Block Procedure for Solving Stiff First Order Initial Value Problems Using Chebyshev Polynomials

A Thesis Submitted to the Department of Mathematics in Partial Fulfillment for the Requirements of the Degree of Masters of Science in Mathematics

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October, 2019
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Declaration

I undersigned declare that, this Thesis paper entitled "**a block procedure for solving stiff first order initial value problems using Chebyshev polynomials** " is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged.
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Acknowledgment

First of all, I am indebted to my almighty God who gave me long life, strength and supported me to reach this precious time. Next, I would like to express my deepest gratitude to my advisor Dr. Alemayehu shiferaw and co-advisor Mr. Solomon Gebregiorgis for their unreserved support, unlimited advice, constructive comments and immediate responses that helped me in developing this thesis.

Also I was very much happy to express my heartfelt gratitude to my wife Zebiba Ayalaw for her moral support for the successful completion of this study.

Next, I would like to thank Jimma University College of Natural Science Departments of Mathematics for sponsoring for this thesis.

Finally, it was my pleasure to thank Samara University for giving the opportunity to study my M.Sc. degree and support financially.

Abstract

In this study, discrete fourth order implicit linear multistep methods (LMMs) in block form for the solution of stiff first order initial value problems (IVPs) was presented using power series as a basis and the Chebyshev polynomials. The method is based on collocation of the differential equation and interpolation of the approximate solution of power series at the grid points. The procedure yields four consistent implicit linear multistep schemes which are combined as simultaneous numerical integrators to form block method. The basic properties of the method such as order, error constant, zero stability, consistency, convergence,and accuracy are investigated. The accuracy of the method is tested with two stiff first order initial value problems. The results are compared with fourth order Runge-Kutta (RK4) method, and Berhan et al. (2019). All numerical examples are solved with the aid of MATLAB software and showed that our proposed method produced better accuracy.

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Chapter 1

Introduction

1.1 Background of the study

In science and engineering, usually mathematical models are developed to help us the understanding of physical phenomena. These models often yield equations that contain some derivatives of an unknown function of one or several variables. Such equations are called differential equations. A differential equation in which the unknown function is a functions of only one independent variable are called ordinary differential equations (ODEs). ODEs are very basic and useful mathematical models in many areas, such as economics, geology, engineering, social science, physics, chemistry, biology, and so on. Many of these ODEs are known as stiff ODE (Yu, 2004). The exact analytical solutions of such problems, except a few, are difficult to obtain, so it is common to seek approximate solutions by means of numerical methods.

In this study we shall be concerned with the construction and analysis of an efficient numerical method for the approximate solution of the general first order differential equations of the form:

$$y'(x) = f(x, y(x)), \quad y(a) = y_0, \quad x \in [a, b] \tag{1.1}$$

In mathematics, a differential equation of the form Eq.(1.1) is said to be a stiff equation if for solving a differential equation certain numerical methods are numerically unstable, unless the step size is taken to be extremely small and characterized as those whose exact solution has a term of the form e^{-cx} where c is a large positive constant (Suli & Mayers, 2003). Stiff problems are problems where certain implicit methods perform better than explicit ones. Stiffness is a subtle, difficult, and important concept in the numerical solution of ODEs. It depends on the differential equation, the initial conditions, and the numerical method (Aliyu et al., 2014).

The developments of numerical methods for the solution of IVPs of ODEs of the form of Eq.(1.1) has given rise to two major discrete methods (Anake, 2011). One of those discrete numerical methods used to solve stiff IVP is multistep methods especially linear multistep methods (LMMs).

LMM is a computational procedure where by a numerical approximation y_{n+j} to the exact solution $y(n+j)$ of the first order Initial Value Problems (IVPs) of Eq. (1.1) is obtained. LMMs are very popular for solving first order IVPs. They are also applied to solve higher order ODE. LMMs are not self-starting hence, need starting values from single-step methods like Euler's method and family of Runge-Kutta methods. The general k-step LMM for the solution of Eq. (1.1) is given by Lambert (1991) as follows:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \tag{1.2}$$

where $\alpha_j's$ and $\beta_j's$ are constants, $\alpha_k \neq 0$ and at least one of the coefficients α_0 and β_0 is non zero. If $\beta_k = 0$, then y_{n+k} is obtained from previous value of y_{n+j}, h, and f_{n+j}, then the k-step method is explicit. If $\beta_k \neq 0$, then y_{n+k} appears both on the left and right hand side of the equation, then the k-step method is implicit method.

Several continuous LMMs have been derived using interpolation and collocation points for the solution of Eq.(1.1) with constant step size using Chebyshev polynomials as a basis. Chebyshev polynomials are a well-known family of orthogonal polynomials on the interval $[-1,1]$ associated with certain weight function. Chebyshev polynomials are of great importance in many areas of mathematics, particularly in approximation theory. There are several kinds of Chebyshev polynomials. The most important kinds of the Chebyshev polynomials is Chebyshev polynomials $T_n(x)$ of first kind. The Chebyshev polynomial $T_n(x)$ of the first kind is a polynomial in x of degree n, defined by the relation by Rivlin (1990) as follows:

$$T_n(x) = \cos(n \arccos x), \quad \text{where} \quad n \geq 0, \text{and}, x \in [-1,1] \tag{1.3}$$

Adeniyi et al. (2006) develop a continuous formulation of some classical initial value problems by non-perturbed multistep collocation approach using Chebyshev polynomial as basis function. Adeniyi & Alabi (2007) proposed Continuous formulation of a class accurate implicit linear multistep methods with Chebyshev basis function in a collocation technique.

Much researchs have been done by the scientific community on developing numerical methods which permit an approximate solution to Eq.(1.1). The most commonly used numerical method is block method. Block methods have been firstly proposed by Milne (1953) to be used only as a means of obtaining starting values for predictor-corrector methods. It is a method which obtained concurrently a block of new values by computing k number of blocks. It is less expensive in terms of the number of function evaluations compared to the linear multistep methods. Block methods are self starting, thus avoiding the use of other methods to get starting solutions, except the initial condition from the problem.

The development of block methods for solving Eq.(1.1) has been studied by various researchers. For instance, Mohammed & Yahaya (2010) developed fully implicit four points block backward difference formula for solving first-order IVPs. They compared the accuracy of the developed method from the exact solution by taking one simple example of ODE, but they did not compare the accuracy of the method with the other existing methods. Akinfenwa et al. (2013) derived Continuous block backward differentiation formula for solving stiff ordinary differential equations. They used polynomial as a basis function to drive the method. James et al. (2013) developed a half-step continuous block method using the approach of collocation of the differential system and interpolation of the power series approximate solution at the grid and off grid points. Their new method was tested on real life problems namely: Growth model and Mixture Model. Their results were found to compete favorably with the existing methods in terms of accuracy and error bound.

Furthermore, many researchers had developed block procedure with LMMs using different basis functions. One of the basis function is power series. Power series is a series in which each term is a power function. It can be used to approximate any function. Any polynomial function can be expressed a power series. Assuming that x_0 is an ordinary point of the differential equation, the solution in a powers of x_0 actually do exist, we denote such a solution:

$$y=\sum_{n=0}^{\infty} C_n(x-x_0)^n \tag{1.4}$$

where $C_n's$ are a constants called the coefficient of the series and x is a variable.

Abualnaja (2015) constructed a block procedure with LMMs (for $k=1,2,$and 3) using Legendre polynomial as a basis function for solving first order ODEs. She gave discrete methods used in block and implement it for solving the non-stiff initial value problems, being the continuous interpolant derived and collocated at grid and off-grid points. The results are compared with RK4, but not compared with other existing methods.

Suleiman et al. (2015) proposed an implicit 2-point block extended backward differentiation formula for integration of stiff initial value problems and checked the performance of their methods by considering stiff problems. Their derived method is better in terms of accuracy than the two-point block backward differentiation formula (2BBDF). However, in terms of computation time, the time taken to complete the integration using the 2BBDF method is better than that in their derived method. The implementation of their proposed method is based on Newton iteration.

Sunday et al. (2015) developed Chebyshevian basis function-type block method for the solution of first order IVPs with oscillating solutions. They develop a block method using Chebyshev polynomial basis function and use it to produce discrete methods which are simultaneously applied as numerical integrators by assembling them into a block method. The efficiency of the method tested by two sampled oscillatory problems and when compared the results their method performed better than exact solution.

Yakusak & Adeniyi (2015) proposed a four step hybrid block method for first order IVPs in ODEs by collocation and interpolation techniques and with Chebyshev polynomial of the first kind as basis function. Their developed method is by introducing four off step points. However, this off step points selected to guarantee zero stability to generate the method for solving ODEs. Their method is complicated to programing, but when compared with existing scheme it yielded better accuracy.

Okedayo et al. (2018) developed modified Legendre collocation block method for solving IVPs of first order ODEs. They proposed block procedure for some k-step LMMs using the Legendre polynomials as the basis function. Discrete methods were given which were used in block and implemented for solving the initial value problems, being continuous interpolant derived and collocated at grid points. Their developed method is implemented without the need for the development of correctors.

Nweze et al. (2018) developed a class of block procedure with the implicit LMM in block form for $k = 1,2$ and 3 using Chebyshev polynomials as a basis function for solving non stiff initial value problem in ODEs. The accuracy of their method compared to the exact solution by taking non stiff problems.

Recently, Berhan et al. (2019) extends and Modified the work of Abualnaja (2015). They constructed a block procedure with implicit LMMs for some k-step ($k = 1,2,3$ and 4) using Legendre polynomials for solving ODEs. It is an implicit method and used for solving stiff initial value problems. Their method depends on the perturbed collocation approximation with shifted Legendre polynomials as perturbation term. However, the solution of stiff first order IVPs with implicit LMM in block form using Chebyshev polynomial as a perturbed term has not developed in the forgoing literature.

Perturbation theory is applicable if the problem at hand cannot be solved exactly, but can be formulated by adding a 'small' term to the mathematical description of the exactly solvable problem. It leads to an expression for the desired solution in terms of a power series in some 'small' parameter known as a perturbation series, that quantifies the deviation from the exactly solvable problem. The leading term in this power series is the solution of the exactly solvable problem (Romero, 2013).

Thus, the aim of this study was to construct block method for some k-step LMMs ($k = 1,2,3$, and 4) for the solutions of a general first order IVP in stiff ODEs based on collocation of the differenial equation and interpolation of the approximate solution using chebyshev polynomial of first kind as a perturbed term, which is the extension and modification of the work Nweze et al. (2018). The method is an implicit fourth order block method which is self-starting and solves stiff ODEs. It improves the accuracy of the existing method proposed by Berhan et al. (2019) for solving stiff problems.

1.2 Objectives of the study

1.2.1 General objective

The general objective of this study was to formulate a block procedure with implicit fourth order LMM using Chebyshev polynomials for solutions of stiff first order IVPs.

1.2.2 Specific objectives

The specific objectives of the present study were:

- To derive a block procedure with implicit LMM using the methods of collocation and interpolation using Chebyshev polynomial as a perturbed term.
- To show the convergence of the present method by checking its consistency and zero stability.
- To verify the method using the existing examples.

1.3 Significance of the study

The outcomes of this study have the following importance:

- To find an alternative numerical solution for first order ODE.
- It may give research skills & scientific research procedures for graduate students.
- It may serve as a reference material for interested scholars to conduct their Thesis on this area.

1.4 Delimitation of the Study

This study was delimited to the construction of an efficient numerical method for the numerical solution of general first order IVPs given in the following form:

$$y\prime(x) = f(x, y(x)), \quad y(a) = y_0, \quad a \leq x \leq b. \tag{1.5}$$

Chapter 2

Review of Related literatures

2.1 Stiff Ordinary Differential Equation

Many scientific and engineering problems which arise in real-life applications are in the form of ordinary differential equations (ODEs), where the analytic solution is unknown. Many of these ODEs are known as stiff ODEs. It is difficult to define stiffness in a mathematically rigorous manner, various more or less successful attempts at this may be found in the literature on the subject (Jackiewicz, 2009).

The earliest pioneering of stiffness in differential problems, presented by Curtiss & Hirschfelder (1952), was apparently far in advance of its time. They named the phenomenon and spotted the nature of stiffness (stability requirement dictates the choice of the step size to be very small). To resolve the problem they recommended possible methods such as Backward Differentiation Formulas (BDF) for numerical integration. They also gave a definition of stiffness as follows: *"Stiff equations are equations where certain implicit methods ,in particular BDF, perform better, usually tremendous better, than explicit ones"*.

Butcher (1985) , who points out that *"systems whose solutions contain rapidly decaying components are referred to as stiff differential equations"*. He adds that such problems are important in numerical analysis because they frequently arise in practical problems and because they are difficult to solve by traditional numerical methods.

Burrage (1989) observe that *"stiffness is a difficult concept to define since it manifests itself in so many different ways but the crucial point is that while the solution to be computed is slowly changing, there exist perturbations that are rapidly damped but which complicate computation of the slowly changing solution"*.

Lambert (1991) point out that stiffness occurs when stability requirements rather than those of accuracy constrain the step length, and that stiffness occurs when some components of the solution decay much more rapidly than others. Then he propose a definition that relates to what we observe in practice:*"If a numerical method with a finite* *region of absolute stability, applied to a system with any initial condition, is forced to use in a certain interval of integration a step length which is excessively small in relation to the smoothness of the exact solution in that interval, then the system is said to be stiff in that interval".*

There have been some strong indications that the theory which underpins stiff computation is now quite well understood, and, in particular, the excellent text of Hairer & Wanner (1996) has helped put this theory on a firm basis. They also give the definition of stiffness as follows: *"Stiff equations are problems for which explicit methods don't work".*

LeVeque (2007) observe that *"the difficulty in integrating stiff systems arises from the fact that many numerical methods, including all explicit methods, are unstable in the sense of absolute stability unless the time step is small relative to the time scale of the rapid transient, which in a stiff problem is much smaller than the time scale of the solution we are trying to compute".*

The development of numerical methods for the solution of IVPs of ODEs of the form $y' = f(x,y), y(a) = y_0$ on the interval $[a,b]$ has given rise to two major discrete methods (Anake, 2011). One of those discrete numerical methods used to solve stiff initial value problem is Multistep methods especially implicit linear multistep methods.

2.2 Linear Multistep Methods(LMMs)

The numerical method for the solution of the differential equation $y\prime(t) = f(t,y)$, $y(t_0) = y_0$, $t \in [t_0,b]$ are called linear multistep methods if the value of $y(t)$ at $t = t_{n+l}$ use the values of dependent variable and its derivative at more than one grid or mesh points. An examples of LMM are Adam-Bashforth method, Adam-Moulton method, and Numerov method. The idea of multistep methods appears when some information from previous points has been gained using single step methods. After gaining the information, the value of $f(t,y)$ at the next step is based on interpolation over that information. Computation using these multistep methods is in general more accurate than those in single step methods. LMMs are not as efficient, in terms of function evaluations, as the one step method and also require some values to start the integration process (Anake, 2011). The modern theory of linear multistep methods was developed in large measure (Dahlquist, 1956), and has become widely known through the exposition by Henrici (1962). The general linear multistep methods or k-step methods is given by Lambert (1991) in the form:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \tag{2.1}$$

where $\alpha_j' s$ and $\beta_j' s$ are constants,$\alpha_k \neq 0$ and at least one of the coefficients α_0 and β_0 is non zero.

2.3 Block Method

A block method is formulated in terms of linear multistep methods. Block method have been firstly proposed by Milne (1953). It preserves the traditional advantage of one step methods, of being self-starting and permitting easy change of step length (Lambert, 1991). Their advantage over Runge-Kutta methods lies in the fact that they are less expensive in terms of the number of functions evaluation for a given order. The method generates simultaneous solutions at all grid points.

A substantial amount of research work has been carried out globally on the application of block method to solve numerically the IVPs in ODEs. The techniques for the derivation of block methods for direct solution of IVPs in ODEs have been discussed in literature over the years and these include, among others collocation, interpolation, integral collocation formulation. Basis functions such as, power series, Chebyshev polynomials, trigonometric functions, the Hermite polynomials, Legendre polynomials with collocation approach have been employed (Berhan et al., 2019).

2.4 Chebyshev Polynomial

Chebyshev polynomials are a well-known family of orthogonal polynomials on the interval $[-1,1]$ whose properties and applications were discovered a century ago by the Russian mathematician Patnuty Lvovich Chebyshev $(1821-1894)$. Their importance for practical computations was rediscovered 60 years ago by Comelious Lanczos the father of Numerical Analysis (Boyd, 2000). There are several kinds of Chebyshev polynomials. The most important kinds of Chebyshev polynomial is the Chebyshev polynomials of the first kind over the interval $[-1,1]$. The Chebyshev polynomial possess the following properties by Sunday et al. (2015). Firstly, $|T_n(x)| \leq 1, x \in [-1,1]$. Secondly, $T_n(x)$ is a polynomial of degree n. If n is even, $T_n(x)$ is an even polynomial and if n is odd, $T_n(x)$ is an odd polynomial. Thirdly, $T_n(x)$ is orthogonal with respect to the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$. Here, a list of works briefly detailing their scope & magnitude of block method based on a family of orthogonal polynomials as a perturbed.

Perturbation theory is applicable if the problem at hand cannot be solved exactly, but can be formulated by adding a "small" term to the mathematical description of the exactly solvable problem. It leads to an expression for the desired solution in terms of a power series in some 'small' parameter known as a perturbation series, that quantifies the deviation from the exactly solvable problem. The leading term in this power series is the solution of the exactly solvable problem, while further terms describe the deviation in the solution, due to the deviation from the initial problem (Romero, 2013).

Abualnaja (2015) constructed a block procedure with LMMs (for $k = 1,2$,and 3) using Legendre polynomial for solving first order ODEs. She gave discrete methods used in block and implement it for solving the non-stiff initial value problems, being the continuous interpolant derived and collocated at grid and off-grid points. The results are compared with RK4, but not compared with other existing methods.

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Ajileye et al. (2018) derived hybrid block method algorithms for solution of first Order IVPs using Power series as a basis. They adopted the method of collocation and interpolation of power series approximation to generate the continuous formula. Tested the accuracy of their method by considering two examples.

Nweze et al. (2018) developed a class of block procedure with the implicit LMM in block form for $k = 1,2$ and 3 using Chebyshev polynomials as a basis function for solving non stiff initial value problem in ODEs. The accuracy of their method compared to the exact solution by taking non stiff problems.

Recently, Berhan et al. (2019) extends and Modified the work of Abualnaja (2015). They constructed a block procedure with implicit LMMs for some k-step ($k = 1,2,3$ and 4) using Legendre polynomials for solving stiff IVPs. Their method depends on the perturbed collocation approximation with shifted Legendre polynomials as perturbation term.

As introduced in the literature, there are a growing research works on the development of different numerical methods to get a better approximate solution for stiff first order ODEs with a family of orthogonal polynomial functions,as there is no one best method for all types of problems. While the central activity of numerical analysts is providing accurate and efficient general purpose numerical methods and algorithms, there has always been a realization that some problem types have distinctive features that they will need their own special theory and techniques (Butcher, 2000). Owing this, in this study we proposed a discrete implicit LMM in block form for the solution of stiff first ODEs using the power series as a basis function and Chebyshev polynomial as a perturbed term. The derived method was implemented in block mode which has the advantages of being self-starting, zero-stable, consistent and convergent.

Chapter 3

Methodology

3.1 Study period and site

The study was conducted in Jimma University under the department of Mathematics from September 2018 to October 2019 G.C.

3.2 Study Design

The study employed documentary review design and experimental design use of MATLAB.

3.3 Source of Information

The relevant sources of information for the study were books, published articles, related studies from Internet.

3.4 Procedure of the Study

In order to achieve the stated objectives, the study followed the following steps:

1. Discretization of the interval $[x_k, x_{n+k}]$.
2. Define the power series solution of first order IVPs.
3. Truncate the obtained power series.
4. Transform the given interval $[x_k, x_{n+k}]$ into the interval $[-1, 1]$.

5. Apply the derivative for the obtain truncated power series and then add a perturbed term using Chebyshev polynomial and step 3.

6. Formulate the system of equations using the techniques of interpolation and collocation by applying on step 2 and step 4 respectively, at different grid points.

7. Solve the resulting system of equation.

8. Write MATLAB code for the method.

9. Validate the schemes by using examples and then compare with other results.

Chapter 4

Mathematical Formulation, Result and Discussion

4.1 Preliminaries

In this section, we shall be concerned with the construction and the analysis of numerical methods for stiff first order ordinary differential equations of the form:

$$y'(x) = f\left(x, y(x)\right), \quad y(a) = y_0, \quad x \in [a, b] \tag{4.1}$$

The following terms and concepts have been used in the formulation of the proposed method.

Definition 4.1 (Rivlin, 1990)

The Chebyshev polynomials $T_n(x)$ of the first kind is a polynomial in x of degree n, defined for $x \in [-1, 1]$ by the relation:

$$T_n(x) = \cos(n \arccos x), \quad n = 0, 1, \cdots . \tag{4.2}$$

By combining the trigonometric identity with the above definition we obtain the fundamental recurrence relation :

$$T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0, \quad T_0(x) = 1, \quad T_1(x) = x \tag{4.3}$$

for $n \geq 0$.

We may immediately deduce from (4.3), that the first six Chebyshev polynomials of the first kind are:

$$\begin{aligned}
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1 \\
T_5(x) &= 16x^5 - 20x^3 + 5x \\
T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1
\end{aligned} \tag{4.4}$$

Definition 4.2 Change of range
If a function is defined on $[a,b]$, it is sometimes necessary for the applications to expand the function in a series of orthogonal polynomials in this interval. Clearly the substitution

$$x = \frac{2}{b-a}[t - \frac{b+a}{2}], \qquad a < b \tag{4.5}$$

transforms the interval $[a,b]$, of the t- axis in to the interval $[-1,1]$, of the x- axis (Suli & Mayers, 2003). The Chebyshev polynomials of first kind is orthogonal polynomial in the interval $[-1,1]$.

Definition 4.3 Power series
The power series solution of a function is given in the form:

$$y(x) = \sum_{j=0}^{\infty} a_j (x - x_0)^j, \text{ where, } x_0 \text{ is a constant.} \tag{4.6}$$

If $x_0 = 0$, we have

$$y(x) = \sum_{j=0}^{\infty} a_j x^j \tag{4.7}$$

4.2 Derivation of the proposed method

In this section, we drive the discrete method to solve Eq.(4.1) at a sequence of nodal points $x_n = x_0 + nh$ where h is the step length and defined by $h = x_{n+j} - x_{n+j-1}$ for $j = 0, 1, \cdots, k$ and n is the number of steps which is a positive integer.
Let the power series solutions of the Eq. (4.1) be

$$y(x) = \sum_{j=0}^{\infty} a_j x^j. \tag{4.8}$$

then the approximate solution will be:

$$y(x)=\sum_{j=0}^{k}a_jx^j \quad ,x_n\le x\le x_{n+k}. \tag{4.9}$$

The first derivative of Eq. (4.9) is given by

$$y'(x)=\sum_{j=0}^{k}ja_jx^{j-1} \quad ,x_n\le x\le x_{n+k}. \tag{4.10}$$

Substituting Eq.(4.10) into Eq.(4.1), we obtained

$$y'(x)=\sum_{j=0}^{k}ja_jx^{j-1}\approx f(x,y). \tag{4.11}$$

Now, by adding the perturbed term $\tau T_k(x_{n+j})$ for $j=0(1)k$ to Eq.(4.11), we obtained:

$$\sum_{j=0}^{k}ja_jx^{j-1}=f(x,y)+\tau T_k(x_{n+j}). \tag{4.12}$$

where τ is a perturbed parameter (determined by the values of f_{n+k}) and $T_k(x_{n+j})$ is the k^{th} Chebyshev polynomial obtained by the recursive formula:

$$T_0(x)=1,T_1(x)=x \quad \& \quad T_{k+1}(x)-2xT_k(x)+T_{k-1}(x)=0. \tag{4.13}$$

From Eq.(4.5) we have

$$x=\frac{2}{b-a}[t-\frac{b+a}{2}], \quad a<b. \tag{4.14}$$

This implies that:

$$x=\frac{2t-(b+a)}{b-a}. \tag{4.15}$$

, or equivalently it is the same as

$$x=\frac{2x_{n+j}-(x_{n+k}+x_n)}{x_{n+k}-x_n} \quad k=1,2,3,4\cdots. \tag{4.16}$$

To derive the proposed method for each k, $k=1,2,3,4\cdots$ we should follow the following steps. First, we take the Chebyshev polynomials Eq. (4.4) and use Eq. (4.16) to convert in to the range $[-1,1]$. Using Eq. (4.16) collocate each $T_k(x)$ at $x_{n+j}, j=0,1,2,\cdots k$ to obtain $T_k(x_{n+j})$, where $T_k(x_{n+j})$ is the Chebyshev polynomial at X_{n+j} such that

$-1 \leq T_k(x_{n+j}) \leq 1$
from Eq.(4.11) we deduce that:

$$a_1 + 2xa_2 + 3x^2a_3 + ... + kx^{k-1}a_k = f(x,y) + \tau T_k(x_{n+j}). \tag{4.17}$$

Interpolating Eq.(4.9) at $x = x_n$, collocating Eq.(4.17) at the collocating points x_{n+j} for $j = 0(1)k$ and substituting the relation $x_{n+k} = x_n + kh$, we get a system of $(k+2)$ equations with $(k+2)$ parameters as shown below:

$$\begin{cases} a_0 + a_1x_n + a_2x_n^2 + \cdots + a_kx_n^k & = y_n \\ a_1 + 2a_2x_n + 3a_3x_n^2 + \cdots + ka_kx_n^{k-1} - \tau T_k(x_n) & = f_n \\ a_1 + 2a_2(x_n+h) + 3a_3(x_n+h)^2 + \cdots + ka_k(x_n+h)^{k-1} - \tau T_k(x_{n+1}) & = f_{n+1} \\ & \vdots \\ a_1 + 2a_2(x_n+kh) + 3a_3(x_n+kh)^2 + \cdots + ka_{n+k}(x_n+kh)^{k-1} - \tau T_k(x_{n+k}) & = f_{n+k} \end{cases} \tag{4.18}$$

Eq. (4.18) is a square matrix in the form

$$AX = b \tag{4.19}$$

where

$$A = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & \dots & x_n^k \\ 0 & 1 & 2x_n & 3x_n^2 & \dots & kx_n^{k-1} - T_kx_n \\ 0 & 1 & 2(x_n+h) & 3(x_n+h)^2 & \dots & k(x_n+h)^{k-1} - T_k(x_n+h) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 2(x_n+kh) & 3(x_n+kh)^2 & \dots & k(x_n+kh)^{k-1} - T_k(x_n+kh) \end{pmatrix},$$

$X = [a_0, a_1, a_2, \cdots a_k, \tau]^T$, and $b = [y_n, f_n, f_{n+1}, \cdots f_{n+k}]^T$
Now, the required numerical scheme of the proposed method will be obtained, if we interpolate Eq. (4.9) at x_{n+k} as follows:

$$y_{n+k} = a_0 + a_1x_{n+k} + a_2x_{n+k}^2 + ... + a_kx_{n+k}^k \tag{4.20}$$

and substitute the values of the parameters $\tau, a_0, a_1, a_2, ...,$ and a_k in Eq.(4.20). Now in this study,we will drive the proposed block implicit LMM only for $k = 1, 2, 3, 4$

4.2.1 Derivation of the method for k=1

Using Eq.(4.13) the chebyshev polynomial is $T_1(x) = x$ and by applying Eq.(4.16) at collocating points x_n and x_{n+j}, we get:

$$
\begin{aligned}
T_1(x_n) = T_1(-1) &= \frac{2x_n - x_{n+1} - x_n}{x_{n+1} - x_n} = -1 \\
T_1(x_{n+1}) = T_1(1) &= \frac{2x_{n+1} - x_{n+1} - x_n}{x_{n+1} - x_n} = 1
\end{aligned} \tag{4.21}
$$

Thus, Eq.(4.18) becomes:

$$
\begin{aligned}
a_0 + a_1 x_n = &\quad y_n \\
a_1 + \tau = &\quad f_n \\
a_1 - \tau = &\quad f_{n+1}
\end{aligned} \tag{4.22}
$$

which gives the matrix form

$$
\begin{pmatrix} 1 & x_n & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \tau \end{pmatrix} = \begin{pmatrix} y_n \\ f_n \\ f_{n+1} \end{pmatrix} \tag{4.23}
$$

Solving the matrix above gives the value

$$
\begin{aligned}
\tau = &\quad \frac{1}{2}(f_n - f_{n+1}) \\
a_0 = &\quad y_n - \frac{1}{2}(f_n + f_{n+1}) x_n \\
a_1 = &\quad \frac{1}{2}(f_n + f_{n+1})
\end{aligned} \tag{4.24}
$$

Substituting this values in Eq.(4.20), we obtain:

$$
y_{n+1} = y_n + \frac{h}{2}(f_n + f_{n+1}) \tag{4.25}
$$

Therefore, Eq.(4.25) is the numerical scheme when $k = 1$, which is the well-known trapezoidal rule.

4.2.2 Derivation of the method for k=2

Using Eq.(4.13) the Chebyshev polynomial for $k=2$ is $T_2(x)=2x^2-1$ and by applying Eq.(4.16) at collocating points x_n ,x_{n+1} and x_{n+2}, we get:

$$\begin{aligned} T_2(x_n) &= T_2(-1) = 1 \\ T_2(x_{n+1}) &= T_2(0) = -1 \\ T_2(x_{n+2}) &= T_2(1) = 1 \end{aligned} \tag{4.26}$$

Thus, Eq.(4.18) becomes:

$$\begin{aligned} a_0 + a_1 x_n + a_2 x_n^2 = &\quad y_n \\ a_1 + 2a_2 x_n - \tau = &\quad f_n \\ a_1 + 2a_2 x_{n+1} + \tau = &\quad f_{n+1} \\ a_1 + 2a_2 x_{n+2} - \tau = &\quad f_{n+2} \end{aligned} \tag{4.27}$$

which gives the matrix form

$$\begin{pmatrix} 1 & x_n & x_n^2 & 0 \\ 0 & 1 & 2x_n & -1 \\ 0 & 1 & 2x_{n+1} & 1 \\ 0 & 1 & 2x_{n+2} & -1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \tau \end{pmatrix} = \begin{pmatrix} y_n \\ f_n \\ f_{n+1} \\ f_{n+2} \end{pmatrix} \tag{4.28}$$

Solving the matrix above gives the value

$$\begin{aligned} \tau = &\quad \frac{1}{4}(-f_n + 2f_{n+1} - f_{n+2}) \\ a_0 = &\quad -\frac{1}{4h}(3hf_n x_n + 2hf_{n+1}x_n - hf_{n+2}x_n + f_n x_n^2 - f_{n+2}x_n^2 - 4hy_n) \\ a_1 = &\quad \frac{1}{4h}(3hf_n + 2hf_{n+1} - hf_{n+2} + 2f_n x_n - 2f_{n+2}x_n) \\ a2 = &\quad -\frac{1}{4h}(f_n - f_{n+2}) \end{aligned} \tag{4.29}$$

Substituting this values in Eq.(4.20), we get:

$$y_{n+2} = y_{n+1} + \frac{h}{2}(f_{n+1} + f_{n+2}) \tag{4.30}$$

Therefore, Eq.(4.30) is the numerical scheme when $k=2$.

4.2.3 Derivation of the method for k=3

Using Eq.(4.13) the Chebyshev polynomial for $k=3$ is $T_3(x)=4x^3-3x$ and by applying Eq.(4.16) at collocating points x_n ,x_{n+1} , x_{n+2} and x_{n+3}, we get:

$$\begin{aligned} T_3(x_n) = T_3(-1) &= -1 \\ T_3(x_{n+1}) = T_3(\frac{-1}{3}) &= \frac{23}{27} \\ T_3(x_{n+2}) = T_3(\frac{1}{3}) &= \frac{-23}{27} \\ T_3(x_{n+3}) = T_3(1) &= 1 \end{aligned} \tag{4.31}$$

Thus, Eq.(4.18) becomes:

$$\begin{aligned} a_0+a_1x_n+a_2x_n^2+a_3x_n^3 = &\quad y_n \\ a_1+2a_2x_n+3a_3x_n^3+\tau = &\quad f_n \\ a_1+2a_2x_{n+1}+3a_3x_{n+1}^3-\tfrac{23}{27}\tau = &\quad f_{n+1} \\ a_1+2a_2x_{n+2}+3a_3x_{n+2}^3+\tfrac{23}{27}\tau = &\quad f_{n+2} \\ a_1+2a_2x_{n+3}+3a_3x_{n+3}^3-\tau = &\quad f_{n+3} \end{aligned} \tag{4.32}$$

which gives the matrix form

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & 0 \\ 0 & 1 & 2x_n & 3x_n^2 & 1 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & -\frac{23}{27} \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & \frac{23}{27} \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & -1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \tau \end{pmatrix} = \begin{pmatrix} y_n \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{pmatrix} \tag{4.33}$$

Solving the matrix above gives the value

$$\begin{cases} \tau = \frac{1}{64}(9f_n - 27f_{n+1} + 27f_{n+2} - 9f_{n+3}) \\ a_0 = -\frac{1}{192h^2}(165h^2 f_n x_n + 81h^2 f_{n+1}x_n - 81h^2 f_{n+2}x_n + 27h^2 f_{n+3}x_n + 95hf_n x_n^2 - 45hf_{n+1}x_n^2 \\ -99hf_{n+2}x_n^2 + 49hf_{n+3}x_n^2 + 16f_n x_n^3 - 16f_{n+1}x_n^3 - 16f_{n+2}x_n^3 + 16f_{n+3}x_n^3 - 192h^2 y_n) \\ a_1 = \frac{1}{192h^2}(165h^2 f_n + 81h^2 f_{n+1} - 81h^2 f_{n+2} + 27h^2 f_{n+3} + 190hf_n x_n - 90hf_{n+1}x_n \\ -198hf_{n+2}x_n + 98hf_{n+3}x_n + 48f_n x_n^2 - 48f_{n+1}x_n^2 - 48f_{n+2}x_n^2 + 48f_{n+3}x_n^2) \\ a_2 = \frac{1}{192h^2}(-95hf_n - 45hf_{n+1} - 99hf_{n+2} + 49hf_{n+3} + 48f_n x_n - 48f_{n+1}x_n - 48f_{n+2}x_n \\ +48f_{n+3}x_n) \\ a_3 = \frac{1}{12h^2}(f_n - f_{n+1} - f_{n+2} + f_{n+3}) \end{cases} \tag{4.34}$$

Substituting this values in Eq.(4.20), we get:

$$y_{n+3} = y_{n+2} + \frac{h}{96}(-3f_n + f_{n+1} + 55f_{n+2} + 43f_{n+3}) \tag{4.35}$$

Therefore, Eq.(4.35) is the numerical scheme when $k = 3$.

4.2.4 Derivation of the method for k=4

Using Eq.(4.13) the Chebyshev polynomial for $k = 4$ is $T_4(x) = 8x^4 - 8x^2 + 1$ and by applying Eq.(4.16) at collocating points x_n ,x_{n+1} , x_{n+2} , x_{n+3} and x_{n+4} ,we get:

$$\begin{aligned} T_4(x_n) &= T_4(-1) = 1 \\ T_4(x_{n+1}) &= T_4(\frac{-1}{2}) = \frac{-1}{2} \\ C_4(x_{n+3}) &= T_4(0) = 1 \\ T_4(x_{n+2}) &= T_4(\frac{1}{2}) = \frac{-1}{2} \\ T_4(x_{n+4}) &= T_4(1) = 1 \end{aligned} \tag{4.36}$$

Thus,Eq.(4.18) becomes:

$$\begin{aligned} a_0 + a_1x_n + a_2x_n^2 + a_3x_n^3 + a_4x_n^4 &= y_n \\ a_1 + 2a_2x_n + 3a_3x_n^2 + 4a_4x_n^3 - \tau &= f_n \\ a_1 + 2a_2x_{n+1} + 3a_3x_{n+1}^2 + a_4x_{n+1}^3 + \frac{1}{2}\tau &= f_{n+1} \\ a_1 + 2a_2x_{n+2} + 3a_3x_{n+2}^2 + a_4x_{n+2}^3 - \tau &= f_{n+2} \\ a_1 + 2a_2x_{n+3} + 3a_3x_{n+3}^2 + a_4x_{n+3}^3 + \frac{1}{2}\tau &= f_{n+3} \\ a_1 + 2a_2x_{n+4} + 3a_3x_{n+4}^2 + a_4x_{n+4}^3 - \tau &= f_{n+4} \end{aligned} \tag{4.37}$$

which gives the matrix form

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & 0 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & -1 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & \frac{1}{2} \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & -1 \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & \frac{1}{2} \\ 0 & 1 & 2x_{n+4} & 3x_{n+4}^2 & 4x_{n+4}^3 & -1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \tau \end{pmatrix} = \begin{pmatrix} y_n \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{pmatrix} \tag{4.38}$$

Solving the above system of equation with a suitable method ,we obtain:

$$\begin{cases} \tau = \frac{1}{12}(f_{n+3} - f_n + f_{n+1} - f_{n+2} - f_{n+4}) \\ a_0 = -\frac{1}{48h^3}(44h^3 f_n x_n + 16h^3 f_{n+1} x_n - 24h^3 f_{n+2} x_n + 16h^3 f_{n+3} x_n - 4h^3 f_{n+4} x_n + 34h^2 f_n x_n^2 \\ -32h^2 f_{n+1} x_n^2 - 24h^2 f_{n+2} x_n^2 + 32h^2 f_{n+3} x_n^2 - 10h^2 f_{n+4} x_n^2 + 10h f_n x_n^3 - 16h f_{n+1} x_n^3 \\ -4h f_{n+2} x_n^3 + 16h f_{n+3} x_n^3 - 6h f_{n+4} x_n^3 + f_n x_n^4 - 2f_{n+1} x_n^4 + 2f_{n+3} x_n^4 - f_{n+4} x_n^4 - 48h^3 y_n) \\ a_1 = \frac{1}{24h^3}(22h^3 f_n + 8h^3 f_{n+1} - 12h^3 f_{n+2} + 8h^3 f_{n+3} - 2h^3 f_{n+4} + 34h^2 f_n x_n - 32h^2 f_{n+1} x_n \\ -24h^2 f_{n+2} x_n + 32h^2 f_{n+3} x_n - 10h^2 f_{n+4} x_n + 15h f_n x_n^2 - 24h f_{n+1} x_n^2 - 6h f_{n+2} x_n^2 \\ +24h f_{n+3} x_n^2 - 9h f_{n+4} x_n^2 + 2f_n x_n^3 - 4f_{n+1} x_n^3 + 4f_{n+3} x_n^3 - 2f_{n+4} x_n^3) \\ a2 = -\frac{1}{24h^3}(17h^2 f_n - 16h^2 f_{n+1} - 12h^2 f_{n+2} + 16h^2 f_{n+3} - 5h^2 f_{n+4} + 15h f_n x_n - 24h f_{n+1} x_n \\ -6h f_{n+2} x_n + 24h f_{n+3} x_n - 9h f_{n+4} x_n + 3f_n x_n^2 - 6f_{n+1} x_n^2 + 6f_{n+3} x_n^2 - 3f_{n+4} x_n^2) \\ a_3 = \frac{1}{24h^3}(5h f_n - 8h f_{n+1} - 2h f_{n+2} + 8h f_{n+3} - 3h f_{n+4} + 2f_n x_n - 4f_{n+1} x_n + 4f_{n+3} x_n \\ -2f_{n+4} x_n) \\ a_4 = -\frac{1}{48h^3}(f_n - 2f_{n+1} + 2f_{n+3} - f_{n+4}) \end{cases} \tag{4.39}$$

Substituting this values in Eq.(4.20), we get:

$$y_{n+4} = y_{n+3} + \frac{h}{48}(f_n - 2f_{n+1} - 4f_{n+2} + 34f_{n+3} + 19f_{n+4}) \tag{4.40}$$

Therefore, Eq.(4.40) is the numerical scheme when $k = 4$.

Generally, the proposed block procedure with the implicit LMM is given by:

$$\begin{aligned}
y_{n+1} &= y_n + \frac{h}{2}(f_n + f_{n+1}) \\
y_{n+2} &= y_{n+1} + \frac{h}{2}(f_{n+1} + f_{n+2}) \\
y_{n+3} &= y_{n+2} + \frac{h}{96}(-3f_n + f_{n+1} + 55f_{n+2} + 43f_{n+3}) \\
y_{n+4} &= y_{n+3} + \frac{h}{48}(f_n - 2f_{n+1} - 4f_{n+2} + 34f_{n+3} + 19f_{n+4})
\end{aligned} \tag{4.41}$$

4.3 Analysis of the Method

4.3.1 Order and error constant of the Method

According to Lambert (1991) the general K-step LMM for solving Eq.(4.1) is written in the form:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \tag{4.42}$$

where $\alpha_j' s$ and $\beta_j' s$ are coefficients of the method to be uniquely determined, h is a constant step size.
By using the substitutions $y_{n+j} = z^j$ and $hf_{n+j} = z^j$ where z is a variable and $j = 0, 1, 2, \cdots, k$, for the LMMs given in Eq. (4.42), we introduce at this point a polynomial which is known as characteristic polynomials:

$$\rho(z) = \sum_{j=0}^{k} \alpha_j z^j \quad and \quad \sigma(z) = \sum_{j=0}^{k} \beta_j z^j \tag{4.43}$$

The polynomial $\rho(z)$ and $\sigma(z)$ respectively the first and second characteristics polynomials (Lambert, 1991).
Moreover, following Henrici (1962), the approach developed in Lambert (1991) and Suli & Mayers (2003), we define the local truncation error associated with Eq.(4.42) by the difference operator:

$$\ell[y(x):h] = \frac{1}{h\sum_{j=0}^{k}\beta_j}\left(\sum_{j=0}^{k}[\alpha_j y(x_n + jh) - h\beta_j f(x_n + jh)]\right). \tag{4.44}$$

Assuming $y(x)$ is sufficiently differentiable solutions and expanding the test function

$y(x+jh)$ and its derivative $y(x+jh)$ in Taylor series about x give us:

$$y(x+jh) = y(x) + (jh)y'(x) + \frac{(jh)^2 y''(x)}{2!} + \cdots + \frac{(jh)^p y^{(p)}(x)}{p!} + \cdots. \quad (4.45)$$

$$y'(x+jh) = y'(x) + (jh)y''(x) + \frac{(jh)^2 y'''(x)}{2!} + \cdots + \frac{(jh)^{p-1} y^{(p)}(x)}{(p-1)!} + \cdots. \quad (4.46)$$

substituting Eqs. (4.45) and (4.46) into Eq. (4.44) after collecting like terms we obtain:

$$\ell[y(x):h] = \frac{1}{h\sigma(1)}[C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \cdots C_{p+1} h^{(p+1)} y^{(p+1)}(x)] \quad (4.47)$$

where, $\sigma(1)$ is the value of the second characteristic polynomial at $z=1$ and

$$\begin{aligned} C_0 &= \sum_{j=0}^{k} \alpha_j \\ C_1 &= \sum_{j=0}^{k} \left(j\alpha_j - \beta_j\right) \\ C_p &= \sum_{j=0}^{k} \left(\frac{j^p}{p!}\alpha_j - \frac{j^{p-1}}{(p-1)!}\beta_j\right), \text{ for } p \geq 2. \end{aligned} \quad (4.48)$$

According to Lambert (1991), Eq.(4.41) is order p, if in Eq. (4.47)

$$C_0 = C_1, = \cdots = C_p = 0, \quad \& \quad C_{P+1} \neq 0.$$

In this case, the number $\frac{C_{p+1}}{\sigma(1)}$ is called the error constant of the method.

By using the above properties, let us drive the order and error constants of the proposed method for each steps.

Now, for $k=1$, we have $\alpha_0 = -1, \alpha_1 = 1, \beta_0 = \frac{1}{2}$, and $\beta_1 = \frac{1}{2}$. Thus, from Eq. (4.48), we obtain:

$$\begin{aligned} C_0 &= \alpha_0 + \alpha_1 = 0 \\ C_1 &= \alpha_1 - (\beta_0 + \beta_1) = 0 \\ C_2 &= \frac{\alpha_1}{2} - \beta_1 = 0 \\ C_3 &= \frac{\alpha_1}{6} - \frac{\beta_1}{2} = -\frac{1}{12} \neq 0 \end{aligned}$$

Therefore, for $k=1$ the order is 2 and the error constant is $-\frac{1}{12}$.

For $k=2$, we have $\alpha_0=0, \alpha_1=-1, \alpha_2=1, \beta_0=0, \beta_1=\frac{1}{2}$, and $\beta_2=\frac{1}{2}$. since, from Eq. (4.48), we get:

$$\begin{aligned}
C_0 =& \quad \alpha_0+\alpha_1+\alpha_2=0 \\
C_1 =& \quad \alpha_1+2\alpha_2-(\beta_0+\beta_1+\beta_2)=0 \\
C_2 =& \quad \frac{\alpha_1}{2}+2\alpha_2-(\beta_1+2\beta_2)=0 \\
C_3 =& \quad \frac{\alpha_1}{6}+\frac{8\alpha_2}{6}-(\frac{\beta_1}{2}+2\beta_2)=-\frac{1}{12}\neq 0
\end{aligned}$$

Hence, for $k=2$ the order is 2 and the error constant is $-\frac{1}{12}$

For $k=3$, we have $\alpha_0=0, \alpha_1=0, \alpha_2=-1, \alpha_3=1, \beta_0=-\frac{3}{96}, \beta_1=\frac{1}{96}, \beta_2=\frac{55}{96}$, and $\beta_3=\frac{43}{96}$. since, from Eq. (4.48), we get:

$$\begin{aligned}
C_0 =& \quad \alpha_0+\alpha_1+\alpha_2+\alpha_3=0 \\
C_1 =& \quad \alpha_1+2\alpha_2+3\alpha_3-(\beta_0+\beta_1+\beta_2+\beta_3)=0 \\
C_2 =& \quad \frac{\alpha_1}{2}+\frac{4\alpha_2}{2}+\frac{9\alpha_3}{2}-(\beta_1+2\beta_2+3\beta_3)=0 \\
C_3 =& \quad \frac{\alpha_1}{6}+\frac{8\alpha_2}{6}+\frac{27\alpha_3}{6}-(\frac{\beta_1}{2}+\frac{4\beta_2}{2}+\frac{9\beta_3}{2})=0 \\
C_4 =& \quad \frac{\alpha_1}{24}+\frac{16\alpha_2}{24}+\frac{81\alpha_3}{24}-(\frac{\beta_1}{6}+\frac{8\beta_2}{6}+\frac{27\beta_3}{6})=-\frac{7}{96}\neq 0
\end{aligned}$$

Since for $k=3$ the order is 3 and error constant is $-\frac{7}{96}$.

For $k=4$, we have $\alpha_0=0, \alpha_1=0, \alpha_2=0, \alpha_3=-1, \alpha_4=1, \beta_0=\frac{1}{48}, \beta_1=-\frac{2}{48}, \beta_2=-\frac{4}{48}, \beta_3=\frac{34}{48}$, and $\beta_4=\frac{19}{48}$. Since, from Eq. (4.48), we get:

$$\begin{aligned}
C_0 =& \quad \alpha_0+\alpha_1+\alpha_2+\alpha_3+\alpha_4=0 \\
C_1 =& \quad \alpha_1+2\alpha_2+3\alpha_3+4\alpha_4-(\beta_0+\beta_1+\beta_2+\beta_3+\beta_4)=0 \\
C_2 =& \quad \frac{\alpha_1}{2}+\frac{4\alpha_2}{2}+\frac{9\alpha_3}{2}+\frac{16\alpha_4}{2}-(\beta_1+2\beta_2+3\beta_3+4\beta_4)=0 \\
C_3 =& \quad \frac{\alpha_1}{6}+\frac{8\alpha_2}{6}+\frac{27\alpha_3}{6}+\frac{81\alpha_4}{6}-(\frac{\beta_1}{2}+\frac{4\beta_2}{2}+\frac{9\beta_3}{2}+\frac{16\beta_4}{2})=0 \\
C_4 =& \quad \frac{\alpha_1}{24}+\frac{16\alpha_2}{24}+\frac{81\alpha_3}{24}+\frac{256\alpha_4}{24}-(\frac{\beta_1}{6}+\frac{8\beta_2}{6}+\frac{27\beta_3}{6}+\frac{64\beta_4}{6})=0 \\
C_5 =& \quad \frac{\alpha_1}{120}+\frac{32\alpha_2}{120}+\frac{243\alpha_3}{120}+\frac{1024\alpha_4}{120}-(\frac{\beta_1}{24}+\frac{16\beta_2}{24}+\frac{81\beta_3}{24}+\frac{256\beta_4}{24})=-\frac{17}{360}\neq 0
\end{aligned}$$

Hence, for $k=4$ the order is 4 and error constant is $-\frac{17}{360}$

Therefore, the order and the error constant of Eq.(4.41) are given by table below.

Table 4.1: Order and Erroe constant of the scheme

step size	Error	Error constant
$k = 1$	2	$-\frac{1}{12}$
$k = 2$	2	$-\frac{1}{12}$
$k = 3$	3	$-\frac{7}{96}$
$k = 4$	4	$-\frac{17}{360}$

4.3.2 Stability Analysis of the block Method

This section presents the stability analysis of the method, that is Eq.(4.41). We begins by presenting the definition of zero-stability taken from (Suli & Mayers, 2003).

Definition 4.4:- A linear multistep method (LMM) is said to be zero stable if no root of the first characteristics polynomial has modulus greater than one and that any root with modulus one is simple.
In other words an LMM is said to be zero stable if

$$|z| \leq 1 \quad \text{and, if z is a repeted root, then} \quad |z| < 1 \tag{4.49}$$

Based on the definition 4.4, let us clarify the zero stability of the block method for each scheme as follows:
For $k = 1$, we have

$$\rho(z) = z - 1 = 0, \text{ implies } \quad z = 1, \text{and} \quad |z| \leq 1 \text{ is satisfied.}$$

For $k = 2$, we have

$$\rho(z) = z^2 - 1 = 0, \text{which implies} \quad z = 1, \& \quad z = -1.$$
$$\text{since in both cases } |z| \leq 1 \text{ is satisfied.}$$

For $k = 3$, we have

$$\rho(z) = z^3 - z^2 = z^2(z-1) = 0, \text{which implies} \quad z = 0, \& \quad z = 1.$$
$$\text{since in both cases } |z| \leq 1 \text{ is satisfied.}$$

For $k = 4$, we have

$$\rho(z) = z^4 - z^3 = z^3(z-1) = 0, \text{which implies} \quad z = 0, \& \quad z = 1.$$
$$\text{since in both cases } |z| \leq 1 \text{ is satisfied.}$$

Therefore, our method is zero stable.

4.3.3 Convergence of the Method

Convergence is an essential feature that every acceptable LMM must possess. This section discuss the convergence of the proposed block method.
Definition 4.5 (Lambert, 1991)
A LMM is said to be consistent if it has order at least one.
In other words, from E.q (4.48), we verify that

$$C_0 = \rho(1) = \sum_{j=0}^{k} \alpha_j, \quad \text{and} \quad C_1 = \rho'(1) - \sigma(1) = \sum_{j=0}^{k} \left(j\alpha_j - \beta_j \right) \tag{4.50}$$

So, According to Faul (2018) consistence is expressed by the relations :

$$\rho(1) = 0, \text{ and } \quad \rho'(1) = \sigma(1) \tag{4.51}$$

Owing to this expression , the method is consistent.

Definition 4.6 (Dahlquist, 1974)
Consistency and zero stability are the necessary and sufficient conditions for the convergence of any numerical schemes. That means:

convergent= consistency + zero stability.

Since our method is both consistent and zero stable, it is thus converges.

4.4 Numerical Example

The mode of implementation of our method is by combining the schemes Eq.(4.41) as a block for solving Eq.(4.1). It is a simultaneous integrator without requiring the starting values except the initial condition from the problem. To assess the performance of the proposed block method, we consider two stiff first order initial value problems. The maximum absolute errors (MAXAE) of the proposed method is compared with that of fourth order Runge-Kutta (RK4), and the method developed by Berhan et al. (2019) namely block procedure with implicit method using shifted Legendre polynomial as a perturbed term. All calculations are carried out with the aid of MATLAB software.

Example 1: Consider the first order stiff ODE, LeVeque (2007)

$$y'(x) = -2100(y - \cos(x)) - \sin(x), \quad y(0) = 1, \quad x \in [0, 1] \tag{4.52}$$

The exact solution is

$$y(x) = \cos(x)$$

Example 2: Consider the first order stiff ODE, Faul (2018).

$$y'(t) = -1000000(y - \frac{1}{t}) - \frac{1}{t^2}, \quad y(1) = 1, \quad x \in [1,2] \tag{4.53}$$

The exact solution is

$$y(t) = \frac{1}{t}$$

Table 4.2: Maximum Absolute errors of RK4, Berhan et al. (2019) and the present method for Example 1.

h	$RK4$	$Berhanet\ al.$ (2019)	$PresentMethod$
10^{-1}	$1.22516e{+}74$	$1.22516e{-}5$	$5.86307e{-}7$
10^{-2}	$2.41053e{+}304$	$9.67880e{-}8$	$5.71593e{-}9$
10^{-3}	$1.53563e{-}7$	$6.46040e{-}11$	$3.33170e{-}11$
10^{-4}	$5.09304e{-}12$	$3.33844e{-}13$	$3.33844e{-}13$
10^{-5}	$1.22125e{-}15$	$4.10783e{-}15$	$4.10783e{-}15$

Table 4.3: Maximum Absolute errors of RK4, and the present method for Example 2.

h	$RK4$	$presentMethod$
10^{-1}	$7.507779e{+}178$	$1.265948e{-}8$
10^{-2}	$5.115563e{+}298$	$1.129138e{-}10$
10^{-3}	$4.207597e{+}298$	$9.959812e{-}13$
10^{-4}	$1.091337e{+}300$	$9.769963e{-}15$
10^{-5}	$5.898859e{+}301$	$2.220446{-}16$
10^{-6}	$3.352874e{-}14$	$2.220446{-}16$

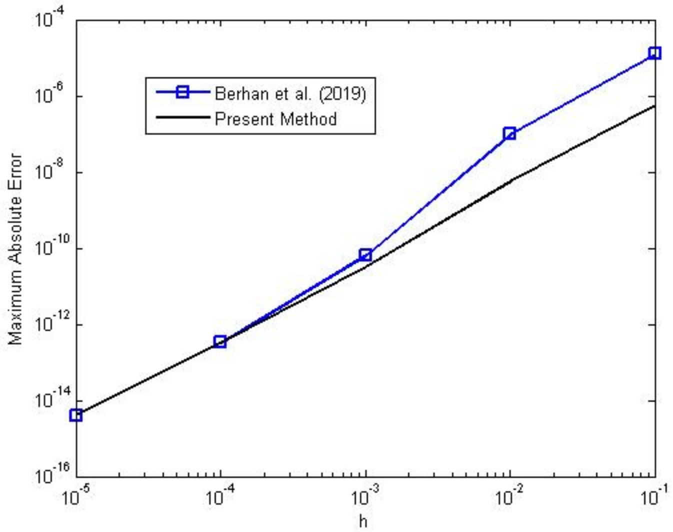

Figure 4.1: Maximum absolute Error of Berhan et al. (2019) and the present method

4.5 Discussion

The block procedure that we have presented in this thesis showed that it solved stiff ODEs numerically. The method gave an accurate result for the general first order ODE and the approximate solution is obtained from the solved problem showed that the efficiency of the numerical method. This method used polynomial interpolation point along with collocation point to solve stiff ODEs. In doing this two stiff problems have been solved and the approximate solution is obtained for this two expriments elucidates the accuracy of the proposed method.

It is observed from the tables that, as the step size h decreases, the method is more accurate (as shown in table 4.1 and 4.2) than RK4 and Berhan et al. (2019). i.e., In table 4.1 for $h = 10^{-1}$ RK4 divergent, Berhan et al. (2019) convergent but the present method is more converge. for $h = 10^{-2}$ RK4 divergent and Berhan et al. (2019) convergent but the present method is more convergent. for $h = 10^{-3}$ RK4 convergent and Berhan et al. (2019) convergent but the present method is more convergent, for $h = 10^{-4}$ RK4 convergent and both Berhan et al. (2019) and the present method are similarly convergent, for $h = 10^{-5}$ RK4 more convergent than Berhan et al. (2019) and the present method but Berhan et al. (2019) and the present method similar to convergent. In table 4.2 for $h = 10^{-1}$ to $h = 10^{-5}$ RK4 divergent but the present method increase the convergent, for $h = 10^{-6}$ RK4 convergent but the present method more convergent. From the graph we also observed that the maximum absolute error of the proposed method is more accurate than the maximum absolute error of Berhan et al. (2019).

Generally, the performances of our method as shown in tables 4.1 and 4.2, are better than the existing methods for the same step size.

4.5.1 Conclusion

This study presented a block procedure with implicit linear multistep method based on power series used as a basis and Chebyshev polynomials used as a perturbed term for solving stiff first order IVPs. A collocation approach along with interpolation at some grid points which produces a family of maximal fourth order schemes have been proposed for the numerical solution of stiff problems in ODEs. The properties of the Chebyshev polynomials are used to introduce the proposed problems to system of equations which are solved by a suitable method. The desirable property of a numerical solution is to be have like the theoretical solution of the problem which can be seen in the above experimental results. The method is tested and found to be consistent, zero stable and convergent. We implement the method on two numerical examples and the numerical evidences shows that the proposed method perform favorable when compared with existing scheme as it yielded better accuracy and effective for stiff problems. Therefore, our method is simple and better to solve stiff IVPs when compared with the existed methods.

4.5.2 Future Scope

In this study, a collocation approach which produces a block procedure with implicit LMM based on power series used as a basis and Chebyshev polynomials using as a perturbed term for solving stiff first order IVPs with non uniform orders has been proposed. Hence, further research should be performed to enhance the order of the differential equation and taking into consideration the off grid points to produce a method of uniform orders. Also performed for solving nonlinear differential equations.

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