ANALYTICAL SOLUTION OF ONE DIMENSIONAL TIME FRACTIONAL PARABOLIC BEAM EQUATION BY USING FRACTIONAL REDUCED DIFFERENTIAL TRANSFORM METHOD

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS, COLLEGE OF NATURAL SCIENCES, JIMMA UNIVERSITY IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS

BY:

DESSALEGN MEKONNEN

ADVISOR: - YESUF OBSIE (PhD)

CO-ADVISOR: - ADEME KEBEDE (M.Sc.)

OCTOBER, 2019

JIMMA, ETHIOPIA

DECLARATION

I, the undersigned declare that, the research entitled "Analytical Solution of One Dimensional Time Fractional Parabolic Beam Equation by Using Fractional Reduced Differential Transform Method " is original and it has not been submitted to any institution elsewhere for the award of any degree or
like, where other sources of information that have been used, they have been acknowledge.

Name: Dessalegn Mekonnen

Signature ________________________

Date _____________________________

Advisor: Yesuf Obsie (PhD)

Signature ________________________

Date ___________________________

Co-advisor: Ademe Kebede (M.Sc.)

Signature ________________________

Date ___________________________

ABSTRACT

In this study Fractional Reduced Differential Transform Method (FRDTM) is presented for solving one dimensional parabolic beam equation. FRDTM is an effective tool to solve partial differential equations analytically. This method provides the solution in the form of a convergent series with easily calculable terms. The efficacy and accuracy of FRDTM is demonstrated by examples, which indicate that the presented method is very effective, simple and easy to implement. The plotted graphs illustrate the behavior of the solution for different values of time fractional order α .

KEYWORDS

Partial differential equation , Caputo time-fractional derivative, Fractional reduced differential transform.

ACKNOWLEDGMENT

Frist of all, I am indebted to my Almighty God who gave me long life and helps me to pass through different obstacles to reach this time. Next, I would like to express my special heartfelt thanks to my advisor, Dr.Yesuf Obsie and co-advisor Mr, Ademe Kebede for their unreserved support, their constructive comment, advice and guidance though out the process of conducting this research.

At last, but not least, I would like to thank my wife Getenash Mekonnen, two of my sons Falmata and Moybon for their support while I was conducting the research.

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ACRONYMS

ADM - Adomian Decomposition Method

DTM - Differential Transform Method

FEM - Finite Element Method

FRDTM-Fractional Reduced Differential Transform Method

HPM - Homotopy Perturbation Method

IVPs - Initial Value Problems

PDEs - Partial Differential Equations

RDTM - Reduced Differential Transform Method

VIM - Variational Iteration Method

CHAPTER ONE

INTRODUCTION

1.1 Background of the study

There are many problems arising in science and engineering modeled using linear or nonlinear partial differential equations. Initial value and Boundary value problem in PDEs occur in fluid mechanics, mathematical physics, astrophysics, biology, materials science, electromagnetism, image processing, computer graphics, etc. These PDEs describe various physical phenomenon including deformation of beams, viscoelastic and inelastic flows, transverse vibrations of a homogeneous beam, plate deflection theory, engineering and applied sciences (IBiş and Yeşilyurt, 2014).

The concept of the differential transform was first proposed by (Zhou, 1986) and its main applications are to solve both linear and nonlinear initial value problems in electric circuit analysis. This method constructs a semi-analytical numerical technique that uses Taylor series expansion for the solution of differential equations in the form of a polynomial.

It is different from the n-order Taylor series method which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method is computationally time-consuming especially for order equation. The differential transform is an iterative procedure for obtaining analytic Taylor series solutions of equations. It can be said that differential transform method is a universal one, and is able to solve various kinds of equations (Biazar *et al*., 2010).

Transform method is a mathematical technique that is applied in various fields. This technique generates the solutions of partial differential equations; relates solutions of difficult partial differential equations to well-known equations and applies to integrable equations. For example, Riccati equation is employed to construct generalized solutions for ordinary and partial differential equations. Various practical transforms for solving various problems were materialized in open literature, such as the Laplace transform, the Fourier transform, the traveling wave transform, the Bäcklund transformation, the integral transform, the fractional integral transforms, the fractional complex transform.

Fractional differential equations are viewed as option models to nonlinear differential equation. Varieties of them play important roles and tools, not only in mathematics but also in physics, dynamical systems, control systems and engineering, to create the mathematical modeling of many physical phenomena. Furthermore, they are employed in social science such as food supplement, climate and economics. Fractional differential equations concerning the Riemann-Liouville fractional operators or Caputo derivative have been recommended by many authors (Ibraim and Dayun, 2014).Determining approximate, numerical and exact solutions for fractional differential equations plays a significant role. Numerical solutions or analytic solutions are typically difficult to be computed. It is therefore, required to impose a process to solve the problem of nonlinear fractional differential equations. Recently, one of the most essential and useful methods for fractional calculus appeared as complex fractional transform (integral and derivative).

Fractional partial differential equations are one of the topics in the analysis of fractional calculus theory. And they are differential equations which can be obtained from the standard partial differential equations by replacing the integer order time derivative by fractional derivative (Masomi *et al.*, 2014) some of these are time fractional heat equations, time fractional wave equations , time fractional telegraphic equation ,time fractional airy's equation and so on .

In 1695, L'Hospital asked the question as to the meaning of $\frac{d^n y}{dx^n}$ "if $n = \frac{1}{2}$; that is" what if n is fractional?" Leibniz replied that" $d^{\frac{1}{2}}x$ will be equal to $x\sqrt{dx : x}$" .It is generally known that integer-order derivatives and integrals have clear physical and geometric interpretations. However, in case of fractional-order integration and differentiation, which represent a rapidly growing field both in theory and in applications to real world problems; it is not so (Dalir and Bashour, 2010). Since the appearance of the idea of and integration of arbitrary (not necessary integer) order there was not any acceptable geometric and physical interpretation of these operations for more than 300 years. In (Podlubny, 2002) it is shown that geometric interpretation of fractional integration is "Shadows on the walls" and its Physical interpretation is 'Shadows of the past".

Beam equations have historical importance, as they have been the focus of attention for prominent scientists such as Leonardo daVinci (14^{th} C) and Daniel Bernoulli ($18^{th\ C}$). Practical applications of the Beam equations are evident in mechanical structures built under the premise of beam theory. The importance of Beam theory has been outlined in the literature over the years (Gunakala *et al.*, 2012).

Examples include the construction of high-rise buildings, bridges across the rivers, air craft and heavy motor vehicles. In these structures, beams are used as the basis of supporting structures or as the main-frame foundation in axles. Without a proper knowledge of beam theory, the successful manufacture of such structures would be unfeasible and unsafe. The Euler-Bernoulli beam theory, sometimes called the classical beam theory, is the most commonly used. It is simple and provides reasonable engineering approximations for many problems. The Finite Element Method (FEM) is one of the most powerful tools used in structural analysis. Finite Element Analysis is based on the premise that an approximate solution to any complex engineering problem can be reached by subdividing a larger complex structure into smaller non-overlapping components of simple geometry called finite elements or elements Complex partial differential equations that describe these structures can be reduced to a set of linear equations that can easily be solved using this method.

The fractional calculus (fractional derivatives and fractional integral) involves different definitions of fractional operators. For example, Grunwald-letnikov fractional derivative, Riemann-Liouville fractional derivative, and rise fractional derivative and caputo fractional derivatives. Here in this study, we will consider only Caputo fractional derivatives definition of for its certain advantages when trying to model real world phenomena with traditional differential equations.

Mathematical model is a simplified description of physical reality expressed in mathematical terms. Thus, the investigation of the exact or approximate solution helps us to understand the means of this mathematical model. Many powerful and efficient methods have been proposed to obtain numerical solutions and exact solution of fractional partial differential equations. These methods include the Adomain Decomposition Method, the Variation iterative Method, the Homotopy perturbation Method, the Differential transformation Method, the finite difference method, the finite element method, the fractional Riccati equation method and so on. In these investigations, we note that many authors have sought exact and numerical solutions for fractional partial differential equations (Cui *et al.*, 2013).

Reduced differential transform method for finding a new approximate analytic solution of fractional partial differential equations has been proposed by (Keskin and Oturanc; 2010). After, seminar work of Keskin and Oturanc, FRDTM has been adopted to solve Vigorous type of differential equation arising in mathematics and other fields of science (Singh and Kumar, 2016) .However, the solution of initial value problems (IVPs) of one dimensional time fractional parabolic beam equation is not studied by FRDTM in the existing literature. Therefore, this study considers the following one dimensional time fractional parabolic beam equations:-

$$\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}u(x,t)=-\beta(x)\frac{\partial^{4}}{\partial x^{4}}u(x,t)+f(x,t),t>0,x\in\Re,m-1<\alpha\leq m \tag{1.1}$$

Subject to initial conditions: $u(x,0)=f(x),u_t(x,0)=g(x)$ (1.2)

where $\beta(x)$>0 is the ratio of flexural rigidity of the beam to its mass per unit length and $f(x,t)$ is a function of the variables **x** and **t.**

1.2. Statement of the problem

Even though time-fractional parabolic beam equations can be found in a wide variety of engineering and scientific applications, analytical solutions of one dimensional time fractional parabolic beam equations by applying fractional reduced differential transform method is not presumably presented in the existing literature. As a result, this study mainly focuses on the following problems related to one dimensional time fractional parabolic beam equations given by Eq. (1.1).

As a result, this study mainly focused on:

- Employing the reduced differential transformed method on time fractional parabolic beam equation.
- Demonstrating the applicability of the method using specific examples.

1.3. Objectives of the Study

1.3.1. General Objective

The general objective of this thesis is to study solutions of one dimensional time fractional parabolic beam equations by fractional reduced differential transform method.

1.3.2. Specific Objectives

The specific objectives of the study were:-

- To use fractional reduced differential transform method to obtain the solution of one dimensional time fractional parabolic beam equation.
- To demonstrate the applicability of the method using specific examples.

1.4. Significance of the Study

This study is believed to have the following significances:-

- It provides techniques of solving initial value problems of one dimensional time fractional parabolic beam equation by using reduced differential transform method.
- It helps to develop other researchers' skill in doing scientific research in Mathematics.
- It will be used as reference material for anyone who works on similar area.

1.5. Delimitation of the Study

This study was delimited to find solution one dimensional linear non homogenous time fractional parabolic beam equation by using fractional reduced differential transform method. The fractional derivative used here is in sense of Caputo fractional derivatives.

CHAPTER TWO

LITERATURE REVIEW

In 2011, many physical problems can be described by mathematical models that involve partial differential equations. A mathematical model is a simplified description of physical reality expressed in mathematical terms. Thus, the investigation of the exact or approximate solution helps us to understand the means of these mathematical models (Taha, 2011).Several numerical methods were developed for solving partial differential equations with variable coefficients such us He's Polynomials (Mohyud-Din, 2009), the homotopy perturbation method (Jin, 2008) homotopy analysis method (Alomari *et al*., 2008) and the modified variational iteration method (Noor and Mohyud-Din, 2008).

The Variational Iteration Method has been applied to hand various kinds of nonlinear problems, for example fractional differential equations nonlinear differential equations nonlinear thermo elasticity and nonlinear wave equations. Adomin Decomposition Method, Homotopy Perturbation Method, Homotopy Analysis method and Variation of Parameter Method are successfully applied to obtain the exact solution of differential equations (Jafaril *et al*., 2014) and (Cui *et al*., 2013).

The beam son elastic foundation has widely been used in plenty of engineering areas. For instance, railway engineering, pipes used in liquid and gas conduction lines, off-shore and port foundations some applications in airports, plane- space and petrochemical industries biomechanical and dentistry. The importance of beam theory has been outlined in the literature over the years. Examples include the construction of high-rise buildings, bridges across the rivers, air craft and heavy motor vehicles (Gunakala *et al*., 2012).

In these structures, beams are used as the basis of supporting structures or as the main-frame foundation in axles with the proper knowledge of beam theory. In recent years, numerous works have focused on the development of more advanced and efficient methods for beam equations such as the Finite Element Method, the classical method, the Generalized Integral Transform Technique (GITT), He Parameter Expanding Method (HPEM). In the last several years authors have discussed about solution of beam equation. For example, (Gunakala *et al*., 2012) were successes about beam equation in using Finite Element method to solve the beam equation with aid of MATLAB.

In 2010, another improved reduced differential transform method for finding a new approximate analytic solution of fractional partial differential equations has been proposed recently been used by Keskin and Oturance.

In 2014, Al-Amr developed the new application reduced differential transform method for the fractional differential equations and showed that RDTM is the easily usable semi analytical method and gives the exact solution for both the linear and nonlinear differential equations). It is possible to find exact solution or a closed approximate solution of a differential equation by using RDTM successfully to solve time fractional heat equations, time fractional wave equation, time fraction telegraphic equations, and so on. However, solutions of Initial Value Problem of fractional linear non- homogeneous beam equations by applying the reduced Differential transform method is yet not found in the existing literature. Consequently, this study applied this method to find the solutions of one dimensional time fraction parabolic beam equation.

CHAPTER THREE

METHODOLOGY

3.1. Study area and period

The study was conducted in the department of Mathematics, college of Natural sciences, Jimma University from September, 2018 to September, 2019.

3.2. Study design

The study was designed to be done analytically.

3.3. Source of data

Important data for this study was collected from books, internets and published research articles.

3.4. Mathematical procedure of the study

In order to achieve the objective of the study the following basic steps were carried out:

1. Apply the fractional reduced differential transform method to both sides of Eq. (1.1) and (1.2), and obtain a recursion relation for the unknown functions $U_0(x), U_1(x), U_2(x), U_3(x)...$
2. Use the inverse fractional reduced differential transform method to obtain the solution of one dimensional time fractional parabolic beam equation.
3. Mathematica software was used to sketch the solution curves of one dimensional time fractional Beam equation for different values of the fractional order derivative α .

CHAPTER FOUR

RESULT AND DISCUSSION

4.1. Preliminaries

This section presents basic definitions and operations or properties related to fractional calculus theory.

4.1.1. Gamma function

Definition4.1. The Euler - Gamma function, $\Gamma(\mathrm{Z})$ which is an extension of the fractional function to complex and real number arguments as in (Dalir *et al*., 2010) is defined by

$$\Gamma(\mathrm{Z}) = \int_0^\infty e^{-1} t^{z-1} dt, \operatorname{Re}(\mathrm{Z}) > 0, \Gamma(\mathrm{Z}) \tag{4.1}$$

For all $\mathrm{Z} > 0$ with $\operatorname{Re}(\mathrm{Z}) > 0$ and $\forall n > ¥$ then the following holds:

- $\Gamma(\mathrm{Z}+1) = \mathrm{Z}\Gamma(\mathrm{Z})$
- $\Gamma(n) = (n-1)!$ .In particular, $\Gamma(1) = 1$

4.1.2. Basic definitions and notations of Fractional Calculus theory

Some essential definitions of fractional order integrals and derivatives that are presented in this study are respectively given by Riemann-Liouville and Caputo.

Definition 4.2 let $\mu \in \Re, m \in ¥$. A function $f : \Re^{+} \to \Re$ belongs the space C_μ if there exists a real number $k \in \Re$ with $k > \mu$ such that $f(t) = t^p g(t)$, where $g(t) \in C[0,\infty)$.

Moreover, $C_\alpha \subset C_\beta$ whenever $\beta \le \alpha$ and $f \in £_\mu^m$ if $f^{(m)} \in £_\mu$ (4.2)

Definition 4.3 let J_x^α be Riemann-Liouville fractional integral operator and $f \in £_\mu$ then

I. $$J_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \alpha > 0 \tag{4.3}$$

II. $$J_t^0 f(t) = f(t) \tag{4.4}$$

For $f \in C_\mu, \mu \ge -1, \alpha, \beta \ge 0$, *and* $\gamma > -1$ the operator J_x^α satisfy the following properties:

I. $$J_x^\alpha J_x^\beta f(x) = J_x^{\alpha+\beta} f(x) = J_x^\beta J_x^\alpha f(x) \tag{4.5}$$

II. $$J_x^\alpha X^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} X^{\alpha+\gamma}, X > 0 \tag{4.6}$$

Remark: The Riemann-Liouville derivative has certain limitations when someone tries to model some real physical problems. In their work, Caputo & Mainardi proposed a modified fractional differential operator D_x^{α} to the theory of viscoelasticity to overcome the inconsistency of Riemann-Liouville derivative. The proposed Caputo fractional derivative permits us to use initial and boundary conditions involving integer order derivatives, which have clear physical interpretations in formulation of problem

Definition 4.4 If $m-1<\alpha \le m, m\in \mathbb{N}, t>0,$ then Caputo fractional derivative of $f \in C_{\mu}$ **(**Carpinteri and Mainaridi, 1997) is defined as

$$D_x^{\alpha} f\left(x\right)=J_x^{m-\alpha} D_x^{m} f\left(x\right)=\frac{1}{\Gamma\left(m-\alpha\right)}\int_0^x \left(x-t\right)^{m-\alpha-1} f^{(m)}\left(t\right)dt \tag{4.7}$$

The basic properties of the Caputo fractional derivative D_x^{α} are presented in the following Lemma.

Lemma: - If $m-1<\alpha \le m, m\in \mathbb{N}$ *and* $f\left(x\right)\in C_{\mu}^{m}, \mu \ge -1,$ then

1. $D_t^{\alpha} D_t^{\beta} f\left(t\right)=D_t^{\alpha+\beta} f\left(t\right)=D_t^{\beta} D_t^{\alpha} f\left(t\right)$ (4.8)
2. $D_t^{\alpha} t^{\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} t^{\gamma-\alpha}, t>0$ (4.9)
3. $D_t^{\alpha} J_t^{\alpha} f\left(t\right)= f\left(t\right), t>0$ (4.10)

4.1.3. Fractional Reduced Differentia Transform Method (FRDTM)

In this section, the basic properties of the fractional reduced differential transform method are described.The FRDTM is the most easily implemented analytical method which provides the exact solution for both linear and nonlinear fractional differential equations, is very effective, reliable and efficient, and very powerful analytical approach, refer (Gupta, 2011); (Srivastava *et al*., 2013); (Srivastava, *et al*., 2014) ;(Singh *et al*., 2013) and (Singh and Kumar, 2016).

Therefore, this study presents the solution of time fractional parabolic beam equation by using FRDTM. Consider a function of two variables $u\left(x,t\right)$ and suppose that it can be represented as a product of two single-valued functions, i.e. $u\left(x,t\right) = f\left(x\right) g\left(t\right)$.

Based on the properties of one-dimensional differential transform method, the function $u\left(x,t\right)$ can be represented as:

$$u(x,t)=\left(\sum_{i=0}^{\infty}F(i)x^{i}\right)\left(\sum_{j=0}^{\infty}G(j)t^{j}\right)=\sum_{k=0}^{\infty}U_k(x)t^k \qquad (4.\ 11)$$

where $U_k(x)$ is called t-dimensional spectrum function of $u(x,t)$ which is also called the reduced transformed function of $u(x,t)$

In fact, the above definition shows that, the concept of fractional reduced differential transform is derived from the power series expansion (Keskin and Oturanc, 2010).

The basic definition and operation of FRDTM as introduced in (Srivastava *et al*., 2013); (Babaei and pour, 2015) and (Miller and Ross, 1993) were given bellow:-

Definition 4.5 If $u(x,t)$ is analytic and continuously differentiable with respect to space variable **x** and time variable **t** in the domain of interest, then the t-dimensional spectrum function or the fractional reduced transformed function of $u(x,t)$ is given by

$$R_D\left[u(x,t)\right]=U_k(x)=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,t)\right]_{t=t_0} \qquad (4.12)$$

where α is a parameter which describes the order of time fractional derivative in a Caputo sense and $U_k(x)$ is the transformed function of the $u(x,t)$.

Definition 4.6 The inverse FRDT of $U_k(x)$ is defined as

$$R_{D^{-1}}\left[U_K(x)\right]=u(x,t)=\sum_{k=0}^{\infty}U_K(x)(t-t_o)^{k\alpha} \qquad (4.13)$$

Now combining Eq. (4.12**)** and (4.13**)**, we obtain:

$$u(x,t)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,t)\right]_{t=t_0}(t-t_o)^{k\alpha} \qquad (4.14)$$

In particular, if $t_o=0$ equation (4.14) becomes

$$u(x,t)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,t)\right]_{t=0}t^{k\alpha} \qquad (4.15)$$

Moreover, if α=0 the FRDTM of Eq. (4.15) reduce to classical RDTM.

Applying the fractional reduced differential transformed operator on both sides of equation $u(x,0)=f(x)$ and $u_t(x,0)=g(x)$, we get respectively $U_0(x)=f(x)$ and $U_1(x)=g(x)$

Hence using equation (4.13) the function $u(x,t)$ can therefore be written in a finite series as $u_n(x,t) = \sum_{k=0}^{n} U_k(x)(t-t_0)^{k\alpha} + R_n(x,t)$ where n represents order of estimated solution. Here the tail function $R_n(x,t)$ is negligibly small. In particular, if $t_0 = 0$ this equation takes the form $u_n(x,t) = \sum_{k=0}^{n} U_k(x) t^{k\alpha}$.

Finally, the accurate solution is found by taking limit of the function, i.e.

$$\lim_{n\to\infty} u_n(x,t) = u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^{k\alpha} = U_0(x) + U_1(x)t^{\alpha} + U_2(x)t^{2\alpha} + U_3(x)t^{3\alpha} + \ldots \qquad (4.16)$$

Based on the definition and properties of time fractional reduced differential transform of one dimensional beam equation we have the following results (Theorems).

Table 4.1 Basic properties of one dimensional fractional reduced differential transform, (srivastava *et al*., 2013) and (Abuteen *et al*., 2016)

No	Original Function	Transformed function (FRDTM)
1	$u(x,t)$	$U_{k(x)} = \frac{1}{\Gamma(k\alpha+1)}\left(\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,t)\right)_{t=t_0}$
2	$w(x,t)=u(x,t)\pm v(x,t)$	$W_k(x)=U_K(x)\pm V_K(x)$
3	$w(x,t)=au(x,t)$	$W_k(x)=aU_k(x)$, for arbitrary constant a
4	$f(x,t)=x^m\sin(\eta x+\theta t)$	$F_k(x)=x^m\frac{\theta^k}{k!}\sin\left(\eta x+\frac{\Pi k}{2}\right)$ η, and, θ are constants
5	$f(x,t)=x^m\cos(\eta x+\theta t)$	$F_k(x)=x^m\frac{\theta^k}{k!}\cos\left(\eta x+\frac{\Pi k}{2}\right)$ η and θ are constants
6	$w(x,t)=\frac{\partial^r}{\partial t^r}u(x,t)$	$W_k(x)=(k+1)(k+2)...(k+r)U_{k+1}(x)=\frac{(k+r)!}{k!}U_{k+r}(x)$
7	$w(x,t)=\frac{\partial}{\partial t}u(x,t)$	$W_k(x)=(k+1)U_{k+1}(x)$
8	$w(x,t)=\frac{\partial}{\partial x}u(x,t)$	$W_k(x)=\frac{\partial}{\partial x}U_k(x)$
9	$w(x,t)=\frac{\partial^r}{\partial x^r}u(x,t)$	$W_k(x)=\frac{\partial^r}{\partial x^r}U_k(x)$
10	$w(x,t)=\frac{\partial^{N\alpha}}{\partial t^{N\alpha}}u(x,t)$	$W_k=\frac{\Gamma(k\alpha+N\alpha+1)}{\Gamma(k\alpha+1)}U_{k+N}$

Here we have the detail of some of the theorems with their proofs from table 4.1.

Theorem4.1 If $w(x,t)=\frac{\partial^r}{\partial x^r}u(x,t)$, then, $W_k(x)=\frac{\partial^r}{\partial x^r}U_k(x)$

Proof let $W_k(x)$ and $U_k(x)$ **t**-dimensional spectrum functions of $w(x,t)$ and $u(x,t)$ respectively and is analytic and **K**-time continuous differentiable function with respect to time **t** and **x** in the domain of our interest. Now applying **FRDT** operator to the left side of the equation $w(x,t)=\frac{\partial^r}{\partial x^r}u(x,t)$ we get

$$W_k(x)=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}w(x,t)\right]_{t=t_o}=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\frac{\partial^{r}}{\partial x^{r}}u(x,t)\right]_{t=t_o}$$

$$=\frac{\partial^{r}}{\partial x^{r}}\left(\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,t)\right]_{t=t_o}\right)=\frac{\partial^{r}}{\partial x^{r}}U_k(x)$$

Hence $W_k(x)=\frac{\partial^{r}}{\partial x^{r}}U_k(x)$.This completes the proofs of the theorem.

Theorem 4.2. If $w(x,t)=\frac{\partial^{N\alpha}}{\partial t^{N\alpha}}u(x,t)$, then $W_k(x)=\frac{\Gamma[k\alpha+N\alpha+1]}{\Gamma[k\alpha+1]}U_{k+N}(x)$

Proof: let $W_K(x)$ and $U_K(x)$ be **t** -dimensional spectrum functions of $w(x,t)$ and $u(x,t)$ respectively and is analytic and **K**-time continuous differentiable function with respect to time **t** and **x** in the domain of our interest. Applying FRDT operator to the left side equation $w(x,t)=\frac{\partial^{N\alpha}}{\partial t^{N\alpha}}u(x,t)$, we obtain

$$W_k(x)=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}w(x,t)\right]_{t=t_o}$$

$$=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\left(\frac{\partial^{N\alpha}}{\partial t^{N\alpha}}u(x,t)\right)\right]$$

$$=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha+N\alpha}}{\partial t^{k\alpha+N\alpha}}u(x,t)\right]_{t=t_o}$$

$$=\frac{\Gamma(k\alpha+N\alpha+1)}{\Gamma(k\alpha+N\alpha+1)}\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha+N\alpha}}{\partial t^{k\alpha+N\alpha}}U(x,t)\right]_{t=t_o}$$

$$=\frac{\Gamma(k\alpha+N\alpha+1)}{\Gamma(k\alpha+1)}\underbrace{\frac{1}{\Gamma(k\alpha+N\alpha+1)}\left[\frac{\partial^{\alpha(K+N)}}{\partial t^{\alpha(K+N)}}U(x,t)\right]_{t=t_o}}_{U_{k+N}(x)}$$

$$W_k(x)=\frac{\Gamma(k\alpha+N\alpha+1)}{\Gamma(k\alpha+1)}U_{K+N}(x)$$

4.2. Main result

The aim of this study is to obtain analytical solution of one dimensional time fractional parabolic beam equation by using fractional reduced differential transform method. This is done based on the works of (Keskin and Oturanc, 2010) that was used to solve fractional partial differential equations. So, the definitions, theorems and some derivations related to FRDTM mentioned in the preceding section were applied here.

I. Consider the one dimensional time fractional homogeneous parabolic beam equation in Caputo sense:

$$\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}u(x,t) = -\beta(x)\frac{\partial^4}{\partial x^4}u(x,t), x \in \Re, t > 0, 0 < \alpha \le 1 \quad (4.17)$$

Subjected to the initial conditions:

$$u(x,0) = f(x), and, u_t(x,0) = g(x), x \in \Re \quad (4.18)$$

where $\beta(x)$ is constant coefficient and $f(x,t) = 0$.

Applying FRDTM on Eq. (4.17**)**, we get the following recurrence relation:

$$\frac{\Gamma(k\alpha + 2\alpha + 1)}{\Gamma(k\alpha + 1)}U_{k+2}(x) = -\beta(x)\frac{\partial^4}{\partial x^4}U_k(x)$$

$$U_{k+2}(x) = -\frac{\Gamma(k\alpha + 1)}{\Gamma(k\alpha + 2\alpha + 1)}\beta(x)\left(\frac{\partial^4}{\partial x^4}U_k(x)\right) \quad (4.19)$$

Again applying FRDTM on both sides Eq. (4.18), we obtain

$$U_0(x) = f(x), U_1(x) = g(x) \quad (4.20)$$

Using Eqs. (4.19) and (4.20), we get the values of $U_k(x)$ for different values of k recursively .i.e,

For **k=0**, $U_2(x) = -\frac{1}{\Gamma(2\alpha + 1)}\beta(x)\frac{\partial^4}{\partial x^4}U_0(x) = -\frac{1}{\Gamma(2\alpha + 1)}\beta(x)\frac{\partial^4}{\partial x^4}f(x)$

For **k=1**, $U_3(x) = -\frac{\Gamma(\alpha + 1)}{\Gamma(3\alpha + 1)}\beta(x)\frac{\partial^4}{\partial x^4}U_1(x) = -\frac{\Gamma(\alpha + 1)}{\Gamma(3\alpha + 1)}\beta(x)\frac{\partial^4}{\partial x^4}g(x)$

For**k=2,** $U_4(x) = -\frac{\Gamma(2\alpha + 1)}{\Gamma(4\alpha + 1)}\beta(x)\frac{\partial^4}{\partial x^4}U_2(x) = \frac{1}{\Gamma(4\alpha + 1)}\beta^2(x)\left(\frac{\partial^8}{\partial x^8}f(x)\right)$

For **k=3**

$$U_5(x) = -\frac{\Gamma(3\alpha+1)}{\Gamma(5\alpha+1)}\beta(x)\left(\frac{\partial^4}{\partial x^4}U_3(x)\right) = \frac{\Gamma(\alpha+1)}{\Gamma(5\alpha+1)}\beta(x)\left[\frac{\partial^4}{\partial x^4}\left(\beta(x)\frac{\partial^4}{\partial x^4}U_1(x)\right)\right]$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma(5\alpha+1)}\beta(x)^2\frac{\partial^8}{\partial x^8}g(x)$$ and continue this way.

Applying inverse FRDTM on $U_k(x)$, we find

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^{k\alpha}$$

$$= U_0(x) + U_1(x)t^{\alpha} + U_2(x)t^{2\alpha} + U_3(x)t^{3\alpha} + U_4(x)t^{4\alpha} + \ldots$$

$$= f(x) + g(x)t^{\alpha} - \frac{1}{\Gamma(2\alpha+1)}\beta(x)\frac{\partial^4}{\partial x^4}f(x)t^{2\alpha} - \frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}\beta(x)\frac{\partial^4}{\partial x^4}g(x)t^{3\alpha} + \ldots \quad (4.21)$$

II. Consider the one dimensional non-homogeneous parabolic beam equation.

The derivation of solution of non-homogenous one-dimensional time fractional beam equation using FRDTM is also treated as in the case of homogenous equation with little modification and is shown below.

$$\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}u(x,t) = -\beta(x)\frac{\partial^4}{\partial x^4}u(x,t) + f(x,t), \alpha \in (0,1], t > 0, 0 < x \le 1 \quad (4.22)$$

Subject to initial conditions

$u(x,0) = f(x), u_t(x,0) = g(x)$ Where $\beta(x) > 0$ and $f(x,t)$ is a continuous function of the variables **x** and **t**. (4.23)

Applying FRDTM on Eqs.(4.22) and (4.23), we get respectively the following recurrence relation:-

$$U_{k+2}(x) = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+2\alpha+1)}\left(-\beta(x)\frac{\partial^4}{\partial x^4}U_k(x) + F_k(x)\right) \quad (4.24)$$

Again applying FRDTM on both sides Eq. (4.23), we obtain

$$U_0(x) = f(x), U_1(x) = g(x) \quad (4.25)$$

Using Eqs. (4.24) and (4.25), we get the values of $U_k(x)$ for different values of k recursively. i.e.

For k=0,

$$U_2(x)=\frac{\Gamma(1)}{\Gamma(2\alpha+1)}\left(-\beta(x)\frac{\partial^4}{\partial x^4}U_0(x)+F_0(x)\right)=\frac{\Gamma(1)}{\Gamma(2\alpha+1)}\left(-\beta(x)\frac{\partial^4}{\partial x^4}f(x)+F_0(x)\right)$$

For k=1,

$$U_3(x)=\frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}\left(-\beta(x)\frac{\partial^4}{\partial x^4}U_1(x)+F_1(x)\right)=\frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}\left(-\beta(x)\frac{\partial^4}{\partial x^4}g(x)+F_1(x)\right)$$

For k=2

$$U_4(x)=\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\left[-\beta(x)\frac{\partial^4}{\partial x^4}\left(\frac{\Gamma(1)}{\Gamma(2\alpha+1)}\left(-\beta(x)\frac{\partial^4}{\partial x^4}f(x)+F_0(x)\right)+F_2(x)\right)\right]$$

For k=3

$$U_5(x)=\frac{\Gamma(3\alpha+1)}{\Gamma(5\alpha+1)}\left(-\beta(x)\frac{\partial^4}{\partial x^4}\frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}\left(-\beta(x)\frac{\partial^4}{\partial x^4}g(x)+F_1(x)\right)+F_3(x)\right)$$

For k=4,

$$U_6(x)=\frac{\Gamma(4\alpha+1)}{\Gamma(6\alpha+1)}\left(\begin{array}{l}-\beta(x)\frac{\partial^4}{\partial x^4}\left(\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\left(-\beta(x)\frac{\partial^4}{\partial x^4}\left(\begin{array}{l}\frac{\Gamma(1)}{\Gamma(2\alpha+1)}\left(-\beta(x)\frac{\partial^4}{\partial x^4}f(x)+F_0(x)\right)\\+F_2(x)\end{array}\right)\right)\right)\\+F_4(x)\end{array}\right)$$

Similarly $U_k(x)$ for $k\geq 5$ can be founds.

Applying inverse FRDTM to $U_k(x)$ we obtain the solution in a series form.

i.e. $u(x,t)=\sum_{k=0}^{\infty}U_k(x)t^{k\alpha}$

$$=U_0(x)+U_1(x)t^{\alpha}+U_2(x)t^{2\alpha}+U_3(x)t^{3\alpha}+U_4(x)t^{4\alpha}+U_5(x)t^{5\alpha}\ldots$$

$$\begin{aligned}&= f(x)+g(x)t^{\alpha}+\frac{\Gamma(1)}{\Gamma(2\alpha+1)}\left(-\beta(x)\frac{\partial^4}{\partial x^4}f(x)+F_0(x)\right)t^{2\alpha}+\\&\frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}\left(-\beta(x)\frac{\partial^4}{\partial x^4}g(x)+F_1(x)\right)t^{3\alpha}+\\&\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\left[-\beta(x)\frac{\partial^4}{\partial x^4}\left(\frac{\Gamma(1)}{\Gamma(2\alpha+1)}\left(-\beta(x)\frac{\partial^4}{\partial x^4}f(x)+F_0(x)\right)+F_2(x)\right)\right]t^{4\alpha}+\\&\frac{\Gamma(3\alpha+1)}{\Gamma(5\alpha+1)}\left(-\beta(x)\frac{\partial^4}{\partial x^4}\frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}\left(-\beta(x)\frac{\partial^4}{\partial x^4}g(x)+F_1(x)\right)+F_3(x)\right)+\ldots\end{aligned} \tag{4.26}$$

Hence, as described in Eq. (4.16) we conclude that the above result gives the analytical solution of the one-dimensional homogenous and non- homogenous time fractional beam equation.

4.3. Illustrative examples

In this part we deal with some examples to show the efficiency and accuracy of fractional reduced differential transform method (FRDTM**)** explained in the above sections for homogeneous and non-homogenous time fractional beam equation.

Example 4.1consider one dimensional constant coefficient homogeneous time fractional parabolic beam equation

$$\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}u(x,t)=-\frac{\partial^4}{\partial x^4}u(x,t), x\in ¡ \ ,t>0, 0<\alpha\le 1, \beta(x)=1 \qquad (4.27)$$

Subjected to the initial condition

$$u(x,0)=\cos x, u_t(x,0)=-\sin x \qquad (4.28)$$

Solution: Applying FRDT operator to both sides of the Eq. (4.27) we obtain the following iteration as follow.

$$R_D\left(\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}u(x,t)=-\frac{\partial^4}{\partial x^4}u(x,t)\right)=\frac{\Gamma(k\alpha+2\alpha+1)}{\Gamma(k\alpha+1)}U_{k+2}(x)=-\frac{\partial^4}{\partial x^4}U_k(x)$$

$$U_{K+2}(x)=-\frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+2\alpha+1)}\frac{\partial^4}{\partial x^4}U_k(x) \qquad (4.29)$$

Again applying FRDTM operator to both sides of the Eq. (4.28) we obtain the following iteration as follow.

$$R_D u(x,0)=U_0(x)=\cos x, R_D u_t(x,0)=U_1(x)=-\sin x \qquad (4.30)$$

Using Eqs. (4.27) and (4.28), we get the following recursive relation:-

For k=0, $U_2(x)=-\frac{\Gamma(1)}{\Gamma(2\alpha+1)}\frac{\partial^4}{\partial x^4}(\cos x)=\frac{-\cos x}{\Gamma(2\alpha+1)} \ U_2(x)=\frac{-\cos x}{\Gamma(2\alpha+1)}$

For k=1 $U_3(x)=-\frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}\frac{\partial^4}{\partial x^4}(U_1(x))=\frac{-\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}\frac{\partial^4}{\partial x^4}(-\sin x)=\frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}\sin x$

For k=2

$$U_4(x)=\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\frac{\partial^4}{\partial x^4}(U_2(x))=-\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\frac{1}{\Gamma(2\alpha+1)}\frac{\partial^4}{\partial x^4}(-\cos x)=\frac{\cos x}{\Gamma(4\alpha+1)}$$

$$U_4(x) = \frac{\cos x}{\Gamma(4\alpha+1)}$$

For k=3

$$U_5(x) = \frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\frac{\partial^4}{\partial x^4}\left(U_2(x)\right) = -\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\frac{1}{\Gamma(2\alpha+1)}\frac{\partial^4}{\partial x^4}(-\cos x) = \frac{\cos x}{\Gamma(4\alpha+1)}$$

$$U_5(x) = \frac{\Gamma(\alpha+1)}{\Gamma(5\alpha+1)}\sin x$$

Applying inverse FRDTM to $U_k(x)$, it yields

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^{k\alpha} = U_0(x) + U_1(x)t^{\alpha} + U_2(x)t^{2\alpha} + U_3(x)t^{3\alpha} + U_4(x)t^{4\alpha} + U_5(x)t^{5\alpha} \ldots$$

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^{k\alpha} = \cos x + (-\sin x)t^{\alpha} + \left(\frac{-\cos x}{\Gamma(2\alpha+1)}\right)t^{2\alpha} + \left(\frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}\sin x\right)t^{3\alpha} + \left(\frac{\cos x}{\Gamma(4\alpha+1)}\right)t^{4\alpha} + \left(\frac{\Gamma(\alpha+1)}{\Gamma(5\alpha+1)}\sin x\right)t^{5\alpha} \ldots \tag{4.31}$$

When $\alpha = 1$, equation (4.31) becomes

$$u(x,t) = \cos x + -\sin x\, t + \frac{-\cos x}{\Gamma(3)}t^2 + \frac{\sin x\, \Gamma(2)}{\Gamma(4)}t^3 + \frac{\cos x}{\Gamma(5)}t^4 + \frac{-\sin x\, \Gamma(2)}{\Gamma(6)}t^5 + \ldots$$

$$u(x,t) = \cos x + -\sin x\, t + \frac{-\cos x}{2!}t^2 + \frac{\sin x}{3!}t^3 + \frac{\cos x}{4!}t^4 + \frac{-\sin x}{5!}t^5 + \ldots$$

$$u(x,t) = \cos x\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots\right) - \sin x\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^5}{5!} + \ldots\right)$$

$u(x,t) = \cos x \cos t - \sin x \sin t = \cos(x+t)$ which is the exact solution of equation (4.27).

Example 4.2 considers one dimensional constant coefficient homogeneous time fractional parabolic beam equation with constant coefficient.

$$\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}u(x,t) = -\beta(x)\frac{\partial^4}{\partial x^4}u(x,t), x \in \Re, t > 0, 0 < \alpha \le 1, \beta(x) = 1 \tag{4.32}$$

Subjected to the initial condition $u(x,0) = \sin x, u_t(x,0) = 0$ (4.33)

Solution: - applying FRDTM operator to both side of the Eqs. (4.32) & (4.33) we obtain the following iteration:-

$$R_D\left(\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}u(x,t)=-\frac{\partial^4}{\partial x^4}u(x,t)\right)$$

$$\frac{\Gamma(k\alpha+2\alpha+1)}{\Gamma(k\alpha+1)}U_{k+2}(x)=-\frac{\partial^4}{\partial t^4}U_k(x)$$

$$U_{K+2}(x)=-\frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+2\alpha+1)}\frac{\partial^4}{\partial x^4}U_k(x) \tag{4.34}$$

$$R_D u(x,0)=U_0(x)=\sin x, R_D u_t(x,0)=U_1(x)=0 \tag{4.35}$$

From (4.34) and (4.35), we get the following recursive relation.

Fork=0, $U_2(x)=-\frac{\Gamma(1)}{\Gamma(2\alpha+1)}\frac{\partial^4}{\partial x^4}U_0(x)=-\frac{1}{\Gamma(2\alpha+1)}\frac{\partial^4}{\partial x^4}\sin x=-\frac{\sin x}{\Gamma(2\alpha+1)}$

Thus, $U_2(x)=-\frac{\sin x}{\Gamma(2\alpha+1)}$

For k=1 $U_3(x)=-\frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}\frac{\partial^4}{\partial x^4}U_1(x)=-\frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}\frac{\partial^4}{\partial x^4}(0)=0$

For k=2 $U_4(x)=-\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\frac{\partial^4}{\partial x^4}U_2(x)=-\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\frac{\partial^4}{\partial x^4}\left(-\frac{\sin x}{\Gamma(2\alpha+1)}\right)=\frac{\sin x}{\Gamma(4\alpha+1)}$

$$U_4(x)=\frac{\sin x}{\Gamma(4\alpha+1)}$$

For k=3 $U_5(x)=\frac{-\Gamma(4\alpha+1)}{\Gamma(5\alpha+1)}\frac{\partial^4}{\partial x^4}U_3(x)=\frac{-\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)}\frac{\partial^4}{\partial x^4}(0)=0$

The differential inverse transform $U_k(x)$:-

$$u(x,t)=\sum_{k=0}^{\infty}U_k(x)t^{k\alpha}$$

$$u(x,t)=U_0(x)+U_1(x)t^{\alpha}+U_2(x)t^{2\alpha}+U_3(x)t^{3\alpha}+U_4(x)t^{4\alpha}+...$$

$$u(x,t)=\sin x+-\frac{\sin x}{\Gamma(2\alpha+1)}t^{2\alpha}+\frac{\sin x}{\Gamma(4\alpha+1)}t^{4\alpha}-\text{K}$$

$$u(x,t)=\sin x\left(1-\frac{1}{\Gamma(2\alpha+1)}t^{2\alpha}+\frac{1}{\Gamma(4\alpha+1)}t^{4\alpha}-......\right) \tag{4.36}$$

$$u(x,t)=\sin x\left(1-\frac{1}{\Gamma(3)}t^2+\frac{1}{\Gamma(5)}t^4-......\right)$$

When $\alpha=1$ is substituted in equation (36) we get the following result:

$$u(x,t)=\sin x\left(1-\frac{1}{2!}t^2+\frac{1}{4!}t^4-......\right)=\sin x\cos t$$

this is exactly the exact solution the given problem Eq. (4.32).

Example 4.3 solves the fourth order parabolic beam equation with variable coefficient by using FRDTM.

$$\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}u(x,t)=-(x+1)\frac{\partial^4}{\partial x^4}u(x,t)+\left(x^4+x^3-\frac{6}{7!}x^7\right)\cos t,0<x\le 1,t>0,\alpha\in(0,1] \tag{4.37}$$

Subjected to the initial condition

$$u(x,0)=\frac{6}{7!}x^7 and\, u_t(x,0)=0,0<x<1,t>0 \tag{4.38}$$

On equation **(**4.37**)** applying FRDTM, we get the following recurrence relation**:**

$$\frac{\Gamma(k\alpha+2\alpha+1)}{\Gamma(k\alpha+1)}U_{k+2}(x)=-(x+1)\frac{\partial^4}{\partial x^4}U_k(x)+\left(x^4+x^3-\frac{6}{7!}x^7\right)\frac{1}{k!}\cos\left(\frac{\Pi k}{2}\right)$$

$$U_{k+2}(x)=\frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+2\alpha+1)}\left(-(x+1)\frac{\partial^4}{\partial x^4}U_k(x)+\left(x^4+x^3-\frac{6}{7!}x^7\right)\frac{1}{k!}\cos\left(\frac{\Pi k}{2}\right)\right) \tag{4.39}$$

where $F_k(x)=\left(x^4+x^3-\frac{6}{7!}x^7\right)\frac{1}{k!}\cos\left(\frac{\Pi k}{2}\right)$.

Again applying FRDTM on both sides equation (4.38), we obtain

$$U_0(x)=\frac{6}{7!}x^7 and\, U_1(x)=0,0<x<1 \tag{4.40}$$

Using Eqs. (4.39) and (4.40**)**, we get the values of $U_k(x)$ recursively.

For **k=0**, $U_2(x)=\frac{\Gamma(1)}{\Gamma(2\alpha+1)}\left(-(x+1)\frac{\partial^4}{\partial x^4}U_0(x)+\left(x^4+x^3-\frac{6}{7!}x^7\right)\frac{1}{0!}\cos\left(\frac{\Pi(0)}{2}\right)\right)$

$$=\frac{\Gamma(1)}{\Gamma(2\alpha+1)}\left[-(x+1)\frac{\partial^4}{\partial x^4}\left(\frac{6}{7!}x^7\right)+\left(x^4+x^3-\frac{6}{7!}x^7\right)\frac{1}{0!}\cos\left(\frac{\Pi(0)}{2}\right)\right]$$

$$=\frac{\Gamma(1)}{\Gamma(2\alpha+1)}\left[-(x+1)\left(x^{3}\right)+\left(x^{4}+x^{3}-\frac{6}{7!}x^{7}\right)\right]$$

$$=\frac{\Gamma(1)}{\Gamma(2\alpha+1)}\left[-x^{4}-x^{3}+x^{4}+x^{3}-\frac{6}{7!}x^{7}\right]=\frac{\Gamma(1)}{\Gamma(2\alpha+1)}\left[-\frac{6}{7!}x^{7}\right]$$

$$U_{2}(x)=\frac{1}{\Gamma(2\alpha+1)}\left(-\frac{6}{7!}x^{7}\right)$$

For **k=1**, $U_{3}(x)=\frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}\left(-(x+1)\frac{\partial^{4}}{\partial x^{4}}U_{1}(x)+\left(x^{4}+x^{3}-\frac{6}{7!}x^{7}\right)\frac{1}{1!}\cos\left(\frac{\Pi}{2}\right)\right)$

$$=\frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}\left(-(x+1)\frac{\partial^{4}}{\partial x^{4}}(0)+\left(x^{4}+x^{3}-\frac{6}{7!}x^{7}\right)(0)\right)$$

$$U_{3}(x)=0$$

For **k=2** $U_{4}(x)=\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\left(-(x+1)\frac{\partial^{4}}{\partial x^{4}}U_{2}(x)+\left(x^{4}+x^{3}-\frac{6}{7!}x^{7}\right)\frac{1}{2!}\cos(\Pi)\right)$

$$=\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\left(-(x+1)\frac{\Gamma(1)}{\Gamma(2\alpha+1)}\frac{\partial^{4}}{\partial x^{4}}\left(-\frac{6}{7!}x^{7}\right)-\frac{\left(x^{4}+x^{3}-\frac{6}{7!}x^{7}\right)}{2!}\right)$$

$$=\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\left(\frac{\Gamma(1)}{\Gamma(2\alpha+1)}\left(-(x+1)\left(-x^{3}\right)\right)-\frac{\left(x^{4}+x^{3}-\frac{6}{7!}x^{7}\right)}{2!}\right)$$

$$=\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\left(\frac{x^{4}+x^{3}}{\Gamma(2\alpha+1)}-\left(\frac{x^{4}+x^{3}}{2!}-\frac{6}{7!}x^{7}\right)\right)$$

$$U_{4}(x)=\frac{x^{4}+x^{3}}{\Gamma(4\alpha+1)}-\frac{\Gamma(2\alpha+1)\left(x^{4}+x^{3}\right)}{2!\Gamma(4\alpha+1)}+\frac{6x^{7}\Gamma(2\alpha+1)}{7!2!\Gamma(4\alpha+1)}$$

For **k=3** $U_{5}(x)=\frac{\Gamma(3\alpha+1)}{\Gamma(5\alpha+1)}\left(-(x+1)\frac{\partial^{4}}{\partial x^{4}}(0)+\left(x^{4}+x^{3}-\frac{6}{7!}x^{7}\right)\frac{1}{3!}(0)\right)$

$$U_{5}(x)=0$$

For **k=4**

$$U_6(x)=\frac{-24(x+1)}{\Gamma(6\alpha+1)}+\frac{12(x+1)\Gamma(2\alpha+1)}{\Gamma(6\alpha+1)}-\frac{(x^4+x^3)\Gamma(2\alpha+1)}{2\Gamma(6\alpha+1)}+\frac{(x^4+x^3)\Gamma(4\alpha+1)}{4\Gamma(6\alpha+1)}$$
$$-\frac{6x^7\Gamma(4\alpha+1)}{7!4!\Gamma(6\alpha+1)}$$

For **k=5** $U_7(x)=0$

For **k=6**

$$U_8(x)=\frac{12(x+1)\Gamma(2\alpha+1)}{\Gamma(8\alpha+1)}-\frac{(x+1)\Gamma(4\alpha+1)}{\Gamma(8\alpha+1)}+\frac{(x^4+x^3)\Gamma(4\alpha+1)}{4!\Gamma(8\alpha+1)}-\frac{(x^4+x^3)\Gamma(6\alpha+1)}{6!\Gamma(8\alpha+1)}+$$
$$\frac{6x^7\Gamma(6\alpha+1)}{7!6!\Gamma(8\alpha+1)}\ldots\ldots$$

Applying inverse FRDTM to $U_k(x)$ us can find that:

$$u(x,t)=\sum_{k=0}^{\infty}U_k(x)t^{k\alpha}=U_0(x)+U_1(x)t^{\alpha}+U_2(x)t^{2\alpha}+U_3(x)t^{3\alpha}+U_4(x)t^{4\alpha}+$$
$$U_5(x)t^{5\alpha}+U_6(x)t^{6\alpha}+U_7(x)t^{7\alpha}+U_8(x)t^{8\alpha}+\ldots$$
$$u(x,t)=U_0(x)+U_2(x)t^{2\alpha}+U_4(x)t^{4\alpha}+U_6(x)t^{6\alpha}+U_8(x)t^{8\alpha}+\ldots$$

$$u(x,t)=\frac{6x^7}{7!}-\frac{6x^7}{7!\Gamma(2\alpha+1)}t^{2\alpha}+\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\left(\frac{x^4+x^3}{\Gamma(2\alpha+1)}+\frac{-x^4-x^3+\frac{6}{7!}x^7}{2!}\right)t^{4\alpha}-$$
$$\frac{\Gamma(4\alpha+1)}{\Gamma(6\alpha+1)}\left(\left(-(x+1)\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\left(\frac{24}{\Gamma(2\alpha+1)}-\frac{24}{2!}+\frac{x^3}{2!}\right)\right)+\frac{x^4}{4!}+\frac{x^3}{4!}-\frac{6x^7}{7!4!}\right)t^{6\alpha}+ \quad 4.41$$
$$\left(\begin{array}{l}\frac{12(x+1)\Gamma(2\alpha+1)}{\Gamma(8\alpha+1)}-\frac{(x+1)\Gamma(4\alpha+1)}{\Gamma(8\alpha+1)}+\frac{(x^4+x^3)\Gamma(4\alpha+1)}{4!\Gamma(8\alpha+1)}-\\ \frac{(x^4+x^3)\Gamma(6\alpha+1)}{6!\Gamma(8\alpha+1)}+\frac{6x^7\Gamma(6\alpha+1)}{7!6!\Gamma(8\alpha+1)}\ldots\ldots\end{array}\right)t^8+\ldots$$

When $\alpha=1$ in Eq. (4.41), we get the exact solution of question (4.37) as shown below. i.e,

$$u(x,t)=\frac{6x^7}{7!}-\frac{6x^7}{7!\Gamma(3)}t^2+\frac{\Gamma(3)}{\Gamma(5)}\left(\frac{x^4+x^3}{\Gamma(3)}+\frac{-x^4-x^3+\frac{6}{7!}x^7}{2!}\right)t^4-$$

$$\frac{\Gamma(5)}{\Gamma(7)}\left(\left(-(x+1)\frac{\Gamma(3)}{\Gamma(5)}\left(\frac{24}{\Gamma(3)}-\frac{24}{2!}+\frac{x^3}{2!}\right)\right)+\frac{x^4}{4!}+\frac{x^3}{4!}-\frac{6x^7}{7!4!}\right)t^6+$$

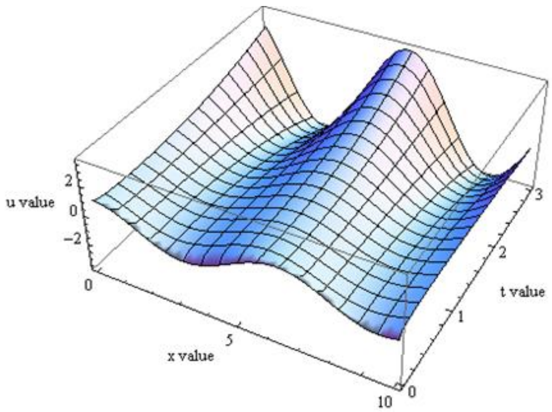
$$\left(\frac{12(x+1)\Gamma(3)}{\Gamma(9)}-\frac{(x+1)\Gamma(5)}{\Gamma(9)}+\frac{(x^4+x^3)\Gamma(5)}{4!\Gamma(9)}-\frac{(x^4+x^3)\Gamma(7)}{6!\Gamma(9)}+\frac{6x^7\Gamma(7)}{7!6!\Gamma(9)}\right)+\ldots$$

$$u(x,t)=\frac{6x^7}{7!}-\frac{6x^7}{7!2!}t^2+\frac{6}{7!4!}x^7t^4-\frac{6x^7}{7!6!}t^6+\frac{6x^7}{7!8!}t^8+\ldots$$

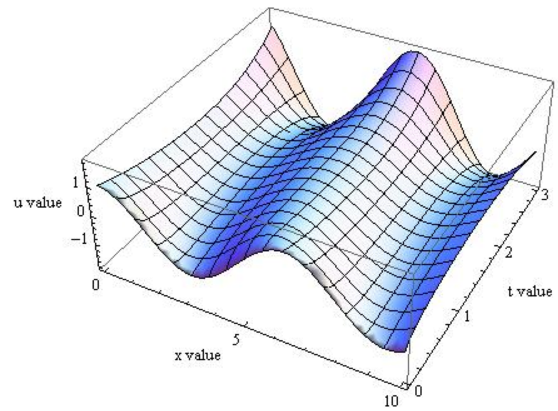
$$=\frac{6x^7}{7!}\left(1-\frac{t^2}{2!}+\frac{t^4}{4!}-\frac{t^6}{6!}+\frac{t^8}{8!}\right)\ldots$$

$$u(x,t)=\frac{6x^7}{7!}\cos t\,.$$

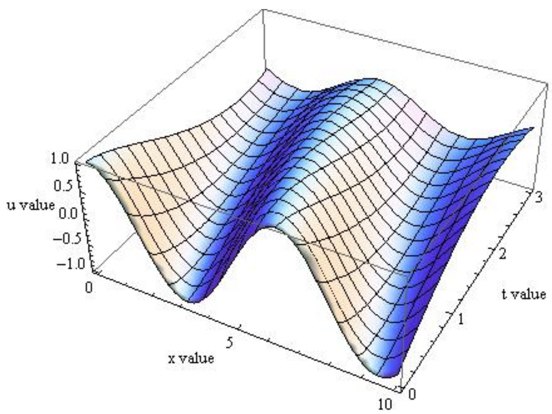
The solution curves of one dimensional time fractional homogenous and non-homogenous beam equations given in Examples 4.1, 4.2 and4.3 for different values of time fractional order α are depicted below in **Figure 4.1,Figure4.2** and **Figure 4.3** respectively.



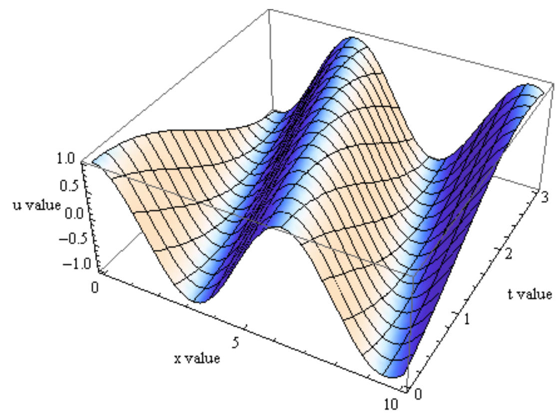
For $\boldsymbol{\alpha}$ = 0.25



For $\boldsymbol{\alpha}$ = 0.5

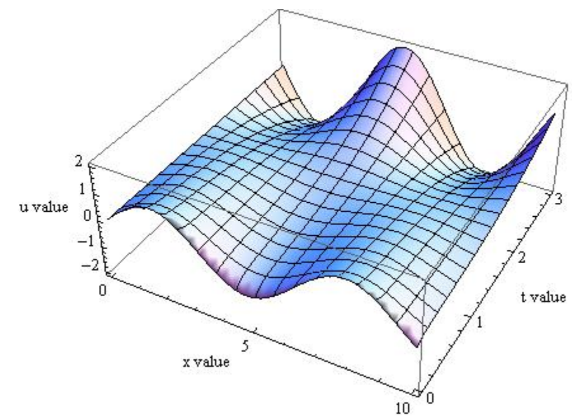


For $\boldsymbol{\alpha}$ = 0.75

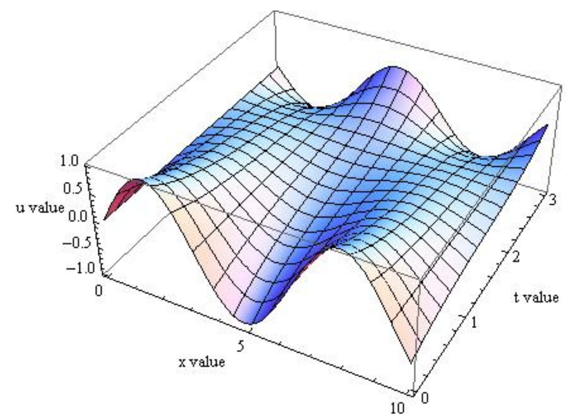


For $\boldsymbol{\alpha}$ = 1.00

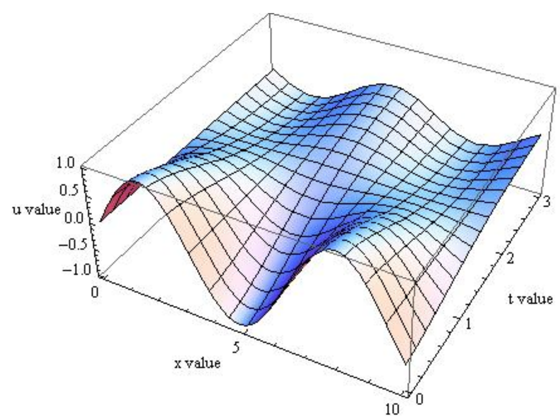
Figure 4.1: 3D plots of the solution of one dimensional time fractional beam equation (Example 4.1)



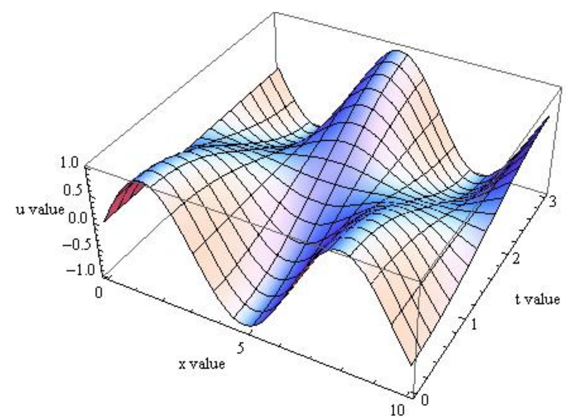
For $\alpha = 0.25$



For $\alpha = 0.5$

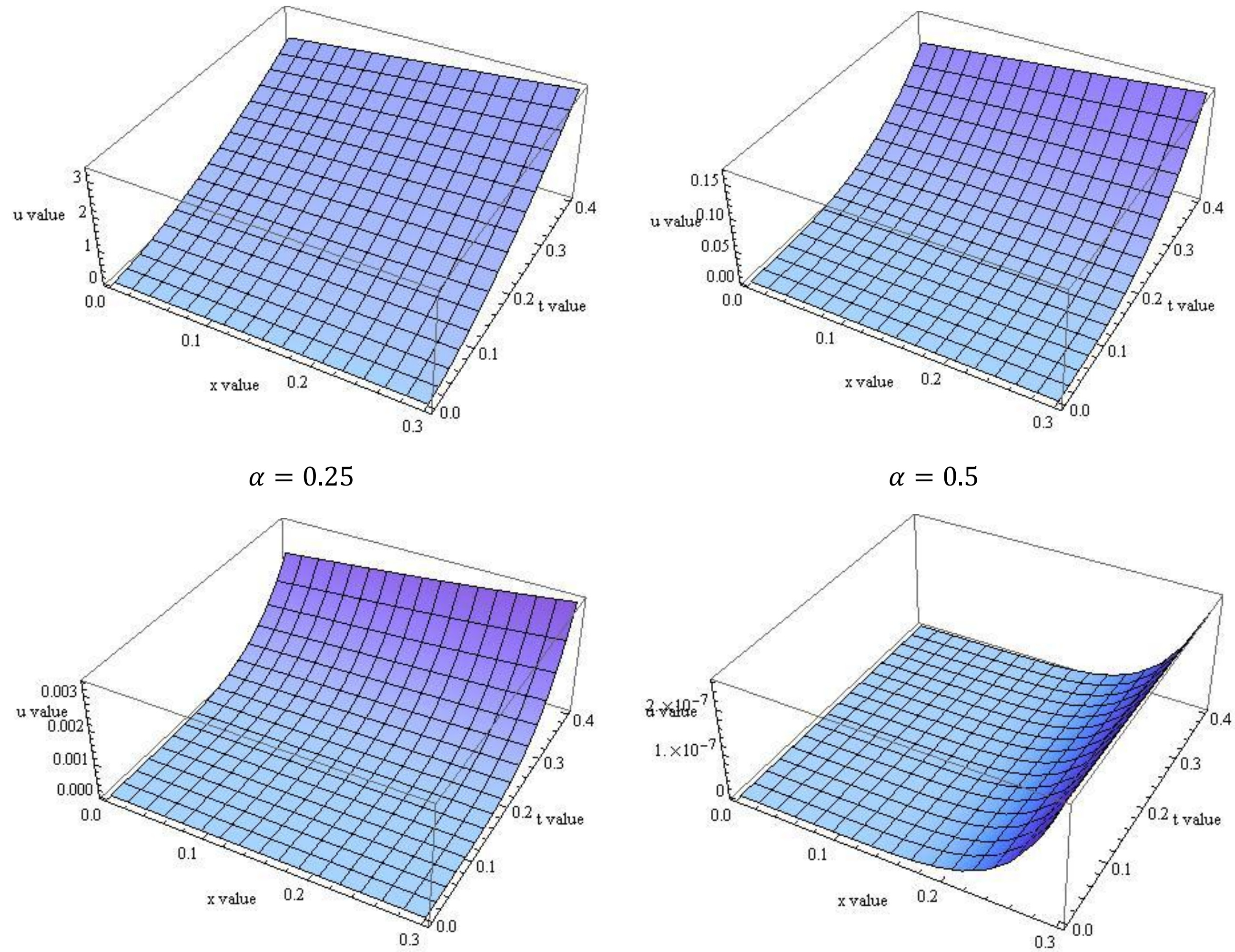


For $\alpha = 0.75$



For $\alpha = 1.00$

Figure 4.2: 3D plots of the solution of one dimensional time fractional beam equation (Example 4.2)



$\alpha = 0.25$ $\alpha = 0.5$

$\alpha = 0.75$ $\alpha = 1$

Figure 4.3: 3D plots of the solution of one dimensional time fractional beam equation (Example 4.3)

4.4. Discussions

The fractional reduced differential transform method (FRDTM) has been successfully applied in partial deferential equation on one dimensional time fractional parabolic beam equation subjected to the given initial condition which gives rapidly converging series solutions.

The fractional reduced differential transform method was used by straightforward substitution without employing linearization and simply uses iterative recurrence relationships. The efficiency and capability of the present method have been checked via three examples.

The obtained solution was in excellent agreement when compared with the solution done by Variational Iterative Method (VIM) and Adomian Decomposition Method (ADM) (wazwaz, 2009).

The graphs show the results obtained for homogenous and non-homogenous time fractional parabolic beam equation for the different values of order α .When the values of α approaches to one ($\alpha \to 1$) the graphs approaches to the graph of the exact solution. When, $\alpha = 1$ the graph exactly fits with the graph of the exact solution of one dimensional time fractional parabolic beam equation.

CHAPTER FIVE

CONCLUSIONS AND FUTURE SCOPE

5.1. Conclusions

In this study, we presented fractional reduced differential transform method for solving one dimensional parabolic beam equation. The obtained solutions disclose that FRDTM is very effective and convenient. The present study has confirmed that the fractional reduced differential transform method offers great advantages of straight forward applicability, computational efficiency and high accuracy.We also notice that the series solutions obtained by the fractional reduced differential transform method are in excellent agreement with the solution given by the Adomain Decomposition Method (ADM) and the Variation iterative Method (VIM), refer to book Partial Differential Equations and Solitary Waves Theory (Wazwaz, 2009, pp.406-412). Moreover, the performed computations show that the FRDTM is much easier than to apply the Adomain Decomposition Method (ADM) and the Variation iterative Method (VIM) as it takes very tiny amount of computation.

5.2. Future scope

The techniques used in this work can also be applied to solve linear and non-linear time fractional partial differential equation and multi-dimensional physical problems emerging in various fields of engineering and applied sciences.

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