Analysis of MagnetoHydroDynamics (MHD) Flow of Nanofluid Over a Porous Medium of an Exponentially Stretching Sheet with Convective Boundary Condition in Presence of Suction/Injection

A Thesis Submitted to the Department of Mathematics, College of Natural Science,Jimma University in Partial Fulfillment for the Requirements of the Degree of Masters of Science in Mathematics

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Declaration

I, the undersigned declare that, this research paper entitled "**Analysis of the MagnetoHydroDynamics (MHD) flow of nanofluid over a porous medium an exponentially stretching sheet with convective boundary condition in presence of suction/injection**" is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged.

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Abstract

In this thesis, the analysis of the MagnetoHydroDynamics(MHD) flow of nanofluid over a porous medium of an exponentially stretching sheet with convective boundary condition in presence of suction/injection is studied. Using a suitable similarity transformation, the governing partial differential equations are transformed into a system of nonlinear higher order ordinary differential equations. The resulting equations are solved numerically by using implicit finite difference scheme known as Keller box method by implementing in MATLAB. The effects of different parameters such as Brownian motion (Nb), thermophoresis (Nt), Eckert number (Ec),Lewis number (Le), permeability (K_p), Prandtl number(Pr), Chemical reaction(R_c) are demonstrated graphically on velocity, temperature and concentration profiles, skin friction coefficient, surface heat transfer rate and mass transfer rate are presented graphically as well. Numerical results obtained in the values of skin friction coefficient are compared with previously reported cases available from literature and they are found to be in a very good agreement.

Nomenclature

T_∞: Ambient fluid temperature.

C_∞: Ambient nanoparticles volume fraction.

Bi: Biot number.

D_B: Brownian diffusion coefficient.

N_b: Brownian motion parameter.

R_C: Chemical Reaction parameter.

Ec: Eckert number.

Q: Heat source/sink parameter.

Le: Lewis number.

Nu_x: Local Nusselt number.

Sh_x: Local Sherwood number.

$B(x)$: Magnetic field strength.

M: Magnetic parameter.

C: Nanoparticles volume fraction.

K_P: Permeability.

P_r: Prandtl number.

N_r: Radiation parameter

C_{fx}: Skin-friction coefficient.

C_p: Specific heat capacity at constant pressure.

U_w, V_w: Stretching velocities.

s: Suction/injection parameter.

C_W: Surface nanoparticles volume fraction.

T_f: Temperature of heat transfer.

T: Temperature of nanofluid .

k: Thermal conductivity.

D_T: Thermophoresis diffusion coefficient.

N_t: Thermophoresis motion parameter.

Greek Symbols

ρ_f: Density of nanofluid.

θ: Dimensionless fluid temperature.

β: Dimensionless nanoparticles volume fraction.

η: Dimensionless similarity variable.

μ: Dynamic viscosity of the base fluid.

$(\rho c)_f$: Heat capacity of the base fluid.

$(\rho c)_p$: Heat capacity of the nanoparticle.

ν: Kinematic viscosity.

σ^*: Stefan-Boltzmann constant.

ψ: Stream function.

τ: The ratio of the nanoparticle heat capacity and the base fluid heat capacity.

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Chapter 1

Introduction

1.1 Background of the Study

The term nanofluid was proposed by (Choi, 1995), referring to dispersion of nanoparticles in the fluids such as water, ethyle glycol, glycol and propylene. Nanofluid is a dilute liquid suspension of nanoparticles with at least one of their principal dimensions smaller than 100 nm. It has been found to possess enhanced thermo physical properties such as thermal conductivity, thermal diffusivity, viscosity and convective heat transfer coefficient compared with those of base fluids like oil and water. Less weight fractions of nanoparticles, when dispersed and suspended stably in base fluid medium, provide drastic enhancements in the thermal properties of base fluids. The vital role of nanofluids is to attain the highest possible thermal properties with low particle weight fractions by uniform dispersion and stable suspension of nanoparticles in base fluid medium. In order to achieve this goal, it is crucial to determine the enhancement of thermal energy transport in liquids. Several engineers and scientists, in the growing nanofluid era, have performed research breakthrough by investigating unexpected thermal properties of nanofluids and also proposed new mechanisms behind improved thermal properties of nanofluids.

Many industries have a strong need for improved fluids that can transfer heat more efficiently. The inherently poor thermal conductivity of convectional fluids puts a fundamental limit on heat transfer. Therefore, for more than a century scientists and engineers have made great efforts to break this fundamental limit. The proposed conventional way to enhance heat transfer in thermal system is to increase the heat transfer surface area of cooling devices and the flow velocity or to disperse solid particle in heat transfer fluids. The concept of using suspensions of solid particles to enhance thermal conductivity of convectional heat transfer fluids was initiated by (Maxwell, 1981). By dispersing millimeter or micrometer sized particles in liquids, Maxwell was able to enhance the thermo physical properties of base fluids. However, major problems such as sedimentation, abrasion, clogging micro channel and high pressure drop prevented the usual micro particles slurries to be used as heat transfer fluids. Because of these reasons, the millimeter or micrometer sized particles suspensions were rejected in heat transfer application.

The boundary layer concept was first introduced by (Prandtl, 1904), which provides major simplifications. This concept is based on the belief that under special conditions certain terms in the governing equations are much smaller than others and therefore can be neglected without significantly affecting the accuracy of the solution. Most boundary-layer models can be reduced to systems of nonlinear ordinary differential equations which are usually solved by numerical methods. The heat transfer characteristic of viscous fluid over a nonlinear stretching sheet was introduced by (Cortel, 2007). A great number of studies for the boundary layer flow over a nonlinear stretching sheet under different aspects of heat and mass transfer, slip and convective boundary conditions and so on, are presented by (Hayat et al., 2009, Abba's and Hayat, 2011, Rana and Bhargava, 2012, Mabood et al., 2015). (Yohannes Yirga and Daniel Tesfaye, 2015) analyzed heat and mass transfer in MHD flow of nanofluids through a porous media due to a permeable stretching sheet with viscous dissipation and chemical reaction effects. MHD boundary layer flow of nanofluid and heat transfer over a porous exponentially stretching sheet in presence of thermal radiation and chemical reaction with suction was presented by (Babu et al., 2017). Inspired and motivated by results of (Babu et al., 2017) the purpose of this study is to analyze numerical solution of MHD flow of a nanofluid over a porous exponentially stretching sheet with convective boundary condition in the presence of suction/injection by employing the implicit finite difference scheme called Keller Box method.

1.2 Statements of the Problem

During the last many years, the study of boundary layer flow and heat transfer over a stretching surface has achieved a lot of success because of its large number of applications in industry and technology. Few of these applications are in materials manufacturing by polymer extrusion, drawing of copper wires, continuous stretching of plastic films, artificial fibers, hot rolling, wire drawing, glass fiber, metal extrusion and metal spinning etc. After the pioneering work of (Sakiadis, 1961), a large amount of literature is available on boundary layer flow of Newtonian and non-Newtonian fluids over linear and nonlinear stretching surfaces (Khan, 2003, Cortel, 2006, Nadeem et al, 2010). Moreover recently, the study of convective heat transfer in nanofluid has achieved great success in various industrial processes. In this study, we will analyze MHD flow of nanofluid over a porous medium of an exponentially stretching sheet with convective boundary condition in the presence of suction/injection by employing the implicit finite difference scheme called Keller Box method. As a result, this study is attempted to answer the following questions:

- What are the similarity transformations apply to change the system of partial differential equations to higher order ordinary differential equations?
- What are the parameters that affect skin friction coefficient, local Nusselt number and Sherwood number?
- How to formulated the implicit finite difference for solving the governing partial differential equations?
- What are certain physical parameters influence velocity, temperature and species concentration.

1.3 Objectives of the Study

1.3.1 General Objective

The general objective of this study is to analyze numerical solution of MHD flow of a nanofluid over a porous medium of an exponentially stretching sheet with convective boundary condition in the presence of suction/injection by employing the implicit finite difference scheme called Keller Box method.

1.3.2 Specific Objectives

The specific objectives of the study are:

1. To transform the governing partial differential equations to higher order ordinary differential equations using similarity transformations.
2. To apply Keller box method to solve the nonlinear ordinary differential equations obtained from the boundary layer equations.

3. To see the effects of various physical parameters such as Brownian motion (Nb), thermophoresis (Nt), Eckert number (Ec), Lewis number (Le), permeability K_p, Prandtl number(Pr) on velocity, temperature and concentration of fluid flow profile.

4. To identify the parameters that affect local skin friction coefficient, surface heat and mass transfer rate.

1.4 Significance of the Study

The outcomes of this study have the following importance:

- It helps to develop the researchers knowledge on applied mathematics research.

- The result and the method can be used as bench mark for other researchers in related areas.

- It may familiarize the researcher with scientific communication in applied mathematics.

- Mathematical simplification in reduction of order differential equations and reduced number of independent variables.

1.5 Delimitation of the Study

The study is delimited to the governing partial differential equations of laminar boundary layer and focus only on constructing Keller box method to analyze numerical solution for MHD flow of nanofluid over a porous an exponentially stretching sheet with convective boundary conditions in the presence of suction/injection.

1.6 Definition of Some Terms

Magnetohydrodynamics: The study of the interaction between magnetic fields and electrically conducting fluids.
Boundary layer: Is a fluid character that forms in the flow of fluid through a body

of surface.

Laminar Flow: Occurs when a fluid flows in the parallel layers, with no disruption between the layers and no cross currents or eddies perpendicular to direction of flow.

Permeability: Is a measure of the ability of a porous media to transmit fluids.

Similarity transformations: The transformations which reduce the number of independent variables of a system of partial differential equations at least one less than that of the original equation are designated similarity transformations.

A steady Flow: Is a flow in which the various physical phenomena like velocity, pressure and density at any point do not change with time.

Stream Function: Is a function ψ which satisfies continuity equation and defined as:

$\frac{\partial \psi}{\partial y} = u, \quad -\frac{\partial \psi}{\partial x} = v$

where

u: velocity component in $x-$ direction

v: velocity component in $y-$ direction.

Chapter 2

Review of Related Literature

2.1 Magnetohydrodynamics(MHD)

Magnetohydrodynamics is the branch of continuum mechanics which deals with the motion of an electrically conducting fluid in the presence of a magnetic field. The word magneto hydrodynamic (MHD) is derived from: Magneto-meaning magnetic field, Hydro meaning liquid and dynamics which means movement. Other variants of nomenclatures are Hydro magnetics, magneto-fluid dynamics, magneto-gas dynamics and so on. The concept of MHD is largely perceived to have been initiated by (Faraday 1812) when he did the first quantitative observation of magnetohydrodynamics. He did experiments with mercury as a conducting fluid flowing in a glass tube placed in magnetic field and observed that voltage was induced in direction perpendicular to both the direction of flow and magnetic field. He further showed that when an electric field is applied to a conducting fluid in the direction which is perpendicular to magnetic field, a force is exerted on the fluid in the direction perpendicular to both electric field and magnetic field. Since then a lot has been done on MHD and its related fields and (Rao. et al., 1990) studied the heat transfer in porous medium in the presence of transverse magnetic field.

The effects of the heat source parameter and Nusselt number were analyzed. They discovered that the effect of increasing porous parameter is to increase the Nusselt Number. (Kinyanjui et al., 2003) investigated MHD Stokes problem for a vertical infinite plate in dissipative rotating fluid with Hall current as (Sigey et al., 2004) presented an investigation on the numerical study on natural convection turbulent heat transfer in an enclosure. As it is known that, MHD is important branch of fluid dynamics. Many technological problems and natural phenomena are susceptible to MHD analysis. Engineers apply MHD principle, in the design of heat exchangers, in creating novel power generating systems, pumps and flow meters, thermal protection, braking, control and re-entry, in space vehicle propulsion. MHD convection flow problems are also very important in the fields of stellar and planetary magnetosphere's, aeronautics, electronics and chemical engineering. Hydromagnetic flow of Newtonian fluid and heat transfer over continuous moving flat surface with uniform suction has been studied by (Prasad et al., 2010). (Kumar i et al., 1990) studied the effects of induced magnetic field and heat source/sink on flow and heat transfer characteristic over a stretching surface. (Nazar et al., 2004) investigated the boundary layer over a moving continuous flat plate in an electrically conducting ambient fluid with a step change in applied magnetic field.

The magnetohydrodynamics (MHD) equations play an important role in many areas of astrophysics, space physics and engineering. Typical applications in those areas require one to capture flow on a range of scales in a way that is as dissipation-free as possible. As a result, there has been considerable interest in bringing accurate and reliable numerical methods to bear on this problem. The MHD system of equations can be written as a set of hyperbolic conservation laws.

2.2 Boundary Layer Flow

Prandtl introduced boundary layer theory in 1904 to understand the flow behavior of a viscous fluid near a solid boundary. Prandtl gave the concept of a boundary layer in large Reynolds number flows and derived the boundary layer equations by simplifying the Navier-Stokes equations to yield approximate solutions. Prandtls boundary layer equations arise in various physical models of fluid mechanics. The equations of the boundary layer theory have been the subject of considerable interest, since they represent an important simplification of the original Navier-Stokes equations. These equations arise in the study of steady flows produced by wall jets, free jets, and liquid jets, the flow past a stretching plate/surface, flow induced due to a shrinking sheet, and so on. These boundary layer equations are usually solved subject to certain boundary conditions depending upon the specific physical model considered. There are three types of boundary layer flows: velocity boundary layer flow, thermal boundary layer flow and concentration boundary layer flow.

2.2.1 Velocity Boundary Layer Flow

The velocity boundary layer develops whenever there is flow over a surface. It is associated with shear stresses parallel to the surface and results in an increase in velocity through the boundary layer from nearly zero right at the surface to the free stream velocity far from the surface. The boundary layer thickness is by convention defined as the distance from the surface at which the velocity is 99 percent of the free stream velocity.

2.2.2 Thermal Boundary Layer Flow

The thermal boundary layer is associated with temperature gradients near the surface, and develops when there is temperature difference between the fluid free stream and the surface. Right at the fluid-surface interface, heat transfer occurs only through conduction. The thickness of the thermal boundary layer is defined as that point at which the temperature difference between the fluid and surface is 99 percent of the temperature difference between the free stream fluid and the surface.

2.2.3 Concentration Boundary Layer Flow

The concentration boundary layer develops when there is a difference in concentration of a component between the free stream and the surface. A concentration profile develops, and the thickness of the concentration boundary layer is defined as that point at which the difference in concentration between the fluid and the surface is 99 percent of the difference in concentration between the free stream fluid and the surface.
(Blasius, 1908) solved the Prandtls boundary layer equations for a flat moving plate problem and gave a power series solution of the problem. (Sakiadis, 1961) initiated the study of the boundary layer flow over a continuously moving rigid surface with a uniform speed. (Crane, 1970) was the first one who studied the boundary layer flow due to a stretching surface and developed the exact solutions of boundary layer equations with parameter. (Gupta and Gupta, 1977) extended the *Cranes* work and for the first time introduced the concept of heat transfer with the stretching sheet boundary layer flow. The boundary layer thickness, signified by, is simply the thickness of the viscous boundary layer region. Because the main effect of viscosity is to slow the fluid near a wall, the edge of the viscous region is found at the point where the fluid velocity is essentially equal to the free-stream velocity. In a boundary layer, the fluid asymptotically approaches the free-stream velocity as one moves away from the wall, so it never actually equals the free-stream velocity.

Chapter 3

Methodology

This chapter contains study design, source of information and description of the research methodology.

3.1 Study area and period

The study was conducted at Jimma University under the department of mathematics from September, 2018 G.C. to June, 2019 G.C.

3.2 Study Design

The study design was mixed design (i.e., documentary review and numerical simulation design).

3.3 Source of Information

The relevant sources of information for this study were books and published articles related to the area of the study.

3.4 Mathematical Procedure of the Study

Mathematical procedure is the fundamental part of the work in mathematical research. Hence, to achieve the stated objectives, the following mathematical procedures has been followed:

1. Transforming the governing partial differential equations to nonlinear higher order ordinary differential equations by introducing similarity transformations.

2. Reduce the nonlinear higher order ordinary differential equations to a system of first order equations.

3. Write the finite difference for the reduced nonlinear higher order ordinary differential equations.

4. Linearize the algebraic equations by using Newton's method and write them in vector matrix form.

5. Solve the linear system by the block tridiagonal elimination technique.

6. Finally a sketching the graphs using MATLAB.

Chapter 4

Mathematical Formulation, Numerical Result and Discussion

4.1 Mathematical Formulation

Consider a steady, two dimensional MHD boundary layer flow of a viscous, incompressible, electrically conducting fluid over exponentially stretching surface in a porous medium with permeability K_p subjected to suction/injection in the presence of magnetic field and heat source or sink. The x-axis is along the continuous stretching surface and the y-axis is normal to the surface. u and v are the velocities in the x- and y- directions respectively. T_W and T_∞ are the wall temperature and the temperature far from the surface respectively, and C_W and C_∞ are species concentration at the wall and species concentration far away from the surface respectively. The physical scheme of the flow problem is depicted in the following figure. An

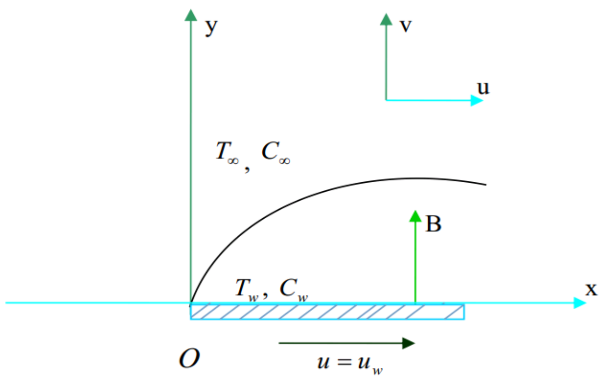

Figure 4.1: physical model and coordinate system of the given scheme

external magnetic field B(x) is applied along the y direction. A non-uniform permeability, radiation heat flux q_r, suction/injection, heat source/sink, thermophoresis effect along with volume fraction of nanoparticles is taken in to account. The boundary layer equation, the steady MHD flow of nanofluid flow over a porous medium of an exponentially stretching sheet with convective boundary condition in the presence of suction/injection are governs the present flow subject to the Boussinesq approximations can be expressed in terms of continuity, momentum, energy and species concentration equations respectively as follows:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{4.1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}-\frac{\delta B^2}{\rho}u-\frac{\mu}{K_p}u \tag{4.2}$$

$$\begin{aligned} u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}&=\alpha\frac{\partial^2 T}{\partial y^2}+\frac{\nu}{c_p}(\frac{\partial u}{\partial y})^2+\tau[D_B\frac{\partial T}{\partial y}\frac{\partial C}{\partial y}+\frac{D_T}{T_\infty}(\frac{\partial T}{\partial y})^2] \\ &-\frac{1}{(\rho_c)_f}\frac{\partial q_r}{\partial y}+\frac{Q_0}{\rho c_p}(T-T_\infty) \end{aligned} \tag{4.3}$$

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D_B\frac{\partial^2 C}{\partial y^2}+\frac{D_T}{T\infty}\frac{\partial^2 T}{\partial y^2}-K_0(C-C_\infty) \tag{4.4}$$

The boundary conditions are given by:

$$\begin{aligned} &U=U_w(x),\ V=V_w(x),\ h_f(T_f-T)=-k\frac{\partial T}{\partial y}, \\ &C=C_w=C_\infty+C_0e^{\frac{x}{2l}} \quad as \quad y\to 0 \\ &U=0,\ T=T_\infty,\ C=C_\infty \quad as \quad y\to\infty \end{aligned} \tag{4.5}$$

Where u and v the velocity components taken in x and y directions respectively. The stretching velocity U_w is given by $U_w=U_0e^{\frac{x}{l}}$ where $U_0>0$ is stretching constant and here $V_w(x)$ is the variable wall mass transfer velocity and is given by $V_w(x)=V_0e^{\frac{x}{2l}}$ with $V_w(x)<0$ for suction and $V_w(x)>0$ for injection respectively. The radiative heat flux in the x-direction is considered negligible as compared to y-direction. Hence, by Using Rosseland approximation for radiation, the radiative heat flux q_r is given by $q_r=-\frac{4\sigma^*}{3k^*}\frac{\partial T^4}{\partial y}$

where k^* is the mean absorption coefficient, σ^* is the Stefan-Boltzmann constant, we assume that the temperature difference with in the flow sufficiently small such that the term T^4 may be expanded as a linear function of temperature. This is done by expanding T^4 in a Taylor series about a free stream temperature T_∞ and neglecting higher order terms we get $T^4 = 4T_\infty^3 T - 3T_\infty^4$

Hence, $q_r = -\frac{16\sigma^* T_\infty^3}{3k^*}\frac{\partial T}{\partial y}$

The variable magnetic field B(x), is taken in the form $B(x) = B_0\ e^{\frac{x}{2l}}$, where B_0 is a constant. The permeability K_P of the porous medium takes the following form $K(x) = K_0\ e^{\frac{-x}{l}}$ where K_0 is reference permeability.

The x axis is taken along the stretching sheet and y axis is perpendicular to the fluid. The fluid is electrically conducting and the magnetic field is assumed to be applied in the y direction.

Similarity Transformation

In order to investigate the velocity, temperature and concentration distribution of the MHD flow of nanofluid over a porous medium exponentially stretching sheet with convective boundary condition in the presence of suction/injection, the following dimensionless similarity variables are introduced:

$$\psi = (2U_0 \nu l)^{\frac{1}{2}} e^{\frac{x}{2l}} f(\eta) \tag{4.6}$$

$$\eta = y(\frac{U_0}{2\nu l})^{\frac{1}{2}} e^{\frac{x}{2l}} \tag{4.7}$$

$$\theta(\eta) = \frac{T - T_\infty}{T_f - T_\infty} \tag{4.8}$$

$$\beta(\eta) = \frac{C - C_\infty}{C_w - C_\infty} \tag{4.9}$$

with the velocity components

$$\begin{aligned} u &= \frac{\partial \psi}{\partial y} = U_0 e^{\frac{x}{l}} f'(\eta) \quad and \\ v &= -\frac{\partial \psi}{\partial x} = -(\frac{\nu U_0}{2l})^{\frac{1}{2}} e^{\frac{x}{2l}} [f(\eta) + \eta f'(\eta)] \end{aligned} \tag{4.10}$$

Where $\psi(x,y)$ is the stream function, $f(\eta)$ is the dimensionless stream function, $f'(\eta)$ is the velocity profile, $\theta(\eta)$ is temperature profile, $\beta(\eta)$ is concentration profile and η is similarity variables.

Using Equation (4.10) the continuity equation is satisfied as:

$$
\begin{aligned}
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} &= \frac{\partial}{\partial x}(U_0 e^{\frac{x}{l}} f'(\eta))+\frac{\partial}{\partial y}(-(\frac{\nu U_0}{2l})^{\frac{1}{2}} e^{\frac{x}{2l}}[f(\eta)+\eta f'(\eta)]) \\
&= \frac{U_0}{l}e^{\frac{x}{l}}[f'(\eta)+\frac{1}{2}\eta f''(\eta)]-\frac{U_0}{l}e^{\frac{x}{l}}[f'(\eta)+\frac{1}{2}\eta f''(\eta)] \\
&= 0
\end{aligned}
$$

And by using Equations (4.7) – (4.10) the governing partial differential equations are transformed into non linear higher order ordinary differential equations as follows:

$$
u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}-\frac{\delta B^2}{\rho}u-\frac{\mu}{K_p}u
$$

$$
\begin{aligned}
&U_0 e^{\frac{x}{l}} f'(\eta)\frac{U_0}{l}e^{\frac{x}{l}}[f'(\eta)+\frac{1}{2}\eta f''(\eta)]+U_0 e^{\frac{x}{l}} f'(\eta)(-\frac{U_0}{l}e^{\frac{x}{l}}[f'(\eta)+\frac{1}{2}\eta f''(\eta)]) \\
&=\frac{U_0^2}{2l}e^{\frac{2x}{l}}f'''-U_0 e^{\frac{2x}{l}}[\frac{\delta B_0^2}{\rho}+\frac{\mu}{K_0}]f'
\end{aligned}
$$

$$
\frac{U_0^2}{l}e^{\frac{2x}{l}}[f'^2-\frac{1}{2}ff'']=\frac{U_0^2}{2l}e^{\frac{2x}{l}}[f^{(3)}-(\frac{2\delta B_0^2}{\rho}+\frac{2\mu}{K_0})f']
$$

$$
f'''+ff''-2f'^2-(\frac{2\delta B_0^2}{\rho}+\frac{2\mu}{K_0})f'=0
$$

$$
f'''-2f'^2+ff''-(M+K_P)f'=0
$$

Equation (4.3) transformed as follow:

$$
u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}+\frac{\nu}{c_p}(\frac{\partial u}{\partial y})^2+\tau[D_B\frac{\partial T}{\partial y}\frac{\partial C}{\partial y}+\frac{D_T}{T_\infty}(\frac{\partial T}{\partial y})^2]-\frac{1}{(\rho_c)_f}\frac{\partial q_r}{\partial y}+\frac{Q_0}{\rho c_p}(T_f-T_\infty)
$$

$$\frac{U_0}{2l}\eta\theta'(\eta)f'(\eta)(T_f-T_\infty)e^{\frac{x}{l}}-[\frac{U_0}{2l}\theta'(\eta)f(\eta)(T_f-T_\infty)e^{\frac{x}{l}}+\frac{U_0}{2l}\eta\theta'(\eta)f'(\eta)(T_f-T_\infty)e^{\frac{x}{l}}]$$
$$=\alpha\theta''e^{\frac{x}{l}}\frac{U_0}{2\nu l}(T_f-T_\infty)+\frac{\nu}{c_p}(\frac{U_0}{l})^2e^{\frac{2x}{l}}f''^2\frac{U_0}{2\nu l}e^{\frac{x}{l}}+\tau[D_B\theta'\beta'e^{\frac{x}{l}}\frac{U_0}{2\nu l}(T_f-T_\infty)(C_w-C_\infty)$$
$$+\frac{D_T}{T_\infty}(\theta')^2e^{\frac{x}{l}}\frac{U_0}{2\nu l}(T_f-T_\infty)^2]+\frac{16\delta^*}{3k^*}\frac{T_\infty^3}{(\rho_c)_f}\theta''e^{\frac{x}{l}}\frac{U_0}{2\nu l}(T_f-T_\infty)+\frac{Q_0}{\rho c_p}\theta(T_f-T_\infty).$$

$$-\frac{U_0}{2l}e^{\frac{x}{l}}f\theta'(T_f-T_\infty)=\frac{U_0}{2\nu l}e^{\frac{x}{l}}[\alpha\theta''(T_f-T_\infty)+\frac{\nu}{c_p}(\frac{U_0}{l})^2f''^2e^{\frac{2x}{l}}+\tau[D_B\theta'\beta'(T_f-T_\infty)(C_w-C_\infty)$$
$$+\frac{D_T}{T_\infty}(\theta')^2(T_f-T_\infty)^2+\frac{16\delta^*}{3k^*}\frac{T_\infty^3}{(\rho_c)_f}\theta''(T_f-T_\infty)+\frac{Q_0}{\rho c_p e^{\frac{x}{l}}}\theta(T_f-T_\infty)]$$

$$\alpha\theta''+\frac{\nu}{c_p}(\frac{U_0}{l})^2f''^2e^{\frac{2x}{l}}\frac{1}{(T_f-T_\infty)}+\tau D_B\theta'\beta'(C_w-C_\infty)+\tau\frac{D_T}{T_\infty}(\theta')^2(T_f-T_\infty)$$
$$+\quad\frac{16\delta^*}{3k^*}\frac{T_\infty^3}{(\rho_c)_f}\theta''+\frac{Q_0}{\rho c_p e^{\frac{x}{l}}}\theta+\nu f\theta'=0$$

$$\theta''+\frac{\nu}{c_p\alpha}(\frac{U_0}{l})^2f''^2e^{\frac{2x}{l}}\frac{1}{(T_f-T_\infty)}+\frac{1}{\alpha}\tau D_B\theta'\beta'(C_w-C_\infty)+\frac{1}{\alpha}\tau\frac{D_T}{T_\infty}(\theta')^2(T_f-T_\infty)$$
$$+\frac{16\delta^*}{3k^*\alpha}\frac{T_\infty^3}{(\rho_c)_f}\theta''+\frac{Q_0}{\rho c_p\alpha e^{\frac{x}{l}}}\theta+\nu\frac{1}{\alpha}f\theta'=0$$

$$\theta''+\frac{N_r}{N_r+1}P_r[f\theta'+ECf''^2N_b\theta'\beta'+N_t(\theta')^2+Q\theta]=0$$

$$\theta''=-\frac{N_r}{N_r+1}P_r[f\theta'+ECf''^2N_b\theta'\beta'+N_t(\theta')^2+Q\theta] \tag{4.11}$$

Equation (4.5) transformed as follow:

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D_B\frac{\partial^2 C}{\partial y^2}+\frac{D_T}{T\infty}\frac{\partial^2 T}{\partial y^2}-K_0(C-C_\infty)$$

$$-\frac{U_0}{2l}f\beta'e^{\frac{x}{l}}(C_w-C_\infty)=\frac{U_0}{2\delta l}e^{\frac{x}{l}}D_B\beta''(C_w-C_\infty)+\frac{U_0}{2\delta l}e^{\frac{x}{l}}\frac{D_T}{T_\infty}\theta''(T_f-T_\infty)$$
$$-K_0\beta(C_w-C_\infty)$$

$$-\frac{U_0}{2l}f\beta' = \frac{U_0}{2\nu l}D_B\beta''\frac{(T_f - T_\infty)}{(C_w - C_\infty)} + \frac{U_0}{2\nu l}\frac{D_T}{T_\infty}\theta''(T_f - T_\infty) - \frac{K_0}{e^{\frac{x}{l}}}\beta$$

$$\beta'' + Lef\beta' - \frac{N_r}{N_r+1}P_r\frac{N_t}{N_b}[f\theta' + ECf''^2N_b\theta'\beta' + N_t(\theta')^2 + Q\theta]$$
$$-LeR_c\beta = 0$$

Finally, by using similarity transformations equations (4.6) – (4.9), Equations (4.2) – (4.4) reduced to non linear higher order ordinary differential equations given by :

$$f''' - 2f'^2 + ff'' - (M + K_p)f' = 0 \tag{4.12}$$

$$\theta'' + \frac{N_r}{N_r+1}P_r[f\theta' + ECf''^2N_b\theta'\beta' + N_t(\theta')^2 + Q\theta] = 0 \tag{4.13}$$

$$\beta'' + Lef\beta' - \frac{N_r}{N_r+1}P_r\frac{N_t}{N_b}[f\theta' + ECf''^2N_b\theta'\beta' + N_t(\theta')^2 + Q\theta] - LeR_c\beta = 0 \tag{4.14}$$

Where primes denote differentiation with respect to η. The Magnetic field parameter (M), the radiation parameter (N_r), the Prandtl number (P_r), the Eckert number (Ec), heat source/sink parameter (Q), the wall mass transfer at the sheet (s), the Lewis number (Le), the chemical radiation parameter (R_c), the Biot number (Bi), the Permeability (K_p) are given by:

$$M = 2\delta B_0^2, \;\; K_p = \frac{2\mu}{K_0}, \;\; Q = \frac{Q_0}{\nu\rho c_p e^{\frac{x}{l}}}, \;\; Le = \frac{\nu}{D_B}$$

$$N_r = \frac{3k^*(\rho_c)_f}{16\delta^* T_\infty^3} \quad , \quad Ec = \frac{1}{c_p}(\frac{U_0}{l})^2\frac{e^{\frac{x}{l}}}{(T_f - T_\infty)}, \;\; U_w = U_0 e\frac{x}{l}$$

$$R_c = \frac{D_B 2lK_0}{U_w}, \;\; Bi = -\frac{c}{k}(\frac{2\nu l}{U_0})^{\frac{1}{2}} \;\; and \;\; s = -V_0(\frac{2l}{\nu U_0})^{\frac{1}{2}}$$

Where $s > 0$ $(v_0 < 0)$ corresponds to mass suction and $s < 0$ $(v_0 > 0)$ corresponds to mass injection. With boundary conditions:

$$\begin{aligned} &f(0) = s,\ f'(0) = 1,\ f'(\infty) = 0,\ \theta(\infty) = 0, \\ &\theta'(0) = -Bi[1-\theta(0)],\ \beta(0) = 1,\ \beta(\infty) = 0 \end{aligned} \tag{4.15}$$

The physical quantities of interest in this problem are the skin - friction parameter (C_{fx}), Local Nusselt number (Nu_x) and Sherwood number (Sh_x) defined as follow:

$$f''(0) = (2ReC_{fx})^{\frac{1}{2}},\ \ \frac{Nu_x}{(2Re)^{\frac{1}{2}}} = -(\frac{x}{2l})^{\frac{1}{2}}\theta'(0),\quad \frac{Sh_x}{(2Re)^{\frac{1}{2}}} = -(\frac{x}{2l})^{\frac{1}{2}}\beta'(0)$$

4.2 Method of Solution

Equations (4.12) – (4.14) subject to the boundary conditions (4.15) were solved numerically by Keller box method which is implemented in matlab. The Keller box method is an implicit finite difference method that can be used to solve differential equations. This method has four fundamental steps.

1. First step is converting Equations (4.12) – (4.14) into a system of first order ordinary differential equations.

2. The second step is approximating the derivatives in system of first order equations with central difference approximations.

3. The third step is linearizing the nonlinear algebraic equations with $Newton's$ method and then casting as the matrix vector form.

4. The fourth step is solving the system of linear equations using block tridiagonal elimination scheme with the suitable initial solution.

In this method the transformed differential equations (4.12), (4.13) and (4.14) are written in terms of first order system (Mitiku and Devaraj, 2017) for that introduce new dependent variable u, v, g and k such that

$$\begin{aligned} f' = u \quad &, \quad u' = v \qquad , \\ \theta' = g \quad &, \quad \beta' = k \end{aligned} \tag{4.16}$$

$$v' + fv - 2u^2 - (M + K_p)u = 0 \tag{4.17}$$

$$g' + \frac{N_r}{N_r+1} P_r [fg + Ecu^2 N_b gk + N_t g^2 + Q\theta] = 0 \tag{4.18}$$

$$k' + Lefk - \frac{N_r}{N_r+1} P_r \frac{N_t}{N_b} [fg + Ecu^2 N_b gk + N_t g^2 + Q\theta] - LeR_c\beta = 0 \tag{4.19}$$

with new boundary conditions

$$\begin{aligned} &u(0) = 1,\ \theta(0) = 1 + \frac{\theta'(0)}{Bi},\ u(\infty) = 0, \\ &f(0) = s,\ \theta(\infty) = 0,\ \beta(0) = 1,\ \beta(\infty) = 0 \end{aligned} \tag{4.20}$$

Now, consider the net rectangle in x–η plane as show in figure 4.2 and the net points are defined as follow:

$$\begin{aligned} &x_0 = 0,\ x_n = x_{n-1} + k_n,\ n = 1, 2, ..., N \\ &\eta_0 = 0,\ \eta_j = \eta_{j-1} + h_j,\ j = 1, 2, ..., J \end{aligned} \tag{4.21}$$

Where k_n is the $\triangle$ *x–spacing* and h_j is the $\triangle$ η–*spacing*
Here η and j are the sequence of number that indicate the coordinate location. Now write the finite difference approximation of the ODE, equation (4.16) for the

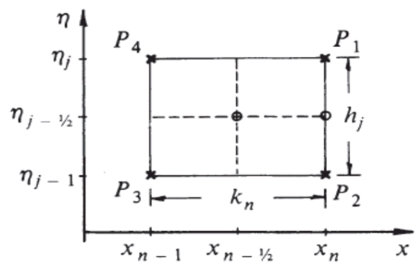

Figure 4.2: Finite difference grid for the Keller Box method

mid point $(x_n, \eta_{j-\frac{1}{2}})$ of the segment P_1P_2 using centered difference derivatives, this called centering about $(x_n, \eta_{j-\frac{1}{2}})$

$$\frac{f_j - f_{j-1}}{h_j} = u_{j-\frac{1}{2}} \quad , \quad \frac{u_j - u_{j-1}}{h_j} = v_{j-\frac{1}{2}} \qquad ,$$

$$\frac{\theta_j - \theta_{j-\frac{1}{2}}}{h_j} = g_{j-\frac{1}{2}} \quad , \quad \frac{\beta_j - \beta_{j-1}}{h_j} = k_{j-\frac{1}{2}} \tag{4.22}$$

An ODE of Equations (4.17), (4.18) and (4.19) are approximated by the centering about the mid point $(x_{n-\frac{1}{2}}, \eta_{j-\frac{1}{2}})$ of the rectangle $P_1P_2P_3P_4$

$$\frac{v_j - v_{j-1}}{h_j} + f_{j-\frac{1}{2}} v_{j-\frac{1}{2}} - 2u^2_{j-\frac{1}{2}} - (M + K_p) u_{j-\frac{1}{2}} = 0 \tag{4.23}$$

$$\frac{g_j - g_{j-1}}{h_j} + \frac{N_r}{N_r + 1} P_r [f_{j-\frac{1}{2}} g_{j-\frac{1}{2}} + Ecu^2_{j-\frac{1}{2}} + N_b g_{j-\frac{1}{2}} k_{j-\frac{1}{2}} + N_t g^2_{j-\frac{1}{2}} + Q\theta_{j-\frac{1}{2}}] = 0 \tag{4.24}$$

$$\frac{k_j - k_{j-1}}{h_j} + Lef_{j-\frac{1}{2}} k_{j-\frac{1}{2}} - \frac{N_r}{N_r + 1} P_r \frac{N_t}{N_b} [f_{j-\frac{1}{2}} g_{j-\frac{1}{2}} + Ec\, u^2_{j-\frac{1}{2}} + N_b g_{j-\frac{1}{2}} k_{j-\frac{1}{2}} + N_t g^2_{j-\frac{1}{2}} + Q\, \theta_{j-\frac{1}{2}}] - Le\, R_c\, \beta_{j-\frac{1}{2}} = 0 \tag{4.25}$$

$$Here\ f_{j-\frac{1}{2}} = \frac{f_j + f_{j-1}}{2} \quad , \quad g_{j-\frac{1}{2}} = \frac{g_j + g_{j-1}}{2},\ k_{j-\frac{1}{2}} = \frac{k_j + k_{j-1}}{2} \; , \; \ldots \; \beta_{j-\frac{1}{2}} = \frac{\beta_j + \beta_{j-1}}{2} \tag{4.26}$$

By using equation (4.26), equations (4.23) – (4.25) becomes

$$f_j - f_{j-1} - \frac{h_j}{2}(u_j + u_{j-1}) = 0 \tag{4.27}$$

$$u_j - u_{j-1} - \frac{h_j}{2}(v_j + v_{j-1}) = 0 \tag{4.28}$$

$$\theta_j - \theta_{j-1} - \frac{h_j}{2}(g_j + g_{j-1}) = 0 \tag{4.29}$$

$$\beta_j - \beta_{j-1} - \frac{h_j}{2}(k_j + k_{j-1}) = 0 \tag{4.30}$$

$$v_j - v_{j-1} + \frac{h_j}{4}(f_j + f_{j-1})(v_j + v_{j-1}) - \frac{h_j}{2}(u_j + u_{j-1})^2$$
$$- \frac{h_j}{2}(M + K_p)(u_j + u_{j-1}) = 0 \tag{4.31}$$

$$g_j - g_{j-1} + \frac{N_r}{N_r+1} P_r \frac{h_j}{4}[(f_j + f_{j-1})(g_j + g_{j-1}) + Ec(u_j + u_{j-1})^2$$
$$+ N_b(g_j + g_{j-1})(k_j + k_{j-1}) + N_t(g_j + g_{j-1})^2]$$
$$+ \frac{N_r}{N_r+1} P_r \frac{h_j}{2} Q(\theta_j + \theta_{j-1}) = 0. \tag{4.32}$$

$$k_j - k_{j-1} + Le\frac{h_j}{4}(f_j + f_{j-1})(k_j + k_{j-1}) - \frac{N_r}{N_r+1} P_r \frac{h_j}{4}[(f_j + f_{j-1})(g_j + g_{j-1})$$
$$+ Ec(u_j + u_{j-1})^2 + N_b(g_j + g_{j-1})(k_j + k_{j-1}) + N_t(g_j + g_{j-1})^2]$$
$$+ \frac{N_r}{N_r+1} P_r \frac{h_j}{2} Q(\theta_j + \theta_{j-1}) - LeR_c\frac{h_j}{2}(\beta_j + \beta_{j-1}) = 0 \tag{4.33}$$

Now linearize the non linear system of equation (4.27) –(4.33) using the $Newton's$ linearization scheme. That is, we assume for $(i+1)^{th}$ iterations

$$f_j^{i+1} = f_j^i + \delta f_j^i \quad etc. \tag{4.34}$$

Substituting equation (4.34) into the above equations and dropping the quadratic terms in δf_j^i, δu_j^i, δv_j^i, δg_j^i, δk_j^i, $\delta \theta_j^i$ and $\delta \beta_j^i$, we obtain a tridiagonal system of algebraic equations.

$$\delta f_j - \delta f_{j-1} - \frac{h_j}{2}(\delta u_j + \delta u_{j-1}) = (r_1)_j \tag{4.35}$$

$$\delta u_j - \delta u_{j-1} - \frac{h_j}{2}(\delta v_j + \delta v_{j-1}) = (r_2)_j \tag{4.36}$$

$$\delta \theta_j - \delta \theta_{j-1} - \frac{h_j}{2}(\delta g_j + \delta g_{j-1}) = (r_3)_j \tag{4.37}$$

$$\delta \beta_j - \delta \beta_{j-1} - \frac{h_j}{2}(\delta k_j + \delta k_{j-1}) = (r_4)_j \tag{4.38}$$

$$(a_1)_j\delta v_j+(a_2)_j\delta v_{j-1}+(a_3)_j\delta f_j+(a_4)_j\delta f_{j-1}+(a_5)_j\delta u_j \\ +(a_6)_j\delta u_{j-1}=(r_5)_j \tag{4.39}$$

$$(b_1)_j\delta g_j+(b_2)_j\delta g_{j-1}+(b_3)_j\delta f_j+(b_4)_j\delta f_{j-1}+(b_5)_j\delta v_j \\ +(b_6)_j\delta v_{j-1}+(b_7)_j\delta k_j+(b_8)_j\delta k_{j-1}+(b_9)_j\delta\theta_j \\ +(b_{10})_j\delta\theta_{j-1}=(r_6)_j \tag{4.40}$$

$$(c_1)_j\delta k_j+(c_2)_j\delta k_{j-1}+(c_3)_j\delta f_j+(c_4)_j\delta f_{j-1}+(c_5)_j\delta g_j+(c_6)_j\delta g_{j-1} \\ +(c_7)_j\delta v_j+(c_8)_j\delta v_{j-1}+(c_9)_j\delta\theta_j+(c_{10})_j\delta\theta_{j-1}+(c_{11})_j\delta\beta_j \\ +(c_{12})_j\delta\beta_{j-1}=(r_7)_j \tag{4.41}$$

Where

$$(a_1)_j=1+\frac{h_j}{4}(f_j+f_{j-1}) \quad , \quad (a_2)_j=-1+\frac{h_j}{4}(f_j+f_{j-1})$$

$$(a_3)_j=(a_4)_j=\frac{h_j}{4}(v_j+v_{j-1}) \quad , \quad (a_5)_j=(a_6)_j=-\frac{h_j}{2}[2(u_j+u_{j-1})+(M+K_p)]$$

$$(b_1)_j=1+\frac{N_r}{N_r+1}P_r\frac{h_j}{4}[(f_j+f_{j-1})+N_b(k_j+k_{j-1})+2N_t(g_j+g_{j-1})]$$

$$(b_2)_j=-1+\frac{N_r}{N_r+1}P_r\frac{h_j}{4}[(f_j+f_{j-1})+N_b(k_j+k_{j-1})+2N_t(g_j+g_{j-1})]$$

$$(b_3)_j=(b_4)_j=\frac{N_r}{N_r+1}P_r\frac{h_j}{4}[(g_j+g_{j-1}) \quad , \quad (b_5)_j=(b_6)_j=\frac{N_r}{N_r+1}P_rEc\frac{h_j}{2}(v_j+v_{j-1})$$

$$(b_7)_j=(b_8)_j=\frac{N_r}{N_r+1}P_rN_b\frac{h_j}{4}(g_j+g_{j-1}) \quad , \quad (b_9)_j=(b_{10})_j=\frac{N_r}{N_r+1}P_rQ\frac{h_j}{2}$$

$$(c_1)_j = 1 + Le\frac{h_j}{4}(f_j + f_{j-1}) - \frac{N_r}{N_r+1}P_rN_t\frac{h_j}{4}(g_j + g_{j-1})$$

$$(c_2)_j = -1 + Le\frac{h_j}{4}(f_j + f_{j-1}) - \frac{N_r}{N_r+1}P_rN_t\frac{h_j}{4}(g_j + g_{j-1})$$

$$(c_3)_j = (c_4)_j = Le\frac{h_j}{4}(k_j + k_{j-1}) - \frac{N_r}{N_r+1}P_r\frac{N_t}{N_b}\frac{h_j}{4}(g_j + g_{j-1})$$

$$(c_5)_j = (c_6)_j = -\frac{N_r}{N_r+1}P_r\frac{h_j}{4}[(f_j + f_{j-1}) + N_b(k_j + k_{j-1})2N_t(g_j + g_{j-1})]$$

$$(c_7)_j = (c_8)_j = -\frac{N_r}{N_r+1}P_r\frac{N_t}{N_b}\frac{h_j}{2}(v_j + v_{j-1}) \quad , \quad (c_9)_j = (c_{10})_j = -\frac{N_r}{N_r+1}P_rQ\frac{h_j}{2}$$

$$(c_{11})_j = (c_{12})_j = Le\ \ R_c\frac{h_j}{2} \qquad and$$

$$(r_1)_j = f_{j-1} - f_j + \frac{h_j}{2}(u_j + u_{j-1})$$

$$(r_2)_j = u_{j-1} - u_j + \frac{h_j}{2}(v_j + v_{j-1})$$

$$(r_3)_j = \theta_{j-1} - \theta_j + \frac{h_j}{2}(g_j + g_{j-1})$$

$$(r_4)_j = \beta_{j-1} - \beta_j + \frac{h_j}{2}(k_j + k_{j-1})$$

$$(r_5)_j = v_{j-1} - v_j - \frac{h_j}{4}(f_j + f_{j-1})(v_j + v_{j-1}) + \frac{h_j}{2}(M + K_p)(u_j + u_{j-1})$$
$$+\frac{h_j}{2}(u_j + u_{j-1})^2$$

$$(r_6)_j = g_{j-1} - g_j - \frac{N_r}{N_r+1} P_r \frac{h_j}{4}[(f_j + f_{j-1})(g_j + g_{j-1}) + Ec(u_j + u_{j-1})^2$$
$$+N_b(g_j + g_{j-1})(k_j + k_{j-1}) + N_t(g_j + g_{j-1})^2] - \frac{R}{R+1} P_r \frac{h_j}{2} Q(\theta_j + \theta_{j-1})$$

$$(r_7)_j = k_{j-1} - k_j - Le\frac{h_j}{4}(f_j + f_{j-1})(k_j + k_{j-1}) + \frac{N_r}{N_r+1} P_r \frac{h_j}{4}[(f_j + f_{j-1})(g_j + g_{j-1})$$
$$+Ec(v_j + v_{j-1})^2 + N_b(g_j + g_{j-1})(k_j + k_{j-1}) + N_t(g_j + g_{j-1})^2 + 2Q(\theta_j + \theta_{j-1})]$$
$$+LeR_c \frac{h_j}{2}(\beta_j + \beta_{j-1})$$

With the boundary conditions

$$\delta U_0 = 1, \;\; \delta U_J = 0, \;\; \delta f_0 = s, \qquad \delta\theta_0 = 1 + \frac{\theta'(0)}{Bi}$$

$$\delta\theta_J = 0, \;\; \delta\beta_0 = 1, \quad \delta\beta_J = 0$$

Hence the linearized system of equations (4.35) – (4.41) can be written in the matrix form as

$$A\delta = r \tag{4.42}$$

Where

$$A = \begin{pmatrix} [A_1] & [C_1] & & & \\ [B_2] & [A_2] & [C_2] & & \\ & \ddots & \ddots & \ddots & \\ & [B_{J-1}] & & [A_{J-1}] & [C_{J-1}] \\ & & & [B_J] & [A_J] \end{pmatrix}$$

$$\delta = \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \\ \delta_7 \end{pmatrix} \quad , \quad r = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \end{pmatrix}$$

The elements of the matrix A are block matrix of order 7×7

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -d_1 & 0 & 0 & 0 & -d_1 & 0 & 0 \\ 0 & -d_1 & 0 & 0 & 0 & -d_1 & 0 \\ 0 & 0 & -d_1 & 0 & 0 & 0 & -d_1 \\ (a_2)_1 & 0 & 0 & (a_3)_1 & (a_1)_1 & 0 & 0 \\ (b_6)_1 & (b_2)_1 & (b_8)_1 & (b_3)_1 & (b_5)_1 & (b_1)_1 & (b_7)_1 \\ (c_8)_1 & (c_6)_1 & (c_2)_1 & (c_3)_1 & (c_7)_1 & (c_5)_1 & (c_1)_1 \end{pmatrix}, for \;\; j = 1$$

$$A_j = \begin{pmatrix} -d_j & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -d_j & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -d_j & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -d_j \\ (a_6)_j & 0 & 0 & (a_3)_j & (a_1)_j & 0 & 0 \\ 0 & (b_{10})_1 & 0 & (b_3)_j & (b_5)_j & (b_1)_j & (b_7)_j \\ 0 & (c_{10})_j & (c_{12})_j & (c_3)_j & (c_7)_j & (c_5)_j & (c_1)_j \end{pmatrix}, for \;\; 2 \le j \le J$$

$$B_j = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -d_j & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -d_j & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -d_j \\ 0 & 0 & 0 & (a_4)_j & (a_2)_j & 0 & 0 \\ 0 & 0 & 0 & (b_4)_j & (b_6)_j & (b_2)_j & (b_8)_j \\ 0 & 0 & 0 & (c_4)_j & (c_8)_j & (c_6)_j & (c_2)_j \end{pmatrix}, for \;\; 2 \le j \le J$$

$$
C_j = \begin{pmatrix} -d_j & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -d_j \\ (a_5)_j & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (b_9)_j & 0 & 0 & 0 & 0 & 0 \\ 0 & (c_9)_j & (c_{11})_j & 0 & 0 & 0 & 0 \end{pmatrix}, for \quad 1 \leq j \leq J-1
$$

Here, $d_j = \frac{h_j}{2}$ and

$$
\delta_1 = \begin{pmatrix} \delta v_0 \\ \delta g_0 \\ \delta k_0 \\ \delta f_1 \\ \delta v_1 \\ \delta g_1 \\ \delta k_1 \end{pmatrix} \quad , \quad \delta_j = \begin{pmatrix} \delta u_{j-1} \\ \delta \theta_{j-1} \\ \delta \beta_{j-1} \\ \delta f_j \\ \delta v_j \\ \delta g_j \\ \delta k_j \end{pmatrix} for \quad 2 \leq j \leq J
$$

$$
r_j = \begin{pmatrix} (r_1)_j \\ (r_2)_j \\ (r_3)_j \\ (r_4)_j \\ (r_5)_j \\ (r_6)_j \\ (r_7)_j \end{pmatrix}, for \quad 1 \leq j \leq J
$$

The solution of equation (4.42) can be obtained using block elimination method, which consist of forward and backward sweeps

Forward sweep:

To solve equation (4.42), we use LU factorization for decomposing matrix A into a product of a lower triangular matrix L and an upper triangular matrix U as follows:

$$
A = LU \tag{4.43}
$$

Where

$$L = \begin{pmatrix} [\gamma_1] & & & & \\ [\beta_2] & [\gamma_2] & & & \\ & \ddots & \ddots & & \\ & & [\beta_{J-1}] & [\gamma_{J-1}] & \\ & & & [\beta_J] & [\gamma_J] \end{pmatrix},$$

$$U = \begin{pmatrix} [I] & [\Gamma_1] & & & \\ & [I] & [\Gamma_2] & & \\ & & \ddots & \ddots & \\ & & & [I] & [\Gamma_{J-1}] \\ & & & & [I] \end{pmatrix}$$

[I] is the identity matrix of order 7×7 and $[\gamma_j]$and $[\Gamma_j]$ are 7×7 which elements are determined by the following equations:

$$[\gamma_1] = [A_1] \tag{4.44}$$

$$[A_1][\Gamma_1] = [C_1] \tag{4.45}$$

$$[\gamma_j] = [A_j] - [B_j][\Gamma_{j-1}],\ j = 2,3,...,J \tag{4.46}$$

$$[\gamma_j][\Gamma_j] = [C_j],\ j = 2,3,...,J-1 \tag{4.47}$$

Backward sweep

Substituting equation (4.43) into equation (4.42), we get

$$LU\delta = r \tag{4.48}$$

If we define,

$$U\delta = W \tag{4.49}$$

Then equation (4.48) becomes

$$LW = r \tag{4.50}$$

Where $W = [w_1, w_2, ..., w_{j-1}, w_J]^T$, w_j are the 7×1 column matrices.
The elements W can be found by solving equation (4.50)

$$[\gamma_1][w_1] = [r_1] \tag{4.51}$$

$$[\gamma_j][w_j] = [r_j] - [\beta_j][w_{j-1}], \; for \;\; 2 \le j \le J \tag{4.52}$$

Once the elements of W are found, we can find the solution of equation (4.49) by using recursion relations

$$[\delta_J] = [w_J] \tag{4.53}$$

$$[\delta_j] = [w_j] - [\beta_j][w_{j-1}], \; for \;\; 1 \le j \le J-1 \tag{4.54}$$

These calculation are repeated until convergence criterion is satisfied and calculations are stopped when

$$|\delta v_0^{(i)}| < \varepsilon$$

Where ε is desired level of accuracy. In this study, the value of $\varepsilon = 10^{-6}$.

4.3 Results and Discussion

The comparison of values of $-f''(0)$ for M = 0, N_r = K_P = s = 0, Ec=0 and $R_C = 0$

	Magyari and Keller	Bhatt. and Layek	Babu et.al	**Present**
$-f''(0)$	1.281808	1.28180838	1.281807536	1.2818061586

In order to validate the numerical results, we first compare the present result with the results obtained by Magyari and Keller, Bhattacharyya and G.C.Layek and Babu et.al as shown in above Table without the presence of magnetic field parameter, chemical reaction parameter, viscous dissipation, radiation parameter and nonporous stretching sheet. The values of $-f''(0)$ from the table are found to be in a good agreement and moreover, the present method improves the works of the aforementioned scholars.

In this study, the system of nonlinear higher order ordinary differential equations (4.12)–(4.14) subject to the boundary condition (4.15) were solved numerically employing Keller box method. By taking the step size $\triangle\,\eta = 0.01$ in η and within the interval $[0, \eta_\infty]$, the effect of different parameters like Magnetic field parameter (M), Biot number(Bi), Permeability (K_p), Eckert number (Ec), Heat source/sink parameter (Q), Thermophoresis (N_t), Lewis number (Le), Chemical reaction parameter (R_C), Prandtl number(P_r), Radiation parameter (N_r) on velocity, temperature and concentration profiles, skin friction coefficient, surface heat transfer rate and mass transfer rate have been analyzed graphically.

For instance, Figures 4.3– 4.5 show the effects of magnetic field parameter, Biot number and Permeability on velocity profile respectively, while the other parameters are being kept constant. From Figure 4.3 it is observed that the velocity profile decreases for the positive values of Magnetic field parameter, due to an increase in resisting force and consequently, velocity declines in the $\eta-$ direction with boundary layer thickness. As an increase of magnetic field parameter, the fluid velocity showing higher in injection than suction. From Figure 4.4, one can observe that increasing Biot number has no effect on velocity profile. This might be due to the fact that Biot number is a ratio of convective to conductive heat transfer. Figure 4.5 illustrates that velocity profile decrease for higher values of the Permeability parameter. That is the boundary layer thickness decreases for large values of porosity parameter. Figures 4.6 – 4.9 show the effects of Biot number, Eckert number, Heat source/sink parameter and Thermophoresis parameter on temperature profile

respectively, provided that the other parameters are constant. The effect of Biot number on temperature profile is significant as it is depicted in Figure 4.6. From the Figure one can observe that increasing Biot number increases the fluid temperature in suction. However, increasing Biot number decreases the fluid temperature in injection. Figure 4.7 reveals that enhancement of Eckert number rises fluid temperature and also the fluid temperature is higher in injection situation than suction. Figure 4.8 depicts the presence of heat source or heat generation effect ($Q > 0$), which shows an increase the thermal state of the fluid causing the thermal boundary layer to increase. In the event that the strength of the heat source is relatively large, the maximum fluid temperature does not occur at the wall but rather it occurs in the fluid region close to it. Conversely, the presence of heat sink or heat absorption effect ($Q < 0$) causes a reduction in the thermal state of the fluid, thus producing lower thermal boundary layer. From Figure 4.9, it is observed that increasing thermophoresis parameter rises temperature of the fluid with enhancement of it in injection than suction.
Effects of Lewis number and Chemical reaction parameter on concentration of the nanoparticles are displayed in Figures 4.10 and 4.11 respectively. From both Figures it is indicated that increasing each of the parameters reduces concentration of the nanoparticles. As we have seen earlier that Biot number has no effect on velocity of the nanofluid, similar effects of the Biot number on species concentration of the nanoparticles has been observed as it is depicted in Figure 4.12. Figure 4.13 describes the effect of Magnetic parameter and Permeability on Skin friction coefficient. As we observe from the figure, increasing in magnetic field parameter and Permeability increases the skin friction coefficient and showing higher in suction than injection. Figure 4.14 shows the effect of Eckert number and Thermophoresis parameter on Nusselt number. The Figure tell us the surface heat transfer rate decreases with increasing both Eckert number and Thermophoresis parameter. Besides the surface heat transfer is higher in suction than injection. Figure 4.15 indicates the effect of Permeability and Chemical reaction parameter on Sherwood number. The surface mass transfer rate is higher in suction than injection.

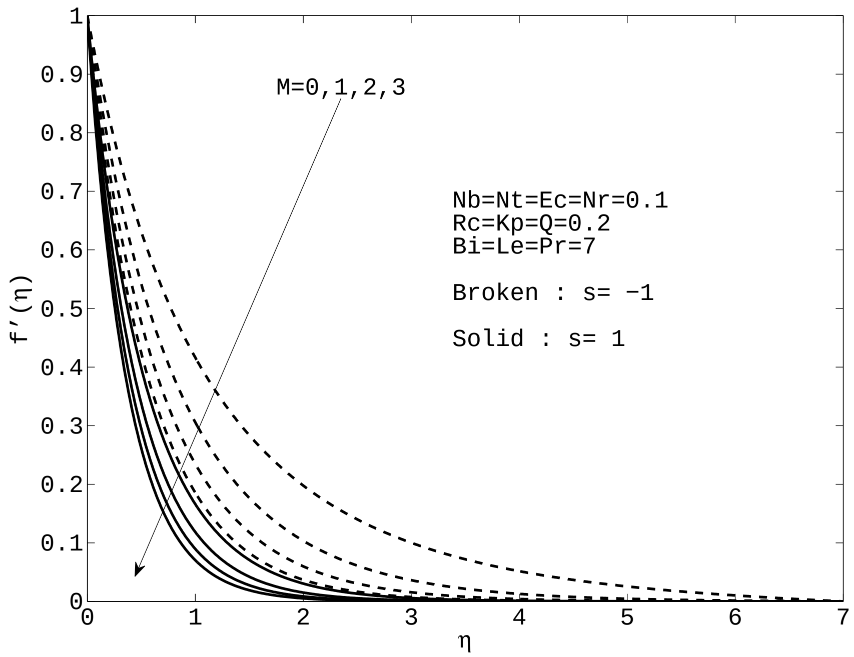

Figure 4.3: Effect of the parameter *M* on velocity profile.

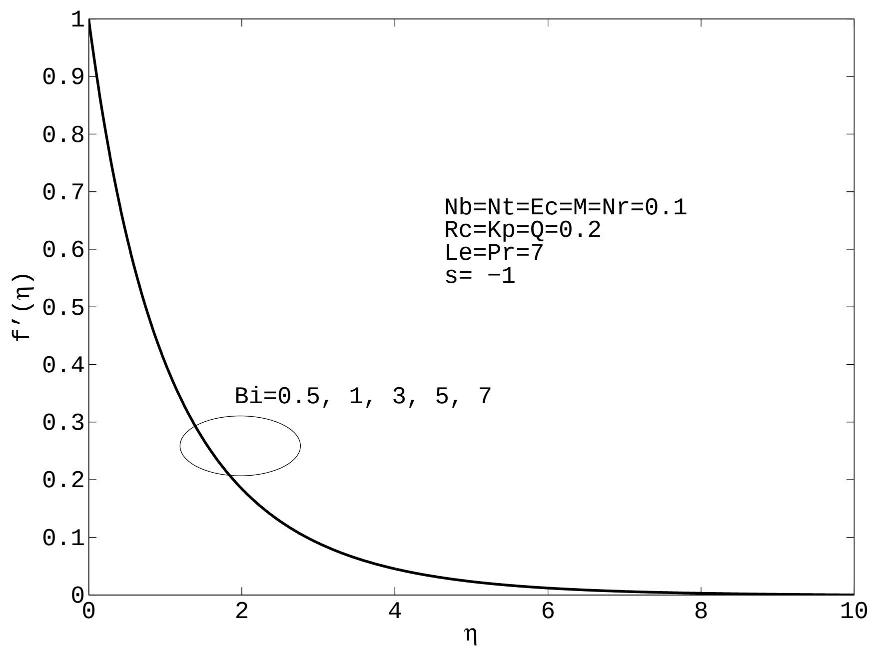

Figure 4.4: Effect of the Biot number *Bi* on velocity profile.

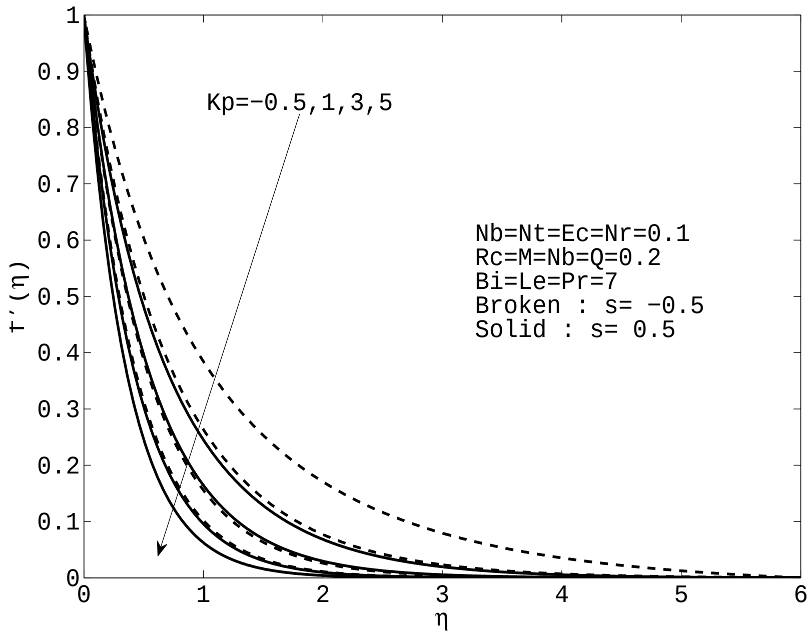

Figure 4.5: Effect of permeability parameter Kp on velocity profile.

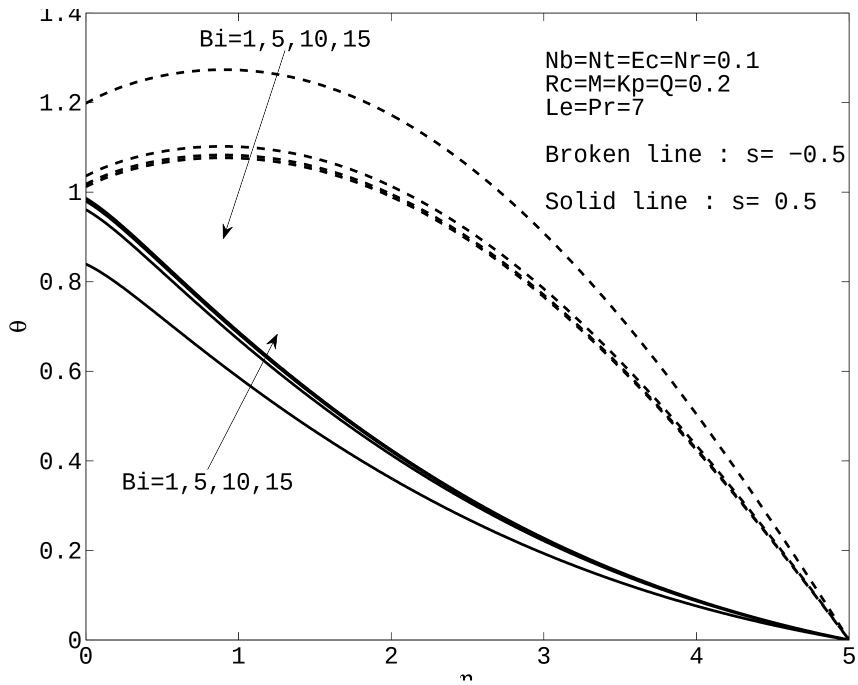

Figure 4.6: Effect of the Biot number Bi on temperature profile.

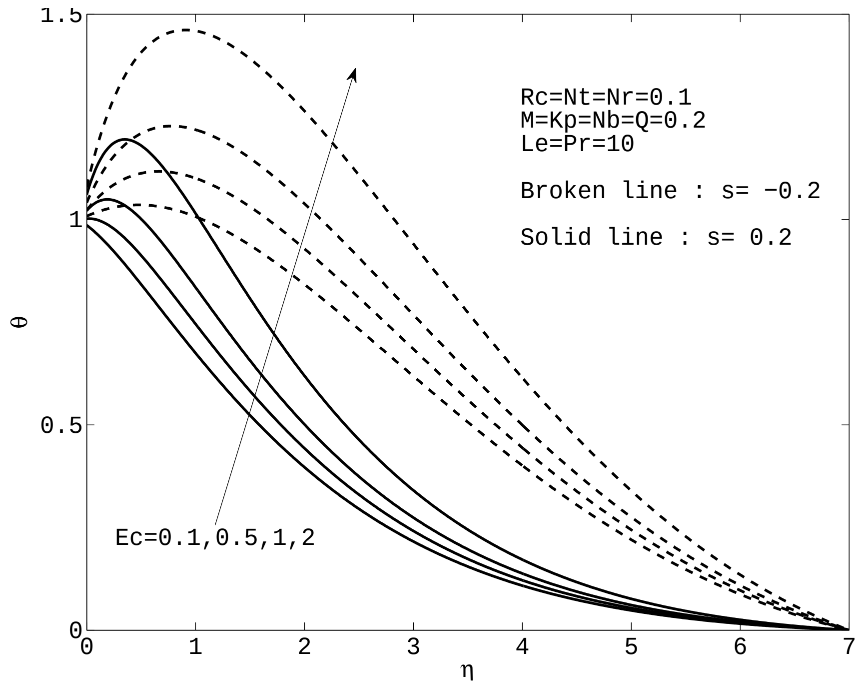

Figure 4.7: Effect of Eckert number Ec on temperature profile.

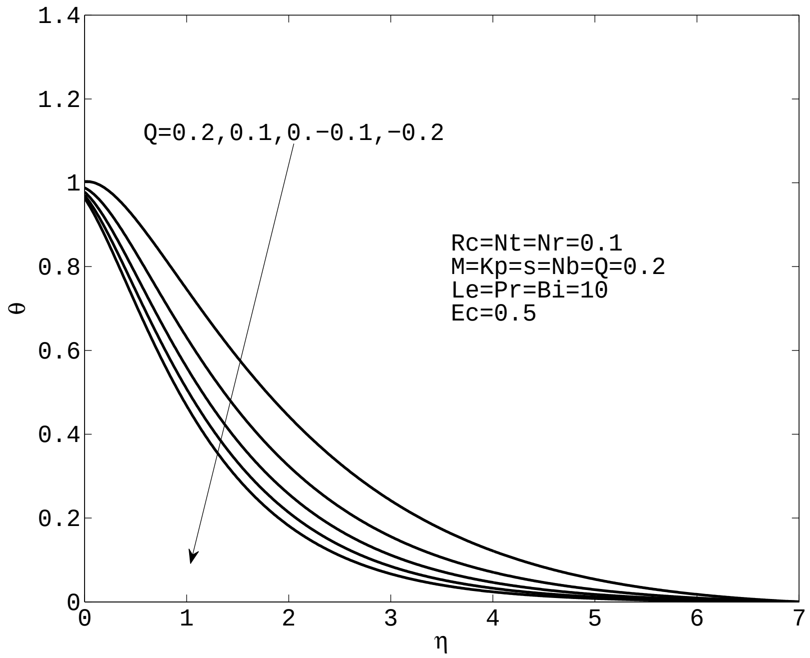

Figure 4.8: Effect of heat source-sink parameter Q on temperature profile.

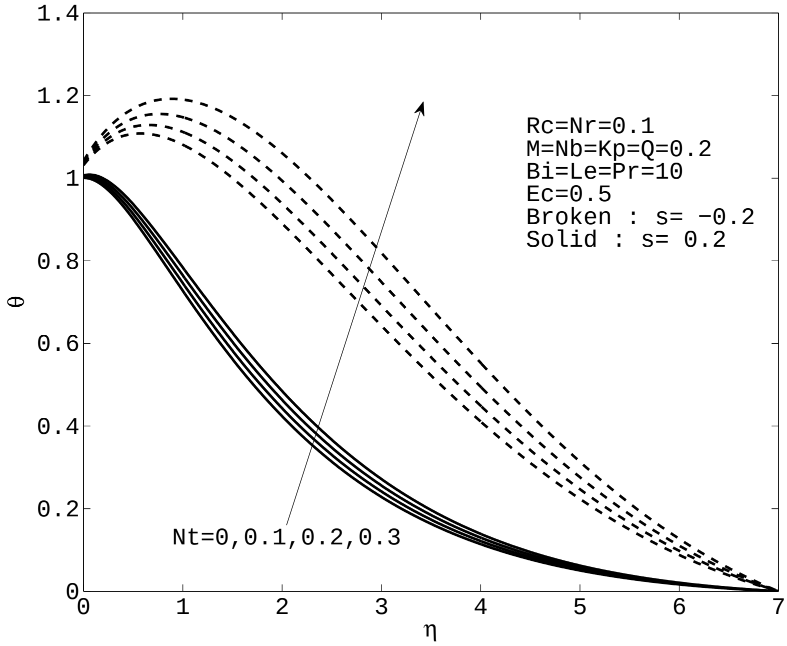

Figure 4.9: Effect of thermophoresis parameter *Nt* on temperature profile.

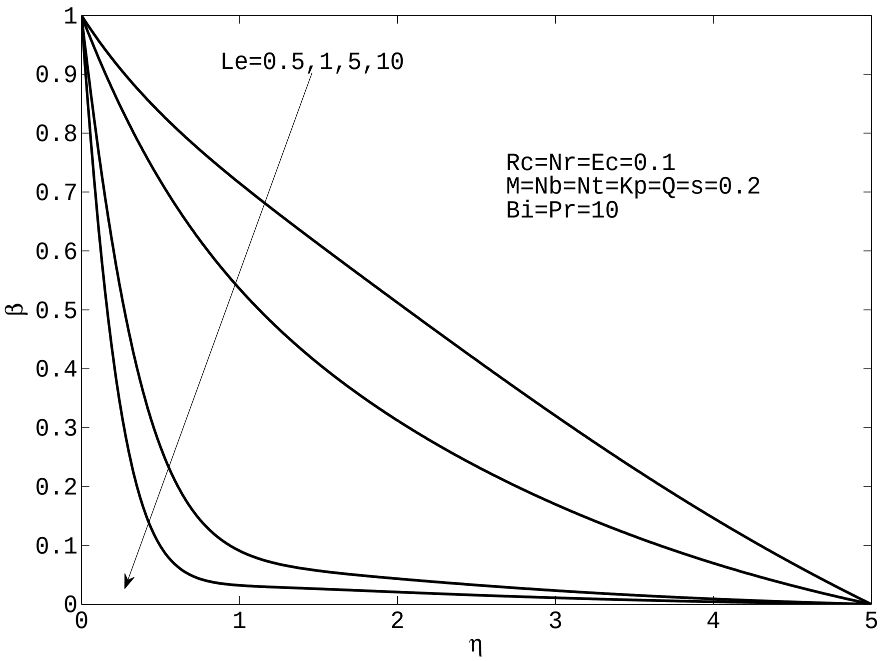

Figure 4.10: Effect of Lewis number *Le* on concentration of nanoparticles.

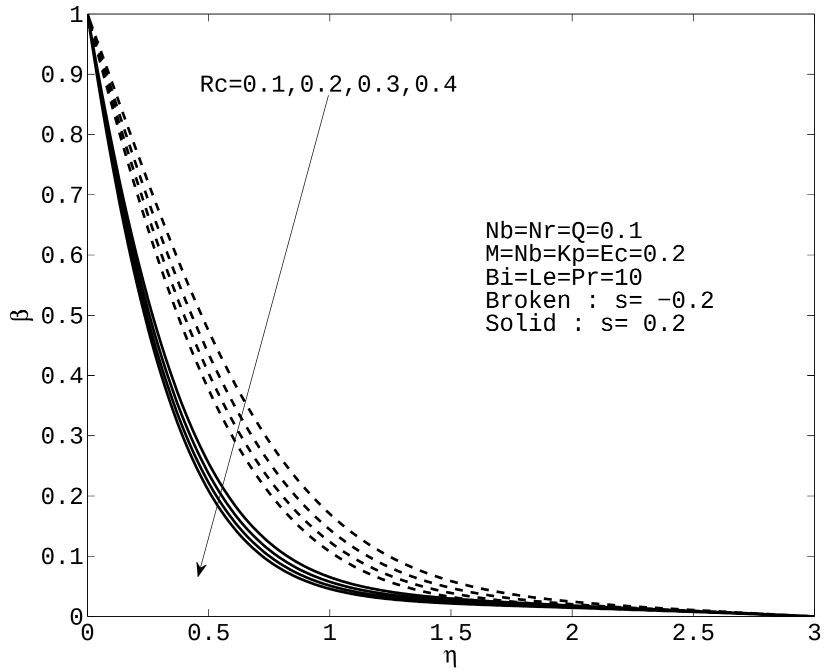

Figure 4.11: Effect of chemical reaction parameter *Rc* on concentration of nanoparticles.

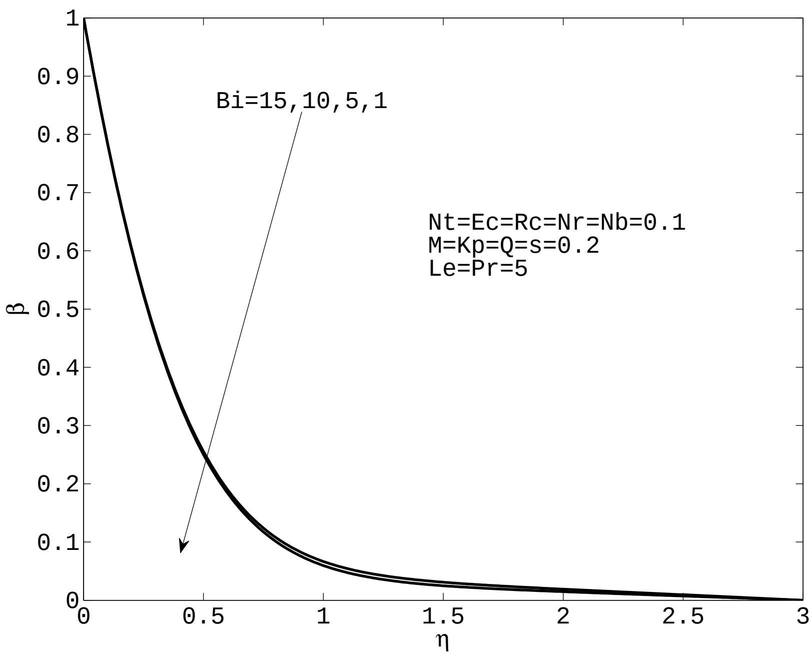

Figure 4.12: Effect of the Biot number *Bi* on concentration of nanoparticles.

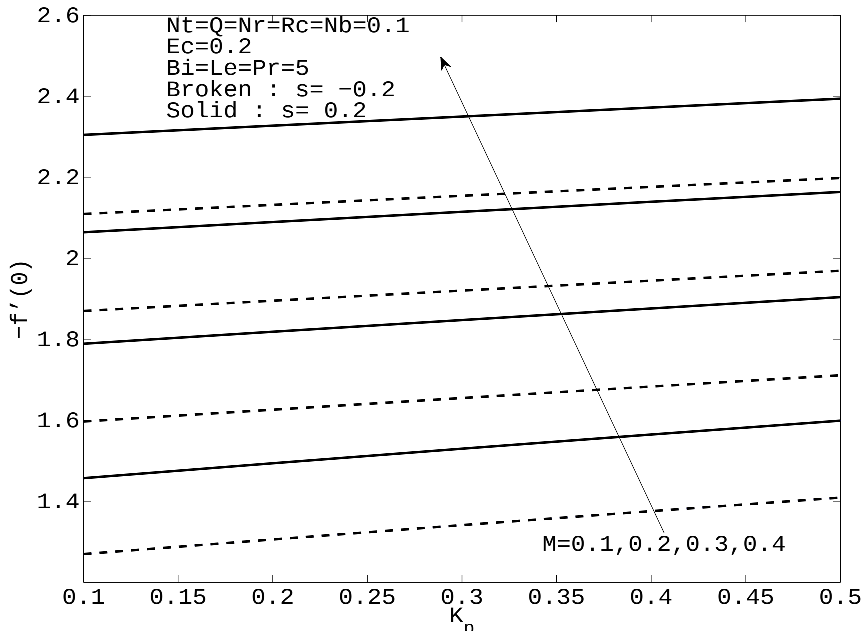

Figure 4.13: Effect of permeability parameter Kp and magnetic parameter M on skin friction coefficient.

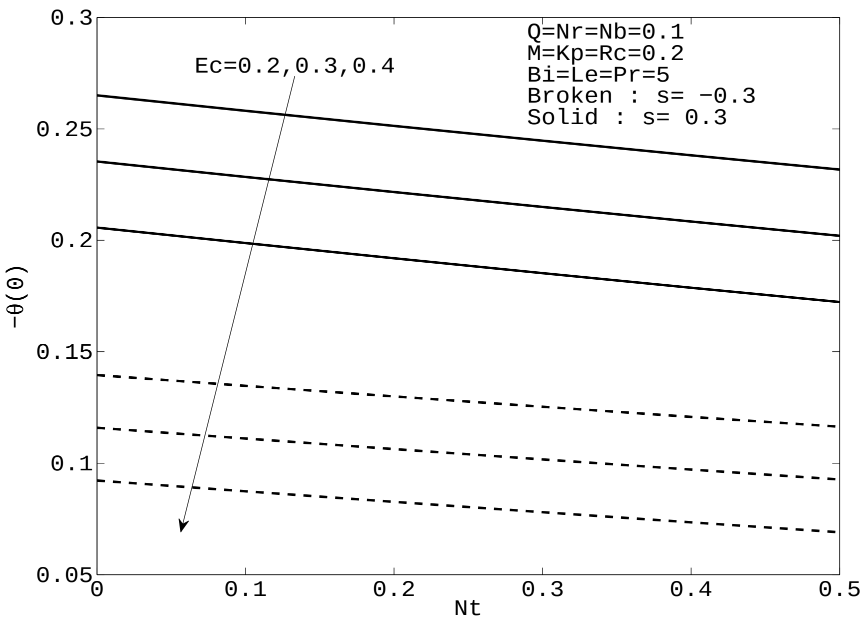

Figure 4.14: Effect of Eckert number Ec and thermophoresis parameter Nt on surface heat transfer rate.

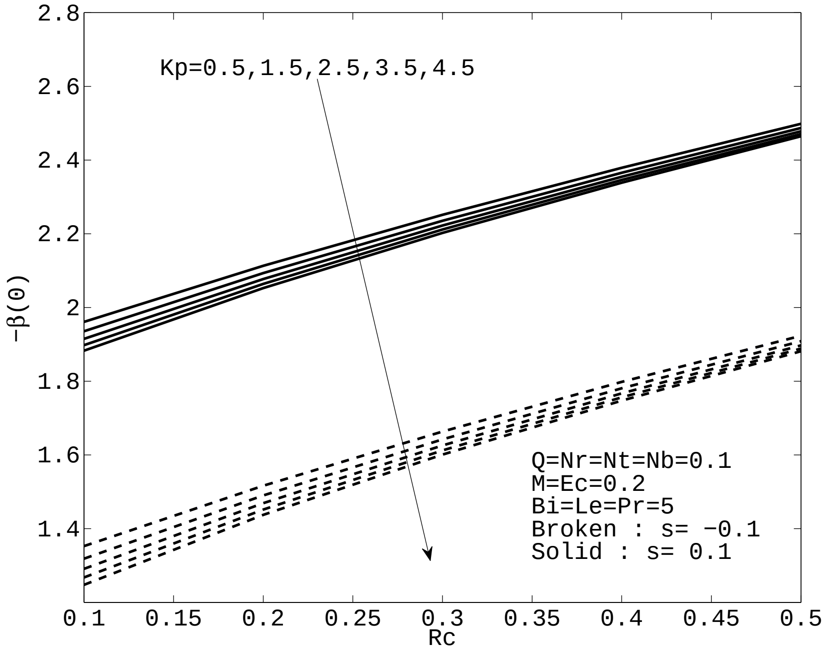

Figure 4.15: Effect of permeability parameter Kp and chemical reaction parameter Rc on surface mass transfer rate.

Chapter 5

Conclusion and Scope for the Future Work

5.1 Conclusion

In this study, analysis of numerical solution of MHD flow of nanofluid over a porous medium of an exponentially stretching sheet with convective boundary condition in the presence of suction/injection was considered. The governing nonlinear partial differential equation were transformed into higher order nonlinear ordinary differential equation and solved numerically by Keller box method. The velocity, temperature, and concentration profiles along the porous medium of an exponentially stretching sheet with convective boundary condition in the presence of suction/injection were studied and the results were shown graphically. The skin-friction coefficient, the rate of heat transfer and mass transfer were analyzed.
From the present study, we found that:

1. Increasing the magnetic parameter M and permeability parameter K_p reduces velocity of the nanofluid, but showing higher in injection than suction.

2. Effect of Biot number on velocity and concentration of nanoparticles is very less. But significant effect of Biot number on temperature of the the fluid is observed.

3. Enhancement of Eckert number (Ec), heat source/sink parameter (Q) and thermophoresis parameter N_t rise temperature of the nanofluid.

4. Increasing Lewis number (Le) and chemical reaction parameter (R_c) reduces concentration of the nanoparticles.

5. Increasing magnetic field parameter (M) and permeability parameter (K_p), the skin friction coefficient increases higher in suction than injection.

6. Rising of Eckert number (Ec) and thermophoresis parameter (N_t) reduce surface heat transfer rate, but being higher in suction than injection.

7. Increasing chemical reaction (R_c) and permeability parameter (K_p) rises the mass transfer rate with higher in suction than injection.

5.2 Scope for the Future Work

In the this thesis, numerical solutions obtained for Magnetohydrodynamics (MHD) flow of nanofluid over a porous medium of an exponentially stretching sheet with convective boundary condition in the presence of suction/injection was studied by employing the implicit finite difference scheme called Keller Box method. So, one can find the solution of the problem discussed above by considering unsteady, turbulent flow, two dimensional MHD boundary layer flow of a viscous, incompressible, electrically conducting fluid over exponentially stretching surface in a porous medium with permeability K_p subjected to suction/injection in the presence of nonuniform transverse magnetic field.

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