ANALYTIC SOLUTIONS OF INTIAL VALUE PROBLEMS OF HOMOGENEOUS TIME FRACTIONAL HEAT-LIKE EQUATIONS USING THE REDUCED DIFFERENTIAL TRANSFORM METHOD

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(Differential Equations)

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ABSTRACT

The main purpose of this study was to develop a scheme to find analytic solutions of multi-dimensional homogeneous time fractional heat like equations under initial conditions by using reduced differential transform method. Analytic solutions based on the iteration technique were proposed (designed) to solve the homogeneous time-fractional heat-like equations in n-dimensions using Reduced Differential Transform Method subjected to the appropriate initial condition. The Reduced Differential Transform Method procedures in one, two, three and more than three dimensions were developed and introduced to obtain the analytic solutions of multi-dimensional homogeneous time fractional heat-like equations.

To see the effectiveness and applicability of the newly introduced procedures of the Reduced Differential Transform Method to obtain analytic solutions of initial value problems of homogeneous time fractional heat-like equations in n-dimensional space ($n \in \mathbb{N}$ *), four test examples were presented. The results show that Reduced Differential Transform Method is successfully implemented to obtain analytic solutions of multi-dimensional homogeneous time fractional linear heat-like equations. Therefore, it can be concluded that the proposed method can be extended to other fractional partial differential equations which can arise in physics and engineering.*

CHAPTER ONE

1. INTRODUCTION

1.1. Back ground of the study

The fractional calculus is the theory of integrals and derivatives of arbitrary order, which unifies and generalizes the concepts of integer-order differentiation and n-fold integration, G. Wang and T. Xu [1]. That is, it is a generalization of ordinary (standard) differentiation and integration to arbitrary (non-integer) order.

As it has been explained in M. Ishteva, et al. [2], the beginning of the fractional calculus is considered to be the Leibniz's letter which raised a question: "Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?" to L'Hospital in 1695, where the notation for differentiation of non-integer order, was discussed.

As in G. Wang and T. Xu [1], even though, fractional calculus is three centuries old as the conventional calculus, it is not very popular among science and/or engineering community. But, the subject has the beauty that fractional derivatives as well as fractional integrals are not a local (or point) property (or quantity). Hence, this reflects the history and non-local distributed effects. Meaning, this subject can translate the reality of nature better! Therefore, making this subject accessible as prevalent subject to science and engineering community adds another dimension to understand or describe basic nature in a better way. Perhaps fractional calculus is what nature understands, and to talk with nature in this language is therefore efficient. In general, Fractional calculus is a branch of mathematical analysis that studies the possibility of taking real number, or even complex number, powers of the differential operator $D=\frac{d}{dx}$ and the integration operator J. (Usually J is used in favor of I to avoid with other I-like identities).

As it has been discussed in A. Secer [3], there are well-known definitions of a fractional derivative and integrals of order, a real number $\alpha > 0$ such as Riemann-Liouville, Grunwald-Letnikow, Caputo, and generalized functions approach from fractional calculus. The most commonly used definitions are the Riemann-Liouville and Caputo. The Riemann-Liouville fractional derivative is mostly used by mathematicians but it is not suitable for physical problems of the real world since it requires the definition of fractional order initial conditions which have no physically meaningful explanation yet. Caputo introduced an alternative definition which has the advantage of defining integer order initial conditions for fractional order differential equations. The Caputo fractional derivative is important because it allows traditional initial and boundary conditions to be included in the formulation of the problem. So, this Caputo fractional derivative is the base for fractional differential equations with integer order initial conditions such as time fractional partial differential equations with integer order initial conditions.

As it was stated in A. Aghili and M.R. Masomi [4], time fractional partial differential equations are differential equations which can be obtained from the standard partial differential equations by replacing the integer order time derivative by a fractional derivative. Some of these are time fractional heat equations, time fractional heat-like equations, time fractional wave equations and so on.

Fractional order partial differential equations, as generalizations of classical integer order partial differential equations, have been used to model problems in fluid flow and other areas of application, in A. Secer [3]. For example, in order to formulate certain electrochemical problems, half-order derivatives and integrals are more useful than the classical models, A. Secer [3].

Fractional derivatives provide an excellent instrument for the descriptive and hereditary properties of various materials and processes. So solving fractional partial differential equations (FPDEs) is completely important in the circumstance of Applied Mathematics, Theoretical Physics and Engineering Sciences, in M. Sohail and S.T. Mohyud-Din [5]. In order to better understand time fractional differential equations as well as further apply them in practical scientific research, it is important to find their exact solutions, in M. Sohail and S.T .Mohyud-Din [5].

Mathematical (solution) methods for partial differential equations are varied, and depend on such equations characteristics, linearity and order. For PDEs, the mathematical (solution) methods are divided into two general classes which are Analytical methods that strive to find exact formula for the dependent variable as a function of all independent variables, and numerical methods which result in approximate values of the dependent variable at prescribed and discrete locations within a finite domain of the independent variables, B. Richard [6]. But, there are mathematical methods which can be neither of the two methods. These methods are semi-analytical methods or semi- numerical methods.

For example, Reduced Differential Transform Method is semi-analytical method, V.K. Srivastava, et al. [7]. It is an iterative procedure for obtaining Taylor series solution of differential equations, M. Sohail and S.T. Mohyud-Din [5]. It was first proposed by Keskin in 1986 and successfully employed to solve many types of nonlinear partial differential equations. As in M. Sohail and S.T. Mohyud-Din [5], Reduced Differential Transform Method successfully applied to solve multi-dimensional time-fractional heat equations. But, nothing was discussed about how to solve initial value problems (IVPs) of multi-dimensional homogeneous time fractional heat-like equations by applying the RDTM in the existing literature in the time before this study. Motivated by the gap, this study was conducted by extending the works of M. Sohail and S.T. Mohyud-Din [5] to multi-dimensional homogeneous time fractional heat like-equations only in the case of finding analytic solutions.

The main purpose of this study was to develop scheme to find analytic solutions of multi-dimensional homogeneous time fractional heat-like equations of the form:

$$\frac{\partial^{\alpha} u(x_1, x_2, \cdots, x_n, t)}{\partial t^{\alpha}} = f_1(x_1, x_2, \cdots, x_n) u_{x_1 x_1} + f_2(x_1, x_2, \cdots, x_n) u_{x_2 x_2} + \cdots + f_n(x_1, x_2, \cdots, x_n) u_{x_n x_n}$$

$$(x_1, x_2, \cdots, x_n) \in \Omega \subseteq \mathbb{R}^n, \quad t > 0, \quad 0 < \alpha \leq 1$$

Subject to the initial condition:

$$u(x_1, x_2, \cdots, x_n, 0) = g_1(x_1, x_2, \cdots, x_n), (x_1, x_2, \cdots, x_n) \in \Omega \subseteq \mathbb{R}^n$$

where $u(x_1, x_2, \cdots, x_n, t)$ such that $u(x_1, x_2, \cdots, x_n, t) = q(x_1, x_2, \cdots, x_n) g(t)$ is analytic and k-times continuously differentiable with respect to time, t and variables: $x_1, x_2, \cdots,$ and x_n in the domain of interest, $\Omega \subseteq \mathbb{R}^n$ which is closed set , $f_i(x_1, x_2, \cdots, x_n) \forall i = 1, 2, \cdots, n$ is a function and α is order of time fractional derivative, by the Reduced Differential Transform Method. The study outlined that the Reduced Differential Transform Method is very effective, simple and powerful mathematical tool for solving multi-dimensional initial value problems of homogeneous time fractional heat-like equations analytically.

1.2. Statements of the problem

Engineering and other areas of sciences can be successfully modeled by the use of fractional derivatives. A. Aghili and A. Motahhari [8] were stated that in reality, the future state of a physical phenomenon, which might depend on its current state as well as its historical state (non-local property), can be successfully modeled by using the theory of derivatives and integrals of fractional order (fractional calculus). But, solving initial value problems of multi-dimensional time fractional homogeneous heat-like equations by applying Reduced Differential Transform Method was not presumably presented in the existing literature. As a result, the study was aimed to fill the gap, and it was intended to answer the following questions:

1. How can we define reduced differential transformed function and its reduced differential inverse transform in n-dimensions $(n \in \mathbb{N})$ for solving initial value problems of multi-dimensional homogeneous time fractional heat-like equations by applying Reduced Differential Transform Method?
2. What theorems can we give by using definitions of reduced differential transformed function and its reduced differential inverse transform in n-dimensions $(n \in \mathbb{N})$ for solving initial value problems of multi-dimensional homogeneous time fractional heat-like equations by applying Reduced Differential Transform Method?
3. How can we find analytic solutions of initial value problems of multi-dimensional homogeneous time fractional heat-like equations in infinite power series form (open form) using Reduced Differential Transform Method ?
4. How can we determine exact solutions of multi-dimensional initial value problems of homogeneous time fractional heat-like equations applying Reduced Differential Transform Method?

1.3. Objectives of the study

1.3.1. General objective

The general objective of this research was to develop a scheme to find analytic solutions of multi-dimensional homogeneous time fractional heat-like equations under initial conditions by reduced differential transform method.

1.3.2. Specific objectives

The specific objectives of the study were:

- ✓ To define reduced differential transformed function and its reduced differential inverse transform in n-dimensions for solving initial value problems of multi-dimensional time-fractional homogeneous heat-like equations by applying Reduced Differential Transform Method
- ✓ To give theorems (mathematical operations) by using definitions of reduced differential transformed function and its reduced differential inverse transform in n-dimensions for solving initial value problems of multi-dimensional homogeneous time-fractional heat-like equations by applying Reduced Differential Transform Method
- ✓ To find analytic solutions of IVPs of multi-dimensional initial value problems of homogeneous time fractional heat-like equations in infinite power series (open) form by using Reduced Differential Transform Method
- ✓ To determine exact solutions of initial value problems of multi-dimensional homogeneous time fractional heat-like equations after applying Reduced Differential Transform Method

1.4. Significance of the study

This research is considered of vital importance for the following reasons:

1. It will develop the researcher skill on conducting scientific research, especially mathematical research.
2. It will familiarize the researcher with the scientific communication in mathematics.
3. It will provide techniques of solving initial value problems (IVPs) of multi-dimensional homogeneous time-fractional heat-like equations by using Reduced Differential Transform Method for readers.

4. It will be used as reference material for anyone who will work on similar study.

1.5. Delimitation of the study

Even though, there were different types of time fractional partial differential equations which can be solved by different analytical, numerical and semi-numerical (or semi-analytical) methods, the study was delimited to initial value problems of multi-dimensional homogeneous time fractional heat-like equations and focused only on developing a scheme to find analytic solutions of multi-dimensional homogeneous time fractional heat-like equations under initial conditions by the Reduced Differential Transform Method, which is semi-numerical or semi-analytical method.

CHAPTER TWO

2. LITERATURE REVIEW

Many phenomena in engineering, physics, chemistry and other sciences can be described very successfully by models using mathematical tools from fractional calculus, i.e. the theory of derivatives and integrals of fractional non-integer order, M.S.M. Noorani, et al. [9]. Fractional differential equations have gained much attention recently due to exact description of nonlinear phenomena. No analytical method was available before 1998 for linear fractional differential equations.

As it was stated in M.S.M. Noorani, et al. [9], the variational iteration method (VIM), which is analytical method, was first proposed in 1998 by He to solve fractional differential equations (seepage flow with fractional derivatives in porous media) and then after it also used to solve more complex fractional differential equations such as linear and nonlinear viscoelastic models with fractional derivatives, nonlinear differential equations of fractional order, linear fractional partial differential equations arising in fluid mechanics ,and the space-and time-fractional KdV equation. The variational iteration method successfully employed to obtain the approximate solution of the fractional heat and wave-like equations with variable coefficients.

In 2007, the homotopy perturbation method (HPM) was applied to both non-linear and linear fractional differential equations and it was showed that HPM is an alternative analytical method for fractional differential equations. HPM also used to solve the fractional heat- and wave-like equations with variable coefficients, in M.S.M. Noorani, et al. [9].

In addition, Differential Transform Method (DTM), which is a semi-analytical or semi numerical technique, was successfully employed to obtain the approximate solutions of the fractional heat- and wave-like equations with variable coefficients, in M. Mohseni and H. Saeedi [10].

In M. Sohail and S.T. Mohyud-Din [5], Reduced Differential Transform Method was applied to solve multi-dimensional time-fractional heat equations. But, nothing was discussed about initial value problems (IVPs) of multi-dimensional homogeneous time-fractional heat-like equations by applying the RDTM in the existing literature. Motivated by the gap, the works of M. Sohail and S.T. Mohyud-Din [5] were extended analytically to initial value problems of multi-dimensional homogeneous time fractional heat like-equations to find analytic solutions.

Therefore, this study was targeted to develop a scheme to find analytic solutions of multi-dimensional homogeneous time fractional heat-like equations under initial conditions by the Reduced Differential Transform Method.

CHAPTER THREE

3. METHODOLOGY

3.1. Study Site, Area and Period

This research was conducted to develop a scheme to find analytic solutions of multi-dimensional homogeneous time fractional heat-like equations under initial conditions by Reduced Differential Transform Method under Differential Equation Stream of Mathematics Department in Jimma University from December, 2013 to June, 2014.

3.2. Study Design

The design of the study was analytical design.

3.3. Sources of Data

The information or data which were related to the topic of the study was collected from secondary sources such as reference books, internet and published research articles (or Journals).

3.4. Administration and Instrumentation of Information or Data

The collection of information or data from secondary sources such as reference books, internet and published research articles (or Journals) was administered by the researcher. During conducting this research, consultation for the researcher was administered by the persons whose field of specialization is related to the study area.

3.5. Procedure of the Study

In order to achieve the objectives of this study, iteration technique, which was used by M. Sohail and S.T. Mohyud-Din [5], was the standard technique (procedure) of the study.

3.6. Ethical Issues

To collect related data at the place where they were available and to process other related supports, cooperation request letters were written to the concerned bodies by officials of Jimma University Natural Science College. In addition, the cooperation request letter from Mathematics department of Jimma University was taken by the researcher to the institute(s) where these materials are available to get consent from them. Moreover, rules and regulations of the institute(s), from which information was collected, were kept by the researcher.

CHAPTER FOUR

4. DISCUSION AND RESULTS

4.1 Preliminaries

4.1.1 Gamma and Beta Functions

Definition 4.1.1.1 The Gamma function $\Gamma(\gamma)$ is a function which is defined in R. Bronson [11] as:

$$\Gamma(\gamma)=\int_{0..}^{\infty} t^{\gamma-1}e^{-t}dt,\ \gamma>0$$

Some properties of the Gamma function $\Gamma(\gamma)$ are the following (**Proof Rf. R. Bronson [11]**):

i. $\Gamma(1)=1$.

ii. $\Gamma(\gamma+1)=\gamma\Gamma(\gamma), \forall\, \gamma>0$.

iii. When $\gamma\in N, \Gamma(\gamma+1)=\gamma!$

Definition 4.1.1.2 The Beta function $B(z,\ w)$ in two variables $z,w\in\mathbb{C}$ is defined by

$$B(z,w)=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.$$

As stated in M. Weilbeer [12], the Beta function possesses the following property:

$$B(z,w)=\int_0^1 t^{z-1}(1-t)^{w-1}dt=\int_0^{\infty}\frac{t^{z-1}}{(1+t)^{z+w}}dt \tag{1}$$

4.1.2 Fractional Calculus, Some basic definitions, properties and theorems

Definition 4.1.2.1 As in M. Bayram & M. Kurulay [13], a real function $f(x), x>0$ is said to be in the space C_μ, $\mu\epsilon\ \mathbb{R}$ if there exists a real number $(p>\mu)$ such that $f(x)=x^p f_1(x)$, where $f_1(x)\epsilon\, C[0,\ \infty)$ and it is Said to be in the space C_μ^m if and only if $\frac{d^m f(x)}{dx^m}=f^{(m)}\in C_\mu, m\epsilon\, N$.

Example 4.1.2.1

$f(x) = x^4 + 2x^2,\ x > 0$

Because $f(x) = x^2(x^2 + 2)$ where $f_1(x) = x^2 + 2,\ f(x) = x^2 f_1(x).$

$\Rightarrow 2 > \mu,$ where $\mu \in (2,\ -\infty)$ and $f_1(x) \in C[0,\ \infty)$

$\therefore\ f(x) = x^4 + 2x^2 \in\ C_\mu(0,\ \infty)$

And $f(x) \in C_\mu^4(0,\ -\infty) \Leftrightarrow \dfrac{d^4 f(x)}{dx^4} = f^{(4)} \in C_\mu(0,\ \infty).$

Definition 4.1.2.2 The Reimann-Liouville fractional integral operator of order of a function $f(x)\epsilon C_\mu,\ \mu > -1$ is defined in S. Momani, et al. [14] as:

$$J_a^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^x (x-t)^{\alpha-1} f(\mathrm{t})d\mathrm{t},\ \alpha > 0 \\ J_a^0 f(x) = f(x), \alpha = 0 \end{cases} \tag{2}$$

Definition4.1.2.3. For the smallest positive integer m, that exceeds α , the Caputo fractional derivative of order α , in S. Momani, et al. [14], is defined as**:**

$$D_a^\alpha f(x) = J_a^{m-\alpha} f^{(m)}(x)$$

$$= \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, for\ f\epsilon\ C_{-1}^m, m-1 < \alpha < m, x > a \\ \dfrac{d^m f(x)}{dx^m} \quad , for\ \alpha = m \end{cases} \tag{3}$$

Definition 4.1.2.4 For the smallest positive integer m, that exceeds α , the Caputo time fractional derivative operator of order $\alpha > 0$ is defined in M. Bayram & M. Kurulay [13] as:

$$D_{0*x}^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t (t-\xi)^{m-\alpha-1} \dfrac{\partial^m u(x,\xi)}{\partial \xi^m} d\xi, for\ m-1 < \alpha < m, \\ \dfrac{\partial^m u(x,t)}{\partial t^m} \quad , for\ \alpha = m \in N \end{cases}$$

As in S. Momani, et al. [14] for fractional derivative of order α and β such that $\alpha, \beta > 0$, $m-1 < \alpha \le m$ and $\gamma > -1, a \ge 0$, we have the following properties (**proof Rf. M. Weilbeer [12]**):

1. $$\left(J_a^{\alpha} J_a^{\beta} f\right)(\mathrm{x}) = \left(J_a^{\beta} J_a^{\alpha} f\right)(\mathrm{x}) = \left(J_a^{\alpha+\beta} f\right)(\mathrm{x}) \tag{4}$$

2. $$J_a^{\alpha}(t-a)^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}(t-a)^{\gamma+\alpha} \ (\textit{with out proof}\ ) \tag{5}$$

3. $$\left(J_a^{\alpha} D_a^{\alpha} f\right)(x) = \left(J_a^{m} D_a^{m} f\right)(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(a)\frac{(x-a)^k}{k!} \quad ,x > a \tag{6}$$

As in S. Momani, et al. [14], the Reimann-Liouville derivative has certain difficulties to model real world problems with fractional differential equations since it does not allow the utilization of derivatives of initial and boundary conditions involving integer order derivatives, which have clear physical interpretations. But, the Caputo fractional derivative allows the utilization of initial and boundary conditions involving integer order derivatives, which have clear physical interpretations. In addition, the derivative of a constant is zero and we can define properly the initial conditions for the fractional differential equations which can be handled by using an analogy with the classical integer case. For these reasons, Caputo fractional derivative was preferred for this study.

Theorem 4.1.2.1 (Integral Mean Value Theorem)

Let $f(x)$ be continuous on [a, b]. Then there is a number ξ in $[\mathrm{a},\ \mathrm{b}]$ such that

$$\int_a^b f(x)dx = f(\xi)(b-a)\left(\text{ proof \ Rf. R. Ellis \& D. Gulick } [15]\right) \tag{7}$$

Theorem 4.1.2.2 (General Mean Value Theorem)

As in S. Momani, et al. [14], suppose that $\mathrm{f}(\mathrm{x}) \epsilon\, C([a,b])$ and $D_a^{\alpha} f(x) \in C([a,b])$ for $0 < \alpha \le 1$, then we have:

$$\mathrm{f}(\mathrm{x}) = \mathrm{f}(\mathrm{a}) + \frac{1}{\Gamma(\alpha)}(D_a^{\alpha} f(\xi))(x-a)^{\alpha} \tag{8}$$

With $0 \le \xi \le x, \forall x \epsilon\ [a,b]$ and D^{α} is the Caputo fractional derivative of order $\alpha > 0$.

Proof:

By (2),

$$(J_a^{\alpha} D_a^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} D_a^{\alpha} f(\mathrm{t})\, d\mathrm{t} \tag{9}$$

Using the integral mean value theorem, (7), we get

$$(J_a^{\alpha} D_a^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} D_a^{\alpha} f(\xi) \int_a^x (x-t)^{\alpha-1}\, d\mathrm{t} = \frac{1}{\Gamma(\alpha)} D_a^{\alpha} f(\xi)(x-a)^{\alpha} \; for\, 0 \le \xi \le x \tag{10}$$

On the other hand, from (6), we have

$$(J_a^{\alpha} D_a^{\alpha} f)(x) = f(x) - f(a) \tag{11}$$

So from (9) and (10), (7) is obtained

Hence,

$$f(x) = f(a) + \frac{1}{\Gamma(\alpha)} D_a^{\alpha} f(\ \xi)(t-a)^{\alpha}$$

Theorem 4.1.2.3 As in S. Momani, et al. [14], suppose that $(D_a^{\alpha})^n f(x),\ (D_a^{\alpha})^{n+1} f(x) \epsilon\, C(a,b]$, for $0 < \alpha \le 1$, then we have

$$(J_a^{n\alpha} (D_a^{\alpha})^n f)(x) - (J_a^{(n+1)\alpha} (D_a^{\alpha})^{n+1} f)(x) = \frac{(x-a)^{n\alpha}}{\Gamma(n\alpha+1)} ((D_a^{\alpha})^n f)(a) \tag{12}$$

Where $(D_a^{\alpha})^n = D_a^{\alpha}.D_a^{\alpha}.D_a^{\alpha} \cdots D_a^{\alpha}$ (n-times)

Proof:

From (4), we have

$$(J_a^{n\alpha} (D_a^{\alpha})^n f)(x) - (J_a^{(n+1)\alpha} (D_a^{\alpha})^{n+1} f)(x)$$

$$= J_a^{n\alpha} [((D_a^{\alpha})^n f)(x) - (J_a^{\alpha} (D_a^{\alpha})^{n+1} f)(x)]$$

$$= J_a^{n\alpha} [((D_a^{\alpha})^n f)(x) - (J_a^{\alpha} D_a^{\alpha})((D_a^{\alpha})^n f)(x)]$$

$$= J_a^{n\alpha} [(D_a^{\alpha})^n [f(x) - (J_a^{\alpha} D_a^{\alpha}) f(x)]]$$

$$= J_a^{n\alpha} [(D_a^{\alpha})^n [f(x) - (f(x) - f(a))]], \text{ (By using (6))}$$

$$= J_a^{n\alpha} (((D_a^{\alpha})^n f)(a))$$

$$= \frac{(t-a)^{n\alpha}}{\Gamma(n\alpha+1)} (((D_a^{\alpha})^n f)(a)) \text{ , (By using (5))}$$

Hence,

$$(J_a^{n\alpha}(D_a^\alpha)^n f)(x) - (J_a^{(n+1)\alpha}(D_a^\alpha)^{n+1} f)(x) = \frac{(x-a)^{n\alpha}}{\Gamma(n\alpha+1)}((D_a^\alpha)^n f)(a)$$

Theorem 4.1.2.4 (Generalized Taylor's Formula)

As in S. Momani, et al. [14], suppose that $(D_0^\alpha)^k f(x) \epsilon\ C(a,b]$ for $k = 0,1,2,\cdots,n+1$, where $0 < \alpha \le 1$, then we have

$$\mathrm{f(x)} = \sum_{i=1}^{n} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)}((D_a^\alpha)^i f)(\mathrm{a}) + \frac{(D_a^\alpha)^{n+1} f)(\xi)}{\Gamma((n+1)\alpha+1)}.(x-a)^{(n+1)\alpha} \tag{13}$$

$$,\ a \le \xi \le x, \forall x \epsilon (a,b]$$

Proof:

From (12), we have

$$\sum_{i=1}^{n}(J_a^{i\alpha}(D_a^\alpha)^i f)(\mathrm{x}) - J_a^{(i+1)\alpha}(D_a^\alpha)^{i+1} f)(\mathrm{x}) = \sum_{i=1}^{n} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)}((D_a^\alpha)^i f)(\mathrm{a}) \tag{14}$$

That is,

$$f(x) - (J_a^{(n+1)\alpha}(D_a^\alpha)^{n+1} f)(x) = \sum_{i=1}^{n} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)}((D_a^\alpha)^i f)(\mathrm{a}) \tag{15}$$

By using (2),

$$(J_a^{(n+1)\alpha}(D_a^\alpha)^{n+1} f)(x) = \frac{1}{\Gamma((n+1)\alpha+1)}\int_a^x (x-t)^{(n+1)\alpha}((D_a^\alpha)^{n+1} f)(\mathrm{t})\,d\mathrm{t}$$

By the integral mean value theorem (or equation (7)),

$$(J_a^{(n+1)\alpha}(D_a^\alpha)^{n+1} f)(x) = \frac{1}{\Gamma((n+1)\alpha+1)}((D_a^\alpha)^{n+1} f)(\xi)\int_a^x (x-t)^{(n+1)\alpha}\,d\mathrm{t}$$

$$(J_a^{(n+1)\alpha}(D_a^\alpha)^{n+1} f)(x) = \frac{(D_a^\alpha)^{n+1} f)(\xi)}{\Gamma((n+1)\alpha+1)}(x-a)^{(n+1)\alpha} \tag{16}$$

From (15) and (16), the generalized Taylor's formula (13) is obtained

Therefore;

$$\mathrm{f(x)} = \sum_{i=1}^{n} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)}((D_a^\alpha)^i f)(\mathrm{a}) + \frac{(D_a^\alpha)^{n+1} f)(\xi)}{\Gamma((n+1)\alpha+1)}.(x-a)^{(n+1)\alpha}$$

For $\alpha = 1$, this Caputo generalized Taylor's formula reduces to the standard (classical) Taylor's formula,

$$\mathrm{f}(\mathrm{x}) = \sum_{i=1}^{n} \frac{(x-a)^i}{i!} f^{(i)}(\mathrm{a}) + \frac{f^{(n+1)}(\xi)}{(n+1)!}.(x-a)^{(n+1)}.$$

The radius of convergence, R for the generalized Taylor's series,

$$\sum_{i=1}^{\infty} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)} ((D_a^\alpha)^i f)(\mathrm{a}) \tag{17}$$

depends on $f(x)$ and a, and is given by:

$$R = |t-a|^\alpha \lim_{n\to\infty} \left| \frac{\Gamma(n\alpha+1)}{\Gamma((n+1)\alpha+1)} \frac{((D_a^\alpha)^{n+1} f)(\mathrm{a})}{((D_a^\alpha)^n f)(\mathrm{a})} \right| \tag{18}$$

Theorem 4.1.2.5 As in S. Momani, et al. [14], suppose $((D_a^\alpha)^k f)(\mathrm{x}) \in C(a,b])$, for $k = 0,1,2,\cdots,n+1$ where $0 < \alpha \le 1$. If $x \in [a,b]$, then

$$f(\mathrm{x}) \cong P_N^\alpha(x) = \sum_{i=1}^{N} \frac{((D_a^\alpha)^{i\alpha} f)(a)}{\Gamma(i\alpha+1)} (x-a)^{i\alpha} \tag{19}$$

In addition, there is a value ξ with $a \le \xi \le x$ so that the error term $R_N^\alpha(x)$ has the form:

$$R_N^\alpha(x) = \frac{((D_a^\alpha)^{N+1} f)(\xi)}{\Gamma((N+1)\alpha+1)}.(x-a)^{(N+1)\alpha} \tag{20}$$

Proof:

Let $((D_a^\alpha)^k f)(\mathrm{x}) \in C(a,b])$ for $k = 0,1,2,\cdots,n+1$ where $0 < \alpha \le 1$.

Let $x \in [a,b]$.

Take $N \in \mathbb{N}$ such that $k = 0,1,2,\cdots,N+1$.

Then by (13), we have

$$\mathrm{f}(\mathrm{x}) = \sum_{i=1}^{N} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)} ((D_a^\alpha)^i f)(\mathrm{a}) + \frac{(D_a^\alpha)^{N+1} f)(\xi)}{\Gamma((N+1)\alpha+1)}.(x-a)^{(N+1)\alpha}$$

Assume $N \in \mathbb{N}$ to be large enough. Then

$$\frac{(D_a^{\alpha})^{N+1}f)(\xi)}{\Gamma\left(\left(N+1\right)\alpha+1\right)}.\left(x-a\right)^{(N+1)\alpha}$$ is negligible. That is, it is almost zero.

So,

$$f\left(\mathrm{x}\right)\cong P_N^{\alpha}\left(x\right)=\sum_{i=1}^{N}\frac{((D_a^{\alpha})^{i\alpha}f)(a)}{\Gamma\left(i\alpha+1\right)}.(x-a)^{i\alpha}\ .$$

Then for a value ξ such that $a\le\xi\le x$, the error term becomes

$$R_N^{\alpha}\left(x\right)=\frac{((D_a^{\alpha})^{N+1}f)(\xi)}{\Gamma\left((N+1)\alpha+1\right)}.(x-a)^{(N+1)\alpha}.$$

The accuracy of $P_N^{\alpha}\left(x\right)$ increases when we choose large N and it decreases as the value of x moves away from a. Hence we must choose N large enough so that the error does not exceed a specified bound. In the following theorem, we find precise condition under which the exponents hold for arbitrary fractional operators. This result is very useful on our approach for solving differential equations of fractional order.

Theorem 4.1.2.6 As it was stated in S. Momani, et al. [14], suppose that $f(x)=x^{\lambda}g(x)$ where $\lambda>-1$ and $g(x)$ has the generalized Taylor's series $g(x)=\sum_{n=0}^{\infty}a_n(x-a)^{n\alpha}$ with radius of convergence, $R>0$, $<\alpha\le 1$.

Then

$$D_a^{\gamma}D_a^{\beta}f\left(x\right)=D_a^{\gamma+\beta}f\left(x\right) \tag{21}$$

for $x\ \epsilon\ (0,R)$ if:

a) $\beta<\lambda+1$ and α is arbitrary or

b) $\beta>\lambda+1$ and γ is arbitrary, and a_k for $k=0,1,2,\cdots,m-1<\beta\le m.$

Proof:

a) In case of $\beta<\lambda+1$, from definition of Caputo fractional deferential operator (3) and from property (5), we have

$$D_a^{\beta}f(x)=\sum_{n=0}^{\infty}a_nD_a^{\beta}(x-a)^{n\alpha+\lambda}=\sum_{n=0}^{\infty}a_n\frac{\Gamma\left(n\alpha+\lambda+1\right)}{\Gamma\left(n\alpha+\lambda-\beta+1\right)}(x-a)^{n\alpha+\lambda-\beta} \tag{22}$$

Since $\lambda-\beta>-1,$ and

$$D_a^{\gamma} D_a^{\beta} f(x) = \sum_{n=0}^{\infty} a_n \frac{\Gamma(n\alpha+\lambda+1)}{\Gamma(n\alpha+\lambda-\beta+1)} D_a^{\gamma} (x-a)^{n\alpha+\lambda-\beta}$$

$$= \sum_{n=0}^{\infty} a_n \frac{\Gamma(n\alpha+\lambda+1)}{\Gamma(n\alpha+\lambda-\beta+1)} \frac{\Gamma(n\alpha+\lambda-\beta+1)}{\Gamma(n\alpha+\lambda-\beta-\gamma+1)} (x-a)^{n\alpha+\lambda-\beta-\gamma}$$

$$= \sum_{n=0}^{\infty} a_n \frac{\Gamma(n\alpha+\lambda+1)}{\Gamma(n\alpha+\lambda-\beta+1)} \frac{\Gamma(n\alpha+\lambda-\beta+1)}{\Gamma(n\alpha+\lambda-\beta-\gamma+1)} x^{n\alpha+\lambda-\beta-\gamma} \tag{23}$$

$$= D_a^{\gamma+\beta} f(x)$$

b) For the other case $\beta > \lambda + 1$, from definition of Caputo fractional deferential operator (3) and from property (5), we have

$$D_a^{\beta} f(x) = \sum_{n=0}^{\infty} a_n D_a^{\beta} (x-a)^{n\alpha+\lambda} = \sum_{n=0}^{\infty} a_n \frac{\Gamma(n\alpha+\lambda+1)}{\Gamma(n\alpha+\lambda-\beta+1)} (x-a)^{n\alpha+\lambda-\beta},$$

Since $\lambda - \beta < -1$ and

$$D_a^{\gamma} D_a^{\beta} f(x) = \sum_{n=0}^{\infty} a_n \frac{\Gamma(n\alpha+\lambda+1)}{\Gamma(n\alpha+\lambda-\beta+1)} D_0^{\gamma} (x-a)^{n\alpha+\lambda-\beta}$$

$$= \sum_{n=0}^{\infty} a_n \frac{\Gamma(n\alpha+\lambda+1)}{\Gamma(n\alpha+\lambda-\beta+1)} \frac{\Gamma(n\alpha+\lambda-\beta+1)}{\Gamma(n\alpha+\lambda-\beta-\gamma+1)} (x-a)^{n\alpha+\lambda-\beta-\gamma}$$

$$= \sum_{n=0}^{\infty} a_n \frac{\Gamma(n\alpha+\lambda+1)}{\Gamma(n\alpha+\lambda-\beta+1)} \frac{\Gamma(n\alpha+\lambda-\beta+1)}{\Gamma(n\alpha+\lambda-\beta-\gamma+1)} (x-a)^{n\alpha+\lambda-\beta-\gamma}$$

$$= D_a^{\gamma+\beta} f(x)$$

So, based on the generalized Taylor's formula, the generalized differential transform of the kth derivative of function $f(x)$ in one variable, $F_\alpha(k)$ and the differential inverse transform of $F_\alpha(k)$, $f(x)$, where $f(x)$ is analytic and continuously differentiable, are defined in S. Momani, et al. [14] as follows.

Definition 4.1.2.5. If the function $f(x)$ is analytic and k-times differentiable continuously with respect to in the domain of interest, then the generalized differential transform, $F_\alpha(k)$ and is defined in S. Momani, et al. [14] as:

$$F_\alpha(k)=\frac{1}{\Gamma(k\alpha+1)}\left[(D_{x_0}^{\alpha})^k f(x)\right]_{x_0=a}=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} f(x)\right]_{x_0=a} \quad (24)$$

Definition 4.1.2.6.The differential inverses transform of $U_k(x)$, $f(x)$ is defined in S. Momani, et al. [13] as:

$$f(x)=\sum_{k=0}^{\infty} U_k(x)(x-a)^{K\alpha} \quad (25)$$

Substituting $\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} f(x)\right]_{t=0}$ for $U_k(x)$ from (24) in (25) using (17) one can obtain that:

$$f(x)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(K\alpha+1)}\left[\frac{\partial^{K\alpha}}{\partial t^{K\alpha}} f(x)\right]_{x_0=a}(x-a)^{K\alpha} \quad (26)$$

4.1.3 Reduced Differential Transform Method

The basic definitions of the reduced differential transform and differential inverse transform in [5, 7, 17 and 18] were discussed as follows.

Let RDT denotes the reduced differential transform operator and denotes the inverse reduced differential transform operator.

Definition 4.1.3.1 As in [5, 17, 18], if the function $u(x,t)$ is analytic and differentiable continuously with respect to time variable t and variable in the domain of interest, then the reduced transformed function is defined as:

$$RDT\left[u(x,t)\right]=U_k(x)=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x,t)\right]_{t=0} \quad (27)$$

Where α is a parameter describing the order of the time fractional derivative in Caputo sense and $U_K(x)$ is transformed function of $u(x,t)$.

Definition4.1.3.2. The reduced differential inverses transform of $U_k(x)$, $RDT^{-1}\left[u(x,t)\right] or\ u(x,t)$ is defined as follows in $[5,\ 17,18]$:

$$RDT^{-1}\left[u(x,t)\right] = u(x,t) = \sum_{k=0}^{\infty} \mathrm{U_k}(x)\,\mathrm{t}^{\mathrm{K}\alpha} \tag{28}$$

Substituting $\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,t)\right]_{t=0}$ for $U_k(x)$ from equation (27) in equation (28), one can obtain that:

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(K\alpha+1)}\left[\frac{\partial^{\mathrm{K}\alpha}}{\partial \mathrm{t}^{\mathrm{K}\alpha}}\mathrm{u}(x,\mathrm{t})\right]_{\mathrm{t}=0} \mathrm{t}^{\mathrm{K}\alpha} \tag{29}$$

Definitions 4.1.3.1 and 4.1.3.2 were stated in [5] and [17] for solving time fractional heat equations and time fractional non-linear evolution equations having time fractional derivative order, α such that $0 < \alpha \le 1$ respectively. These definitions were also stated in [17] for solving Caputo time fractional-order hyperbolic telegraph equation having time fractional derivative order, such that $0 < \alpha \le 2$.

Even though, the definitions of t-dimensional spectrum function (or the Reduced Transformed function) and the reduced differential inverse transform of the transformed function and the mathematical operations (theorems) of Reduced Differential Transform Method in two dimensions were not stated in M. Sohail and S.T. Mohyud-Din [5] and V.K. Srivastava et al. [7], they were used for solving time fractional heat equations and two dimensional time-fractional telegraph equations respectively.

Definition 4.1.3.3 As in V.K. Srivastava, et al.[7], if the function $u(x,y,z,t)$ is analytic and differentiated continuously with respect to time variable t and space variables $x, y\, and\, z$ in the domain of interest, then the t-dimensional spectrum function (or the reduced transformed function), $U_k(\mathrm{x},\ \mathrm{y},\ \mathrm{z})$ is defined as:

$$RDT\left[u(x,y,z,t)\right] = U_k(x,y,z) = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^k}{\partial t^k}u(x,y,z,t)\right]_{t_0=0} \tag{30}$$

Where α is a parameter describing the order of the time fractional derivative in Caputo sense and $U_K(x,y,z)$ is t-dimensional spectrum function of $u(x,y,z,t)$.

Definition 4.1.3.4. The differential inverses transform of $\mathrm{U_k}(x,y,z)$, $RDT^{-1}\left[u(x,y,z,t)\right]$ *or* $u(x,y,z,t)$ is defined as follows in V.K. Srivastava, et al [7]:

$$RDT^{-1}\left[u(x,y,z,t)\right] = u(x,y,z,t) = \sum_{k=0}^{\infty} \mathrm{U_k}(x,y,z)\,\mathrm{t}^{\mathrm{K}\alpha} \qquad (31)$$

Substituting $\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^k}{\partial t^k} u(x,y,z,t)\right]_{t=0}$ for $U_k(x,y,z)$ from equation (30) in equation (31), one can obtain that:

$$u(x,y,z,t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(K\alpha+1)}\left[\frac{\partial^{K\alpha}}{\partial t^{K\alpha}} u(x,y,z,t)\right]_{t_0=0} t^{K\alpha} \qquad (32)$$

Definitions 4.1.3.3 and 4.1.3.4 were stated in V.K. Srivastava, et al. [7] for solving two and three dimensional time-fractional telegraphic equations having time fractional derivative order, α such that $0 < \alpha \leq 2$. But, nothing was said about the definition of t-dimensional spectrum function (the reduced transformed function), the definition of the reduced differential inverse transform of the reduced transformed function and mathematical operations (theorems) of Reduced Differential Transform Method in two dimensions in V.K. Srivastava, et al. [7] even though their idea was used for solving two dimensional time-fractional telegraph equations.

In addition, nothing was said about the definition of t-dimensional spectrum function (the reduced transformed function), the definition of the reduced differential inverse transform of the reduced transformed function and mathematical operations (theorems) of Reduced Differential Transform Method in three dimensional space in M. Sohail and S.T. Mohyud-Din[5] but their idea was used for solving time fractional heat equations.

Some of the mathematical operations (theorems) in one dimension performed by reduced differential transform method $[5,\ 17,\ 18]$ were stated and discussed as follows.

Theorem 4.1.3.1 If $w(x,t), u(x,t)$ and $v(x,t)$ be analytic and k-times continuously differentiable functions with respect to time t and x in the domain of interest, $\Omega \in \mathbb{R}$ which is closed set such that

$w(x,t)=u(x,t)\pm v(x,t)$, then $W_k(\mathrm{x})=U_k(\mathrm{x})\pm V_k(\mathrm{x})$, *where* $W_k(\mathrm{x}), U_k(\mathrm{x})$ *and* $V_k(\mathrm{x})$ are reduced differential transform of $w(x,t), u(x,t)$ *and* $v(x,t)$ respectively.

Proof:

Let $w(x,t), u(x,t)$ and $v(x,t)$ be analytic and k-times continuously differentiable functions with respect to time t in the domain of interest, $\Omega \in \mathbb{R}$ which is closed set such that $w(x,t)=u(x,t)\pm v(x,t)$, where $k=1,2,\cdots$

Let $W_k(\mathrm{x})$, $U_k(\mathrm{x})$ $V_k(\mathrm{x})$ and $V_k(\mathrm{x})$ be t-dimensional spectrum functions of $w(x,t), u(x,t)$ and $v(x,t)$ respectively.

Now we want to show that

$$W_k(\mathrm{x})=U_k(\mathrm{x})\pm V_k(\mathrm{x}).$$

Then

$$RDT\left[w(x,t)\right]=RDT\left[u(x,t)\pm v(x,t)\right]$$

where RDT denotes the reduced differential transform operator.

By definition 4.1.3.1, $\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}w(x,t)\right]_{t=0}=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}(\ u(x,t)\pm v(x,t))\right]_{t_0=0}$.

$$\Rightarrow \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}w(x,t)\right]_{t=0}=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,t)\right]_{t_0=0}\pm\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}v(x,t)\right]_{t_0=0}.$$

Then

$$W_k(\mathrm{x})=U_k(\mathrm{x})\pm V_k(\mathrm{x}) \text{ , since } W_k(\mathrm{x})=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}w(x,t)\right]_{t_0=0}.$$

By definition 4.1.3.1,

$$U_k(\mathrm{x})=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,t)\right]_{t_0=0}\text{, and } V_K(x)=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}v(x,t)\right]_{t_0=0}.$$

Thus,

$$w(x,t)=u(x,t)\pm v(x,t)\Rightarrow W_k(\mathrm{x})=U_k(\mathrm{x})\pm V_k(\mathrm{x}).$$

Therefore, if $w(x,t)=u(x,t)\pm v(x,t)$, then $W_k(\mathrm{x})=U_k(\mathrm{x})\pm V_k(\mathrm{x})$

Theorem4.1.3.2. If $w(x,t)=\alpha u(x,t)$, then $W_k(\mathrm{x})=\alpha U_k(\mathrm{x})$ *, where* $W_k(\mathrm{x})$ *and* $U_k(\mathrm{x})$ are reduced differential transform of $w(x,t)$ *and* $u(x,t)$ respectively.

Proof:

Let $w(x,t)$ and $u(x,t)$ be analytic and k-times continuously differentiable functions with respect to time t and x in the domain of interest, $\Omega\in\mathbb{R}$ which is closed set such that $w(x,t)=\beta(x,t)$, where β is constant and $k=1,2,\cdots$

Let $W_k(\mathrm{x})$ and $U_k(\mathrm{x})$ be t-dimensional spectrum function of $w(x,t)$ and $u(x,t)$ respectively.

Now we want to show that $W_k(\mathrm{x})=\beta U_k(\mathrm{x})$.

Then $RDT\left[w(x,t)\right]=RDT\left[\beta u(x,t)\right]$, where RDT denotes the reduced differential transform operator.

Then,

$$RDT\left[w(x,t)\right]=\beta RDT\left[u(x,t)\right].$$

By definition 4.1.3.1,

$$\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}w(x,t)\right]_{t_0=0}=\beta\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,t)\right]_{t_0=0}.$$

But by definition 4.1.3.1,

$$W_k(\mathrm{x})=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}w(x,t)\right]_{t_0=0},\text{ and } U_k(\mathrm{x})=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,t)\right]_{t_0=0.}$$

Then

$$W_k(\mathrm{x})=\beta U_k(\mathrm{x}).$$

Thus,

$$w(x,t)=\beta u(x,t)\Rightarrow W_k(\mathrm{x})=\beta U_k(\mathrm{x}).$$

Theorem 4.1.3.3 If $\mathrm{w}(\mathrm{x},\ \mathrm{t})=\frac{\partial^{\mathrm{n}}}{\partial \mathrm{t}^{\mathrm{n}}}\mathrm{u}(\mathrm{x},\mathrm{t})$, then $\mathrm{W}_{\mathrm{k}}(\mathrm{x})=(\mathrm{k}+1)(\mathrm{k}+2)\cdots(\mathrm{k}+\mathrm{n})\mathrm{U}_{\mathrm{k}+\mathrm{n}}(\mathrm{x})$

Proof:

Let $w(x,t)$ and $u(x,t)$ be analytic and k-times continuously differentiable functions with respect to time t and in the domain of interest $\Omega \in \mathbb{R}$ which is closed set such that $w(x,t)=\frac{\partial^{\mathrm{n}}}{\partial \mathrm{t}^{\mathrm{n}}}u(x,t)$, where $k=1,2,\cdots$

Let $W_k(\mathrm{x})$ and $U_k(\mathrm{x})$ be t-dimensional spectrum functions of $w(x,t)$ and $u(x,t)$ respectively.

By definition 4.1.3.1, we get

$$RDT\left[w(x,t)\right]=W_K(x)=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}w(x,t)\right]_{t_0=0}$$

Where RDT denotes the reduced differential transform operator

Since

$$w(x,t)=\frac{\partial^{\mathrm{n}}}{\partial \mathrm{t}^{\mathrm{n}}}u(x,t),\ W_K(x)=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\left(\frac{\partial^{\mathrm{n}}}{\partial \mathrm{t}^{\mathrm{n}}}u(x,t)\right)\right]_{t_0=0}.$$

For

$$\alpha=1,\ W_K(x)=\frac{1}{\mathrm{k}!}\left[\frac{\partial^{k}}{\partial t^{k}}\left(\frac{\partial^{\mathrm{n}}}{\partial \mathrm{t}^{\mathrm{n}}}u(x,t)\right)\right]_{t_0=0}.$$

$$W_K(x)=\frac{1}{\mathrm{k}!}\left[\frac{\partial^{k}}{\partial t^{k}}\left(\frac{\partial^{\mathrm{n}}}{\partial \mathrm{t}^{\mathrm{n}}}u(x,t)\right)\right]_{t_0=0}.$$

$$W_K(x)=\frac{1}{\mathrm{k}!}\left[\frac{\partial^{k+n}}{\partial t^{k+n}}\left(u(x,t)\right)\right]_{t_0=0}.$$

$$W_K(x)=\frac{(k+n)!}{\mathrm{k}!(k+n)!}\left[\left(\frac{\partial^{\mathrm{k+n}}}{\partial \mathrm{t}^{\mathrm{k+n}}}u(x,t)\right)\right]_{t_0=0}.$$

$$W_K(x)=\frac{(k+1)(k+2)(K+3)\cdots(k+n)k!}{\mathrm{k}!(k+n)!}\left[\left(\frac{\partial^{\mathrm{k+n}}}{\partial \mathrm{t}^{\mathrm{k+n}}}u(x,t)\right)\right]_{t_0=0}.$$

$$W_K(x)=(k+1)(k+2)(K+3)\cdots(k+n)\left(\frac{1}{(k+n)!}\left[\left(\frac{\partial^{\mathrm{k+n}}}{\partial \mathrm{t}^{\mathrm{k+n}}}u(x,t)\right)\right]_{t_0=0}\right).$$

By definition4.1.3.1,

$$W_K(x) = (k+1)(k+2)(K+3)\cdots(k+n)U_{k+n}(x).$$

Thus,

$$\mathrm{w}(\mathrm{x},\mathrm{t}) = \frac{\partial^{\mathrm{n}}}{\partial \mathrm{t}^{\mathrm{n}}}\mathrm{u}(\mathrm{x},\mathrm{t}) \Rightarrow W_K(x) = (k+1)(k+2)(K+3)\cdots(k+n)U_{k+n}(x).$$

Therefore, if $\mathrm{w}(\mathrm{x},\mathrm{t}) = \frac{\partial^{\mathrm{n}}}{\partial \mathrm{t}^{\mathrm{n}}}\mathrm{u}(\mathrm{x},\mathrm{t})$ then $W_K(x) = (k+1)(k+2)(K+3)\cdots(k+n)U_{k+n}(x)$.

Theorem 4.1.3.4 If $w(x,t) = \frac{\partial^{\mathrm{N}\alpha}}{\partial \mathrm{t}^{\mathrm{N}\alpha}}u(x,t)$, then $W_k(x) = \frac{\Gamma(K\alpha+N\alpha+1)}{\Gamma(K\alpha+1)}\mathrm{U}_{\mathrm{K+N}}(\mathrm{x})$

Proof:

Let $w(x,t)$ and $u(x,t)$ be analytic and k-times continuously differentiable functions with respect to time t and in the domain of interest $\Omega \in \mathbb{R}$ which is closed set such that $w(x,t) = \frac{\partial^{\mathrm{N}\alpha}}{\partial \mathrm{t}^{\mathrm{N}\alpha}}u(x,t)$, where $k = 1,2,\cdots$

Let $W_k(\mathrm{x})$ and $U_k(\mathrm{x})$ be t-dimensional spectrum function of $w(x,t)$ and $u(x,t)$ respectively.

By definition 4.1.3.1, we get

$$RDT\left[w(x,t)\right] = W_K(x) = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}w(x,t)\right]_{t_0=0},$$

Where RDT denotes the reduced differential transform operator.

But

$$w(x,t) = \frac{\partial^{\mathrm{N}\alpha}}{\partial \mathrm{t}^{\mathrm{N}\alpha}}u(x,t)$$

Then

$$W_K(x) = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\left(\frac{\partial^{\mathrm{N}\alpha}}{\partial \mathrm{t}^{\mathrm{N}\alpha}}u(x,t)\right)\right]_{t_0=0}.$$

By multiplying with

$$\frac{\Gamma(K\alpha+N\alpha+1)}{\Gamma(K\alpha+N\alpha+1)}, \quad W_K(x) = \frac{\Gamma(K\alpha+N\alpha+1)}{\Gamma(k\alpha+1)\Gamma(K\alpha+N\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\left(\frac{\partial^{\mathrm{N}\alpha}}{\partial \mathrm{t}^{\mathrm{N}\alpha}}u(x,t)\right)\right]_{t_0=0}.$$

$$W_K(x)=\frac{\varGamma(K\alpha+N\alpha+1)}{\Gamma(k\alpha+1)}\left(\frac{1}{\varGamma(K\alpha+N\alpha+1)}\left[\frac{\partial^{k\alpha+\mathrm{N}\alpha}}{\partial t^{k\alpha+\mathrm{N}\alpha}}(u(x,t))\right]_{t_0=0}\right).$$

By definition 4.1.3.1,

$$U_{k+N}(x)=\frac{1}{\varGamma(K\alpha+N\alpha+1)}\left[\frac{\partial^{k\alpha+\mathrm{N}\alpha}}{\partial t^{k\alpha+\mathrm{N}\alpha}}(u(x,t))\right]_{t_0=0}$$

$$\Rightarrow\ W_K(x)=\frac{\varGamma(K\alpha+N\alpha+1)}{\Gamma(k\alpha+1)}U_{k+N}(x).$$

Thus, $\mathrm{w}(\mathrm{x},\mathrm{t})=\frac{\partial^{\mathrm{N}\alpha}}{\partial \mathrm{t}^{\mathrm{N}\alpha}}u(x,t)\Rightarrow W_K(x)=\frac{\varGamma(K\alpha+N\alpha+1)}{\Gamma(k\alpha+1)}U_{k+N}(x).$

Theorem 4.1.3.5

If $w(x,t)=x^m t^n$, then $W_k(\mathrm{x})=x^m\ \delta(k-n), where\ x^m\delta(k-n)=\begin{cases}1, if\ k=n\\ 0,\ if\ k\neq n\end{cases}$

Proof:

From (28),

$w(x,t)=x^m t^n$, can be written as $w(x,t)=\sum_{k=0}^{\infty}x^m\ \delta(k-n)t^{k\alpha}$

Where $n=k\alpha\Rightarrow\alpha=1$ (since both k & n are natural numbers).

Now, taking the reduced differential transform (27) of $\left(w(x,t)=\sum_{k=0}^{\infty}x^m\ \delta(k-n)t^{k\alpha}\right)$, we get

$$W_k(\mathrm{x})=x^m\ \delta(k-n), where\ x^m\delta(k-n)=\begin{cases}1, if\ k=n\\ 0,\ if\ k\neq n\end{cases}$$

Hence, the theorem holds true.

Theorem 4.1.3.6 If $w(x,t)=\frac{\partial^2}{\partial \mathrm{x}^2}u(x,t)$, then $W_k(x)=\frac{\partial^2}{\partial \mathrm{x}^2}\mathrm{U}_{\mathrm{K}}(\mathrm{x})$.

Proof:

Let $w(x,t)\, and\, u(x,t)$ be analytic and k-times continuously differentiable functions with respect to time t and x in the domain of interest, $\Omega \in \mathbb{R}$ which is closed set such that $w(x,t) = \frac{\partial^2}{\partial \mathrm{x}^2} u(x,t)$, where $k = 1,2,\cdots$

Let $W_k(\mathrm{x})$ and $U_k(\mathrm{x})$ be t-dimensional spectrum function of $w(x,t)$ and $u(x,t)$ respectively.

By definition 4.1.3.1, we get

$$RDT\left[w(x,t)\right] = W_K(x) = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} w(x,t)\right]_{t=0}$$

Where RDT denotes the reduced differential transform operator

Then

$$W_K(x) = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\left(\frac{\partial^2}{\partial \mathrm{x}^2} u(x,t)\right)\right]_{t=0}, \text{ since } w(x,t) = \frac{\partial^2}{\partial \mathrm{x}^2} u(x,t)$$

$$W_K(x) = \frac{\partial^2}{\partial \mathrm{x}^2}\left(\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\left(u(x,t)\right)\right]_{t=0}\right)$$

But from definition 4.1.3.1,

$$U_k(x) = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\left(u(x,t)\right)\right]_{t=0}$$

Then

$$W_K(x) = \frac{\partial^2}{\partial \mathrm{x}^2} U_k(x)$$

Thus, if $w(x,t) = \frac{\partial^2}{\partial \mathrm{x}^2} u(x,t), then\ W_k(x) = \frac{\partial^2}{\partial \mathrm{x}^2} \mathrm{U_K}(\mathrm{x})$.

4.2 Main Results

To the best of the researcher knowledge till now no one has developed scheme (procedures) to find analytic solutions of multi-dimensional homogeneous time fractional heat-like equations under initial conditions by Reduced Differential Transform Method. Due to this, the work of M. Sohail and S.T. Mohyud-Din [5] were extended by the researcher to develop procedures to find analytic solutions of one, two, three and more than three dimensional homogeneous time fractional heat-like equations separately under initial conditions by Reduced Differential Transform Method only in analytical case.

4.2.1 Some new Basic Definitions, Properties and Theorems in n-Dimensional Space where $n \in \mathbb{N}$

In this sub-section, definitions: 4.1.2.1, 4.1.2.2 and 4.1.2.3 in one dimension were extended to n-dimensions $(n \in \mathbb{N})$ and new definitions: 4.2.1.1, 4.2.1.2 and 4.2.1.3 were introduced to n-dimensions $(n \in \mathbb{N})$ respectively. In addition, theorems: 4.1.2.2, 4.1.2.3, 4.1.2.4, 4.1.2.5 and 4.1.2.6 were extended to n-dimensions $(n \in \mathbb{N})$ and new theorems: 4.2.1.1, 4.2.1.2, 4.2.1.3, 4.2.1.4 and 4.2.1.5 were introduced in n-dimensions $(n \in \mathbb{N})$ respectively.

Definition 4.2.1.1 Let a function $u(x_1, x_2, \cdots, x_n, \mathrm{t})$ be analytic and k-times continuously differentiable with respect to time, t and $x_1, x_2, \cdots,$ and x_n such that $u(x_1, x_2, \cdots, x_n, \mathrm{t}) = q(x_1, x_2, \cdots, x_n) g(\mathrm{t})$ where $(x_1, x_2, \cdots, x_n)$ is element of domain of interest, $\Omega \subseteq \mathbb{R}^n$ which is closed set, and $t > 0$.

Then $u(x_1, x_2, \cdots, x_n, \mathrm{t})$ is said to be in the space $C_\mu(\Omega \times (0, \infty))$, $\mu \epsilon \mathbb{R}$ if there exists a real number ($p > \mu$) such that $u(x_1, x_2, \cdots, x_n, \mathrm{t}) = t^p l(t) q(x_1, x_2, \cdots, x_n)$ and $g(\mathrm{t}) = t^p l(t)$,where $\mathrm{q}(x_1, x_2, \cdots, x_n) \in C(\Omega), l(t) \in [0, \infty)$ and also $u(x_1, x_2, \cdots, x_n, \mathrm{t})$ is Said to be in the space $C_\mu^m(\Omega \times (0, \infty))$ if and only if $\dfrac{\partial^m u(x_1, x_2, \cdots, x_n, \mathrm{t})}{\partial t^m} \in C_\mu$, m ϵ $\mathbb{N}$.

Example 4.2.1.1 $u(x_1, x_2, \cdots, x_n, \mathrm{t}) = (t^8 + 2t^5)(4 + \frac{x_1^3}{6} + \frac{x_2^3}{6} + \frac{x_3^3}{6} + \frac{x_4^3}{6}),\ t > 0.$

Because $u(x_1, x_2, \cdots, x_n, \mathrm{t}) = t^5(t^3 + 2)(4 + \frac{x_1^3}{6} + \frac{x_2^3}{6} + \frac{x_3^3}{6} + \frac{x_4^3}{6})$

where $u_1(x_1, x_2, \cdots, x_n, \mathrm{t}) = (t^3 + 2)(4 + \frac{x_1^3}{6} + \frac{x_2^3}{6} + \frac{x_3^3}{6} + \frac{x_4^3}{6})$,

$f(x) = t^5 u_1(x_1, x_2, \cdots, x_n, \mathrm{t})$.

$\Rightarrow 5 > \mu$, where $\mu \in (5, -\infty)$ and $u_1(x_1, x_2, \cdots, x_n, \mathrm{t}) \in C(\Omega \mathrm{X}[0, \infty))$

$$\therefore\ u(x_1, x_2, \cdots, x_n, t) = (t^8 + 2t^5)(4 + \frac{x_1^{\ 3}}{6} + \frac{x_2^{\ 3}}{6} + \frac{x_3^{\ 3}}{6} + \frac{x_4^{\ 3}}{6}) \in C_\mu(\Omega \text{ X } (0, \infty))$$

And

$$u(x_1, x_2, \cdots, x_n, t) \in C_\mu^8(\Omega \text{ X } (0, -\infty)) \Leftrightarrow \frac{\partial^8 u(x_1, x_2, \cdots, x_n, t)}{\partial t^8} = u^{(8)} \in C_\mu(\Omega \text{ X } (0, \infty)).$$

Definition 4.2.1.2 Let $u(x_1, x_2, \cdots, x_n, t)$ be analytic and k-times continuously differentiable with respect to $t, x_1, x_2, \cdots, and\, x_n$ such that $u(x_1, x_2, \cdots, x_n, t) = q(x_1, x_2, \cdots, x_n)g(t)$, where $(x_1, x_2, \cdots, x_n) \in \Omega \subseteq \mathbb{R}^n$ and $t \in (0, \infty)$. Then the Reimann-Liouville time fractional integral operator of order $\alpha \geq 0$ of the function $u(x_1, x_2, \cdots, x_n, t) \epsilon C_\mu(\Omega \times (0, \infty))$, $\mu > -1$ is defined as:

$$J_0^\alpha u(x_1, x_2, \cdots, x_n, t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-\xi)^{\alpha-1} u(x_1, x_2, \cdots, x_n, \xi) d\xi, \ \alpha > 0 \\ J_0^0 u(x_1, x_2, \cdots, x_n, t) = u(x_1, x_2, \cdots, x_n, t), \alpha = 0 \end{cases} \tag{33}$$

Definition 4.2.1.3 Let $u(x_1, x_2, \cdots, x_n, t)$ be analytic and k-times continuously differentiable with respect to $t, x_1, x_2, \cdots, and\, x_n$ such that $u(x_1, x_2, \cdots, x_n, t) = q(x_1, x_2, \cdots, x_n)g(t)$, where $(x_1, x_2, \cdots, x_n) \in \Omega \subseteq \mathbb{R}^n$ and $t \in (0, \infty)$. Then for the smallest integer, m that exceeds α, Caputo time fractional derivative of order $\alpha > 0$ is defined as**:**

$$D_0^\alpha u(x_1, x_2, \cdots, x_n, t) = \frac{\partial^\alpha u(x_1, x_2, \cdots, x_n, t)}{\partial t^\alpha}$$

$$= \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t (t-\xi)^{m-\alpha-1} \dfrac{\partial^m u(x_1, x_2, \cdots, x_n, t)}{\partial \xi^m} d\xi, for\, m-1 < \alpha < m \\ \dfrac{\partial^m u(x_1, x_2, \cdots, x_n, t)}{\partial t^m} \quad , for\, \alpha = m \end{cases} \tag{34}$$

For time fractional derivative of order α and β such that $\alpha, \beta > 0$, $m-1 < \alpha \leq m$, $m-1 < \beta \leq m$ and $\gamma > -1$, we have the following properties:

1. $$\left(J_0^\alpha J_0^\beta u\right)(x_1, x_2, \cdots, x_n, t) = \left(J_0^\beta J_0^\alpha u\right)(x_1, x_2, \cdots, x_n, t) = \left(J_0^{\alpha+\beta} u\right)(x_1, x_2, \cdots, x_n, t) \tag{35}$$

2. $J_0^{\alpha} t^{\gamma} = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\gamma+\alpha}$ (36)

3. $\left(J_0^{\alpha} D_0^{\alpha} u\right)(x_1, x_2, \cdots, x_n, t) = u(x_1, x_2, \cdots, x_n, t) - \sum_{k=0}^{m-1} \dfrac{\partial^k}{\partial t^k} u(x_1, x_2, \cdots, x_n, 0) \dfrac{t^k}{k!}$ (37)

Proof:

Property1

Let $u(x_1, x_2, \cdots, x_n, \mathrm{t}) = q(x_1, x_2, \cdots, x_n) g(\mathrm{t})$, where $(x_1, x_2, \cdots, x_n) \in \Omega \subseteq \mathbb{R}^n$ and $t \in (0, \infty)$ be analytic and k-times continuously differentiable with respect to $t, x_1, x_2, \cdots, \textit{and } x_n$.

Then

$$\left(J_0^{\alpha} J_0^{\beta} u\right)(x_1, x_2, \cdots, x_n, t) = (J_0^{\alpha} J_0^{\beta})\left(q(x_1, x_2, \cdots, x_n) g(\mathrm{t})\right)$$

$$= q(x_1, x_2, \cdots, x_n)[J_0^{\beta} J_0^{\alpha} g)(\mathrm{t})]$$

$$= q(x_1, x_2, \cdots, x_n)(J_0^{\beta} J_0^{\alpha} g)(\mathrm{t})) \quad \text{(By (4))}$$

$$= (J_0^{\beta} J_0^{\alpha})(q(x_1, x_2, \cdots, x_n)(g(\mathrm{t}))$$

$$\left(J_0^{\alpha} J_0^{\beta} u\right)(x_1, x_2, \cdots, x_n, t) = \left(J_0^{\beta} J_0^{\alpha} u\right)(x_1, x_2, \cdots, x_n, t) \quad (38)$$

And $\left(J_0^{\beta} J_0^{\alpha} u\right)(x_1, x_2, \cdots, x_n, t) = \left(J_0^{\beta} J_0^{\alpha}\right)\left(q(x_1, x_2, \cdots, x_n) g(\mathrm{t})\right)$

$$= q(x_1, x_2, \cdots, x_n)(J_0^{\beta} J_0^{\alpha} g(\mathrm{t}))$$

$$= q(x_1, x_2, \cdots, x_n)(J_0^{\alpha+\beta} g(\mathrm{t})) \quad \text{(By (4)}$$

$$= (J_0^{\alpha+\beta})(q(x_1, x_2, \cdots, x_n) g(\mathrm{t}))$$

$$= (J_0^{\alpha+\beta})(q(x_1, x_2, \cdots, x_n) g(\mathrm{t}))$$

$$= \left(J_0^{\alpha+\beta} u\right)(x_1, x_2, \cdots, x_n, \mathrm{t}) \quad (39)$$

Thus,

$$\left(J_0^{\beta} J_0^{\alpha} u\right)(x_1, x_2, \cdots, x_n, t) = \left(J_0^{\alpha+\beta} u\right)(x_1, x_2, \cdots, x_n, \text{t})$$

Hence by (38) and (39), (35) is obtained, i.e.

$$\left(J_0^{\alpha} J_0^{\beta} u\right)(x_1, x_2, \cdots, x_n, t) = \left(J_0^{\beta} J_0^{\alpha} u\right)(x_1, x_2, \cdots, x_n, t) = \left(J_0^{\alpha+\beta} u\right)(x_1, x_2, \cdots, x_n, \text{t}).$$

Property 2

Let $u(x_1, x_2, \cdots, x_n, \text{t}) = q(x_1, x_2, \cdots, x_n) g(\text{t})$, where $(x_1, x_2, \cdots, x_n) \in \Omega \subseteq \mathbb{R}^n$ and $t \in (0, \infty)$ be analytic and k-times continuously differentiable with respect to $t, x_1, x_2, \cdots,$ and x_n.

Let $f(t) = t^{\gamma}$, with $\gamma + 1 > 0$

Then by equation (2),

$$J_0^{\alpha} t^{\gamma} = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} t^{\gamma} dx$$

$$J_0^{\alpha} t^{\gamma} = \frac{1}{\Gamma(\alpha)} \int_0^t \left(1 - \frac{x}{t}\right)^{\alpha-1} t^{\alpha-1} \left(\frac{x}{t}\right)^{\gamma} t^{\gamma} dx$$

$$J_0^{\alpha} t^{\gamma} = \frac{1}{\Gamma(\alpha)} t^{\alpha-1+\gamma} \int_0^t \left(1 - \frac{x}{t}\right)^{\alpha-1} \left(\frac{x}{t}\right)^{\gamma} dx$$

Let $s = \frac{x}{t}$, with $ds = \frac{dx}{t}$, where $x = 0 \Rightarrow s = 0$, and $x = t \Rightarrow s = 1$.

$$J_0^{\alpha} t^{\gamma} = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} t^{\gamma} \int_0^1 (1-s)^{\alpha-1} s^{\gamma} t ds$$

$$J_0^{\alpha} t^{\gamma} = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} t^{\gamma} t \int_0^1 (1-s)^{\alpha-1} s^{\gamma} ds$$

$$J_0^{\alpha} t^{\gamma} = \frac{1}{\Gamma(\alpha)} t^{\alpha+\gamma} \int_0^1 (1-s)^{\alpha-1} s^{(\gamma+1)-1} ds$$

By equation (1),

$$J_0^{\alpha} t^{\gamma} = \frac{1}{\Gamma(\alpha)} t^{\gamma+\alpha} \frac{\Gamma(\alpha)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}$$

$$= \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\gamma+\alpha}$$

Hence, $J_0^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\gamma+\alpha}$

Property 3

Let $u(x_1, x_2, \cdots, x_n, \mathrm{t}) = q(x_1, x_2, \cdots, x_n) g(\mathrm{t})$, where $(x_1, x_2, \cdots, x_n) \in \Omega \subseteq \mathbb{R}^n$ and $t \in (0, \infty)$ be analytic and k-times continuously differentiable with respect to $t, x_1, x_2, \cdots,$ *and* x_n.

$$\left(J_0^{\alpha} D_0^{\alpha} u\right)(x_1, x_2, \cdots, x_n, t) = (J_0^{\alpha} D_0^{\alpha})\left(q(x_1, x_2, \cdots, x_n) g(\mathrm{t})\right)$$

$$= \left(q(x_1, x_2, \cdots, x_n))(J_0^{\alpha} D_0^{\alpha} g)(\mathrm{t})\right)$$

Since by (6), $(J_0^{\alpha} D_0^{\alpha} g)(\mathrm{t}) = g(t) - \sum_{k=0}^{m-1} g^{(k)}(0) \frac{t^k}{k!}$,

$$\left(q(x_1, x_2, \cdots, x_n))(J_0^{\alpha} D_0^{\alpha} g)(\mathrm{t})\right) = q(x_1, x_2, \cdots, x_n)[g(t) - \sum_{k=0}^{m-1} g^{(k)}(0) \frac{t^k}{k!}]$$

$$= q(x_1, x_2, \cdots, x_n) g(t) - q(x_1, x_2, \cdots, x_n) \sum_{k=0}^{m-1} g^{(k)}(0) \frac{t^k}{k!}$$

$$= q(x_1, x_2, \cdots, x_n) g(t) - \sum_{k=0}^{m-1} q(x_1, x_2, \cdots, x_n) g^{(k)}(0) \frac{t^k}{k!}$$

$$= u(x_1, x_2, \cdots, x_n, t) - \sum_{k=0}^{m-1} \frac{\partial^k}{\partial t^k} u\left(x_1, x_2, \cdots, x_n, 0\right) \frac{t^k}{k!}$$

Hence, $\left(J_0^{\alpha} D_0^{\alpha} u\right)(x_1, x_2, \cdots, x_n, t) = u(x_1, x_2, \cdots, x_n, t) - \sum_{k=0}^{m-1} u^{(k)}(x_1, x_2, \cdots, x_n, 0) \frac{t^k}{k!}$).

Theorem4.2.1.1 Suppose that $u(x_1, x_2, \cdots, x_n, t) \in C(\Omega \times [0,b])$ such that $u(x_1, x_2, \cdots, x_n, t) = q(x_1, x_2, \cdots, x_n) g(\mathrm{t})$ $g(t) \in C[0,b])$, $D_0^{\alpha} u(x_1, x_2, \cdots, x_n, t) \in C(\Omega \times [0,b])$ and $D_0^{\alpha} g(t) \in C[0,b])$, for $0 < \alpha \le 1$, $(x_1, x_2, \cdots, x_n) \in \Omega \subseteq \mathbb{R}^n$ and $t \in [0,b]$, then we have:

$$u(x_1, x_2, \cdots, x_n, t) = u(x_1, x_2, \cdots, x_n, 0) + \frac{1}{\Gamma(\alpha)} (D_0^{\alpha} u(x_1, x_2, \cdots, x_n, \xi)) t^{\alpha} \qquad (40)$$

with $0 \le \xi \le 1$, where $(x_1, x_2, \cdots, x_n) \in \Omega \subseteq \mathbb{R}^n$ and $t \in [0,b]$, and $D^{\alpha} = \frac{\partial^{\alpha} u((x_1, x_2, \cdots, x_n, t)}{\partial t^{\alpha}}$ is the Caputo time fractional derivative of order $\alpha > 0$.

Proof:

From (33), we have

$$(J_0^\alpha D_0^\alpha u)(x_1,x_2,\cdots,x_n,t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\xi)^{\alpha-1}D_0^\alpha u(x_1,x_2,\cdots,x_n,\xi)\,d\xi \tag{41}$$

$$(J_0^\alpha D_0^\alpha u)(x_1,x_2,\cdots,x_n,t)=q(x_1,x_2,\cdots,x_n)\frac{1}{\Gamma(\alpha)}\int_0^t(t-\xi)^{\alpha-1}D_0^\alpha g(\xi)\,d\xi$$

Using the integral mean value theorem (7), we get

$$(J_0^\alpha D_0^\alpha u)(x_1,x_2,\cdots,x_n,t)=q(x_1,x_2,\cdots,x_n)\frac{1}{\Gamma(\alpha)}D_0^\alpha g(\xi)\int_0^t(t-\xi)^{\alpha-1}\,d\xi$$

$$\Rightarrow (J_0^\alpha D_0^\alpha u)(x_1,x_2,\cdots,x_n,t)=\frac{1}{\Gamma(\alpha)}D_0^\alpha u(x_1,x_2,\cdots,x_n,\xi)\int_0^t(t-\xi)^{\alpha-1}\,d\xi$$

$$=\frac{1}{\Gamma(\alpha)}D_0^\alpha u(x_1,x_2,\cdots,x_n,\xi)t^\alpha \ for\ 0\le\xi\le t \tag{42}$$

On the other hand, from (37), we have

$$(J_0^\alpha D_0^\alpha u)(x_1,x_2,\cdots,x_n,t)=u(x_1,x_2,\cdots,x_n,t)-u(x_1,x_2,\cdots,x_n,0) \tag{43}$$

So, from (42) and (43), (40) is obtained

Hence,

$$u(x_1,x_2,\cdots,x_n,t)=u(x_1,x_2,\cdots,x_n,0)+\frac{1}{\Gamma(\alpha)}(D_0^\alpha u(x_1,x_2,\cdots,x_n,\xi))t^\alpha$$

Theorem 4.2.1.2 Suppose that $(D_0^\alpha)^m u(x_1,x_2,\cdots,x_n,t),\ (D_0^\alpha)^{m+1}u(x_1,x_2,\cdots,x_n,t)\epsilon\, C(\Omega\times(0,b])$, for $0<\alpha\le 1$, then we have

$$(J_0^{m\alpha}(D_0^\alpha)^m u)(x_1,x_2,\cdots,x_n,t)-(J_0^{(m+1)\alpha}(D_0^\alpha)^{m+1}u)(x_1,x_2,\cdots,x_n,t)$$

$$=\frac{t^{m\alpha}}{\Gamma(m\alpha+1)}(((D_0^\alpha)^m u)(x_1,x_2,\cdots,x_n,0)) \tag{44}$$

Where $(D_0^\alpha)^m=D_0^\alpha.D_0^\alpha.D_0^\alpha\cdots D_0^\alpha$ (m-times)

Proof:

From (35), we have

$$(J_0^{m\alpha}(D_0^\alpha)^m u)(x_1,x_2,\cdots,x_n,t)-(J_0^{(m+1)\alpha}(D_0^\alpha)^{m+1}u)(x_1,x_2,\cdots,x_n,t)$$

$$=J_0^{m\alpha}(((D_0^\alpha)^m u)(x_1,x_2,\cdots,x_n,t)-(J_0^\alpha(D_0^\alpha)^{m+1}u)(x_1,x_2,\cdots,x_n,t))$$

$$=J_0^{m\alpha}(((D_0^\alpha)^m u)(x_1,x_2,\cdots,x_n,t)-(J_0^\alpha D_0^\alpha)((D_0^\alpha)^m u)(x_1,x_2,\cdots,x_n,t))$$

$$=J_0^{m\alpha}(((D_0^\alpha)^m u)(x_1,x_2,\cdots,x_n,0))\text{by using equation}(37).$$

$$=\frac{t^{m\alpha}}{\Gamma(m\alpha+1)}(((D_0^\alpha)^m u)(x_1,x_2,\cdots,x_n,0))$$ by using (36)

Theorem 4.2.1.3 Suppose that $u(x_1,x_2,\cdots,x_n,\mathrm{t})=q(x_1,x_2,\cdots,x_n)g(\mathrm{t})\in C(\Omega\times[0,b])$, $(D_0^\alpha)^k u(x_1,x_2,\cdots,x_n,t)\in C(\Omega\times(0,b])$, for $k=0,1,2,\cdots,m+1$, where $0<\alpha\le 1$, then we have

$$u(x_1,x_2,\cdots,x_n,t)=\sum_{i=1}^{m}\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}((D_0^\alpha)^i u)(x_1,x_2,\cdots,x_n,0)+\frac{(D_0^\alpha)^{m+1}u)(x_1,x_2,\cdots,x_n,\xi)}{\Gamma((m+1)\alpha+1)}t^{(m+1)\alpha},$$
$$0\le\xi\le t,\forall t\epsilon(0,b] \quad (45)$$

Proof:

From (45), we have

$$\sum_{i=1}^{m}[J_0^{i\alpha}(D_0^\alpha)^i u)(x_1,x_2,\cdots,x_n,t)-J_0^{(i+1)\alpha}(D_0^\alpha)^{i+1}u)(x_1,x_2,\cdots,x_n,t)]$$

$$=\sum_{i=1}^{m}\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}((D_0^\alpha)^i u)(x_1,x_2,\cdots,x_n,0) \quad (46)$$

That is,

$$u(x_1,x_2,\cdots,x_n,t)-(J_0^{(m+1)\alpha}(D_0^\alpha)^{m+1}u)(x_1,x_2,\cdots,x_n,t)$$

$$=\sum_{i=1}^{m'}\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}((D_0^\alpha)^i u)(x_1,x_2,\cdots,x_n,0) \quad (47)$$

Applying the integral mean value theorem yields

$$(J_0^{(m+1)\alpha}(D_0^\alpha)^{m+1}u)(x_1,x_2,\cdots,x_n,t)$$

$$=\frac{1}{\Gamma((m+1)\alpha+1)}\int_0^t(t-\xi)^{(m+1)\alpha}((D_0^\alpha)^{m+1}u)(x_1,x_2,\cdots,x_n,\xi)d\xi$$

$$= \frac{1}{\Gamma((m+1)\alpha+1)}\left((D_0^{\alpha})^{m+1} u\right)(x_1, x_2, \cdots, x_n, \xi)\int_0^t (t-\xi)^{(m+1)\alpha} d\xi$$

$$= \frac{(D_0^{\alpha})^{m+1} u)(x_1, x_2, \cdots, x_n, \xi)}{\Gamma((m+1)\alpha+1)} t^{(m+1)\alpha} \qquad (48)$$

From (47) and (48), equation (45) is obtained.

In case of α=1, equation (45) becomes

$$u(x_1, x_2, \cdots, x_n, t) = \sum_{i=1}^{m} \frac{t^i}{i!}\frac{\partial^i}{\partial t^i} u(x_1, x_2, \cdots, x_n, 0) + \frac{\partial^{m+1}}{\partial t^{m+1}} u(x_1, x_2, \cdots, x_n, \xi)\frac{t^{m+1}}{(m+1)!}$$

, with $0 \le \xi \le t, \forall t \epsilon (0,b]$

The radius of convergence, R for the generalized Taylor's series of $g(t)$,

$$\sum_{i=1}^{\infty} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)}((D_0^{\alpha})^i g)(0) \text{ in}$$

$$\sum_{i=1}^{\infty} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)}((D_0^{\alpha})^i u)(x_1, x_2, \cdots, x_n, 0) \, or \sum_{i=1}^{\infty} g(x_1, x_2, \cdots, x_n)\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}((D_0^{\alpha})^i g)(0) \qquad (49)$$

depends on $g(t)$ and is given by:

$$R = |t|^{\alpha} \lim_{m\to\infty}\left|\frac{\Gamma(m\alpha+1)}{\Gamma((m+1)\alpha+1)} \cdot \frac{((D_0^{\alpha})^{m+1} g)(0)}{((D_0^{\alpha})^{m} g)(0)}\right| \qquad (50)$$

Theorem4.2.1.4 Suppose $((D_0^{\alpha})^k u)(x_1, x_2, \cdots, x_n, t) \epsilon C(\Omega\times(0,b])$, for $k = 0,1,2,\cdots,m+1$, where $0 < \alpha \le 1$. If $(x_1, x_2, \cdots, x_n) \epsilon \Omega \subseteq \mathbb{R}^n$ and $t \epsilon [0,b]$, then

$$u(x_1, x_2, \cdots, x_n, t) \cong P_M^{\alpha}(x_1, x_2, \cdots, x_n, t) = \sum_{i=0}^{M} \frac{((D_0^{\alpha})^i u)(x_1, x_2, \cdots, x_n, 0)}{\Gamma(i\alpha+1)} . t^{i\alpha} \qquad (51)$$

In addition, there is a value ξ with $0 \le \xi \le t$ so that the error term $R_M^{\alpha}(x_1, x_2, \cdots, x_n, t)$ has the form:

$$R_M^{\alpha}(x_1, x_2, \cdots, x_n, t) = \sum_{i=1}^{M} \frac{((D_0^{\alpha})^{M+1} u)(x_1, x_2, \cdots, x_n, 0)}{\Gamma((M+1)\alpha+1)} . t^{(M+1)\alpha} \qquad (52)$$

Proof:

Let $u(x_1, x_2, \cdots, x_n, t)$ be analytic and k-times differentiable with respect to $t, x_1, x_2, \cdots,$ and x_n such that $u(x_1, x_2, \cdots, x_n, t) = q(x_1, x_2, \cdots, x_n)g(t)$ in terms of functions of single variable.

Since

$((D_0^\alpha)^k u(x_1, x_2, \cdots, x_n, t) \epsilon C(\Omega \times (0, b])$, $((D_0^\alpha)^k g)(t) \in C[0, b]$ and

$q(x_1, x_2, \cdots, x_n) \in C(\Omega)$.

From theorem 4.1.1.4 (equation (19)), we know that

$$g(t) \cong P_M^\alpha(t) = \sum_{i=1}^{M} \frac{((D_0^\alpha)^i g)(0)}{\Gamma(i\alpha + 1)} . t^{i\alpha} .$$

Multiplying both sides of this equation by $q(x_1, x_2, \cdots, x_n)$, we get

$$q(x_1, x_2, \cdots, x_n) g(t) \cong q(x_1, x_2, \cdots, x_n) R_N^\alpha(t)$$
$$= q(x_1, x_2, \cdots, x_n) \sum_{i=1}^{M} \frac{((D_0^\alpha)^i u)(x_1, x_2, \cdots, x_n, 0)}{\Gamma(i\alpha + 1)} . t^{i\alpha} .$$

$$\Rightarrow u(x_1, x_2, \cdots, x_n, t) \cong P_M^\alpha(x_1, x_2, \cdots, x_n, t) = \sum_{i=1}^{M} \frac{((D_0^\alpha)^i q(x_1, x_2, \cdots, x_n) g(0)}{\Gamma(i\alpha + 1)} . t^{i\alpha}$$

Since

$$u(x_1, x_2, \cdots, x_n, 0) = q(x_1, x_2, \cdots, x_n) g(0),$$

$$u(x_1, x_2, \cdots, x_n, t) \cong P_M^\alpha(x_1, x_2, \cdots, x_n, t) = \sum_{i=1}^{M} \frac{((D_0^\alpha)^i \; u(x_1, x_2, \cdots, x_n, 0)}{\Gamma(i\alpha + 1)} . t^{i\alpha}$$

, (equation (51))

From theorem 4.1.1.4 (equation (19)),

$$R_M^\alpha(t) = \frac{((D_0^\alpha)^{M+1} g)(\xi)}{\Gamma((M+1)\alpha + 1)} . (t)^{(M+1)\alpha}$$

$$\Rightarrow q(x_1, x_2, \cdots, x_n) R_M^\alpha(t) = q(x_1, x_2, \cdots, x_n) \frac{((D_a^\alpha)^{M+1} g)(\xi)}{\Gamma((M+1)\alpha + 1)} . (t)^{(N+1)\alpha}$$

$$\Rightarrow R_M^\alpha(x_1, x_2, \cdots, x_n, t) = \frac{((D_a^\alpha)^{M+1} q(x_1, x_2, \cdots, x_n) g(\xi)}{\Gamma((M+1)\alpha + 1)} . (t)^{(M+1)\alpha}$$

$$\therefore R_M^\alpha(x_1, x_2, \cdots, x_n, t) = \frac{((D_0^\alpha)^{M+1} u)(x_1, x_2, \cdots, x_n, \xi)}{\Gamma((M+1)\alpha + 1)} . t^{(M+1)\alpha}$$ (Equation 52 holds true).

We must choose M large enough so that the error does not exceed a specified bound. From the next theorem, we find precise condition under which the exponents hold for arbitrary fractional operators. This result is very useful on our approach for solving partial differential equations in n-dimensions of fractional order.

Theorem 4.2.1.5 Suppose that $u(x_1,x_2,\cdots,x_n,t)=q(x_1,x_2,\cdots,x_n)g(\mathrm{t})=q(x_1,x_2,\cdots,x_n)l(\mathrm{t})t^{\lambda}$,

Where $g(\mathrm{t})=l(\mathrm{t})t^{\lambda}$, $\lambda>-1$ and $l(t)$ has the generalized Taylor's series $l(t)=\sum_{i=0}^{\infty}a_i t^{i\alpha}$ with radius of convergence $R>0$, $0<\alpha\le 1$. Then

$$D_0^{\gamma}D_0^{\beta}u(x_1,x_2,\cdots,x_n,\mathrm{t})=D_0^{\gamma+\beta}u(x_1,x_2,\cdots,x_n,\mathrm{t}) \tag{53}$$

, for $t\epsilon\ (0,R)$ if:

a) $\beta<\lambda+1$ and is arbitrary or

b) $\beta>\lambda+1$ and γ is arbitrary, and a_k for $k=0,1,2,\cdots,m-1<\beta\le m.$

Proof:

Let $u(x_1,x_2,\cdots,x_n,t)=q(x_1,x_2,\cdots,x_n)g(\mathrm{t})=q(x_1,x_2,\cdots,x_n)l(\mathrm{t})t^{\lambda}$,

Where $g(\mathrm{t})=l(\mathrm{t})t^{\lambda}$, $\lambda>-1$

Let $l(t)$ has the generalized Taylor's series $l(t)=\sum_{i=0}^{\infty}a_n t^{i\alpha}$ with radius of convergence $R>0$, $0<\alpha\le 1$.

a) In case of $\beta<\lambda+1,\ \beta-\lambda<1$,

$$D_0^{\gamma}D_0^{\beta}u(x_1,x_2\cdots x_n,t)=D_0^{\gamma}D_0^{\beta}(q(x_1,x_2,\cdots,x_n)g(\mathrm{t}))$$

$$=q(x_1,x_2,\cdots,x_n)D_0^{\gamma}D_0^{\beta}(g(\mathrm{t}))$$

$$=q(x_1,x_2,\cdots,x_n)D_0^{\gamma+\beta}(g(t))\text{ , by (22a)}$$

$$=D_0^{\gamma+\beta}[q(x_1,x_2,\cdots,x_n)g(t)]$$

$$=D_0^{\gamma+\beta}u(x_1,x_2\cdots x_n,t)$$

Thus,

$$D_0^{\gamma}D_0^{\beta}u(x_1,x_2\cdots x_n,t)=D_0^{\gamma+\beta}u(x_1,x_2\cdots x_n,t)$$

b) In case of $\beta > \lambda + 1,\ \beta - \lambda > 1$,

$$D_0^{\gamma} D_0^{\beta} u(x_1, x_2 \cdots x_n, t) = D_0^{\gamma} D_0^{\beta} (q(x_1, x_2, \cdots, x_n) g(\mathrm{t}))$$

$$= q(x_1, x_2, \cdots, x_n) D_0^{\gamma} D_0^{\beta} (g(\mathrm{t}))$$

$$= q(x_1, x_2, \cdots, x_n) D_0^{\gamma+\beta} (g(t)) \text{ , by (22b)}$$

$$= D_0^{\gamma+\beta} (q(x_1, x_2, \cdots, x_n) g(t)))$$

$$= D_0^{\gamma+\beta} u(x_1, x_2 \cdots x_n, t)$$

Thus,

$$D_0^{\gamma} D_0^{\beta} u(x_1, x_2 \cdots x_n, t) == D_0^{\gamma+\beta} u(x_1, x_2 \cdots x_n, t)$$

4.2.2 Reduced Differential Transform Method in n-Dimensions ($n \in \mathbb{N}$)

In this section, the Reduced Differential Transform Method [16] that is used to obtain analytic solutions for the initial value problems of homogeneous time fractional heat-like equations in n-dimensions, was newly introduced and discussed.

There is Caputo fractional derivatives of order α such that $0 < \alpha \le 1$ of $u(x_1, x_2, \cdots, x_n, t)$ with respect to time, t at a point $(x_1, x_2, \cdots, x_n, 0)$ in n-dimensions ($n \in \mathbb{N}$) in the initial value problems of homogeneous time fractional heat-like equations in n-dimensions. This makes that the reduced differential transform method can be used to obtain analytic solutions of initial value problems of homogeneous time fractional heat-like equations in n-dimensions. So, based on generalized Taylor's formula of Caputo time fractional derivative in n-dimensions ($n \in \mathbb{N}$) or equation (49), the reduced differential transform of the kth derivative of function $u(x_1, x_2, \cdots, x_n, \mathrm{t})$ with respect to time at $t_0 = 0$ denoted by $F_\alpha(0, 0, \cdots 0, k)\, or\, U_K(x_1, x_2, \cdots, x_n)$ and the reduced differential inverse transform of $F_\alpha(0, 0, \cdots, 0, k)\ or\, U_K(x_1, x_2, \cdots, x_n)$ denoted by $u(x_1, x_2, \cdots, x_n, \mathrm{t})$, where $u(x_1, x_2, \cdots, x_n, \mathrm{t})$ such that $u(x_1, x_2, \cdots, x_n, \mathrm{t}) = q(x_1, x_2, \cdots, x_n) g(t)$ is analytic and continuously differentiable with respect time t and $x_1, x_2, \cdots,$ and x_n in n-dimensional space in the domain of interest(which is closed set), $\Omega \subseteq \mathbb{R}^n$, were defined and introduced as follows:

Definition 4.2.2.1

If the function $u(x_1, x_2, \cdots, x_n, t)$ such that $u(x_1, x_2, \cdots, x_n, \mathrm{t}) = q(x_1, x_2, \cdots, x_n) g(t)$ is analytic and k-times differentiable continuously with respect to time t and variables $x_1, x_2, \cdots,$ and x_n in the domain of interest, then the t-dimensional spectrum function (the reduced transformed function) is denoted by $U_k(x_1, x_2, \cdots, x_n)$ or $F_\alpha(0,0,\cdots,0,k)$ and is defined as:

$$F_\alpha(0,0,\cdots,0,k) = U_k(x_1, x_2, \cdots, x_n) = \frac{1}{\Gamma(k\alpha+1)}\left[(D_{t_0}^{\alpha})^k u(x_1, x_2, \cdots, x_n, t)\right]_{t_0=0}$$

$$= \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x_1, x_2, \cdots, x_n, t)\right]_{t_0=0} \tag{54}$$

, where α such that $0 < \alpha \leq 1$ is a parameter describing the order of the time fractional derivative in Caputo sense and $U_K(x_1, x_2, \cdots, x_n) or F_\alpha(0,0,\cdots,0,k)$ is t-dimensional spectrum function of $u(x_1, x_2, \cdots, x_n, t)$.

Definition 4.2.2.2.The reduced differential inverses transform of $\mathrm{U_k}(x_1, x_2, \cdots, x_n)$ is denoted by $u(x_1, x_2, \cdots, x_n, t)$ and is defined as:

$$u(x_1, x_2, \cdots, x_n, t) = \sum_{k=0}^{\infty} \mathrm{U_k}(x_1, x_2, \cdots, x_n) \mathrm{t}^{\mathrm{K}\alpha} \tag{55}$$

, where α , such that $0 < \alpha \leq 1$ is a parameter describing the order of the time fractional derivative in Caputo sense

Substituting $\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x_1, x_2, \cdots, x_n, t)\right]_{t=0}$ from (54) in (55) using (51), one can obtain that:

$$u(x_1, x_2, \cdots, x_n, t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x_1, x_2, \cdots, x_n, t)\right]_{t=0} \mathrm{t}^{\mathrm{K}\alpha} \tag{55}$$

So, (55) is reduced differential inverses transform of (54). The mathematical operations (or theorems) performed by Reduced Differential Transform Method, which were deduced from equations (54) and (55), were introduced and stated by the researcher as follows.

Theorem 4.2.2.1 If $w(x_1, x_2, \cdots, x_n, t), u_1(x_1, t), u_2(x_2, t), \cdots, and\, u_n(x_n, t)$ be analytic and k-times continuously differentiable functions with respect to time t and $x_1, x_2, \cdots$ and x_n in the domain of interest, $\Omega \in \mathbb{R}^n$ which is closed set such that

$$w(x_1, x_2, \cdots, x_n, t) = u_1(x_1, t) \pm u_2(x_2, t) \pm \cdots \pm u_n(x_n, t)$$

then

$$W_k(x_1, x_2, \cdots, x_n) = U_{1k}(\mathrm{x}_1) \pm U_{2k}(\mathrm{x}_2) \pm \cdots \pm U_{nk}(\mathrm{x}_n).$$

Proof:

Le $w(x_1, x_2, \cdots, x_n, t), u_1(x_1, t), u_2(x_2, t), \cdots,$ and $u_n(x_n, t)$ are analytic and k-times continuously differentiable functions with respect to time t in the domain of interest, $\Omega \in \mathbb{R}^n$ which is closed set such $w(x_1, x_2, \cdots, x_n, t) = u_1(x_1, t) \pm u_2(x_2, t) \pm \cdots \pm u_n(x_n, t)$.

Let $W_k(x_1, x_2, \cdots, x_n), U_{1k}(\mathrm{x}_1), U_{2k}(\mathrm{x}_2), \cdots,$and $U_{nk}(\mathrm{x}_n)$ be t-dimensional spectrum functions of $w(x_1, x_2, \cdots, x_n, t), u_1(x_1, t), u_2(x_2, t), \cdots,andu_n(x_n, t)$respectively.

Now we want to show that $W_k(x_1, x_2, \cdots, x_n) = U_{1k}(\mathrm{x}_1) \pm U_{2k}(\mathrm{x}_2) \pm \cdots \pm U_{nk}(\mathrm{x}_n)$

Take reduced differential transform of both sides of

$$w(x_1, x_2, \cdots, x_n, t) = u_1(x_1, t) \pm u_2(x_2, t) \pm \cdots \pm u_n(x_n, t).$$

That is,

$$RDT\left[w(x_1, x_2, \cdots, x_n, t)\right] = RDT\left[u_1(x_1, t) \pm u_2(x_2, t) \pm \cdots \pm u_n(x_n, t)\right]$$

, where RDT denotes the reduced differential transform operator.

By definition 4.2.2.1 or equation (54),

$$RDT\left[w(x_1, x_2, \cdots, x_n, t)\right] = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} w(x_1, x_2, \cdots, x_n, t)\right]_{t_0=0}$$

$$= \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} [u_1(x_1, t) \pm u_2(x_1, t) \pm \cdots \pm u_n(x_1, t)]\right]_{t_0=0}.$$

$$=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u_1(x_1,t)\right]_{t_0=0}\pm\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u_2(x_1,t)\right]_{t=0}\pm\cdots\pm\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u_n(x_1,t)\right]_{t_0=0}.$$

Then

$W_k(x_1,x_2,\cdots,x_n)=U_{1k}(\mathrm{x}_1)\pm U_{2k}(\mathrm{x}_2)\pm\cdots\pm U_{nk}(\mathrm{x}_n)$, since

$$W_k(x_1,x_2,\cdots,x_n)=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}w(x_1,x_2,\cdots,x_n,t)\right]_{t_0=0}.$$

By definition 4.2.2.1 or equation (54),

$$U_{1k}(\mathrm{x}_1)=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u_1(x_1,t)\right]_{t_0=0},\ U_{2k}(\mathrm{x}_2)=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u_2(x_2,t)\right]_{t_0=0},\cdots,and$$

$$U_{nK}(x_n)=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u_n(x_n,t)\right]_{t_0=0}.$$

Thus,

$$w(x_1,x_2,\cdots,x_n,t)=u_1(x_1,t)\pm u_2(x_2,t)\pm\cdots\pm u_n(x_n,t)\Rightarrow U_{1k}(\mathrm{x}_1)\pm U_{2k}(\mathrm{x}_2)\pm\cdots\pm U_{nk}(\mathrm{x}_n).$$

Theorem 4.2.2.2 If $w(x_1,x_2,\cdots,x_n,t)=\alpha\, u(x_1,x_2,\cdots,x_n,t)$, then $W_k(x_1,x_2,\cdots,x_n,t)=\alpha U_k(x_1,x_2,\cdots,x_n)$.

Proof:

Let $w(x_1,x_2,\cdots,x_n,t)$ and $u(x_1,x_2,\cdots,x_n,t)$ be analytic and k-times continuously differentiable functions with respect to time t, and $x_1,x_2,\cdots,x_n$ in the domain of interest, $\Omega\in\mathbb{R}^n$ which is closed set such that $w(x_1,x_2,\cdots,x_n,t)=\beta u(x_1,x_2,\cdots,x_n,t)$, $\forall\beta$ which are constants and $k=1,2,\cdots$

Let $W_k(x_1,x_2,\cdots,x_n)$ and $U_k(x_1,x_2,\cdots,x_n)$ be t-dimensional spectrum function of $w(x_1,x_2,\cdots,x_n,t)$ and $u(x_1,x_2,\cdots,x_n,t)$ respectively.

Now we want to show that $W_k(x_1,x_2,\cdots,x_n)=\beta U_k(x_1,x_2,\cdots,x_n)$.

Then $RDT\left[w(x_1,x_2,\cdots,x_n,t)\right]=RDT\left[\beta u(x_1,x_2,\cdots,x_n,t)\right]$, where RDT denotes the reduced differential transform operator.

Then

$$,RDT[w(x_1,x_2,\cdots,x_n,t)]=\beta RDT[u(x_1,x_2,\cdots,x_n,t)]$$

By definition 4.2.2.1 or equation (54),

$$\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}w(x_1,x_2,\cdots,x_n,t)\right]_{t_0=0}=\beta\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x_1,x_2,\cdots,x_n,t)\right]_{t_0=0}.$$

But

$$W_k(x_1,x_2,\cdots,x_n)=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}w(x_1,x_2,\cdots,x_n,t)\right]_{t_0=0}\quad\text{, and}$$

$$U_k(x_1,x_2,\cdots,x_n)=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x_1,x_2,\cdots,x_n,t)\right]_{t_0=0}$$

Then

$$W_k(x_1,x_2,\cdots,x_n)=\beta U_k(x_1,x_2,\cdots,x_n).$$

$$w(x_1,x_2,\cdots,x_n,t)=\beta u(x_1,x_2,\cdots,x_n,t)\Rightarrow W_k(x_1,x_2,\cdots,x_n)=\beta U_k(x_1,x_2,\cdots,x_n).$$

Therefore,

if $w(x_1,x_2,\cdots,x_n,t)=\beta u(x_1,x_2,\cdots,x_n,t)$, then $W_k(x_1,x_2,\cdots,x_n)=\beta U_k(x_1,x_2,\cdots,x_n)$

Theorem 4.2.2.3 If $w(x_1,x_2,\cdots,x_n,t)=\frac{\partial^n}{\partial t^n}u(x_1,x_2,\cdots,x_n,t)$, then

$$W_k(x_1,x_2,\cdots,x_n)=(k+1)(k+2)\cdots(k+n)U_{k+n}(x_1,x_2,\cdots,x_n).$$

Proof:

Let $w(x_1,x_2,\cdots,x_n,t)$ and $u(x_1,x_2,\cdots,x_n,t)$ be analytic and k-times continuously differentiable functions with respect to time t and the variables $x_1,x_2,\cdots$ and x_n in the domain of interest, $\Omega\in\mathbb{R}^n$ which is closed set such that

$$w(x_1,x_2,\cdots,x_n,t)=\frac{\partial^n}{\partial t^n}u(x_1,x_2,\cdots,x_n,t)\text{, where } k=1,2,\cdots$$

Let $W_k(x_1,x_2,\cdots,x_n)$ and $U_k(x_1,x_2,\cdots,x_n)$ be t-dimensional spectrum function of $w(x_1,x_2,\cdots,x_n,t)$ and $u(x_1,x_2,\cdots,x_n,t)$ respectively.

By definition 4.2.2.1 or equation (54), we get

$$RDT\left[w(x_1,x_2,\cdots,x_n,t)\right]=W_K(x_1,x_2,\cdots,x_n)=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}w(x_1,x_2,\cdots,x_n,t)\right]_{t=0},$$

Where RDT denotes the reduced differential transform operator.

Since

$$w(x_1,x_2,\cdots,x_n,t)=\frac{\partial^{\mathrm{n}}}{\partial \mathrm{t}^{\mathrm{n}}}u(x_1,x_2,\cdots,x_n,t),$$

$$W_K(x_1,x_2,\cdots,x_n)=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\left(\frac{\partial^{\mathrm{n}}}{\partial \mathrm{t}^{\mathrm{n}}}u(x_1,x_2,\cdots,x_n,t)\right)\right]_{t=0}$$

For $\alpha=1$, $W_K(x_1,x_2,\cdots,x_n)=\frac{1}{\mathrm{k}!}\left[\frac{\partial^{k}}{\partial t^{k}}\left(\frac{\partial^{\mathrm{n}}}{\partial \mathrm{t}^{\mathrm{n}}}u(x_1,x_2,\cdots,x_n,t)\right)\right]_{t=0}$

$$W_K(x_1,x_2,\cdots,x_n)=\frac{1}{\mathrm{k}!}\left[\frac{\partial^{k}}{\partial t^{k}}\left(\frac{\partial^{\mathrm{n}}}{\partial \mathrm{t}^{\mathrm{n}}}u(x_1,x_2,\cdots,x_n,t)\right)\right]_{t=0}$$

$$W_K(x_1,x_2,\cdots,x_n)=\frac{1}{\mathrm{k}!}\left[\frac{\partial^{k+n}}{\partial t^{k+n}}\left(\frac{\partial^{\mathrm{n}}}{\partial \mathrm{t}^{\mathrm{n}}}u(x_1,x_2,\cdots,x_n,t)\right)\right]_{t=0}$$

$$W_K(x_1,x_2,\cdots,x_n)=\frac{(k+n)!}{\mathrm{k}!(k+n)!}\left[\left(\frac{\partial^{\mathrm{k+n}}}{\partial \mathrm{t}^{\mathrm{k+n}}}u(x_1,x_2,\cdots,x_n,t)\right)\right]_{t=0}$$

$$W_K(x_1,x_2,\cdots,x_n)=\frac{(k+1)(k+2)\cdots(k+n)k!}{\mathrm{k}!(k+n)!}\left[\left(\frac{\partial^{\mathrm{k+n}}}{\partial \mathrm{t}^{\mathrm{k+n}}}u(x_1,x_2,\cdots,x_n,t)\right)\right]_{t=0}$$

$$W_K(x_1,x_2,\cdots,x_n)=(k+1)(k+2)\cdots(k+n)\left(\frac{1}{(k+n)!}\left[\left(\frac{\partial^{\mathrm{k+n}}}{\partial \mathrm{t}^{\mathrm{k+n}}}u(x_1,x_2,\cdots,x_n,t)\right)\right]_{t=0}\right)$$

By definition 4.2.2.1 or equation (54),

$$W_K(x_1,x_2,\cdots,x_n)=(k+1)(k+2)(K+3)\cdots(k+n)U_{k+n}(x_1,x_2,\cdots,x_n).$$

Thus,

$$\mathrm{w}(x_1,x_2,\cdots,x_n,\mathrm{t})=\frac{\partial^{\mathrm{n}}}{\partial \mathrm{t}^{\mathrm{n}}}\mathrm{u}(x_1,x_2,\cdots,x_n,\mathrm{t})$$

$$\Rightarrow W_K(x_1,x_2,\cdots,x_n)=(k+1)(k+2)\cdots(k+n)U_{k+n}(x_1,x_2,\cdots,x_n).$$

Therefore; $\mathrm{w}(x_1,x_2,\cdots,x_n,\mathrm{t})=\frac{\partial^{\mathrm{n}}}{\partial \mathrm{t}^{\mathrm{n}}}\mathrm{u}(x_1,x_2,\cdots,x_n,\mathrm{t})$, then

$$W_K\left(x_1,x_2,\cdots,x_n\right)=\left(k+1\right)\left(k+2\right)\left(K+3\right)\cdots\left(k+n\right)U_{k+n}\left(x_1,x_2,\cdots,x_n\right)$$

Theorem 4.2.2.4 If $w\left(x_1,x_2,\cdots,x_n,t\right)=\frac{\partial^{\mathrm{N\alpha}}}{\partial \mathrm{t}^{\mathrm{N\alpha}}}u\left(x_1,x_2,\cdots,x_n,t\right)$, then

$$W_k\left(x\right)=\frac{\Gamma(K\alpha+N\alpha+1)}{\Gamma(K\alpha+1)}\mathrm{U}_{\mathrm{K+N}}\left(x_1,x_2,\cdots,x_n\right).$$

Proof:

Let $w\left(x_1,x_2,\cdots,x_n,t\right)$ and $u\left(x_1,x_2,\cdots,x_n,t\right)$ be analytic and k-times continuously differentiable functions with respect to time t, and the variables: $x_1,x_2,\cdots$ and x_n in the domain of interest, $\Omega\in\mathbb{R}^n$ which is closed set such that $w\left(x_1,x_2,\cdots,x_n,t\right)=\frac{\partial^{\mathrm{N\alpha}}}{\partial \mathrm{t}^{\mathrm{N\alpha}}}u\left(x_1,x_2,\cdots,x_n,t\right)$, where $k=1,2,\cdots$

Let $W_k\left(x_1,x_2,\cdots,x_n\right)$ and $U_k(x_1,x_2,\cdots,x_n)$ be t-dimensional spectrum function of $w\left(x_1,x_2,\cdots,x_n,t\right)$ and $u(x_1,x_2,\cdots,x_n,t)$ respectively.

By definition 4.2.2.1 or equation (54), we get

$$RDT\left[w\left(x_1,x_2,\cdots,x_n,t\right)\right]=W_K\left(x_1,x_2,\cdots,x_n\right)=\frac{1}{\Gamma\left(k\alpha+1\right)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}w\left(x_1,x_2,\cdots,x_n,t\right)\right]_{t_0=0}$$

Where RDT denotes the reduced differential transform operator.

But $w\left(x_1,x_2,\cdots,x_n,t\right)=\frac{\partial^{\mathrm{N\alpha}}}{\partial \mathrm{t}^{\mathrm{N\alpha}}}u\left(x_1,x_2,\cdots,x_n,t\right)$

Then $W_K(x_1,x_2,\cdots,x_n)=\frac{1}{\Gamma\left(k\alpha+1\right)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\left(\frac{\partial^{\mathrm{N\alpha}}}{\partial \mathrm{t}^{\mathrm{N\alpha}}}u\left(x_1,x_2,\cdots,x_n,t\right)\right)\right]_{t_0=0}$

By multiplying the right hand side with

$$\frac{\Gamma\left(K\alpha+N\alpha+1\right)}{\Gamma\left(K\alpha+N\alpha+1\right)},\ W_K\left(x_1,x_2,\cdots,x_n\right)=\frac{\Gamma\left(K\alpha+N\alpha+1\right)}{\Gamma\left(k\alpha+1\right)\Gamma\left(K\alpha+N\alpha+1\right)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\left(\frac{\partial^{\mathrm{N\alpha}}}{\partial \mathrm{t}^{\mathrm{N\alpha}}}u\left(x_1,x_2,\cdots,x_n,t\right)\right)\right]_{t_0=0}$$

$$W_K(x)=\frac{\Gamma\left(K\alpha+N\alpha+1\right)}{\Gamma\left(k\alpha+1\right)}\left(\frac{1}{\Gamma\left(K\alpha+N\alpha+1\right)}\left[\frac{\partial^{k\alpha+\mathrm{N\alpha}}}{\partial t^{k\alpha+\mathrm{N\alpha}}}\left(u\left(x_1,x_2,\cdots,x_n,t\right)\right)\right]_{t_0=0}\right)$$

Since $U_{k+N}\left(x_1,x_2,\cdots,x_n\right)=\frac{1}{\Gamma\left(K\alpha+N\alpha+1\right)}\left[\frac{\partial^{k\alpha+\mathrm{N\alpha}}}{\partial t^{k\alpha+\mathrm{N\alpha}}}\left(u\left(x_1,x_2,\cdots,x_n,t\right)\right)\right]_{t_0=0}$

, by definition 4.2.2.1 or equation (54), $W_K\left(x_1,x_2,\cdots,x_n\right)=\frac{\Gamma\left(K\alpha+N\alpha+1\right)}{\Gamma\left(k\alpha+1\right)}U_{k+N}\left(x_1,x_2,\cdots,x_n\right).$

Thus,

$$\mathrm{w}\left(x_1,x_2,\cdots,x_n,\mathrm{t}\right)=\frac{\partial^{\mathrm{N}\alpha}}{\partial \mathrm{t}^{\mathrm{N}\alpha}}u\left(x_1,x_2,\cdots,x_n,t\right)$$
$$\Rightarrow W_K\left(x_1,x_2,\cdots,x_n\right)=\frac{\Gamma\left(K\alpha+N\alpha+1\right)}{\Gamma\left(k\alpha+1\right)}U_{k+N}\left(x_1,x_2,\cdots,x_n\right).$$

Therefore, $\mathrm{w}\left(x_1,x_2,\cdots,x_n,\mathrm{t}\right)=$

$\frac{\partial^{\mathrm{N}\alpha}}{\partial \mathrm{t}^{\mathrm{N}\alpha}}u\left(x_1,x_2,\cdots,x_n,t\right)$, then $W_K\left(x_1,x_2,\cdots,x_n\right)=\frac{\Gamma\left(K\alpha+N\alpha+1\right)}{\Gamma\left(k\alpha+1\right)}U_{k+N}(x_1,x_2,\cdots,x_n)$

Theorem 4.2.2.5 If $w\left(x_1,x_2,\cdots,x_n,t\right)=g(x_1,x_2,\cdots,x_n)t^n$, then

$$W_k\left(g(x_1,x_2,\cdots,x_n)t^n\right)=g\left(x_1,x_2,\cdots,x_n\right)\delta\left(k-n\right)=\begin{cases} g\left(x_1,x_2,\cdots,x_n\right), & if\ \delta\left(k-n\right)=1, where\, k=n \\ 0, if\ \delta\left(k-n\right)=0, & where\, k\neq n \end{cases}.$$

Proof:

From definition 4.2.2.1,

$w\left(x_1,x_2,\cdots,x_n,t\right)=g\left(x_1,x_2,\cdots,x_n\right)t^n$, can be written as

$w\left(x_1,x_2,\cdots,x_n,t\right)=\sum_{k=0}^{\infty}g\left(x_1,x_2,\cdots,x_n\right)\delta\left(k-n\right)t^k$, where $n=n\alpha$ for $\alpha=1$,

So, from the definition of reduced differential inverse transform (4.2.2.2) or equation (55), we have

$$RDT^{-1}(w\left(x_1,x_2,\cdots,x_n,t\right))=RDT^{-1}(\sum_{k=0}^{\infty}g\left(x_1,x_2,\cdots,x_n\right)\delta\left(k-n\right)t^k).$$

Then

$$W_K\left(x_1,x_2,\cdots,x_n\right)=g\left(x_1,x_2,\cdots,x_n\right)\delta\left(k-n\right)$$

$$= \begin{cases} g(x_1, x_2, \cdots, x_n), \ if\ \delta(k-n) = 1, where\ k = n \\ 0, \ if\ \delta(k-n) = 0, \ where\ k \neq n \end{cases}$$

Hence, the theorem holds true.

Theorem 4.2.2.6

If $w(x_1, x_2, \cdots, x_n, t) = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} u(x_1, x_2, \cdots, x_n, t)$, then $W_k(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \mathrm{U_K}(x_1, x_2, \cdots, x_n)$,

$i = 1, 2, \cdots, n.$

Proof:

Let $w(x_1, x_2, \cdots, x_n, t)$ *and* $u(x_1, x_2, \cdots, x_n, t)$ be analytic and k-times continuously differentiable functions with respect to time t and variables: $x_1, x_2, \cdots$ and x_n in the domain of interest, $\Omega \in \mathbb{R}^n$ which is closed set such that $w(x_1, x_2, \cdots, x_n, t) = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} u(x_1, x_2, \cdots, x_n, t)$, where $k = 1, 2, \cdots$

Let $W_k(x_1, x_2, \cdots, x_n)$ and $U_k(x_1, x_2, \cdots, x_n$ be t-dimensional spectrum function of $w(x_1, x_2, \cdots, x_n, t)$ and $u(x_1, x_2, \cdots, x_n, t)$ respectively.

By definition 4.2.2.1 or equation (54), we get

$$RDT\left[w(x_1, x_2, \cdots, x_n, t)\right] = W_K(x_1, x_2, \cdots, x_n) = \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} w(x_1, x_2, \cdots, x_n, t)\right]_{t_0=0}$$

Where RDT denotes the reduced differential transform operator

Since $w(x_1, x_2, \cdots, x_n, t) = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} u(x_1, x_2, \cdots, x_n, t)$,

$$W_K(x_1, x_2, \cdots, x_n) = \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}} \left(\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} u(x_1, x_2, \cdots, x_n, t)\right)\right]_{t_0=0}$$

$$W_K\left(x_1, x_2, \cdots, x_n\right)=\sum_{i=1}^{n} \frac{\partial^2}{\partial \chi_i^{\,2}}\left(\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\left(u\left(x_1, x_2, \cdots, x_n, t\right)\right)\right]_{t_0=0}\right)$$

But from definition 4.2.2.1,

$$U_k(x_1, x_2, \cdots, x_n)=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\left(u\left(x_1, x_2, \cdots, x_n, t\right)\right)\right]_{t_0=0}$$

Then

$$W_K\left(x_1, x_2, \cdots, x_n\right)=\sum_{i=1}^{n} \frac{\partial^2}{\partial \chi_i^{\,2}} U_k(x_1, x_2, \cdots, x_n)$$

Thus,

$$w\left(x_1, x_2, \cdots, x_n, t\right)=\sum_{i=1}^{n} \frac{\partial^2}{\partial \chi_i^{\,2}} u\left(x_1, x_2, \cdots, x_n, t\right) \Rightarrow W_k\left(x\right)=\sum_{i=1}^{n} \frac{\partial^2}{\partial \chi_i^{\,2}} \mathrm{U}_{\mathrm{K}}\left(x_1, x_2, \cdots, x_n\right).$$

If $w\left(x_1, x_2, \cdots, x_n, t\right)=\sum_{i=1}^{n} \frac{\partial^2}{\partial \chi_i^{\,2}} u\left(x_1, x_2, \cdots, x_n, t\right)$, *then* $W_k\left(x_1, x_2, \cdots, x_n\right)=\sum_{i=1}^{n} \frac{\partial^2}{\partial \chi_i^{\,2}} \mathrm{U}_{\mathrm{K}}\left(x_1, x_2, \cdots, x_n\right)$

As a result, using definitions 4.2.2.1 and 4.2.2.2, and mathematical operations (theorems) 4.2.2.1, 4.2.2.2, 4.2.2.3, 4.2.2.4, 4.2.2.5 and 4.2.2.6, the Reduced Differential Transform Method [16] procedures for solving initial value problems of one, two, three and more than three dimensional homogeneous time fractional heat-like equations, where $n \in \mathbb{N}$ were newly developed and explained separately by the researcher as in the following four sub-sections.

4.2.2.1 Reduced Differential Transform Method Procedures for Solving One Dimensional Homogeneous Time Fractional Heat-like Equations

Under this sub-section, Reduced Differential Transform Method procedures for solving one dimensional homogeneous time fractional heat-like equations of the following form was newly developed and introduced.

1. Take one dimensional homogeneous time fractional heat-like equations of the form:

$$\frac{\partial^\alpha u\left(x_1, t\right)}{\partial t^\alpha}=f_1\left(x_1\right) u_{x_1 x_1}, \chi_1 \in \Omega \subseteq \mathbb{R}, t>0, 0<\alpha \leq 1, u_{x_1 x_1}=\frac{\partial^2 u}{\partial x_1^2} \tag{57}$$

Subject to initial condition:

$$u\left(x_1, 0\right)=g_1\left(x_1\right), \chi_1 \in \Omega \subseteq \mathbb{R} \tag{58}$$

Where Ω is closed set

2. Apply reduced differential transform in one dimension to both sides of each of equations (57) and (58). That is,

$$RDT\left[\frac{\partial^{\alpha}u(x_1,t)}{\partial t^{\alpha}}\right]=RDT\left[f_1(x_1)u_{x_1x_1}\right],x_1\in\Omega\subseteq\mathbb{R},t>0\ ,0<\alpha\leq 1 \qquad (59)$$

$$RDT[u_0(x_1,0)]=RDT\left[g_1(x_1)\right],\quad x_1\in\Omega\subseteq\mathbb{R}, \qquad (60)$$

Where Ω is closed set

By theorem 4.2.2.4 for

N=1, $RDT\left[\frac{\partial^{\alpha}u(x_1,t)}{\partial t^{\alpha}}\right]=\frac{\Gamma(K\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1}(x_1)$

By theorems 4.2.2.5 and 4.2.2.6,

$$RDT\left[f_1(x_1)u_{x_1x_1}\right]=f_1(x_1)\frac{\partial^2}{\partial x_1{}^2}U_k(x_1)$$

By theorem 4.2.2.5,

$$RDT\left[u(x_1,0)\right]=U_0(x_1)=g_1(x_1)$$

3. After applying definition 4.2.2.1 in one dimension for n=1 and theorems 4.2.2.1−4.2.2.6 in one dimension $(for\, n=1)$ to equations (59) and(60) ,we can obtain iteration formulae:

$$\frac{\Gamma(K\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1}(x_1)=f_1(x_1)\frac{\partial^2}{\partial x_1{}^2}U_k(x_1),x_1\in\Omega\subseteq R,t>0\ ,0<\alpha\leq 1 \qquad (61)$$

$$U_0(x_1)=g_1(x_1),\Omega\subseteq\mathbb{R} \qquad (62)$$

Where Ω is closed set

4. Substituting equation (62) in equation (61) successively by straight forward iteration, we can obtain the $U_k(x_1)$ values, i.e. $U_1(x_1), U_2(x_1), U_3(x_1),\cdots$ values $\forall k=1,2,3,\cdots$ in one dimension.

5. Taking the reduced differential inverse transform of the set of values of $U_k(x_1)$,

$\{U_k(x_1)$: k=0,1,2,$\cdots\}$ by using definition 4.2.2.1 (equation (54)) in one dimension $\forall k = 0,1,2,\cdots$, we can obtain analytic solution of the problem, which can be given in the form of equations (57) and (58) ,as:

$$u(x,t) = U_0(x_1) + U_1(x_1)t^{\alpha} + U_2(x_1)t^{2\alpha} + U_3(x_1)t^{3\alpha}, \cdots \forall k = 1,2,3,\cdots \qquad (63)$$

in infinite power series (open) form. Taking the special case (α=1), the exact solution of the problem which can be given in the form of equations (57) and (58) can be obtained from equation (63) in closed form.

4.2.2.2 Reduced Differential Transform Method Procedures for Solving Two Dimensional Homogeneous Time Fractional Heat-like Equations

Under this sub-section, Reduced Differential Transform Method procedures for solving two dimensional homogeneous time fractional heat-like equations was newly developed and explained as follows.

1. Take two dimensional time fractional homogeneous heat-like equations of the forms:
$$\frac{\partial^{\alpha} u(x_1,x_2,t)}{\partial t^{\alpha}} = f_1(x_1,x_2)u_{x_1x_1} + f_2(x_1,x_2)u_{x_2x_2}, (x_1,x_2) \in \Omega \subseteq \mathbb{R}^2, t > 0, 0 < \alpha \le 1 \quad (64)$$
Subject to the initial condition:
$$\mathrm{u}(x_1,x_2,0) = g_1(x_1,x_2),\ (x_1,x_2) \in \Omega \subseteq \mathbb{R}^2 \qquad (65)$$
Where Ω is closed set
2. Apply reduced differential transform in two dimensions to both sides of each of equations (64) and (65). That is,

$$RDT\left[\frac{\partial^{\alpha} u(x_1,x_2,t)}{\partial t^{\alpha}}\right] = R_D\left[f_1(x_1,x_2)u_{x_1x_1}\right] + R_D\left[f_2(x_1,x_2)u_{x_2x_2}\right] \qquad (66)$$

$$RDT\left[u(x_1,x_2,0)\right] = RDT\left[g_1(x_1,x_2)\right] \qquad (67)$$

By theorem 4.2.2.4 in two dimensions for

$$N=1,\ RDT\left[\frac{\partial^{\alpha}u(x_1,x_2,t)}{\partial t^{\alpha}}\right]=\frac{\Gamma(K\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1}(x_1,x_2)$$

By theorems 4.2.2.5 and 4.2.2.6 in two dimensions,

$$RDT\left[f_1(x_1,x_2)u_{x_1x_1}\right]=f_1(x_1,x_2)\frac{\partial^2}{\partial x_1{}^2}U_k(x_1,x_2)$$

By theorems 4.2.2.5, $R_D\left[u(x_1,x_2,0)\right]=U_0(x_1,x_2)=g_1(x_1,x_2)$

3. After applying definition 4.2.2.1 in one dimension (for n=1)and theorems $4.2.2.1-4.2.2.6$ in two dimensions $(for\, n=1)$ to equations (66) and (67), we can obtain iteration formulae:

$$\frac{\Gamma(K\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1}(x_1,x_2)=f_1(x_1)\frac{\partial^2}{\partial x_1{}^2}U_k(x_1,x_2)+f_2(x_1,x_2)\frac{\partial^2}{\partial x_2{}^2}U_k(x_1,x_2),$$
$$(x_1,x_2)\in\Omega\subseteq\mathbb{R}^2,t>0,0<\alpha\le 1 \qquad (68)$$

$$U_0(x_1,x_2)=g_1(x_1,x_2),\ (x_1,x_2)\in\Omega\subseteq\mathbb{R}^2 \qquad (69)$$

Where Ω is closed set

4. Substituting equation (69) in equation (68) successively by straight forward iteration, we can obtain the $U_k(x_1,x_2)$ values, i.e. $U_1(x_1,x_2),U_2(x_1,x_2),U_3(x_1,x_2),\cdots$ values $\forall k=1,2,3,\cdots$ in two dimensions.
5. Taking the reduced differential inverse transform of the set of values of $U_k(x_1,x_2)$, $\{U_k(x_1,x_2):$ k= $0,1,2,\cdots\}$ by using definition 4.2.2.2 (equation (55)) in two dimensions $\forall k=0,1,2,\cdots$, we can obtain the analytic solution of the problem which can be given in the form of equations$(64)\,and\ (65)$ as:

$$u(x_1,x_2,t)=U_0(x_1,x_2)+U_1(x_1,x_2)t^{\alpha}+U_2(x_1,x_2)t^{2\alpha}+U_3(x_1,x_2)t^{3\alpha},\cdots\forall k=1,2,\cdots \qquad (70)$$

in infinite power series (open form).Taking the special case (α=1), the exact solution of the problem which can be given in the form of equations (64) and (65) can be obtained from equation(70) in closed form.

4.2.2.3 Reduced Differential Transform Method Procedures for Solving Three Dimensional Homogeneous Time Fractional Heat-like Equations

Under this sub-section, Reduced Differential Transform Method Procedures for solving three dimensional homogeneous time fractional heat-like equations of the following form was newly developed and explained.

1. Consider three dimensional time fractional homogeneous heat-like equations of the form:

$$\frac{\partial^\alpha u(x_1,x_2,x_3,t)}{\partial t^\alpha} = f_1(x_1,x_2,x_3)u_{x_1x_1} + f_2(x_1,x_2,x_3)u_{x_2x_2} + f_3(x_1,x_2,x_3)u_{x_3x_3},$$
$$(x_1,x_2,x_3) \in \Omega \subseteq \mathbb{R}^3, t>0, 0<\alpha \le 1 \qquad (71)$$

Subject to the initial condition:

$$u(x_1,x_2,x_3,0) = g_1(x_1,x_2,x_3),\ (x_1,x_2,x_3) \in \Omega \subseteq \mathbb{R}^3 \qquad (72)$$

Where Ω is closed set

2. Apply reduced differential transform in three dimensions to both sides of each of equations (71) and (72). That is,

$$RDT\left[\frac{\partial^\alpha u(x_1,x_2,x_3,t)}{\partial t^\alpha}\right] = RDT\left[f_1(x_1,x_2,x_3)u_{x_1x_1}\right] + RDT\left[f_2(x_1,x_2,x_3)u_{x_2x_2}\right] +$$
$$RDT\left[f_3(x_1,x_2,x_3)u_{x_3x_3}\right] \qquad (73)$$
$$RDT[u(x_1,x_2,x_3,0)] = RDT[g_1(x_1,x_2,x_3)] \qquad (74)$$

By theorem 4.2.2.4 in 3-dimensions for N=1,

$$RDT\left[\frac{\partial^\alpha u(x_1,x_2,x_3,t)}{\partial t^\alpha}\right] = \frac{\Gamma(K\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1}(x_1,x_2,x_3)$$

By theorems 4.2.2.5 and 4.2.2.6 in 3-dimensions,

$$RDT\left[f_2(x_1,x_2,x_3)u_{x_2x_2}\right] = f_2(x_1,x_2,x_3)\frac{\partial^2}{\partial x_2{}^2}U_k(x_1,x_2,x_3)$$

By theorems 4.2.2.6, $RDT\left[u(x_1,x_2,x_3,0)\right] = U_0(x_1,x_2,x_3) = g_1(x_1,x_2,x_3)$

3. After applying definition 4.2.2.1 in three dimensions(n=3) and theorems 4.2.2.1−4.2.2.6 in three dimensions (for n=3) to equations (73) and (74), we obtain iteration formulae:

$$\frac{\Gamma(K\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1}(x_1,x_2,x_3)=f_1(x_1,x_2,x_3)\frac{\partial^2}{\partial x_1^{\,2}}U_k(x_1,x_2,x_3)+f_2(x_1,x_2,x_3)\frac{\partial^2}{\partial x_2^{\,2}}U_k(x_1,x_2,x_3)+$$

$$f_3(x_1,x_2,x_3)\frac{\partial^2}{\partial x_3^{\,2}}U_k(x_1,x_2,x_3),\ (x_1,x_2,x_3)\in\Omega\subseteq\mathbb{R}^3, t>0, 0<\alpha\le 1 \qquad (75)$$

$$U_0(x_1,x_2,x_3)=g_1(x_1,x_2,x_3),\ (x_1,x_2,x_3)\in\Omega\subseteq\mathbb{R}^3 \qquad (76)$$

Where Ω is closed set

4. Substituting equation (76) in equation (75) successively by straight forward iteration, we can obtain the $U_k(x_1,x_2,x_3)$ values, i.e. the values of,

$U_1(x_1,x_2,x_3)$, $U_2(x_1,x_2,x_3)$, $U_3(x_1,x_2,x_3),\cdots$ $\forall k=1,2,3,\cdots$ in three dimensions.

5. Taking the reduced differential inverse transform of the set of values of $U_k(x_1,x_2,x_3)$, $\{U_k(x_1,x_2,x_3)$: k= $0,1,2,\cdots\}$ by using definition 4.2.2.2 (equation (55)) in one dimension $\forall k=0,1,2,\cdots$, we can obtain analytic solution of the problem which can be given in the form of equations(71) and (72) as:

$$u(x_1,x_2,x_3,t)=U_0(x_1,x_2,x_3)+U_1(x_1,x_2,x_3)t^{\alpha}+U_2(x_1,x_2,x_3)t^{2\alpha}+U_3(x_1,x_2,x_3)t^{3\alpha},\cdots\forall k=1,2,3,\cdots \quad (77)$$

in infinite power series (open form).Taking the special case (α=1), the exact solution of the problem which can be given in the form of equations (71) and (72) can be obtained from equation (77) in closed form.

4.2.2.4 Reduced Differential Transform Method Procedures for Solving n-dimensional Time Fractional Homogeneous Heat-like Equations, where $n>3\, and\, n\in\mathbb{N}$

Under this sub-section, Reduced Differential Transform Method Procedures, which is used in ideal world (but not in the physical world), for solving more than three dimensional initial value problems of time fractional homogeneous heat-like equations of the following form was newly developed and explained .

1. Consider n-dimensional time fractional homogeneous heat-like equations of the form:

$$\frac{\partial^{\alpha}u(x_1,x_2,\cdots,x_n\ ,t)}{\partial t^{\alpha}}=f_1(x_1,x_2,\cdots,x_n)u_{x_1x_1}+f_2(x_1,x_2,\cdots,x_n)u_{x_2x_2}+\cdots$$

$$+f_n(x_1,x_2,\cdots,x_n)u_{x_nx_n},(x_1,x_2,\cdots,x_n)\in\Omega\subseteq\mathbb{R}^n\ ,t>0,\ 0<\alpha\le 1 \qquad (78)$$

Subject to the initial condition:

$$u(x_1,x_2,\cdots,x_n,0)=g_1(x_1,x_2,\cdots,x_n),(x_1,x_2,\cdots,x_n)\in\Omega\subseteq\mathbb{R}^n \quad (79)$$

where $n>3$ and $n\in N$, and Ω is closed set.

2. Apply Reduced Differential Transform to both sides of each of equations (78) and (79) in n-dimensions. That is,

$$RDT\left[\frac{\partial^\alpha u(x_1,x_2,\cdots,x_n,t)}{\partial t^\alpha}\right]=RDT\left[f_1(x_1,x_2,\cdots,x_n)u_{x_1x_1}\right]+RDT\left[f_2(x_1,x_2,\cdots,x_n)u_{x_2x_2}\right]+\cdots+$$

$$RDT\left[f_n(x_1,x_2,\cdots,x_n)u_{x_nx_n}\right], \quad (80)$$

$$RDT\left[u(x_1,x_2,\cdots,x_n,0)\right]=RDT\left[g_1(x_1,x_2,\cdots,x_n)\right] \quad (81)$$

By theorem 4.2.2.4 in n-dimensions where $n>3$ and $n\in\mathbb{N}$ for N=1,

$$RDT\left[\frac{\partial^\alpha u(x_1,x_2,\cdots,x_n,t)}{\partial t^\alpha}\right]=\frac{\Gamma(K\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1}(x_1,x_2,\cdots,x_n)$$

By theorems 4.2.2.5 and 4.2.2.6 in n-dimensions, where $n>3$ and $n\in\mathbb{N}$

$$RDT\left[f_1(x_1,x_2,\cdots,x_n)u_{x_1x_1}\right]=f_1(x_1,x_2,\cdots,x_n)\frac{\partial^2}{\partial x_1{}^2}U_k(x_1,x_2,\cdots,x_n).$$

By theorems 4.2.2.5 and 4.2.2.6 in n-dimensions where $n>3$ and $n\in\mathbb{N}$,

$$RDT\left[f_2(x_1,x_2,\cdots,x_n)u_{x_2x_2}\right]=f_2(x_1,x_2,\cdots,x_n)\frac{\partial^2}{\partial x_2{}^2}U_k(x_1,x_2,\cdots,x_n)$$

By theorems 4.2.2.5 and 4.2.2.6 in n-dimensions, where $n>3$ and $n\in\mathbb{N}$,

$$RDT\left[f_3(x_1,x_2,\cdots,x_n)u_{x_3x3}\right]=f_3(x_1,x_2,\cdots,x_n)\frac{\partial^2}{\partial x_3{}^2}U_k(x_1,x_2,\cdots,x_n)$$

$$\vdots$$

$$\vdots$$

By theorems 4.2.2.5 and 4.2.2.6 in n-dimensions, where $n>3$ and $n\in\mathbb{N}$,

$$RDT\left[f_n(x_1,x_2,\cdots,x_n)u_{x_nx_n}\right]=f_n(x_1,x_2,\cdots,x_n)\frac{\partial^2}{\partial x_n{}^2}U_k(x_1,x_2,\cdots,x_n)$$

3. After applying definition 4.2.2.1 in n-dimensions, where $n > 3$ and $n \in \mathbb{N}$,and theorems 4.2.2.1−4.2.2.6 in n-dimensions, where $n > 3$ and $n \in \mathbb{N}$, to equations $(80)\, and\, (81)$, we can obtain iteration formulae:

$$\frac{\Gamma(K\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1}(x_1,x_2,\cdots,x_n) = f_1(x_1,x_2,\cdots,x_n)\frac{\partial^2}{\partial x_1^{\ 2}}U_k(x_1,x_2,\cdots,x_n)+$$
$$f_2(x_1,x_2,\cdots,x_n)\frac{\partial^2}{\partial x_2^{\ 2}}U_k(x_1,x_2,\cdots,x_n)+\cdots+f_n(x_1,x_2,\cdots,x_n)\frac{\partial^2}{\partial x_n^{\ 2}}U_k(x_1,x_2,\cdots,x_n),$$
$$(x_1,x_2,\cdots,x_n)\in\Omega\subseteq\mathbb{R}^n\ ,t>0,\ 0<\alpha\le 1 \qquad (82)$$
$$U_0(x_1,x_2,\cdots,x_n) = g_1(x_1,x_2,\cdots,x_n), \qquad (x_1,x_2,\cdots,x_n)\in\Omega\subseteq\mathbb{R}^n \qquad (83)$$

Where Ω is closed set

4. Substituting equation (83) in equation (82) successively by using straight forward iteration, we can obtain the $U_k(x_1,x_2,\cdots,x_n)$ values, i.e. $U_1(x_1,x_2,\cdots,x_n),\ U_2(x_1,x_2,\cdots,x_n), U_3(x_1,x_2,\cdots,x_n),\cdots values\,\forall k=1,2,3,\cdots$ in n-dimensions, where $n \in \mathbb{N}$.

5. Taking the reduced differential inverses transform of the set of values of $U_k(x_1,x_2,x_3)$, $\{U_k(x_1,x_2,\cdots,x_n):$ k= 0,1,2,⋯} by using definition 4.2.2.2 (equation (55)) in n-dimensions $\forall k = 0,1,2,\cdots$, the analytic solution in infinite power series (open form) for the problem which can be given in the form of equations $(78)\, and\, (79)$ is

$$u(x_1,x_2,\cdots,x_n,t) = U_0(x_1,x_2,\cdots,x_n)+U_1(x_1,x_2,\cdots,x_n)t^{\alpha}+U_2(x_1,x_2,\cdots,x_n)t^{2\alpha}+$$
$$U_3(x_1,x_2,\cdots,x_n)t^{3\alpha},\cdots\ \forall k=1,2,3,\cdots \qquad (84)$$

Taking the special case (α=1), the exact solution of the problem which can be given in the form of equations $(78)\, and\, (79)$ can be obtained from equation (84) in closed form.

4.3 New Applications

In this section, to validate the efficiency and applicability of the newly introduced and explained Reduced Differential Transform Method Procedures under the sub-sections 4.2.2.1, 4.2.2.2, 4.2.2.3 and 4.2.2.4 for solving initial value problems of homogeneous time fractional heat-like equations in the domain of interest, four test examples of which: one in one dimension, one in two dimensions, one in three dimensions and one in 4-dimensions (where $n > 3$ and $n \in \mathbb{N}$) were presented as follows.

Example 4.3.1 Consider the one dimensional homogeneous time fractional heat-like equations

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{1}{12} x^2 \frac{\partial^2 u}{\partial x^2}, x \in [1,100], t > 0, \alpha \in (0,1] \tag{85}$$

Subject to initial condition:

$$u(x,0) = x^4,\ x \in [1,100] \tag{86}$$

Then the analytic solution for the problem is

$$u(x,t) = x^4 \left[1 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \cdots \right]$$

in infinite power series form (or in open form) and hence for special case (α=1), the exact solution of the problem is $u(x,t) = x^4 e^t$ in a closed form.

Solution:

By taking Reduced Differential Transform (RDT) on both sides of equations (85) and (86), we obtain the iteration relation:

$$\frac{\Gamma(K\alpha+\alpha+1)}{\Gamma(K\alpha+1)} \mathrm{U}_{\mathrm{K}+1}(x) = \frac{1}{12} x^2 \left(\frac{\partial^2}{\partial x^2} \mathrm{U}_{\mathrm{K}}(x) \right) \tag{87}$$

$$U_0(x) = x^4 \tag{88}$$

Where is the reduced transformed function, which is the t-dimensional spectrum function

Using equation (88) in equation (87), we obtain successively the values of

$\mathrm{U}_{\mathrm{K}}(x), \forall k = 1,2,3,\ldots.$

For $k = 0$, $\frac{\Gamma(\alpha+\alpha+1)}{\Gamma(\alpha+1)} \mathrm{U}_{\mathrm{K}+1}(x) = \frac{1}{12} x^2 \left(\frac{\partial^2}{\partial x^2} \mathrm{U}_{\mathrm{K}}(x) \right) \Rightarrow \frac{\Gamma(\alpha+1)}{\Gamma(1)} \mathrm{U}_1(\mathrm{x}) = \frac{1}{12} x^2 \left(\frac{\partial^2}{\partial x^2} \mathrm{U}_0(x) \right)$

But, we have $U_0(x) = x^4$ from equation (86).

$$\frac{\Gamma(\alpha+1)}{\Gamma(1)}\mathrm{U}_1(x)=\frac{1}{12}x^2\left(\frac{\partial^2}{\partial x^2}(x^4)\right)\Rightarrow \mathrm{U}_1(\mathrm{x})=\frac{\Gamma(1)x^4}{\Gamma(\alpha+1)}$$

$$\mathrm{U}_1(\mathrm{x})=\frac{x^4}{\Gamma(\alpha+1)}\,[\because \Gamma(1)=1\times\Gamma(1)=\Gamma(1+1)=1!=1]$$

$$\therefore \mathrm{U}_1(\mathrm{x})=\frac{x^4}{\Gamma(\alpha+1)}.$$

For $\boldsymbol{k}=1$,

$$\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}\mathrm{U}_{\mathrm{K}+1}(x)=\frac{1}{12}x^2\left(\frac{\partial^2}{\partial x^2}\mathrm{U}_{\mathrm{k}}(x)\right)\Rightarrow \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}\mathrm{U}_2(x)=\frac{1}{12}x^2\left(\frac{\partial^2}{\partial x^2}\mathrm{U}_1(x)\right)$$

$$\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}\mathrm{U}_2(x)=\frac{1}{12}x^2\left(\frac{\partial^2}{\partial x^2}\left(\frac{x^4}{\Gamma(\alpha+1)}\right)\right)\Rightarrow \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}\mathrm{U}_2(x)=\frac{1}{12}x^2\left(\frac{\partial^2}{\partial x^2}\left(\frac{x^4}{\Gamma(\alpha+1)}\right)\right)\Rightarrow$$
$$\mathrm{U}_2(\mathrm{x})=\frac{1}{\Gamma(2\alpha+1)}(x^4)\ \mathrm{U}_2(\mathrm{x})=\frac{1}{\Gamma(2\alpha+1)}(x^4)$$

$$\therefore \mathrm{U}_2(x)=\frac{x^4}{\Gamma(2\alpha+1)}$$

For $k=2$,

$$\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}\mathrm{U}_{\mathrm{K}+1}(x)=\frac{1}{2}x^2\left(\frac{\partial^2}{\partial x^2}\mathrm{U}_{\mathrm{k}}(x)\right)\Rightarrow \frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)}\mathrm{U}_3(x)=\frac{1}{2}x^2\left(\frac{\partial^2}{\partial x^2}\mathrm{U}_2(x)\right)$$

$$\Rightarrow \frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)}\mathrm{U}_3(x)=\frac{1}{12}x^2\left(\frac{\partial^2}{\partial x^2}\left(\frac{x^4}{\Gamma(2\alpha+1)}\right)\right)\Rightarrow \mathrm{U}_3(x)=\frac{x^4}{\Gamma(3\alpha+1)}$$

$$\therefore \mathrm{U}_3(x)=\frac{x^4}{\Gamma(3\alpha+1)}.$$

For $\boldsymbol{k}=3$,

$$\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}\mathrm{U}_{K+1}(x)=\frac{1}{12}x^2\left(\frac{\partial^2}{\partial x^2}\mathrm{U}_k(x)\right)\Rightarrow\frac{\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)}\mathrm{U}_4(\mathrm{x})=\frac{1}{12}x^2\left(\frac{\partial^2}{\partial x^2}\mathrm{U}_3(x)\right)$$

$$\Rightarrow\frac{\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)}\mathrm{U}_4(x)=\frac{1}{12}x^2\left(\frac{\partial^2}{\partial x^2}\left(\frac{x^4}{\Gamma(3\alpha+1)}\right)\right)\Rightarrow\mathrm{U}_4(x)=\frac{x^4}{\Gamma(4\alpha+1)}$$

$$\therefore\mathrm{U}_4(x)=\frac{x^4}{\Gamma(4\alpha+1)}.$$

$$\vdots$$

Thus,

$$U_0(x)=x^4,U_1(x)=\frac{x^4}{\Gamma(\alpha+1)}U_2(x)=\frac{x^4}{\Gamma(2\alpha+1)},U_3(x)=\frac{x^4}{\Gamma(3\alpha+1)},U_4(x)=\frac{x^4}{\Gamma(4\alpha+1)},\cdots \quad (89)$$

Now by definition 4.2.2.2 (or equation 55) for n=1,

$$u(x,t)=\sum_{k=0}^{\infty}\mathrm{U}_k(x)\mathrm{t}^{K\alpha}=\mathrm{U}_0(x)+U_1(x)t^{\alpha}+U_2(x)t^{2\alpha}+U_3(x)t^{3\alpha}+U_4(x)t^{4\alpha}+\cdots$$

By equation (87), $u(x,t)=x^4+\frac{x^4}{\Gamma(\alpha+1)}t^{\alpha}+\frac{x^4}{\Gamma(2\alpha+1)}t^{2\alpha}+\frac{x^4}{\Gamma(3\alpha+1)}t^{3\alpha}+\frac{x^4}{\Gamma(4\alpha+1)}t^{4\alpha}+\cdots$

Thus, the analytic solution for the problem is

$u(x,t)=x^4\left[1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}+\cdots\right]$ in infinite power series (or open) form.

For the special case α=1, $u(x,t)$ becomes

$u(x,t)=x^4\left[1+\frac{t}{1!}+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots\right]$, since $\Gamma(p+1)=p!=n!,\forall n=1,2,\cdots$

$$u(x,t) = x^4 e^t \left(\because e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3i} + \frac{t^4}{4!} + \cdots \right)$$

Thus, for the special case (α=1), the exact solution of the problem is $u(x,t) = x^4 e^t$ in closed form.

Therefore, the analytic solution for the problem is

$$u(x,t) = x^4 \left[1 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \cdots \right]$$ in infinite power series form (or in open form) and hence for the special case (α=1), the exact solution is $u(x,t) = x^4 e^t$ of equation the problem in a closed form.

Example 4.3.2.Consider the two dimensional homogeneous time fractional heat-like equations

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{1}{2} y^2 \frac{\partial^2 u}{\partial x^2} + \frac{1}{2} x^2 \frac{\partial^2 u}{\partial y^2}, -\sqrt{2} \leq x,\ \mathrm{y} \leq 7,\ t > 0,\ 0 < \alpha \leq 1 \tag{90}$$

With initial condition:

$$u(x, y, 0) = x^2 + y^2 \ , -\sqrt{2} \leq x,\ \mathrm{y} \leq 7 \tag{91}$$

Then the analytic solution for the problem is

$$u(x,y,t) = x^2 + y^2 + \frac{x^2 + y^2}{\Gamma(\alpha+1)} t^{\alpha} + \frac{x^2 + y^2}{\Gamma(2\alpha+1)} t^{2\alpha} + \frac{x^2 + y^2}{\Gamma(3\alpha+1)} t^{3\alpha} + \frac{x^2 + y^2}{\Gamma(4\alpha+1)} t^{4\alpha} + \cdots$$ in infinite power series form (or in open form) and hence for the special case (α=1), the exact solution is $u(x,y,t) = (x^2 + y^2) e^t$ in closed form

Solution:

By taking Reduced Differential Transform (RDT) on both sides of equations (90) and (91) and then applying theorems 4.2.2.1 – 4.2.2.2, we get the iteration relation:

$$\frac{\Gamma\left(K\alpha+\alpha+1\right)}{\Gamma\left(K\alpha+1\right)}U_{K+1}\left(x,y\right)=\frac{1}{2}x^2\left(\frac{\partial^2}{\partial x^2}U_K\left(x,y\right)\right)+\frac{1}{2}y^2\left(\frac{\partial^2}{\partial x^2}U_K\left(x,y\right)\right) \quad (92)$$

$$U_0\left(x,y\right)=x^2+y^2 \quad (93)$$

Where is the t-dimensional spectrum function (the transform function)

Using equation (93) in equation (92) , we successively obtain the values of

$\mathrm{U_K}\left(x,y\right)$, for $\forall k=1,2,3,\ldots$

For $\boldsymbol{k}=0$, $\frac{\Gamma\left(\alpha+1\right)}{\Gamma\left(1\right)}U_1\left(x,y\right)=\frac{1}{2}x^2\left(\frac{\partial^2}{\partial x^2}U_0\left(x,y\right)\right)+\frac{1}{2}y^2\left(\frac{\partial^2}{\partial x^2}U_0\left(x,y\right)\right)$

Since $U_0\left(x\right)=x^2+y^2$ from equation (93) , $U_1\left(x,y\right)=\frac{\Gamma\left(1\right)(x^2+y^2)}{\Gamma(\alpha+1)}$.

$U_1\left(x\right)=\frac{x^2+y^2}{\Gamma(\alpha+1)}(\because\Gamma\left(1\right)=1\times\Gamma\left(1\right)=\Gamma\left(1+1\right)=1!=1)$.

$\therefore \mathrm{U}_1\left(x\right)=\frac{x^2+y^2}{\Gamma(\alpha+1)}$.

For $\boldsymbol{k}=1$, $\frac{\Gamma\left(2\alpha+1\right)}{\Gamma\left(\alpha+1\right)}U_2\left(x,y\right)=\frac{1}{2}x^2\left(\frac{\partial^2}{\partial x^2}U_1\left(x,y\right)\right)+\frac{1}{2}y^2\left(\frac{\partial^2}{\partial x^2}U_1\left(x,y\right)\right)$

$$\frac{\Gamma\left(2\alpha+1\right)}{\Gamma\left(\alpha+1\right)}\mathrm{U}_2\left(x,y\right)=\frac{1}{2}x^2\left(\frac{\partial^2}{\partial x^2}\left(\frac{x^2+y^2}{\Gamma\left(\alpha+1\right)}\right)\right)+\frac{1}{2}y^2\left(\frac{\partial^2}{\partial y^2}\left(\frac{x^2+y^2}{\Gamma\left(\alpha+1\right)}\right)\right)$$

$$\mathrm{U}_2\left(x,y\right)=\frac{\Gamma(\alpha+1))}{\Gamma\left(2\alpha+1\right)}.\frac{x^2+y^2}{\Gamma(\alpha+1)}$$

$$\therefore \mathrm{U}_2\left(x,y\right)=\frac{x^2+y^2}{\Gamma(2\alpha+1)}$$

For $\boldsymbol{k}=2$, $\frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)}\mathrm{U}_3\left(x,y\right)=\frac{1}{2}x^2\left(\frac{\partial^2}{\partial x^2}\mathrm{U}_2\left(x,y\right)\right)+\frac{1}{2}y^2\left(\frac{\partial^2}{\partial x^2}\mathrm{U}_2\left(x,y\right)\right)$

$$\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}\mathrm{U}_3(x,y)=\frac{1}{2}x^2\left(\frac{\partial^2}{\partial x^2}\left(\frac{x^2+y^2}{\Gamma(2\alpha+1)}\right)\right)+\frac{1}{2}y^2\left(\frac{\partial^2}{\partial y^2}\left(\frac{x^2+y^2}{\Gamma(2\alpha+1)}\right)\right)$$

$$\mathrm{U}_3(x,y)=\frac{\Gamma(2\alpha+1))}{\Gamma(3\alpha+1)}\cdot\frac{x^2+y^2}{\Gamma(2\alpha+1)}$$

$$\therefore \mathrm{U}_3(x,y)=\frac{x^2+y^2}{\Gamma(3\alpha+1)}.$$

For $\boldsymbol{k}=3$, $\frac{\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)}\mathrm{U}_4(x,y)=\frac{1}{2}x^2\left(\frac{\partial^2}{\partial x^2}\mathrm{U}_3(x,y)\right)+\frac{1}{2}y^2\left(\frac{\partial^2}{\partial x^2}\mathrm{U}_3(x,y)\right)$

$$\frac{\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)}\mathrm{U}_4(\mathrm{x},\ \mathrm{y})=\frac{1}{2}x^2\left(\frac{\partial^2}{\partial x^2}\left(\frac{x^2+y^2}{\Gamma(3\alpha+1)}\right)\right)+\frac{1}{2}y^2\left(\frac{\partial^2}{\partial y^2}\left(\frac{x^2+y^2}{\Gamma(3\alpha+1)}\right)\right)$$

$$\frac{\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)}\mathrm{U}_4(\mathrm{x},\ \mathrm{y})=\frac{x^2+y^2}{\Gamma(3\alpha+1)}$$

$$\therefore \mathrm{U}_4(\mathrm{x},\ \mathrm{y})=\frac{x^2+y^2}{\Gamma(4\alpha+1)}.$$

$$\vdots$$

Thus,

$$\mathrm{U}_0(x,y)=x^2+y^2,\mathrm{U}_1(\mathrm{x})=\frac{x^2+y^2}{\Gamma(\alpha+1)},\mathrm{U}_2(x,y)=\frac{x^2+y^2}{\Gamma(2\alpha+1)},\mathrm{U}_3(x,y)=\frac{x^2+y^2}{\Gamma(3\alpha+1)},\mathrm{U}_4(x,y)=\frac{x^2+y^2}{\Gamma(4\alpha+1)},\cdots \quad (94)$$

Now by definition 4.2.2.2 (or equation (55) in two dimensions (n=2), $u(x,t)=\sum_{k=0}^{\infty}U_k(x)t^{K\alpha}$ $=U_0(x)+U_1(x)t^{\alpha}+U_2(x)t^{2\alpha}+U_3(x)t^{3\alpha}+U_4(x)t^{4\alpha}+\cdots,\forall k=1,2,\cdots.$

By equation (94),

$$u(x,y,t)=x^2+y^2+\frac{x^2+y^2}{\Gamma(\alpha+1)}t^{\alpha}+\frac{x^2+y^2}{\Gamma(2\alpha+1)}t^{2\alpha}+\frac{x^2+y^2}{\Gamma(3\alpha+1)}t^{3\alpha}+\frac{x^2+y^2}{\Gamma(4\alpha+1)}t^{4\alpha}+\cdots$$

Thus, the analytic solution of the problem is

$u(x,y,t)=x^2+y^2+\frac{x^2+y^2}{\Gamma(\alpha+1)}t^{\alpha}+\frac{x^2+y^2}{\Gamma(2\alpha+1)}t^{2\alpha}+\frac{x^2+y^2}{\Gamma(3\alpha+1)}t^{3\alpha}+\frac{x^2+y^2}{\Gamma(4\alpha+1)}t^{4\alpha}+\cdots$ in infinite power series (or open) form.

For the special case α=1, $u(x,y,t)$ becomes

$$u(x,y,t)=\left(x^2+y^2\right)\left[1+\frac{t}{1!}+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots\right],\text{ since }\Gamma(p+1)=p!=n!,\forall n=1,2,\cdots$$

Since $e^t=1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\frac{t^5}{5!}+\cdots$, we get $u(x,y,t)=\left(x^2+y^2\right)e^t$.

Thus, for the special case (α=1), the exact solution the problem is $u(x,y,t)=\left(x^2+y^2\right)e^t$ in closed form.

Therefore, the analytic solution of the problem is

$$u(x,y,t)=x^2+y^2+\frac{x^2+y^2}{\Gamma(\alpha+1)}t^{\alpha}+\frac{x^2+y^2}{\Gamma(2\alpha+1)}t^{2\alpha}+\frac{x^2+y^2}{\Gamma(3\alpha+1)}t^{3\alpha}+\frac{x^2+y^2}{\Gamma(4\alpha+1)}t^{4\alpha}+\cdots$$

in infinite power series (or open) form and hence for the special case (α=1), the exact solution of the problem is $u(x,y,t)=\left(x^2+y^2\right)e^t$ in closed form.

Example 4.3.3 Consider the three dimensional homogeneous time fractional heat-like equation

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=-\left(\frac{xyz}{3\sin xyz}-\frac{1}{3}\right)\left(\frac{1}{yz}\frac{\partial^2 u}{\partial x^2}+\frac{1}{xz}\frac{\partial^2 u}{\partial y^2}+\frac{1}{xy}\frac{\partial^2 u}{\partial z^2}\right),\frac{\pi}{3}\le x,y,z\le\frac{\pi}{2},t>0\ ,0<\alpha\le 1 \quad (95)$$

Subject to initial condition:

$$u(x,y,z,0) = -(xyz - \sin xyz),\ \frac{\pi}{3} \le x,y,z \le \frac{\pi}{2} \qquad (96)$$

Then the analytic solution is

$$u(x,y,zt) = -(xyz - \sin xyz) + \frac{xyz - \sin xyz}{\Gamma(\alpha+1)} t^{\alpha} + \frac{-(xyz - \sin xyz)}{\Gamma(2\alpha+1)} t^{2\alpha} + \frac{xyz - \sin xyz}{\Gamma(3\alpha+1)} t^{3\alpha} + \frac{-(xyz - \sin xyz)}{\Gamma(4\alpha+1)} t^{4\alpha} + \cdots$$

in infinite power series form (or in open form), and hence for the special case (α=1), the exact solution is

$u(x,y,z,t) = -xyz\cosh t + xyz\sinh t + \sin xyz\cosh t - \sin xyz\sinh t$, in closed form

Solution:

By taking Reduced Differential Transform (RDT) on both sides of equations (95) and (96) and applying theorems 4.2.2.1 – 4.2.2.2, we get the iteration relation

$$\frac{\Gamma(K\alpha+\alpha+1)}{\Gamma(K\alpha+1)} \mathrm{U}_{K+1}(x,y,z) = -\left(\frac{xyz}{3\sin xyz} - \frac{1}{3}\right)\left(\frac{1}{yz}\frac{\partial^2}{\partial x^2}\mathrm{U}_K(x,y,z) + \frac{1}{xz}\frac{\partial^2}{\partial y^2}\mathrm{U}_K(x,y,z)\frac{1}{xy}\frac{\partial^2}{\partial Z^2}\mathrm{U}_K(x,y,z)\right) \qquad (97)$$

$$U_0(x,y,z) = -(xyz - \sin xyz) \qquad (98)$$

Where $\mathrm{U}_K(x,y,z)$ is the t-dimensional spectrum function (the transformed function)

Using equation (98) in equation (97), we successively obtain the values of $\mathrm{U}_K(x,y.z), \forall k = 1,2,3,\ldots$

For $\boldsymbol{k} = 0$,

$$\frac{\Gamma(\alpha+1)}{\Gamma(1)} \mathrm{U}_1(x,y,z) = -\left(\frac{xyz}{3\sin xyz} - \frac{1}{3}\right)\left(\frac{1}{3yz}\frac{\partial^2}{\partial x^2}\mathrm{U}_0(x,y,z) + \frac{1}{3xz}\frac{\partial^2}{\partial y^2}\mathrm{U}_0(x,y,z)\frac{1}{3xy}\frac{\partial^2}{\partial z^2}\mathrm{U}_0(x,y,z)\right)$$

$$\mathrm{U}_1(x,y,z) = \frac{\Gamma(1)xyz - \sin xyz}{\Gamma(\alpha+1)} \qquad (\because \Gamma(1) = 1)$$

$$\therefore \mathrm{U}_1(x,y,z) = \frac{xyz - \sin xyz}{\Gamma(\alpha+1)}$$

For, k=1

$$\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}\mathrm{U}_2(x,y,z)=-\left(\frac{xyz}{3\ \text{sin}xyz}-\frac{1}{3}\right)\left(\frac{1}{yz}\frac{\partial^2}{\partial x^2}\mathrm{U}_1(x,y,z)+\frac{1}{xz}\frac{\partial^2}{\partial y^2}\mathrm{U}_1(x,y,z)\frac{1}{xy}\frac{\partial^2}{\partial z^2}\mathrm{U}_1(x,y,z)\right)$$

$$\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}\mathrm{U}_2(x,y,z)=-\left(\frac{xyz}{3\ \text{sin}xyz}-\frac{1}{3}\right)\left(\begin{array}{l}\frac{1}{yz}\frac{\partial^2}{\partial x^2}\left(\frac{xyz-\sin xyz}{\Gamma(\alpha+1)}\right)+\frac{1}{xz}\frac{\partial^2}{\partial y^2}\left(\frac{xyz-\sin xyz}{\Gamma(\alpha+1)}\right)+\\ \frac{1}{xy}\frac{\partial^2}{\partial z^2}\left(\frac{xyz-\sin xyz}{\Gamma(\alpha+1)}\right)\end{array}\right)$$

$$\therefore \mathrm{U}_2(x,y,z)=\frac{-(xyz-\sin xyz)}{\Gamma(2\alpha+1)}.$$

For $\boldsymbol{k}=2$,

$$\frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)}\mathrm{U}_3(x,y,z)=-\left(\frac{xyz}{3\ \text{sin}xyz}-\frac{1}{3}\right)\left(\frac{1}{yz}\frac{\partial^2}{\partial x^2}\mathrm{U}_2(x,y,z)+\frac{1}{xz}\frac{\partial^2}{\partial y^2}\mathrm{U}_2(x,y,z)\frac{1}{xy}\frac{\partial^2}{\partial x^2}\mathrm{U}_2(x,y,z)\right)$$

$$\frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)}\mathrm{U}_3(x,y,z)=-\left(\frac{xyz}{3\ \text{sin}xyz}-\frac{1}{3}\right)\left(\begin{array}{l}\frac{1}{yz}\frac{\partial^2}{\partial x^2}\left(\frac{-(xyz-\sin xyz)}{\Gamma(2\alpha+1)}\right)+\frac{1}{xz}\frac{\partial^2}{\partial y^2}\left(\frac{-(xyz-\sin xyz)}{\Gamma(2\alpha+1)}\right)+\\ \frac{1}{xy}\frac{\partial^2}{\partial x^2}\left(\frac{-(xyz-\sin xyz)}{\Gamma(2\alpha+1)}\right)\end{array}\right)$$

$$\therefore \mathrm{U}_3(x,y,z)=\frac{xyz-\sin xyz}{\Gamma(3\alpha+1)}.$$

$$\vdots$$

Thus,

$$\mathrm{U}_0(x,y,z)=-(xyz-sinxyz),\mathrm{U}_1(\mathrm{x,y,z})=\frac{xyz-sinxyz}{\Gamma(\alpha+1)},\mathrm{U}_2(\mathrm{x})=\frac{-(xyz-sinxyz)}{\Gamma(2\alpha+1)},\mathrm{U}_3(\mathrm{x,y,z})=$$

$$\frac{xyz-sinxyz}{\Gamma(3\alpha+1)},\cdots \tag{99}$$

Now, by definition 4.2.2.2 (or equation (55) in three dimensions

$$u(\mathrm{x},\mathrm{y},\mathrm{z},t)=\sum_{k=0}^{\infty}\mathrm{U}_{\mathrm{k}}(\mathrm{x},\mathrm{y},\mathrm{z})\mathrm{t}^{\mathrm{K}\alpha}=\mathrm{U}_0(\mathrm{x},\mathrm{y},\mathrm{z})+\mathrm{U}_1(\mathrm{x},\mathrm{y},\mathrm{z})t^{\alpha}+\mathrm{U}_2(\mathrm{x},\mathrm{y},\mathrm{z})t^{2\alpha}+\mathrm{U}_3(\mathrm{x},\mathrm{y},\mathrm{z})t^{3\alpha}+\cdots$$

By equation (99),

$$u(x,y,z,t)=-(xyz-sinxyz)+\frac{xyz-sinxyz}{\Gamma(\alpha+1)}t^{\alpha}+\frac{-(xyz-sinxyz)}{\Gamma(2\alpha+1)}t^{2\alpha}+\frac{xyz-sinxyz}{\Gamma(3\alpha+1)}t^{3\alpha}+\cdots$$

Thus, the analytic solution for the problem is

$$u(x,y,z,t)=-(xyz-sinxyz)+\frac{xyz-sinxyz}{\Gamma(\alpha+1)}t^{\alpha}+\frac{-(xyz-sinxyz)}{\Gamma(2\alpha+1)}t^{2\alpha}+\frac{xyz-sinxyz}{\Gamma(3\alpha+1)}t^{3\alpha}+\frac{-(xyz-\sin xyz)}{\Gamma(4\alpha+1)}t^{4\alpha}+\cdots$$

in infinite power series form (or in open form).

For the special case α=1, $u(x,y,z,t)=(xyz-sinxyz)\left[-1+\frac{t}{1!}-\frac{t^2}{2!}+\frac{t^3}{3!}-\cdots\right]$

$$u(x,y,z,t)=(xyz-sinxyz)\left[\left(-1-\frac{t^2}{2!}-\cdots\right)+\left(t+\frac{t^3}{3!}+\cdots\right)\right]$$

$$u(x,y,z,t)=(xyz-sinxyz)\left[-\left(1+\frac{t^2}{2!}+\cdots\right)+\left(t+\frac{t^3}{3!}+\cdots\right)\right]$$

Since $\cosh t=1+\frac{t^2}{2!}+\cdots$ and $\sinh t=t+\frac{t^3}{3!}+\cdots$, we get

$$u(x,y,z,t)=(xyz-sinxyz)\left[(-\cosh t)+\sinh t\right].$$

Thus, $u(x,y,z,t)=-xyz\cosh t+xyz\sinh t+\sin xyz\cosh t-\sin xyz\sinh t$

Therefore, the exact solution of the problem is

$u(x,y,z,t)=-xyz\cosh t+xyz\sinh t+\sin xyz\cosh t-\sin xyz\sinh t$ in closed form.

Therefore, the analytic solution of the problem is

$$u(x,y,z,t)=-(xyz-sinxyz)+\frac{xyz-sinxyz}{\Gamma(\alpha+1)}t^{\alpha}+\frac{-(xyz-sinxyz)}{\Gamma(2\alpha+1)}t^{2\alpha}+\frac{xyz-sinxyz}{\Gamma(3\alpha+1)}t^{3\alpha}+\frac{-(xyz-\sin xyz)}{\Gamma(4\alpha+1)}t^{4\alpha}+\cdots$$

, in infinite power series form (or in open form), and hence for the special case (α=1), the exact solution of the problem is

$u(x,y,z,t)=-xyz\cosh t+xyz\sinh t+\sin xyz\cosh t-\sin xyz\sinh t$ in closed form.

Example 4.3.4 Consider the four dimensional homogeneous time fractional heat-like equation

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\left(\frac{1}{x_1}+\frac{x_1^{3}}{6}\right)u_{x_1x_1}+\left(\frac{1}{x_2}+\frac{x_2^{3}}{6}\right)u_{x_2x_2}+\left(\frac{1}{x_3}+\frac{x_3^{3}}{6}\right)u_{x_3x_3}+\left(\frac{1}{x_4}+\frac{x_4^{3}}{6}\right)u_{x_4x_4}$$

$$1\le x_1,x_2,x_3,x_4\le 100,\ 0<\alpha\le 1 \qquad (100)$$

With initial condition:

$$u(x_1,x_2,x_3,x_4,0)=4+\frac{x_1^{3}}{6}+\frac{x_2^{3}}{6}+\frac{x_3^{3}}{6}+\frac{x_4^{3}}{6},\quad 1\le x_1,x_2,x_3,x_4\le 100 \qquad (101)$$

Then the analytic solution of the problem is

$$u(x_1,x_2,x_3,x_4,t)=\left(4+\frac{x_1^{3}}{6}+\frac{x_2^{3}}{6}+\frac{x_3^{3}}{6}+\frac{x_4^{3}}{6}\right)\left[1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\cdots\right]$$

in infinite power series form (or open) form, and hence for the special case (α=1), the exact solution of the problem is $u(x_1,x_2,x_3,x_4,t)=\left(4+\frac{x_1^{3}}{6}+\frac{x_2^{3}}{6}+\frac{x_3^{3}}{6}+\frac{x_4^{3}}{6}\right)e^{t}$ in closed form.

Solution:

By taking Reduced Differential Transform (RDT) on both sides of equations (98) and (99) and then applying theorems 4.2.2.1 - 4.2.2.6 for n=4, we obtain the iteration relation:

$$\frac{K\alpha+\alpha+1}{K\alpha+1}\mathrm{U}_{\mathrm{K}+1}\left(x_1,x_2,x_3,x_4\right)=\left(\frac{1}{x_1}+\frac{x_1^{\ 3}}{6}\right)\frac{\partial^2}{\partial x_1^{\ 2}}\mathrm{U}_{\mathrm{K}}\left(x_1,x_2,x_3,x_4\right)+\left(\frac{1}{x_2}+\frac{x_2^{\ 3}}{6}\right)\frac{\partial^2}{\partial x_2^{\ 2}}\mathrm{U}_{\mathrm{K}}\left(x_1,x_2,x_3,x_4\right)$$

$$+\left(\frac{1}{x_3}+\frac{x_3^{\ 3}}{6}\right)\frac{\partial^2}{\partial x_3^{\ 2}}\mathrm{U}_{\mathrm{K}}\left(x_1,x_2,x_3,x_4\right)+\left(\frac{1}{x_4}+\frac{x_4^{\ 3}}{6}\right)\frac{\partial^2}{\partial x_4^{\ 2}}\mathrm{U}_{\mathrm{K}}\left(x_1,x_2,x_3,x_4\right) \qquad (102)$$

$$U_0\left(x_1,x_2,x_3,x_4\right)=4+\frac{x_1^{\ 3}}{6}+\frac{x_2^{\ 3}}{6}+\frac{x_3^{\ 3}}{6}+\frac{x_4^{\ 3}}{6} \qquad (103)$$

, where $\mathrm{U}_{\mathrm{K}}\left(x_1,x_2,x_3,x_4\right)$ is the t-dimensional spectrum function (the transformed function).

Using equation (103) in equation (102), we successively obtain the values of $\mathrm{U}_{\mathrm{K}}\left(x_1,x_2,x_3,x_4\right)$, $\forall k=1,2,3,\ldots$

For $k=0$,

$$\frac{\Gamma(\alpha+1)}{\Gamma(1)}\mathrm{U}_1\left(x_1,x_2,x_3,x_4\right)=\left(\frac{1}{x_1}+\frac{x_1^{\ 3}}{6}\right)\frac{\partial^2}{\partial x_2^{\ 2}}\mathrm{U}_0\left(x_1,x_2,x_3,x_4\right)+\left(\frac{1}{x_2}+\frac{x_2^{\ 3}}{6}\right)\frac{\partial^2}{\partial x_2^{\ 2}}\mathrm{U}_0\left(x_1,x_2,x_3,x_4\right)$$

$$+\left(\frac{1}{x_3}+\frac{x_3^{\ 3}}{6}\right)\frac{\partial^2}{\partial x_3^{\ 2}}\mathrm{U}_0\left(x_1,x_2,x_3,x_4\right)+\left(\frac{1}{x_4}+\frac{x_4^{\ 3}}{6}\right)\frac{\partial^2}{\partial x_4^{\ 2}}\mathrm{U}_0\left(x_1,x_2,x_3,x_4\right)$$

$$\therefore \mathrm{U}_1\left(x_1,x_2,x_3,x_4\right)=\frac{1}{\Gamma(\alpha+1)}\left[4+\frac{x_1^{\ 3}}{6}+\frac{x_2^{\ 3}}{6}+\frac{x_3^{\ 3}}{6}+\frac{x_4^{\ 3}}{6}\right].$$

For $k=1$,

$$\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}\mathrm{U}_2\left(x_1,x_2,x_3,x_4\right)=\left(\frac{1}{x_1}+\frac{x_1^{\ 3}}{6}\right)\frac{\partial^2}{\partial x_2^{\ 2}}\mathrm{U}_1\left(x_1,x_2,x_3,x_4\right)+\left(\frac{1}{x_2}+\frac{x_2^{\ 3}}{6}\right)\frac{\partial^2}{\partial x_2^{\ 2}}\mathrm{U}_1\left(x_1,x_2,x_3,x_4\right)$$

$$+\left(\frac{1}{x_3}+\frac{x_3^{\ 3}}{6}\right)\frac{\partial^2}{\partial x_3^{\ 2}}\mathrm{U}_1\left(x_1,x_2,x_3,x_4\right)+\left(\frac{1}{x_4}+\frac{x_4^{\ 3}}{6}\right)\frac{\partial^2}{\partial x_4^{\ 2}}\mathrm{U}_1\left(x_1,x_2,x_3,x_4\right)$$

$$\therefore \mathrm{U}_2\left(x_1,x_2,x_3,x_4\right)=\frac{1}{\Gamma(2\alpha+1)}\left[4+\frac{x_1^{\ 3}}{6}+\frac{x_2^{\ 3}}{6}+\frac{x_3^{\ 3}}{6}+\frac{x_4^{\ 3}}{6}\right]$$

***For k** $=2$,*

$$\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}\mathrm{U}_3(x_1,x_2,x_3,x_4)=\left(\frac{1}{x_1}+\frac{x_1^{\ 3}}{6}\right)\frac{\partial^2}{\partial x_2^{\ 2}}\mathrm{U}_2(x_1,x_2,x_3,x_4)+\left(\frac{1}{x_2}+\frac{x_2^{\ 3}}{6}\right)\frac{\partial^2}{\partial x_2^{\ 2}}\mathrm{U}_2(x_1,x_2,x_3,x_4)$$

$$+\left(\frac{1}{x_3}+\frac{x_3^{\ 3}}{6}\right)\frac{\partial^2}{\partial x_3^{\ 2}}\mathrm{U}_2(x_1,x_2,x_3,x_4)+\left(\frac{1}{x_4}+\frac{x_4^{\ 3}}{6}\right)\frac{\partial^2}{\partial x_4^{\ 2}}\mathrm{U}_2(x_1,x_2,x_3,x_4)$$

$$\therefore \mathrm{U}_3(x_1,x_2,x_3,x_4)=\frac{1}{\Gamma(3\alpha+1)}\left[4+\frac{x_1^{\ 3}}{6}+\frac{x_2^{\ 3}}{6}+\frac{x_3^{\ 3}}{6}+\frac{x_4^{\ 3}}{6}\right]$$

$$\vdots$$

Thus,

$$\mathrm{U}_0(x_1,x_2,x_3,x_4)=4+\frac{x_1^{\ 3}}{6}+\frac{x_2^{\ 3}}{6}+\frac{x_3^{\ 3}}{6}+\frac{x_4^{\ 3}}{6},\ \ \mathrm{U}_1(x_1,x_2,x_3,x_4)=\frac{1}{\Gamma(\alpha+1)}\left[4+\frac{x_1^{\ 3}}{6}+\frac{x_2^{\ 3}}{6}+\frac{x_3^{\ 3}}{6}+\frac{x_4^{\ 3}}{6}\right]$$

$$,\ \mathrm{U}_2(x_1,x_2,x_3,x_4)=\frac{1}{\Gamma(2\alpha+1)}\left[4+\frac{x_1^{\ 3}}{6}+\frac{x_2^{\ 3}}{6}+\frac{x_3^{\ 3}}{6}+\frac{x_4^{\ 3}}{6}\right],\ \mathrm{U}_3(x_1,x_2,x_3,x_4)=$$

$$\frac{1}{\Gamma(3\alpha+1)}\left[4+\frac{x_1^{\ 3}}{6}+\frac{x_2^{\ 3}}{6}+\frac{x_3^{\ 3}}{6}+\frac{x_4^{\ 3}}{6}\right],\cdots \tag{104}$$

By definition 4.2.2.1 (or equation (54) in 4-dimensions and by equation (104),

$$u(x_1,x_2,x_3,x_4,t)=\sum_{k=0}^{\infty}\mathrm{U}_{\mathrm{K}}(x_1,x_2,x_3,x_4)\,\mathrm{t}^{\mathrm{K}\alpha}.$$

$$u(x_1,x_2,x_3,x_4,t)=4+\frac{x_1^{\ 3}}{6}+\frac{x_2^{\ 3}}{6}+\frac{x_3^{\ 3}}{6}+\frac{x_4^{\ 3}}{6}+\frac{1}{\Gamma(\alpha+1)}\left[4+\frac{x_1^{\ 3}}{6}+\frac{x_2^{\ 3}}{6}+\frac{x_3^{\ 3}}{6}+\frac{x_4^{\ 3}}{6}\right]t^{\alpha}+$$

$$\frac{1}{\Gamma(2\alpha+1)}\left[4+\frac{x_1^{\ 3}}{6}+\frac{x_2^{\ 3}}{6}+\frac{x_3^{\ 3}}{6}+\frac{x_4^{\ 3}}{6}\right]t^{2\alpha}+\frac{1}{\Gamma(3\alpha+1)}\left[4+\frac{x_1^{\ 3}}{6}+\frac{x_2^{\ 3}}{6}+\frac{x_3^{\ 3}}{6}+\frac{x_4^{\ 3}}{6}\right]t^{3\alpha}+\cdots$$

Thus, the analytic solution for the problem is

$$u(x_1,x_2,x_3,x_4,t)=\left(4+\frac{x_1^{\ 3}}{6}+\frac{x_2^{\ 3}}{6}+\frac{x_3^{\ 3}}{6}+\frac{x_4^{\ 3}}{6}\right)\left[1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\cdots\right]$$

in infinite power series (open) form.

For the special case α=1, $u\left(x_1,x_2,x_3,x_4,t\right)=\left(4+\frac{x_1^{\ 3}}{6}+\frac{x_2^{\ 3}}{6}+\frac{x_3^{\ 3}}{6}+\frac{x_4^{\ 3}}{6}\right)\left[1+\frac{t}{1!}+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots\right]$

Thus, $u\left(x_1,x_2,x_3,x_4,t\right)=\left(4+\frac{x_1^{\ 3}}{6}+\frac{x_2^{\ 3}}{6}+\frac{x_3^{\ 3}}{6}+\frac{x_4^{\ 3}}{6}\right)e^t$.

Therefore, the exact solution of the problem is

$u\left(x_1,x_2,x_3,x_4,t\right)=\left(x_1,x_2,x_3,x_4,t\right)=\left(4+\frac{x_1^{\ 3}}{6}+\frac{x_2^{\ 3}}{6}+\frac{x_3^{\ 3}}{6}+\frac{x_4^{\ 3}}{6}\right)e^t$ in closed form.

Therefore, the analytic solution of the problem is

$u\left(x_1,x_2,x_3,x_4,t\right)=\left(4+\frac{x_1^{\ 3}}{6}+\frac{x_2^{\ 3}}{6}+\frac{x_3^{\ 3}}{6}+\frac{x_4^{\ 3}}{6}\right)\left[1+\frac{t}{1!}+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots\right]$ in infinite power series form and hence for the special case $\alpha=1$, the exact solution of the problem is $u\left(x_1,x_2,x_3,x_4,t\right)=\left(4+\frac{x_1^{\ 3}}{6}+\frac{x_2^{\ 3}}{6}+\frac{x_3^{\ 3}}{6}+\frac{x_4^{\ 3}}{6}\right)e^t$ in closed form.

CHAPTER FIVE

5 Conclusion and Future Scopes

In this study, new applications of Reduced Differential Transform Method were introduced to handle multi-dimensional initial value problems of homogeneous time fractional heat-like equations. The definitions of transformed function and the reduced differential inverse transform of the transformed function were developed and introduced in n-dimensional space ($n \in \mathbb{N}$) for solving multi-dimensional initial value problems of homogeneous time fractional heat-like equations with the values of the time fractional derivative of order, α such that $0 < \alpha \leq 1$. Six mathematical operations (theorems) which were performed by the Reduced Differential Transform Method, were deduced from these definitions for solving multi-dimensional initial value problems of homogeneous time fractional heat-like equations in n-dimensional space ($n \in \mathbb{N}$). Consequently, the Reduced Differential Transform Method procedures in one, two, three and more than three dimensions were developed and introduced to obtain analytic solutions of multi-dimensional homogeneous time fractional heat-like equations.

To see the effectiveness and applicability of the Reduced Differential Transform Method through newly introduced procedures to obtain analytic solutions of multi-dimensional initial value problems of homogeneous time fractional heat-like equations in n-dimensional space ($n \in \mathbb{N}$), four test examples were presented. The Reduced Differential Transform Method was successfully implemented to obtain analytic solutions of multi-dimensional homogeneous time fractional heat-like equations. The analytic solutions obtained by reduced differential transform method are infinite power series for appropriate initial conditions in open form while the exact solutions obtained for special case ($(\alpha = 1)$ are in closed form. That is, when $\alpha = 1$, the analytic solutions of multi-dimensional initial value problems of homogeneous time fractional heat-like equations in n-dimensional space ($n \in \mathbb{N}$) in infinite power series become the exact solutions of the standard (ordinary) multi-dimensional heat-like equations. The results show that the Reduced Differential Transform Method is a powerful mathematical tool for solving multi-dimensional initial value problems of homogeneous time fractional heat-like equations analytically. Thus, we conclude that the proposed method can be extended to solve other fractional partial differential equations (specially, time fractional partial differential equations) with variable coefficients which can arise in physics and engineering. That is, the work presented in this study leaves a lot of room for future research. The following are some of the items which deserve further investigation:

- ☞ Reduced Differential Transform Method for numerical solutions of initial value problems of homogeneous time fractional heat-like equations.
- ☞ Reduced Differential Transform Method for initial value problems of homogeneous time fractional non-linear heat-like equations.
- ☞ Reduced Differential Transform Method for initial value problems of non-homogeneous time fractional heat-like equations.
- ☞ Reduced Differential Transform Method for initial boundary value problems of homogeneous time factional heat-like equations.

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