}/1/30$ queue using Poisson, Set 1 MSP and Set 2 MSP as given above. We have taken $\lambda = 2.25$ and $\rho = 0.5$. On can observe from the figure that as N increases, P_{loss} decreases and asymptotically approaches its minimum value. P_{loss} is very high for Set 2 MSP compared to Poisson and Set 1 MSP.

An examination of these figures gives the following results:

- For fixed value of ρ, the system performance increases when a is small and d is large, and for fixed value of λ, larger values of b yields better performance, as observed in practice.

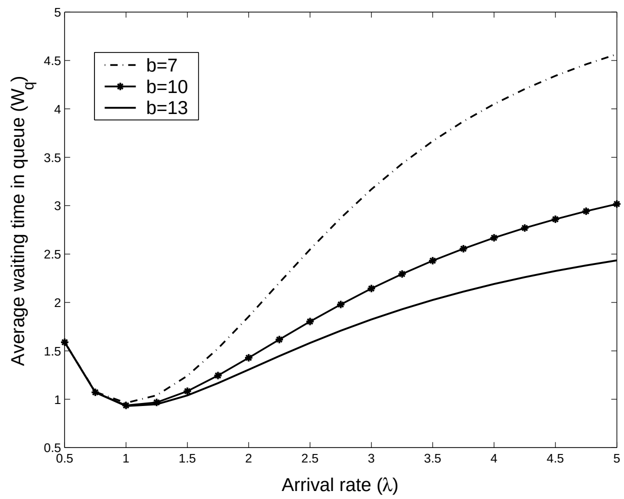

FIGURE 3. Effect of λ on W_q.

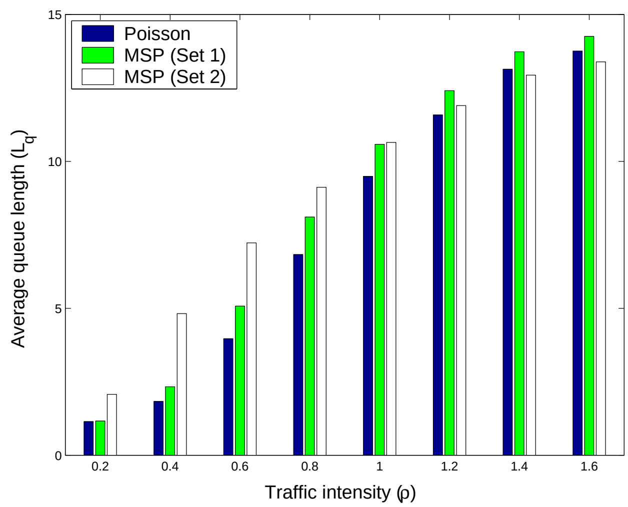

FIGURE 4. Effect of ρ on L_q.

- As λ or ρ increases W_q initially decreases then increases and asymptotically approaches its maximum value, whereas L_q and P_{loss} show a steady increase.
- As N increases P_{loss} decreases and asymptotically approaches to its minimum value for all service time distributions. One may note here that for fixed N, P_{loss} is more for highly correlated service time distribution.

6. **Conclusions.** This paper analyzes a finite buffer single server accessible and non-accessible batch service queue with general input and MSP services. The supplementary variable and the embedded Markov chain techniques have been used to obtain the steady state queue length distributions at pre-arrival and arbitrary epochs. The tables and figures show that the performance of the queueing system

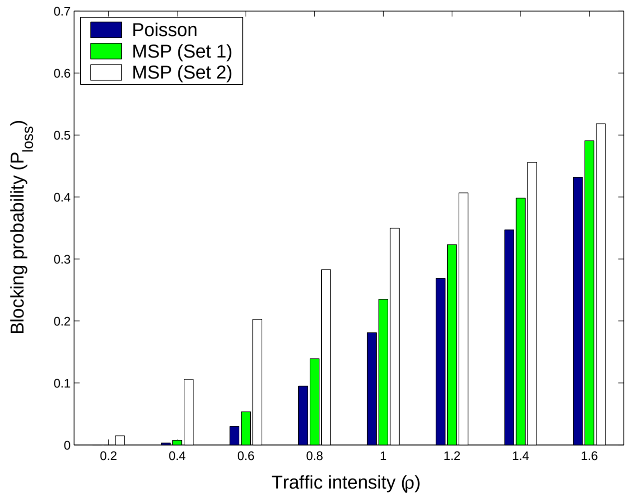

FIGURE 5. Effect of ρ on P_{loss}.

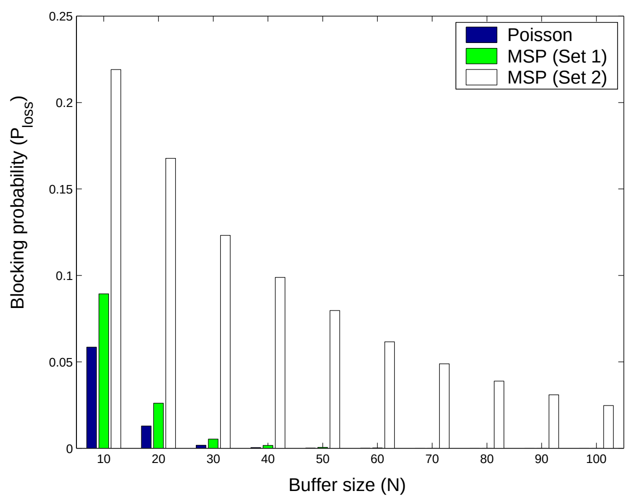

FIGURE 6. Effect of N on P_{loss}.

is not only affected by the arrival and service patterns but also by the correlations among service times of batches of customers. This queueing model has significant applications in the areas of transportation systems, telecommunication systems, computer networks, etc. The techniques used in this paper can be applied to analyze more complex models under batch arrival batch Markovian service process $GI^{[x]}/BMSP^{(a,d,b)}/1$ queue in both finite and infinite buffers. Further, the cost analysis of the models to obtain the optimum limits of the threshold values may become an interesting topic for future investigation.

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E-mail address: vijaya_iit2003@yahoo.co.in
E-mail address: yesufobsie@yahoo.com