Block Procedure with Implicit Sixth Order Linear Multistep Method using Legendre Polynomials for Solving Stiff First Order Initial Value Problems for ODEs.

A Thesis Submitted to the Department of Mathematics, College of Natural Science, Jimma University, for the Partial Fulfillment of the Requirement of Master of Science in Mathematics.

(Numerical Analysis)

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DECLARATION

I undersigned declare that, this research work entitled **"Block Procedure with Implicit Sixth Order Linear Multistep Method using Legendre Polynomials for Solving Stiff First Order Initial Value Problems for ODEs."** is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged.

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Abstract:

In this study, discrete sixth order implicit linear multistep methods (LMM) in block form of uniform step size for the solution of initial value problems (IVPs) for ordinary differential equations (ODEs) was presented using the Legendre polynomials. The method is based on collocation of the differential equation and interpolation of the approximate solution of power series at the grid points. The procedure yields four consistent linear multistep schemes which are combined as simultaneous numerical integrators to form block method. The method is found to be consistent and zero-stable hence convergent. The accuracy of the method is tested with some standard stiff first order initial value problems. The results are compared with fourth order Runge-Kutta and with the implicit backward difference methods 2BBDF and 2BEBDF. All numerical examples show that our proposed method has a better performance over the existing methods.

Keywords: Block Procedure, Collocation, Interpolation, Legendre Polynomials, LMM, Stiff.

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Acronyms:

2BBDF: 2-point Block Backward Difference Formula

2BEBDF: 2-point Block Extended Backward Difference Formula

BDF: Backward Difference Formula

DDE: Delay Differential Equation

IVPs: Initial Value Problems

LMM: Linear Multistep Method

LTE: Local Truncation Error

MAE: Maximum Absolute Error

ODEs: Ordinary Differential Equations

PM: Proposed Method

RK: Runge-Kutta

CHAPTER ONE

Introduction

1.1. Background of the Study

Many problems in science and engineering can be formulated either in the form of the initial value problems or in terms of the boundary value problems. An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation. A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an ordinary differential equation. The order of the highest order derivative involved in a differential equation is called the order of the differential equation (Shepley, 2004). Hence, differential equations will typically be written in the form $y(x) = f(x, y(x)),\quad \text{where} \quad y' = \dfrac{dy}{dx}$.

Ordinary differential equations frequently occur in mathematical models that arise in many branches of science, engineering and economics. Unfortunately it is seldom that these equations have solutions which can be expressed in closed form, so it is common to seek approximate solutions by means of numerical methods. Nowadays this can usually be achieved very inexpensively to high accuracy and with a reliable bound on the error between the analytical solution and its numerical approximation. In this study we are concerned with the construction and analysis of numerical method for solving first-order initial value problems for ordinary differential equations of the form:

$$y'(x) = f(x, y(x))$$

$$y(x_0) = y_0, \qquad (1)$$

The field of numerical analysis not only develops methods but also analyses them by three central concepts such as convergence, stability and order.

Linear Multistep Method (LMM) is a computational procedure where by a numerical approximation y_{n+j} to the exact solution $y(x_{n+j})$ of the first order Initial Value Problems (IVPs) of (1) is obtained. In LMM to find the k^{th} approximate value we use the already calculated previous k approximate values. Given a sequence of equally spaced mesh points x_n with step size h, the general k-step LMM is as given in (Lambert, 1973):

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \qquad (2)$$

where the coefficients $\alpha_0, \alpha_1, \ldots, \alpha_k$ and $\beta_o, \beta_1, \ldots, \beta_k$ are real constants. In order to avoid degenerate cases, we shall assume that $\alpha_k \neq 0$ and that α_0 and β_0 are not both equal to zero. If $\beta_k = 0$, then y_{n+k} is obtained explicitly from previous values of y_j, $f(x_j, y_j)$, where $j = 0, 1, 2, \ldots, k-1$ and the k step method is said to be explicit. On the other hand, if $\beta_k \neq 0$, then appears not only on the left-hand side but also on the right, within $f(x_{n+k,} y_{n+k})$ due to this implicit dependence on y_{n+k} the method is then called implicit. Equation(2) is called linear because it involves only linear combinations of the y_{n+j} and the $f(x_{n+j}, y_{n+j})$, $j = 0, 1, \ldots, k$ for the sake of notational simplicity, henceforth we shall often write f_n instead of $f(x_n, y_n)$ .The linear multistep method in equation (2) generates discrete schemes which are used to solve first-order ODEs. A family of two-steps block generalized Adams methods were developed for the solution of Non-stiff IVPs in ODEs (Abudullahi *et al.,* 2014). However stiff differential equations arise in many areas of science and technology. Their solutions are known to be numerically unstable with many numerical methods, unless the step size taken is extremely small. A linear system of ODEs with constant coefficients is called stiff, when all its eigenvalues have negative real part and its stiffness ratio is large. A system is stiff, if certain components decay much faster than others. For one thing, it is possible even for a scalar problem to be stiff even though for a scalar problem the stiffness ratio is always one since there is only one eigenvalue (Randall, 2004). Brugnano *et al.* (2011) proposed the methods that have unbounded region of absolute stability to overcome the stability restriction on the step size for the solution of stiff IVPs. One of the most popular methods for solving stiff ODEs of the form (1) is the backward differentiation formula (Curtiss and Hirschfelder, 1952), which is a linear multistep method. Suleiman *et al*. (2015) proposed an implicit 2-point block extended backward differentiation formula for integration of stiff initial value problems and checked the performance of their methods by considering stiff problems. A number of researchers have developed LMM of the type (2) for the solution of first order IVPs in ODEs. Moreover, power series has also being extensively used in literature for the same purpose. James *et al.* (2013) developed a continuous block method using the approach of collocation of the differential system and interpolation of the power series approximate solution at the grid and off grid points.

The interpolation problem is to construct a function $q(x)$ that passes through each values of $f(x)$ at grid points $x_{0,} x_1, x_2, \ldots, x_n$. To find a function $q(x)$ the interpolation requirements $q(x_j) = f(x_j), 0 \leq j \leq n$, are satisfied. The points $x_0, x_1, \ldots, x_n$ are called the interpolation points. The property of passing through $f(x_j)$ is referred to us interpolating the data. The function that interpolates the data is an interpolant or an interpolating polynomial (Doron, 2010). If $x_0, x_1, \ldots\ldots, x_n$ are distinct, then for any $f(x_0), f(x_1), \ldots, f(x_n)$ there exists a unique polynomial $q_n(x)$ of degree $\leq n$ such that the interpolation conditions $q_n(x_j) = f(x_j),\ j = 0, 1, \ldots, n$ are satisfied.

A collocation solution u_h to a functional equation (for example an ordinary differential equation) on an interval I is an element from some finite-dimensional function space (the collocation space) which satisfies the equation on an appropriate finite subset of points in I (the set of collocation points) whose cardinality essentially matches the dimension of the collocation space . If initial conditions are present, then u_h will usually be required to fulfill these conditions, too. For initial-value problems in ordinary differential equations such collocation methods were first studied systematically in the late 1960 (Harmann, 2004). Suppose that ODEs of equation (1), is to be solved over the interval $\left[x, x_0 + h\right]$. Choose $0 \leq c_1 < c_2 < \ldots < c_n \leq 1$. The corresponding (polynomial) collocation method approximate the solution y by the polynomial p of degree n which satisfies the initial condition $p(x_0) = y_{0,}$ and the differential equation $p'(x_0) = f(x_{0,} p(x_0))$ at all collocation points $x = x_0 + c_k h$ for $k = 1, 2, \ldots, n$. This gives conditions which matches the $n+1$ parameters needed to specify a polynomial of degree n .

Several continuous LMMs have been derived using interpolation and collocation for the solution of equation (1) with constant step size using Legendre polynomials.

Khader *et al.* (2014) developed an integral collocation approach based on Legendre polynomials for solving Riccati, Logistic and delay differential equations. The properties of the Legendre polynomials helped them to reduce the proposed problems to the solution of nonlinear systems of algebraic equations using Newton iteration method.

Legendre polynomials that were first studied by the French mathematician Darien-Marie Legendre (1752-1833).The Legendre polynomial originated in determining the force of attraction exerted by solids of revolution and their orthogonal properties were established by A.M. Legendre during 1784-1790. Substantial amount of research work has been carried out globally on the application of orthogonal functions to various fields of engineering. The Legendre polynomials are derived from the simple recursive formula (Poularikas, 1999).

The recursive formula is:

$$(n+1)p_{n+1}(x) - (2n+1)xp_n(x) + np_{n-1}(x) = 0 \qquad (4)$$

The Legendre polynomials are orthogonal in [-1, 1] since they satisfied the inner product property which is given by:

$$\int_{-1}^{1} p_n(x)p_m(x)dx = 0 \text{ for } n \neq m. \text{If } m = n, \text{ then } \int_{-1}^{1} [p_n(x)]^2 dx = \frac{2}{2n+1}, n = 0,1,2,\ldots$$

and therefore the set $\varphi_n(x) = \sqrt{\frac{2n+1}{2}}\, p_n(x), n = 0,1,2,\ldots$ is orthonormal.

Yakusak *et al.* (2014) proposed uniform order (all the schemes have the same order) Legendre approach for continuous hybrid methods for the solution of first order ordinary differential equations. They adopted the method of interpolation of the approximation and collocation of its differential equation and with Legendre polynomials of the first kind as basis function. Kumleng *et al.* (2013) proposed a new three and five step block linear methods based on the Adams family for the direct solution of stiff initial value problems (IVPs). Recently, Abualnaja, (2015), constructed a block procedure with linear multistep methods using Legendre polynomials for solving ODEs. The researcher derived a block procedure for some k-step linear multistep methods (k=1, 2 and 3). It is an explicit method and used for solving non-stiff initial value problems. The method depends on the perturbed collocation approximation with shifted Legendre polynomials as perturbation term and with maximum order of four.

In this study, we proposed a block procedure with implicit linear multistep methods for direct solutions of equation (1). It is the method of interpolation of approximate solutions of power series and collocation of its differential equations using shifted Legendre polynomials as perturbed term with uniform step size. The shifted Legendre polynomials are polynomials which are transformed from intervals [a, b] to the interval [-1, 1]. According to Solomon Gebregiorgis and Genanew Gofe, (2016), there is no one best method in terms of accuracy, since the performance of a given method depends on the characteristics of the ODEs we are considering such as stiffness and stability. So the performance of the proposed method will be checked by comparing the numerical results with known stable and convergent methods such as fourth order Runge-Kutta, 2BBDF and 2BEBDF.

1.2. Statement of the Problem

Many researchers in numerical analysis proposed different methods to solve the first order initial value problems in ordinary differential equations. Abualnaja, (2015) constructed a block procedure with linear multistep method using Legendre polynomials. The researcher derived a block procedure for some k-steps linear multistep methods (k=1, 2 and 3). It is an explicit method which solves only non-stiff initial value problems. The method depends on the perturbed collocation approximation with Legendre polynomials as perturbation term and with maximum order of four. Suleiman *et al*. (2015) proposed an implicit 2-point block extended backward differentiation formula for integration of stiff initial value problems and checked the performance of their methods by considering stiff problems. In this study, we are going to formulate the approximate solutions for first order IVPs in ODEs by applying the implicit linear multistep method in block form for $k = 1, 2, 3, 4$ based on Legendre polynomials. To this end, the present study attempted to answer the following basic questions:

- How do we describe the block procedure with implicit linear multistep methods using shifted Legendre polynomials as a perturbed term?
- How do we describe the convergence of the proposed methods?
- How do we validate the scheme using numerical examples?
- How the proposed method compared with the existing methods?
- To what extent the proposed method approximate the exact solutions?

1.3. Objective of the study

1.3.1. General objective

The general objective of this study is to formulate a block procedure with implicit sixth order linear multistep method using Legendre polynomials for solutions of stiff first order initial value problems in ordinary differential equations.

1.3.2. Specific objectives

The specific objective of this study is:

- To derive a block procedure with implicit LMM by the method of collocation and interpolation using shifted Legendre polynomials as a perturbed term.
- To show the consistence, stability and convergence of the present method.
- To get a block procedure schemes for solving stiff problems.
- To approximate the exact solutions with minimal errors.

1.4 Significance of the study

The outcome of this research has the following importance:

- It may provide some background information for other researchers who want to work on similar topics.
- It may have some contribution in solving stiff IVPs for ODEs.
- Further, this research may be useful for the graduate program of the department.

1.5. Delimitation of the study

This study delimited to the numerical solution of stiff first order initial value problems using implicit sixth order block procedure with LMM by considering the shifted Legendre polynomials as a perturbation term.

CHAPTER TWO

2. Review Literature

The idea of extending the Euler method by allowing the approximate solution at a point to depend on the solution values and the derivative values at several previous step values was originally proposed by Bashforth and Adams (1883). Not only was this special type of method, now known as the Adams–Bashforth method, introduced, but a further idea was suggested. This further idea was developed in detail by Moulton (1926). Other special types of linear multistep methods were proposed by Nystrom (1925) and Milne (1926, 1953). The idea of predictor–corrector methods is associated with the name of Milne, especially because of a simple type of error estimate available with such methods. The 'backward difference' methods were introduced by Curtiss and Hirschfelder (1952), and these have a special role in the solution of stiff problems (Butcher, 2008).

The problem of stiffness leads to computational difficulty in many practical problems. In general a problem is called stiff if, roughly speaking, we are attempting to compute a particular solution that is smooth and slowly varying (relative to the time interval of the computation), but in a context where the nearby solution curves are much more rapidly varying. In other words if we perturb the solution slightly at any time, the resulting solution curve through the perturbed data has rapid variation (Randall, 2004).

A substantial amount of research work has been carried out globally on the application of LMM to solve numerically the IVPs in ODEs. The techniques for the derivation of continuous linear multistep methods (LMMs) for direct solution of initial value problems in ordinary differential equations have been discussed in literature over the years and these include, among others collocation, interpolation, block forms as simultaneous numerical integrators , integral collocation formulation. Basis functions such as, power series, Chebyshev polynomials, trigonometric functions, the Hermite polynomials; Legendre polynomials with collocation approach have been employed. Legendre polynomials are a well-known family of orthogonal polynomials on the interval [-1, 1] that have many applications. They are widely used because of their properties in the approximation of functions. Here, a list of works briefly detailing their scope and magnitude of LMM based on Legendre polynomial functions.

Olabode and Momoh (2016) proposed the continuous hybrid multistep methods with Legendre basis function for direct treatment of second order stiff ODEs. They achieved the method by constructing a continuous representation of hybrid multistep schemes via interpolation of the approximate solution and collocation of derivative function with Legendre polynomial as basis functions. The discrete schemes were obtained from the continuous scheme as a by-product and applied in block form as simultaneous numerical integrators to solve initial value problems (IVPs). The resultant schemes are self-starting; do not need the development of separate predictors. They used the Legendre polynomial without perturbation as basis functions for the construction of continuous schemes, which simultaneously generate solution of the differential equations. They developed two block methods with the same order of five. The derived methods were implemented in block mode which have the advantages of being self-starting, stable and convergent.

Khader *et al.* (2014) developed an integral collocation approach based on Legendre polynomials for solving Riccati, Logistic and Delay Differential Equations (DDE). The properties of the Legendre polynomials helped them to reduce the proposed problems and leads to the solution of non-linear system of algebraic equations using Newton iteration method. In their article, an integral collocation approach based on shifted Legendre polynomials on the intervals [0, 1] (using the change of variable $z = 2x - 1$) was introduced and they conclude that their approximation is in agreement with the exact solution.

Recently, perturbation methods have been gaining much popularity. Perturbation theory comprises mathematical methods for finding approximate solutions to a problem, by starting from the exact solution of a related, simpler problem. Perturbation theory is applicable if the problem at hand cannot be solved exactly, but can be formulated by adding a "small" term to the mathematical description of the exactly solvable problem. Perturbation theory leads to an expression for the desired solution in terms of a power series in some ‘small’ parameter- known as a perturbation series-that quantifies the deviation from the exactly solvable problem. The leading term in this power series is the solution of the exactly solvable problem, while further terms describe the deviation in the solution, due to the deviation from the initial problem (Romero, 2013). Perturbation theory was investigated by the classical scholars, as the result of which the computations could be performed with a very high accuracy.

Abualnaja, (2015) constructed a block procedure with linear multistep methods using Legendre polynomials for solving ODEs. The researcher derived a block procedure for some k-steps linear multistep methods (k=1, 2 and 3) using the methods of collocation and interpolation. The method depends on the perturbed collocation approximation by shifting the Legendre polynomials to the intervals [-1, 1] using the change of variable (change of range). The researcher exploited the derivative of power series approximation along with the perturbed Legendre polynomials. By collocating the derivatives of power series at the grid points $x_{n+j},\ j = 0,1,2,\ldots,k$ and interpolating the power series at x_n and generated a system of equations. At last by interpolate the approximate power series solution at x_{n+1} and substituting the values of the parameters developed the method. The order of the scheme is four, the method is explicit and it is for non-stiff IVPs. In our study we tried to develop a block procedure with implicit sixth order LMM which is for solving stiff IVPs.

CHAPTER THREE

3. Methodology

3.1. Study area and period

The study has been conducted in Jimma University under the supervision of the department of mathematics from September 2016 G.C. to October 2017 G.C.

3.2. Study Design

This study employed both documentary review and experimental design on the first order IVPs in ODEs. So the study design is a mixed type.

3.3. Source of information

The relevant sources of information for this study are books, published articles and related studies and the experimental results obtained by using MATLAB software for the present methods (MATLAB version is 2013).

3.4. Mathematical Procedures

Important results and data relevant to the study have been collected by means of documentary review. Then it is further analyzed and extended with the aim of improving the existing method. In order to verify the effectiveness of the proposed method numerical data have been collected & graphs have been sketched by coding the schemes and running using MATLAB software. To do so, the study has followed the following procedures:

1. Define the power series solution of first order IVPs in ODEs.
2. Truncate the approximate solution of power series.
3. Use the change interval (change of range) in order to convert the Legendre polynomials at x_{n+j}, $j = 0,1,2,\dots,k$ to the interval [-1, 1].
4. Apply the derivative of approximate power series solution by introducing a perturbed term, which is the product of perturbation parameter λ and shifted Legendre polynomial, Where λ is determined by the values of f_{n+k} .
5. Formulate the system of equations using the methods of collocation and interpolation and solve them.
6. Check the consistency, stability and convergence of the proposed new method.
7. Write MATLAB code for the block method (MATLAB ver. 2013).
8. Validate the schemes using stiff numerical examples and compare results with Runge-Kutta, 2BBDF and 2BEBDF methods.

CHAPTER FOUR

4. Mathematical formulation, Result and discussion

4.1. Preliminaries

In this section we shall be concerned with the construction and the analysis of numerical methods of first-order IVPs for ordinary differential equations of the form

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0, \tag{4.1}$$

The following terms and concepts have been exploited in the formulation of our proposed method:

A. Legendre polynomials:

The Legendre polynomials $p_n(x)$ are derived from the simple recursive formula:

$$(n+1)p_{n+1}(x) - (2n+1)xp_n(x) + np_{n-1}(x) = 0 \tag{4.3}$$

With $p_0(x) = 1, \; n = 0, 1, 2, \ldots.$

where the first six Legendre polynomials of the first kind are:

$$\begin{aligned}
p_0(x) &= 1 \\
p_1(x) &= x \\
p_2(x) &= \frac{3}{2}x^2 - \frac{1}{2} \\
p_3(x) &= \frac{5}{2}x^3 - \frac{3}{2}x \\
p_4(x) &= \frac{35}{8}x^4 - \frac{30}{8}x^2 + \frac{3}{8} \\
p_5(x) &= \frac{63}{8}x^5 - \frac{70}{8}x^3 + \frac{15}{8}x
\end{aligned} \tag{4.4}$$

B. Change of range: If a function $f(x)$ is defined on [a, b], it is sometimes necessary in the applications to expand the function in a series of orthogonal polynomials in this interval. Clearly the formula given by:

$$x = \frac{2}{b-a}\left[t - \frac{b+a}{2}\right], \qquad a < b \tag{4.5}$$

transforms the interval [a, b] of the t- axis in to the interval [-1, 1] of the x- axis.

C. Power series:

Assuming that x_0 is an ordinary point of the differential equation, the solutions in powers of $x - x_0$ actually do exist (Shepley, 2004); we denote such a solution by

$$y = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \dots$$

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

If $x_0 = 0$, we have

$$y = \sum_{n=0}^{\infty} c_n x^n \qquad (4.6)$$

Since the series converges on an interval $\left| x - x_0 \right| < R$ about x_0, it may be differentiated term by term on this interval

$$\frac{dy}{dx} = c_1 + 2c_2(x - x_0) + 3c_3(x - x_0)^2 + \dots$$

If $x_0 = 0$, we have

$$\frac{dy}{dx} = \sum_{n=1}^{\infty} n c_n x^{n-1} \qquad (4.7)$$

D. Existence of unique solution:

Picard's Theorem (4.1): Suppose that $f(x, y)$ is a continuous function of its arguments in a region U of the (x, y) plane which contains the rectangle

$R = \left\{ (x, y) : x_0 \le x \le X_M , \left| y - y_0 \right| \le Y_M \right\}, where\ X_M > x_0\ and\ Y_M > 0$ are constants.

Suppose also, that there exists a positive constant L such that

$$\left| f(x, y) - f(x, z) \right| \le L \left| y - z \right| \qquad (4.8)$$

holds whenever (x, y) and (x, z) lie in the rectangle R. Finally, letting $M = \max \left\{ \left| f(x, y) \right| : (x, y) \varepsilon R \right\}$, suppose that $M(X_M - x_0) \le Y_M$. Then there exists a unique continuously differentiable function $x \to y(x)$, defined on the closed interval $[x_0, X_M]$, which satisfies equation (1). The condition (4.8) is called a Lipschitz condition, and L is called the Lipschitz constant for f (Suli and Mayers, 2003).

E. The k-step method

The general k -Step method for (4.1) is written in the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \qquad (4.9)$$

where α_j, β_j are coefficients of the method to be uniquely determined and $\alpha_j^2 + \beta_j^2 \ne 0$, h is a constant step size and k is the step number.

F. Local truncation error (LTE)

For LMMs it is easy to derive the general formula for the local truncation error (Suli and Mayers, 2003we have

$$T(x_{n+k}) = \frac{1}{h\sum_{j=0}^{k}\beta_j}\left(\sum_{j=0}^{k}\alpha_j y(x_{n+j}) - h\sum_{j=0}^{k}\beta_j y'(x_{n+j})\right), \qquad f(y(x_{n+j})) = y'(x_{n+j}) \qquad (4.10)$$

where, $y(x)$ is the exact solution. Assuming $y(x)$ is smooth and expanding (4.9) in Taylor series gives,

$$y(x_{n+j}) = y(x_n) + jhy'(x_n) + \frac{1}{2!}(jh)^2 y''(x_n) + \ldots$$

$$y'(x_{n+j}) = y'(x_n) + jhy''(x_n) + \frac{1}{2!}(jh)^2 y'''(x_n) + \ldots$$

and so

$$T(x_{n+k}) = \frac{1}{h\sum_{j=0}^{k}\beta_j}\left(\sum_{j=0}^{k}\alpha_j\right)y(x_n) + \left(\sum_{j=0}^{k}(j\alpha_j - \beta_j)\right)y'(x_n) + h\left(\sum_{j=0}^{k}\left(\frac{1}{2}j^2\alpha_j - j\beta_j\right)\right)y''(x_n)$$

$$+ \ldots + h^{q-1}\left(\sum_{j=0}^{k}\left(\frac{1}{q!}j^q\alpha_j - \frac{1}{(q-1)!}j^{q-1}\beta_j\right)\right)y^q(x_n) + \ldots \qquad (4.11)$$

From equation (4.11), we have

$$T_n = \frac{1}{h\sigma(1)}[c_0 y(x_n) + c_1 hy'(x_n) + c_2 h^2 y''(x_n) + \ldots] \qquad (4.12)$$

where,

$$c_0 = \sum_{j=0}^{k} \alpha_j$$

$$c_1 = \sum_{j=1}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j$$

$$c_2 = \sum_{j=1}^{k} \frac{j^2}{2!}\alpha_j - \sum_{j=1}^{k} j\beta_j, \qquad (4.13)$$

. .

. .

. .

$$c_q = \sum_{j=1}^{k} \frac{j^q}{q!}\alpha_j - \sum_{j=1}^{k} \frac{j^{q-1}}{(q-1)!}\beta_j$$

The method is consistent if $T \to 0 \ as \ h \to 0$, which requires that at least the first two terms in this expansion vanish, that is $\sum_{j=0}^{k} \alpha_j = 0$ and $\sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k} \beta_j$

G. Characteristic polynomials

Given the linear k-step method (4.9) we consider its first and second characteristic polynomials, respectively

$$\rho(z) = \sum_{j=0}^{k} \alpha_j z^j \text{ and } \sigma(z) = \sum_{j=0}^{k} \beta_j z^j \qquad (4.15)$$

Where, as before, we assume that $\alpha_k \neq 0, \alpha_0^2 + \beta_0^2 \neq 0$ (Suli and Mayers, 2003)

Definition 4.1: The numerical method (4.9) is said to have order of accuracy p , if p is the largest positive integer such that, for any sufficiently smooth solution curve in the domain D of the initial value problem (4.1), there exists constants K and h_0 such that $\left|T_n\right| \leq kh_p$ for $0 < h \leq h_0$ for any k+1 points $(x_{n,} y(x_n)),\ldots,(x_{n+k}, y(x_{n+k}))$ on the solution curve.

Thus, we deduce from (4.12) that the method is of order of accuracy p , if and only if, $c_0 = c_1 = \ldots = c_p = 0$ and $c_{p+1} \neq 0$. In this case, the number $\frac{c_{p+1}}{\sigma(1)}$ is called the error constant of the method (Suli and Mayers, 2003). Where $\sigma(1)$ is the value of the second characteristic polynomial at $z = 1$.

Theorem 4.2: (Root Condition): A linear multistep method is zero stable for any initial value problem of the form (4.1), where f satisfies the hypothesis of Picard's theorem, if, and only if, all roots of the first characteristic polynomial of the method are inside the closed unit disc in the complex plane, with any which lie on the unit circle being simple (Suli and Mayers, 2003).

H. Convergence of linear multistep methods:-

Definition 4.2: A linear multistep method defined by (4.9) is said to be convergent in a interval $[x_0, X_M]$ if $\lim_{h \to 0} y_h(x) = y(x), \; x \varepsilon [x_0, X_M]$, Provided only that

$\lim_{h \to 0} y_h(x + jh) = y_0, \; 0 \le j \le k$ here $y_h(x)$ is the numerical solution computed using a step size of h and $y(x)$ is the theoretical solution (Yohanna, 2017).

An important result connecting the concepts of zero-stability, consistency and convergence of a linear multistep method was proved by the Swedish mathematician Germund Dahlquist. Here we stated the theorem without proof.

Theorem 4.3: (Dahlquist's Equivalence Theorem) For a linear k-step method that is consistent with the ordinary differential equation (4.1) where f is assumed to satisfy a Lipschitz coerndition, and with consistent starting values, zero-stability is necessary and sufficient for convergence. Moreover if the solution y has continuous derivative of order p + 1 and truncation error $O(h^p)$, then the global error of the method, $e_n = y(x_n - y_n)$, is also $O(h^p)$ (Suli and Mayers, 2003). Therefore, a multistep method is convergent if and only if it is consistent and stable. By virtue of Dahlquist's theorem, if a linear multistep method is not zero-stable its global error cannot be made arbitrarily small by taking the mesh size h sufficiently small for any sufficiently accurate initial data.

4.2. The derivation of the proposed method:

In this section we drive discrete methods to solve equation (4.1) at a sequence of nodal points $x_n = x_0 + nh$ where $h > 0$ is the step length defined by $h = x_{n+j} - x_{n+j-1}, \; j = 0,1,2,\dots,k$ and $y(x)$ denotes the exact solution to equation (4.1) while the approximate solution is denoted by $\overline{y(x)} = \{y_n, y_{n+1}, \dots, y_N\}$,for some positive number N .The proposed method depends on the perturbed collocation approximation with the Legendre polynomials (4.4) as the perturbation term . In the first consider the approximate solution of the perturbed form of equation (4.1) in the following power series (Abualnaga , 2015).

Let the power series solutions of the equation (4.1) be, $y(x) = \sum_{j=0}^{\infty} c_j x^j$, then the approximate solution will be:

$$y_k(x) = \sum_{j=0}^{k} c_j \phi_j(x), x_n \le x \le x_{n+k} \quad , and \;\; y_k(x) = \overline{y(x)}$$

$$\overline{y(x)} = \sum_{j=0}^{k} c_j \phi_j(x) \tag{4.20}$$

Where,

$$\phi_j(x) = x^j, \; j = 0,1,2,\dots,k \tag{4.21}$$

Substituting from equation (4.20) in equation (4.1) we have,

$$\sum_{j=0}^{k} c_j \phi_j'(x) \approx f(x, y) \tag{4.21a}$$

Using perturbation term $\lambda L_k(x_{n+j})$, $j = 0,1,2,\dots,k$, where λ is a perturbed parameter and it is determined by the values of f_{n+k} , and $L_k(x_{n+j})$ is the k^{th} shifted Legendre polynomial which is a Legendre polynomial converted from $[x_n, x_{n+k}]$ to [-1, 1] by collocating the grid points x_{n+j} using the change of range (4.5).

Now, by adding the perturbed term in (4.21a), we obtained:

$$\sum_{j=0}^{k} c_j \phi_j'(x) = f(x, y) + \lambda L_k(x_{n+j}) \tag{4.22}$$

From equation (4.5), we have:

$x = \frac{2}{b-a}[t - \frac{b+a}{2}], a < b$. This implies that $x = \frac{2t-(b-a)}{b-a}$, or equivalently it is the same as:

$$x = \frac{2t - [x_{n+k} - x_n]}{x_{n+k} - x_n},\ k = 1,2,3,\ldots \qquad (4.23)$$

To derive the proposed method for each $k, k = 1,2,3,\ldots..$ we should follow the following steps.

First, we take the Legendre polynomials (4.4) and use (4.5) to convert in to the range [-1, 1]. Using equation (4.23) collocate each $p_k(x)$ at $x_{n+j}, j = 0,1,2,\ldots,k$ to obtain $L_k(x_{n+j})$, where $L_k(x_{n+j})$ is the shifted Legendre polynomial at x_{n+j} such that $-1 \le L_k(x_{n+j}) \le 1$.

From equation (4.22) we deduce that

$$\sum_{j=0}^{k}[c_j\phi_j]' = \sum_{j=0}^{k}(c_jx^j)' = \sum_{j=0}^{k} jc_jx^{j-1} = 0 + c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \ldots + kc_kx^{k-1} = f(x,y) + \lambda L_k(x)$$

Now equation (4.22) is the same as:

$$c_1 + 2xc_2 + 3x^2c_3 + \ldots + kx^{k-1}c_k = f(x,y) + \lambda L_k(x_{n+j}) \qquad (4.24)$$

We now interpolate (4.20) at $x = x_n$ and collocate equation (4.24) at $x_{n+j}, j = 0,1,2,\ldots,k$, we get a square matrix with the parameters $\lambda, c_0, c_1, c_2, \ldots, c_k$.

And the system of equations is:

$$\begin{array}{l}
c_0 + c_1x_n + c_2x_n^2 + c_3x_n^3 + \ldots + c_kx_n^k = y_n \\
c_1 + 2c_2x_n + 3c_3x_n^2 + \ldots + kc_kx_n^{k-1} - \lambda L_k(x_n) = f_n \\
c_1 + 2c_2x_{n+1} + 3c_3x_{n+1}^2 + \ldots + kc_kx_{n+1}^{k-1} - \lambda L_k(x_{n+1}) = f_{n+1} \\
\vdots \\
c_1 + 2c_2x_{n+k} + 3c_3x_{n+k}^2 + \ldots + kc_{n+k}x_{n+k}^{k-1} - \lambda L_k(x_{n+k}) = f_{n+k}
\end{array} \qquad (4.25)$$

Equation (4.25) is a square matrix in the form:

$$AX = b \qquad (4.26)$$

where

$$A = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & \dots & x_n^k & -0 \\ 0 & 1 & 2x_n & 3x_n^2 & \dots & kx_n^{k-1} & -L_k(x_n) \\ . & . & . & & . & . & \\ . & . & . & & . & . & \\ . & . & . & & . & . & \\ 0 & 1 & 2x_{n+k} & 3x_{n+k}^2 & \dots & kx_{n+k}^{k-1} & -L_k(x_{n+k}) \end{bmatrix}$$

$X = [c_0 \; c_1 \; c_2 \; \dots \; c_k \; \lambda]^T$, $b = [y_n \quad f_n \quad f_{n+1} \; \dots \; f_{n+k}]^T$

Therefore, by using any suitable methods to solve equation (4.26), we will have the values of all the parameters in terms of the step length h , y_n and f_{n+j} . Again, from equation (4.20), we have

$$\overline{y(x)} = c_0 + c_1 x + c_2 x^2 + \dots + c_k x^k \qquad (4.27)A$$

Now, the required numerical scheme of the proposed method will be obtained if we collocates equation (4.27) at x_{n+k} and substitute the values of the parameters in:

$$y_{n+k} = c_0 + c_1 x_{n+k} + c_2 x_{n+k}^2 + \dots + c_k x_{n+k}^k \qquad (4.28)$$

Now in this study, we will drive the proposed block implicit LMM only for $k = 1, 2, 3, 4$.

i. For $k = 1$

We have the Legendre polynomials $P_1(x) = x$ and using (4.23) collocates at x_n, x_{n+1} , we obtain

$$L_1(x_n) = \frac{2x_n - (x_{n+1} - x_n)}{x_{n+1} - x_n} = -1 \,,\; L_1(x_{n+1}) = \frac{2x_{n+1} - (x_{n+1} - x_n)}{x_{n+1} - x_n} = 1$$

In addition, from equation (4.24), we have

$$c_1 = f(x, y) + \lambda L_1(x_{n+j})$$

$$c_1 - \lambda L_1(x_{n+j}) = f(x, y) \qquad (4.29)$$

We now interpolate (4.20) at $x = x_n$ and collocate equation (4.29) at x_{n+j}, $j = 0,1$, we get a system of equations with parameters c_0, c_1 and λ . This implies

$$c_0 + c_1 x_n = y_n$$
$$c_1 + \lambda = f_n \qquad (4.30)$$
$$c_1 - \lambda = f_{n+1}$$

By solving equation (4.30), we have

$$\lambda = \frac{1}{2}(f_n - f_{n+1})$$
$$c_1 = f_n - \lambda = \frac{1}{2}(f_n + f_{n+1}) \qquad (4.31)$$
$$c_0 = y_n - x_n(f_n - \lambda) = y_n - \frac{1}{2}x_n(f_n + f_{n+1})$$

From (4.27) for $k = 1$, we have

$$\overline{y(x)} = c_0 + c_1 x \qquad (4.32)$$

Now interpolate equation (4.32) at $x = x_{n+1}$ and substitute for c_0 and c_1, we have

$$y_{n+1} = c_0 + c_1 x_{n+1}$$
$$y_{n+1} = y_n - \frac{1}{2}(f_n + f_{n+1})x_n + \frac{1}{2}(f_n + f_{n+1})x_{n+1}$$
$$y_{n+1} = y_n + \frac{1}{2}(x_{n+1} - x_n)(f_n + f_{n+1})$$
$$y_{n+1} = y_n + \frac{h}{2}(f_n + f_{n+1}) \qquad (4.33)$$

Therefore, equation (4.33) is the numerical scheme when $k = 1$, which is the well-known trapezoidal rule.

ii) For $k = 2$

From equation (4.4) the Legendre polynomial for $k = 2$ is $P_2(x) = \frac{1}{2}(3x^2 - 1)$ and using equation (4.23) and collocating at $x_{n+j},\ j = 0,1,2$, we obtain

$$\boldsymbol{L_2(x_n) = 1;\ L_2(x_{n+1}) = -\frac{1}{2};\ L_2(x_{n+2}) = 1}$$

From equation (4.24), we have

$$c_1 + 2c_2 x = f(x, y) + \lambda L_2(x_{n+j}) \qquad (4.34)$$

Now interpolate (4.27) at $x = x_n$ and collocate equation (4.34) at $x_{n+j},\ j = 0,1,2$, we get the system of equations with c_0, c_1, c_2 & λ.

$$c_0 + c_1 x_n + c_2 x_n^2 = y_n$$

$$c_1 + 2c_2 x_n - \lambda = f_n$$

$$c_1 + 2c_2 x_{n+1} + \frac{1}{2}\lambda = f_{n+1}$$

$$c_1 + 2c_2 x_{n+2} - \lambda = f_{n+2} \qquad (4.35)$$

Using suitable method, the solution of the system of equations (4.35) is:

$$\lambda = \frac{1}{3}(-f_n + 2f_{n+1} - f_{n+2})$$

$$c_2 = \frac{1}{4h}(-f_n + f_{n+2})$$

$$c_1 = \frac{1}{3}(2f_n + 2f_{n+1}) + \frac{1}{2h}x_n(f_n - f_{n+2})$$

$$c_0 = y_n - \frac{1}{3}x_n(2f_n + 2f_{n+1} - f_{n+2}) + \frac{1}{4h}x_n^2(-f_n + f_{n+2}) \qquad (4.36)$$

Now interpolate equation (4.27) for $k = 2$ at $x = x_{n+2}$ and substitute for c_0, c_1, c_2 & λ, we have

$$\overline{y(x)} = c_0 + c_1 x + c_2 x^2$$

$$y_{n+2} = c_0 + c_1 x_{n+2} + c_2 x_{n+2}^2$$

$$y_{n+2} = [y_n - \frac{1}{3}x_n(2f_n + 2f_{n+1} - f_{n+2}) + \frac{1}{4h}x_n^2(f_{n+2} - f_n)]$$

$$+ [\frac{1}{3}(2f_n + 2f_{n+1} - f_{n+2}) + \frac{1}{2h}x_n(x_{n+2})(f_n - f_{n+2})] + [\frac{1}{4h}x_{n+2}^2(f_{n+2} - f_n)]$$

$$y_{n+2} = y_n + \frac{1}{3}(x_{n+2} - x_n)(2f_n + 2f_{n+1} - f_{n+2}) + \frac{1}{4h}(x_{n+2}^2 - 2x_n x_{n+2} + x_n^2)(f_{n+2} - f_n)$$

$$y_{n+2} = y_n + \frac{2}{3}h(2f_n + 2f_{n+1} - f_{n+2}) + h(f_{n+2} - f_n)$$

$$y_{n+2} = y_n + \frac{h}{3}(f_n + 4f_{n+1} + f_{n+2}) \qquad (4.37)$$

Therefore, equation (4.37) is the implicit scheme for $k = 2$.

iii) For $k = 3$

The Legendre polynomial for $k = 3$ is $P_3(x) = \frac{1}{2}(5x^3 - 3x)$ and use (4.23), collocate $P_3(x)$ at $x = x_{n+j}, \ j = 0,1,2,3$, we obtain

$$L_3(x_n) = -1, \quad L_3(x_{n+1}) = \frac{11}{27}; \quad L_3(x_{n+2}) = -\frac{11}{27}; \quad L_3(x_{n+3}) = 1$$

From equation (4.24) for $k = 3$, we have

$$c_1 + 2c_2 x + 3c_3 x^2 = f(x, y) + \lambda L_3(x_{n+j}) \qquad (4.38)$$

We now interpolate (4.27) at $x = x_n$ and collocate equation (4.38) at $x = x_{n+j}, \ j = 0,1,2,3$, we get a system of equations:

$$\begin{aligned}
&c_0 + x_n c_1 + x_n^2 c_2 + x_n^3 c_3 = y_n \\
&c_1 + 2x_n c_2 + 3x_n^2 c_3 + \lambda = f_n \\
&c_1 + 2x_{n+1} c_2 + 3x_{n+1}^2 c_3 - \frac{11}{27}\lambda = f_{n+1} \\
&c_1 + 2x_{n+2} c_2 + 3x_{n+2}^2 c_3 + \frac{11}{27}\lambda = f_{n+2} \\
&c_1 + 2x_{n+3} c_2 + 3x_{n+3}^2 c_3 - \lambda = f_{n+3}
\end{aligned} \qquad (4.39)$$

By any suitable method we will have the solution of equation (4.39), such that

$$\begin{aligned}
\lambda &= \frac{1}{40}(9f_n - 27f_{n+1} + 27f_{n+2} - 9f_{n+3}) \\
c_3 &= \frac{1}{12h^2}(f_n - f_{n+1} - f_{n+2} + f_{n+3}) \\
c_2 &= \frac{1}{60h}(-28f_n + 9f_{n+1} + 36f_{n+2} - 17f_{n+3}) - \frac{1}{4h^2}x_n(f_n - f_{n+1} - f_{n+2} + f_{n+3}) \\
c_1 &= \frac{1}{40}(31f_n + 27f_{n+1} - 27f_{n+2} + 9f_{n+3}) - \frac{1}{30h}x_n(-28f_n + 9f_{n+1} + 36f_{n+2} - 17f_{n+3}) \\
&\quad + \frac{1}{4h^2}x_n^2(f_n - f_{n+1} - f_{n+2} + f_{n+3}) \\
c_0 &= y_n - \frac{1}{40}x_n(31f_n + 27f_{n+1} - 27f_{n+2} + 9f_{n+3}) + \frac{1}{60h}x_n^2(-28f_n + 9f_{n+1} + 36f_{n+2} - 17f_{n+3}) \\
&\quad - \frac{1}{12h^2}x_n^3(f_n - f_{n+1} - f_{n+2} + f_{n+3})
\end{aligned} \qquad (4.40)$$

Now interpolate equation (4.27) for $k = 3$ at $x = x_{n+3}$ and substitute for c_0, c_1, c_2, c_3 & λ , we have

$$\overline{y(x)} = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$

$$y_{n+3} = c_0 + c_1 x_{n+3} + c_2 x_{n+3}^2 + c_3 x_{n+3}^3$$

$$= [y_n - \frac{1}{40} x_n (31 f_n + 27 f_{n+1} - 27 f_{n+2} + 9 f_{n+3}) + \frac{1}{60h} x_n^2 (-28 f_n + 9 f_{n+1} + 36 f_{n+2} - 17 f_{n+3})$$

$$- \frac{1}{12h^2} x_n^3 (f_n - f_{n+1} - f_{n+2} + f_{n+3}) + (x_{n+3})[\frac{1}{40}(31 f_n + 27 f_{n+1} - 27 f_{n+2} + 9 f_{n+3})$$

$$- \frac{1}{30h} x_n (-28 f_n + 9 f_{n+1} + 36 f_{n+2} - 17 f_{n+3}) + \frac{1}{4h^2} x_n^2 (f_n - f_{n+1} - f_{n+2} + f_{n+3})]$$

$$+ (x_{n+3}^2)[\frac{1}{60h}(-28 f_n + 9 f_{n+1} + 36 f_{n+2} - 17 f_{n+3}) - \frac{1}{4h^2} x_n (f_n - f_{n+1} - f_{n+2} + f_{n+3})]$$

$$+ (x_{n+3}^3)[\frac{1}{12h^2}(f_n - f_{n+1} - f_{n+2} + f_{n+3})]$$

$$= y_n + \frac{1}{40}(x_{n+3} - x_n)(31 f_n + 27 f_{n+1} - 27 f_{n+2} + 9 f_{n+3})$$

$$+ \frac{1}{60h}(x_n^2 - 2 x_n x_{n+3} + x_{n+3}^2)(-28 f_n + 9 f_{n+1} + 36 f_{n+2} - 17 f_{n+3})$$

$$+ \frac{1}{12h^2}(x_{n+3}^3 + 3 x_n^2 x_{n+3} - 3 x_n x_{n+3}^2 - x_n^3)(f_n - f_{n+1} - f_{n+2} + f_{n+3})$$

$$= \frac{3h}{40}(31 f_n + 27 f_{n+1} - 27 f_{n+2} + 9 f_{n+3}) + \frac{3h}{20}(-28 f_n + 9 f_{n+1} + 36 f_{n+2} - 17 f_{n+3})$$

$$+ \frac{9h}{4}(f_n - f_{n+1} - f_{n+2} + f_{n+3})$$

$$= y_n + \frac{h}{40}[(93 - 168 + 90) f_n + (81 + 54 - 90) f_{n+1}$$

$$+ (-81 + 216 - 90) f_{n+2} + (27 - 102 + 90) f_{n+3}]$$

$$= y_n + \frac{h}{40}(15 f_n + 45 f_{n+1} + 45 f_{n+2} + 15 f_{n+3})$$

$$y_{n+3} = y_n + \frac{3h}{8}(f_n + 3 f_{n+1} + 3 f_{n+2} + f_{n+3}) \qquad (4.41)$$

Therefore, equation (4.41) is the implicit scheme when $k = 3$.

iv. For $k = 4$

The Legendre polynomial for $k = 4$ is $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$, using equation (4.23), then collocate $P_4(x)$ at $x_{n+j},\ j = 0,1,2,3,4$, and we have

$$L_4(x_n) = 1;\ L_4(x_{n+1}) = -\frac{37}{128};\ L_4(x_{n+2}) = \frac{3}{8};\ L_4(x_{n+3}) = -\frac{37}{128};\ L_4(x_{n+4}) = 1$$

From equation (4.24) for $k = 4$, we have

$$c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 = f(x,y) + \lambda L_4(x_{n+j}) \qquad (4.42)$$

We now interpolate (4.27) at $x = x_n$ and collocate equation (4.42) at $x = x_{n+j},\ j = 0,1,2,3,4$, we get a system of equations:

$$c_0 + x_nc_1 + x_n^2c_2 + x_n^3c_3 + x_n^4c_4 = y_n$$

$$c_1 + 2x_nc_2 + 3x_n^2c_3 + 4x_n^3c_4 - \lambda = f_n$$

$$c_1 + 2x_{n+1}c_2 + 3x_{n+1}^2c_3 + 4x_{n+1}^3 + \frac{37}{128}\lambda = f_{n+1}$$

$$c_1 + 2x_{n+2}c_2 + 3x_{n+2}^2c_3 + 4x_{n+2}^3 - \frac{3}{8}\lambda = f_{n+2} \qquad (4.43)$$

$$c_1 + 2x_{n+3}c_2 + 3x_{n+3}^2c_3 + 4x_{n+3}^3 + \frac{37}{128}\lambda = f_{n+3}$$

$$c_1 + 2x_{n+4}c_2 + 3x_{n+4}^2c_3 + 4x_{n+4}^3c_4 - \lambda = f_{n+4}$$

By any suitable method we will have the solution of equation (4.43), such that

$$\lambda = \frac{1}{105}(-16 f_n + 64 f_{n+1} - 96 f_{n+2} + 64 f_{n+3} - 16 f_{n+4})$$

$$c_4 = \frac{1}{48h^3}(- f_n + 2 f_{n+1} + 0 * f_{n+2} - 2 f_{n+3} + f_{n+4})$$

$$c_3 = \frac{1}{504h^2}(101 f_n - 152 f_{n+1} - 66 f_{n+2} + 184 f_{n+3} - 67 f_{n+4})$$
$$- \frac{1}{12h^3} x_n (- f_n + 2 f_{n+1} - 2 f_{n+3} + f_{n+4})$$

$$c_2 = \frac{1}{168h}(-111 f_n + 80 f_{n+1} + 132 f_{n+2} - 144 f_{n+3} + 43 f_{n+4})$$
$$- \frac{1}{168h^2} x_n (101 f_n - 152 f_{n+1} - 66 f_{n+2} + 184 f_{n+3} - 67 f_{n+4})$$
$$+ \frac{1}{8h^3} x_n^2 (- f_n + 2 f_{n+1} - 2 f_{n+3} + f_{n+4}) \qquad (4.44)$$

$$c_1 = - \frac{1}{84h} x_n (-111 f_n + 80 f_{n+1} + 132 f_{n+2} - 144 f_{n+3} + 43 f_{n+4})$$
$$+ \frac{1}{168h^2} x_n^2 (101 f_n - 152 f_{n+1} - 66 f_{n+2} + 184 f_{n+3} - 67 f_{n+4})$$
$$- \frac{1}{12h^3} x_n^3 (- f_n + 2 f_{n+1} - 2 f_{n+3} + f_{n+4})$$
$$+ \frac{1}{105}(89 f_n + 64 f_{n+1} - 96 f_{n+2} + 64 f_{n+3} - 16 f_{n+4})$$

$$c_0 = y_n - \frac{1}{105} x_n (89 f_n + 64 f_{n+1} - 96 f_{n+2} + 64 f_{n+3} - 16 f_{n+4})$$
$$+ \frac{1}{168h} x_n^2 (-111 f_n + 80 f_{n+1} + 132 f_{n+2} - 144 f_{n+3} + 43 f_{n+4})$$
$$- \frac{1}{504h^2} x_n^3 (101 f_n - 152 f_{n+1} - 66 f_{n+2} + 184 f_{n+3} - 67 f_{n+4})$$
$$+ \frac{1}{48h^3} x_n^4 (- f_n + 2 f_{n+1} - 2 f_{n+3} + f_{n+4})$$

Now interpolate equation (4.27) for $k = 4$ at $x = x_{n+4}$ and substitute for c_0, c_1, c_2, c_3, c_4 & λ , we have

$$y_{n+4} = c_0 + c_1 x_{n+4} + c_2 x_{n+4}^2 + c_3 x_{n+4}^3 + c_4 x_{n+4}^4 \qquad (4.45)$$

Let

$$r = 89 f_n + 64 f_{n+1} - 96 f_{n+2} + 64 f_{n+3} - 16 f_{n+4}$$

$$s = -111 f_n + 80 f_{n+1} + 132 f_{n+2} - 144 f_{n+3} + 43 f_{n+4}$$

$$t = 101 f_n - 152 f_{n+1} - 66 f_{n+2} + 184 f_{n+3} - 67 f_{n+4}$$

$$u = - f_n + 2 f_{n+1} - 2 f_{n+3} + f_{n+4}$$

Then,

$$y_{n+4} = c_0 + c_1 x_{n+4} + c_2 x_{n+4}^2 + c_3 x_{n+4}^3 + c_4 x_{n+4}^4$$

$$y_{n+4} = y_n + \frac{1}{105}(x_{n+4} - x_n)(r) + \frac{1}{168h}(x_n^2 - 2x_n x_{n+4} + x_{n+4}^2)(s)$$

$$+ \frac{1}{504h^2}(-x_n^3 + 3x_n^2 x_{n+4} - 3x_n x_{n+4}^2 + x_{n+4}^3)(t)$$

$$+ \frac{1}{48h^3}(x_n^4 - 4x_n^3 x_{n+4} + 6x_n^2 x_{n+4}^2 - 4x_n x_{n+4}^3 + x_{n+4}^4)(u)$$

$$y_{n+4} = y_n + \frac{4h}{105}(r) + \frac{16h^2}{168h}(s) + \frac{64h^3}{504h^2}(t) + \frac{256h^4}{48h^3}(u)$$

$$y_{n+4} = y_n + h(\frac{4r}{105} + \frac{2s}{21} + \frac{8t}{63} + \frac{16u}{3})$$

$$y_{n+4} = y_n + \frac{h}{315}(12r + 30s + 40t + 1680u)$$

$$y_{n+4} = y_n + \frac{h}{315}(98 f_n + 448 f_{n+1} + 168 f_{n+2} + 448 f_{n+3} + 98 f_{n+4}) \qquad (4.46)$$

Therefore, equation (4.46) is the implicit scheme when k=4.

4.3 The basic properties of the proposed method:

Order, Error Constant and Consistency

According to definition **4**.1, we have:

For $k = 1, c_0 = c_1 = c_2 = 0$ and $c_3 \neq 0$, so the order of (4.33) is $p = 2$, and its error constant is $\frac{c_3}{\sigma(1)} = \frac{-1}{12}$.

For $k = 2,\ c_0 = c_1 = c_2 = c_3 = c_4 = 0,\ and\ c_5 \neq 0$, so the order of (4.37) is four, and its error constant is

$$\frac{c_{p+1}}{\sigma(1)} = \frac{c_5}{\sigma(1)} = \frac{-1}{180}$$

For $k = 3$, $c_0 = c_1 = c_2 = c_3 = c_4 = 0$, and $c_5 \neq 0$, so the order of (4.41) is four, and its error constant is

$$\frac{c_5}{\sigma(1)} = \frac{-672}{69120}$$

For $k = 4$, $c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0$, and $c_7 \neq 0$, so the order of (4.46) is six, and its error constant is

$$\frac{c_7}{\sigma(1)} = \frac{-96768}{45722880}$$

Therefore, the order of the schemes (4.33), (4.37), (4.41) and (4.46) are 2,4,4,6 respectively. And their error constants are respectively

$$\frac{-1}{12}, \frac{-1}{180}, \frac{-672}{69120} \text{ and } \frac{-96768}{45722880}.$$

Definition 4.3: A linear multistep method of the form (4.9) is said to be consistent if the LMM is of order $p \geq 1$ (Suli and Mayers, 2003)

As we have already determined the order of each scheme and by definition 4.3 above our proposed method is consistent.

Zero-Stability and Convergence:

It is known from the literature that the stability of a LMM determines the manner in which the error is propagated as the numerical computation proceeds. Hence, the investigation of zero-stability property is necessary.

Definition 4.4: According to Lambert (1973), the LMM is said to be zero-stable if no root of the first characteristic polynomial $\rho(z)$ has modulus greater than one, and if every root with modulus one is simple. The investigation carried out on the four schemes (4.33), (4.37), (4.41) and (4.46) revealed that all the roots of the derived schemes are less than or equal to one; hence the proposed method is zero-stable by definition 4.4 above. This is clarified for each scheme as follows:

For equation (4.33), we have

$\rho(z) = z - 1 = 0$, implies $z = 1$, and $|z| \leq 1$ is satisfied.

For equation (4.37), we have

$\rho(z) = z^2 - 1 = 0$, which implies that $z = 1$ and $z = -1$ and hence in both cases $|z| \leq 1$.

For equation (4.41), we have

$\rho(z) = z^3 - 1 = 0,\ \ (z-1)(z^2 + z + 1) = 0$, which implies that

$$z = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}i}{2},$$

$$z_1 = 1, \quad z_2 = \frac{-1 - \sqrt{3}i}{2} \text{ and } z_3 = \frac{-1 + \sqrt{3}i}{2}, \text{ here, } |z_n| \leq 1, n = 1, 2, 3 .$$

For equation (4.46),we have

$$\rho(z) = z^4 - 1 = 0, \left(z^2 - 1\right)(z^2 + 1) = 0$$

$$(z-1)(z+1)(z^2+1) = 0$$

$$z_1 = 1,\ z_2 = -1,\ z_{3/4} = \frac{\pm\sqrt{-4}}{2} = \pm\frac{2i}{2} = \pm i,\ \ |z_n| \leq 1, n = 1, 2, 3, 4$$

Therefore, our method is zero-stable.

The main aim of a numerical method is to produce solution that behaves similar to the theoretical solution at all times. The convergence of the proposed method is considered in the light of the basic properties of Dahlquist theorem. Since the consistency and zero-stable of the schemes have been established, then the proposed block procedure with implicit LMM is convergent.

The proposed block procedure with implicit linear multistep method is of the form

$$y(x) = \alpha_0(x) y_n + h[\beta_0(x) f_n + \beta_1(x) f_{n+1} + \beta_2(x) f_{n+2} + \beta_3(x) f_{n+3} + \beta_4(x) f_{n+4}] \qquad (4.47)$$

And it is given by:

$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{2}(f_n + f_{n+1}) \\ y_{n+2} &= y_n + \frac{h}{3}(f_n + 4 f_{n+1} + f_{n+2}) \\ y_{n+3} &= y_n + \frac{h}{8}(3 f_n + 9 f_{n+1} + 9 f_{n+2} + 3 f_{n+3}) \qquad (4.48) \\ y_{n+4} &= y_n + \frac{h}{315}(98 f_n + 448 f_{n+1} + 168 f_{n+2} + 448 f_{n+3} + 98 f_{n+4}) \end{aligned}$$

Now consider equation (4.9), writing this in block form we have

$$AY_M = Ey_n + hdf(y_n) + hbF(Y_M) \qquad (4.49)$$

The discrete block method (4.48) can be written in the form (4.49) where,

$$Y_M = [y_{n+1} \quad y_{n+2} \quad y_{n+3} \quad y_{n+4}]^T, y_n = [y_{n-3} \quad y_{n-2} \quad y_{n-1} \quad y_n]^T$$

$$F(y_n) = [f_{n+1} \quad f_{n+2} \quad f_{n+3} \quad f_{n+4}]^T, f(y_n) = [f_{n-3} \quad f_{n-2} \quad f_{n-1} \quad f_n]^T \qquad (4.50)$$

Such that:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, E = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, d = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & \frac{3}{8} \\ 0 & 0 & 0 & \frac{98}{315} \end{bmatrix}, b = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \frac{4}{3} & \frac{1}{3} & 0 & 0 \\ \frac{9}{8} & \frac{9}{8} & \frac{3}{8} & 0 \\ \frac{448}{315} & \frac{168}{315} & \frac{448}{315} & \frac{98}{315} \end{bmatrix} \quad (4.51)$$

It is important to note that the block method (4.48) has four function evaluations per step.

Let the linear operator $L[y(x);h]$ associated with the block (4.9) be defined as,

$$L[y(x);h] = AY_M - Ey_n - h^{\mu} df(y_n) - h^{\mu} bF(Y_M) \qquad (4.52)$$

where, $y(x)$ is an arbitrary test function that is continuously differentiable in the interval $[x_n, x_{n+k}]$ and μ is the order of the differential equation. Expanding (4.52) or expanding $y(x_n + jh)$ and $y'(x_n + jh)$ about x_n using Taylor series and collecting like terms in h and y gives:

$$T_n = \frac{1}{h\sigma(1)}[c_0 y(x) + c_1 hy'(x) + c_2 h^2 y''(x) + \ldots + c_p h^p y^p(x) + c_{p+1} h^{p+1} y^{p+1}(x) + \ldots] \qquad (4.53)$$

Thus, we deduce from (4.53) that the method is of order of accuracy p if, and only if, $c_0 = c_1 = \ldots = c_p = 0$, and $c_{p+1} \neq 0$. In this case

$T_n = \frac{1}{h\sigma(1)} h^{p+1} y^{p+1}(x_n) + o(h^{p+1})$ is the truncation error, and the number $\frac{c_{p+1}}{\sigma(1)}$ is called the error constant of the method.

According to Yohanna, (2017), applying (4.52) on (4.48), we obtained:

$$L[y(x);h] = AY_M - Ey_n - h^{\mu} d f(y_n) - h^{\mu} b F(Y_M)$$

$$L[y(x);h] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix}$$

$$- h \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{4}{3} & \frac{1}{3} & 0 & 0 \\ \frac{3}{8} & \frac{9}{8} & \frac{9}{8} & \frac{3}{8} & 0 \\ \frac{98}{315} & \frac{448}{315} & \frac{168}{315} & \frac{448}{315} & \frac{98}{315} \end{bmatrix} \begin{bmatrix} f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} \qquad (4.54)$$

Expanding (4.54) in Taylor series gives:

$$\begin{bmatrix} \sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^{'} - y_n - \frac{1}{2} h y_n^{'} - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left\{ \frac{1}{2}(1)^j \right\} \\ \sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^{j} - y_n - \frac{1}{3} h y_n^{'} - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left\{ \frac{4}{3}(1)^j + \frac{1}{3}(2)^j \right\} \\ \sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^{'} - y_n - \frac{3}{8} h y_n^{'} - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left\{ \frac{9}{8}(1)^j + \frac{9}{8}(2)^j + \frac{3}{8}(3)^j \right\} \\ \sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^{j} - y_n - \frac{98}{315} h y_n^{'} - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left\{ \frac{448}{315}(1)^j + \frac{168}{315}(2)^j + \frac{448}{315}(3)^j + \frac{98}{315}(4)^j \right\} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (4.55)$$

Therefore, from equation (4.55) above, we can determine the order, error constant and consistency of the proposed block method as before.

Definition 4.5: A block method is said to be zero-stable if as $h \to 0$, the roots $z_r, r = 1(1)k$ of the first characteristic polynomial $\rho(z)$ is given by $\rho(z) = \det\left[\sum_{j=0}^{k} A^j z^{k-j} \right] = 0$, satisfies $|z_r| \le 1$, the multiplicity of $|z_r| = 1$ not exceeding the order of the differential equation (Yohanna, 2017)

Therefore, the block method is said to be zero stable if the roots z_r of the first characteristic polynomial $\rho(z)$ of (4.54), defined by $\rho(z) = \det\left[zA - E \right]$ satisfies $|z_r| \le 1$ and every root with $|z_r| = 1$ has the multiplicity not exceeding one.

From equation (4.54), we have

$$\rho(z) = \det\left[zA - E\right] = 0$$

$$\rho(z) = \det\left[z\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right] = \det\begin{bmatrix} z & 0 & 0 & -1 \\ 0 & z & 0 & -1 \\ 0 & 0 & z & -1 \\ 0 & 0 & 0 & z-1 \end{bmatrix} = z^3\left(z-1\right) = 0$$

This implies, $z_1 = z_2 = z_3 = 0, \text{ and } z_4 = 1$. Hence by definition (4.1), (4.3) and (4.5), our block method is consistent and zero-stable. Following theorem4.3, we conclude that our proposed block method is convergent since it is consistent and zero-stable.

4.4. Numerical examples

The mode of implementation of our method is by combining the schemes (4.48) as a block. It is a simultaneous integrator for the IVPs of first order differential equation without requiring the starting values. In order to assess the performance of our block method, we consider four first order initial value problems in ODEs and the problems consist both linear and non-linear. We considered also some high stiff IVPs. All calculations are carried out with the aid of MATLAB software.

Problem4.1: Consider a stiff initial value problem:

$$y'(x) = -10(y - x^3) + 3x^2 \qquad y(0) = 1 \;\; with \;\; h = 0.1 \quad on \;\; [0,1]$$

Whose exact solution is $y(x) = x^3 + e^{-10x}$

[see: (Bolaji and Duromola, 2017)]

The results are as shown in table 1 below

Table 4.1: Exact and approximate values for problem 1.

x	Exact	RK	PM
0	**1**	**1**	**1**
0.1	**3.68879e-001**	**3.76031e-001**	**3.34667e-001**
0.2	**1.43335e-001**	**1.48730e-001**	**1.19556e-001**
0.3	**7.67871e-002**	**7.99302e-002**	**6.45185e-002**
0.4	**8.23156e-002**	**8.40676e-002**	**7.68395e-002**
0.5	**1.31738e-001**	**1.32807e-001**	**1.29613e-001**
0.6	**2.18479e-001**	**2.19271e-001**	**2.17871e-001**
0.7	**3.43912e-001**	**3.446334e-001**	**3.43957e-001**
0.8	**5.12335e-001**	**5.13081e-001**	**5.11979e-001**
0.9	**7.29123e-001**	**7.29937e-001**	**7.29085e-001**
1.0	**1.00005e+000**	**1.00094e+000**	**9.99861e-001**

To give a more visual impact, the graph of table 1 was plotted below in figure 1.

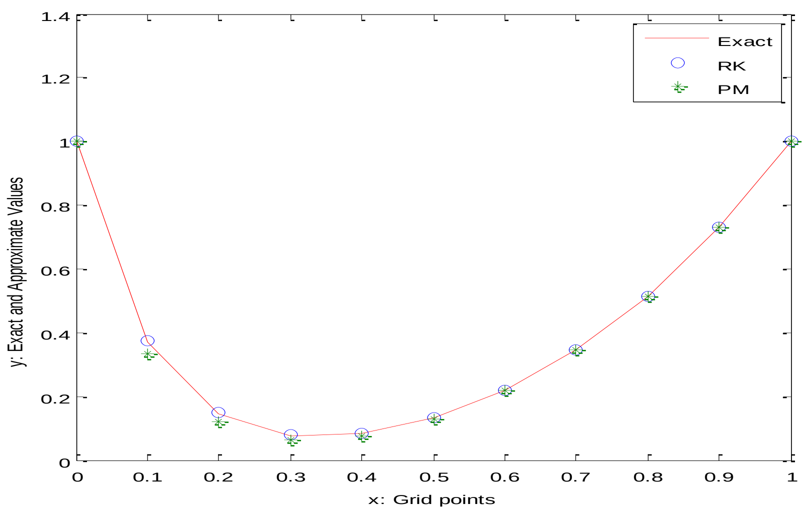

Figure 4.1: Graphs of Exact, RK and PM for problem 1 when h=0.1.

Table4.2: Comparison of Absolute Errors for problem1.

x	RK	PM
0	**0**	**0**
0.1	**7.151808828557704e-003**	**3.421277450477572e-002**
0.2	**5.395185513387274e-003**	**2.377972768105720e-002**
0.3	**3.143107413386068e-003**	**1.226854984934542e-002**
0.4	**1.751927029234573e-003**	**5.476132715894672e-003**
0.5	**1.068640220152822e-003**	**2.124778274805617e-003**
0.6	**7.924680305480825e-004**	**6.076959352396827e-004**
0.7	**7.210756121509565e-004**	**4.513678158774015e-005**
0.8	**7.456464637368709e-004**	**3.569532543316045e-004**
0.9	**8.132561052782705e-004**	**3.836179311267340e-005**
1.0	**8.995997862495386e-004**	**1.839381331677492e-00**

The results from the table 4.2 above of the absolute errors show that for almost half values of $x \varepsilon [0,1]$, the proposed method is better than RK for $h = 0.1$. As $(h \to 0)$, both RK and PM methods converges to the exact solutions as shown in figure4.1 above.

Note: the solutions table and graph of problem 1 for $h = 0.2$ is indicated below:

Table4.3: Exact, RK and PM values for problem 1 when $h = 0.2$

x	Exact	RK	PM
0	**1**	**1**	**1**
0.2	**1.43335e-001**	**3.42667e-001**	**1.00000e-003**
0.4	**8.23156e-002**	**1.80889e-001**	**6.60000e-002**
0.6	**2.18479e-001**	**2.64296e-001**	**2.13200e-001**
0.8	**5.12335e-001**	**5.41432e-001**	**5.13314e-001**
1.0	**1.00005e+000**	**1.02714e+000**	**9.96495e-001**

Table4.4: Comparisons of absolute errors of RK and PM for problem 1 when h=0.2.

x	RK	PM
0	0	0
0.2	1.993313834300540e-001	1.333352832366128e-001
0.4	9.857325000015474e-002	1.631563888873418e-002
0.6	4.581754411963004e-002	5.278752176666318e-003
0.8	2.909663613752955e-002	9.788230863831959e-004
1.0	2.709863299204840e-002	3.549900908236037e-003

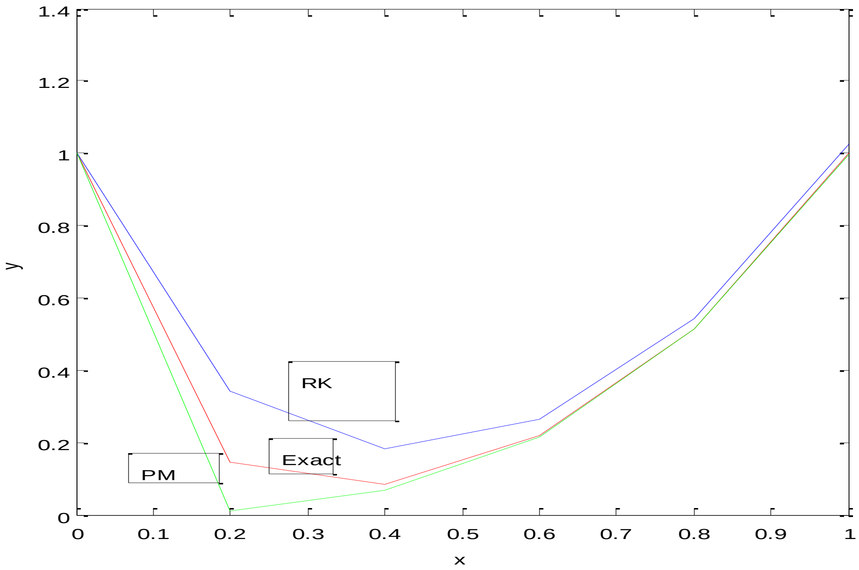

Figure4.2: Graphs of Exact, RK and PM for problem 1 when h=0.2

From table4.4 above we deduce that for large h our method is effective for stiff problems when we compare with Runge-Kutta of order four method, but both RK and PM converges to the exact solution as $h \to 0$

Problem4.2: $y'(x)=\frac{y(1-y)}{2y-1}$ $y(0)=\frac{5}{6}$ $[0,1]$

Exact solution is:

$$y(x)=\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{1}{36}e^{-x}}$$

[see: Suleiman, (2015)]

Table4.5: Maximum Absolute errors of 2BBDF, 2BEBDF and PM for problem 2.

h	Method		
	2BBDF	2BEBDF	PM
10^{-2}	1.47080e-003	6.64937e-004	1.17710e-006
10^{-3}	1.52651e-004	7.05780e-005	1.17799e-008
10^{-4}	1.53220e-005	7.10123e-006	1.17705e-010
10^{-5}	1.53277e-006	7.10560e-007	3.35798e-012
10^{-6}	1.53305e-007	7.10611e-008	1.42827e-014

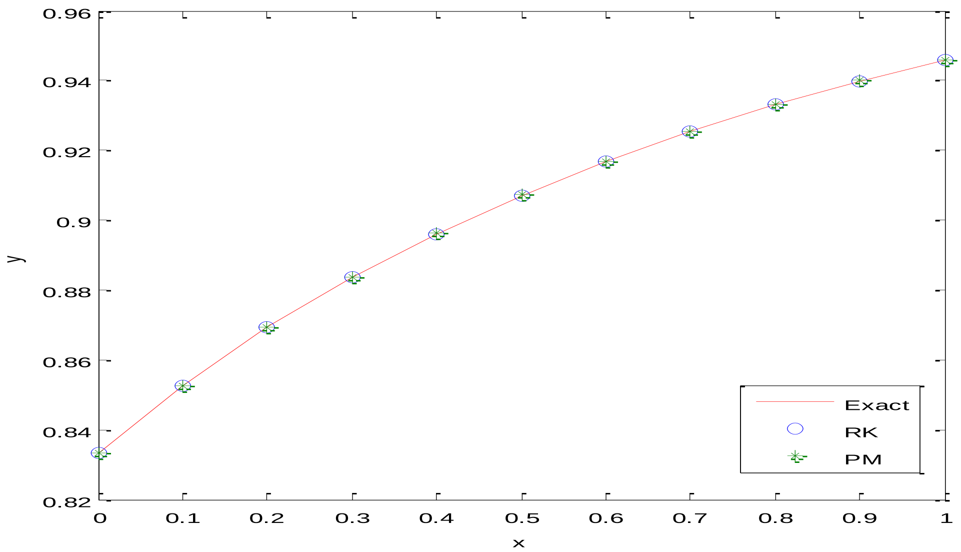

Figure 4.3: Graphs of Exact, RK and PM for Problem 2 for h=0.1.

Problem4.3: $y'(x) = \frac{50}{y} - 50y \qquad y(0) = \sqrt{2} \qquad 0 \le x \le 1$

The exact solution is: $y(x) = (1 + e^{-100x})^{\frac{1}{2}}$

[See, Suleiman, (2015)]

Table4.6: Maximum Absolute Errors of 2BBDF, 2BEBDF and PM for problem 3.

H	Method		
	2BBDF	**2BEBDF**	**PM**
10^{-2}	**1.44729e-001**	**9.24961e-003**	**7.50583e-003**
10^{-3}	**2.15168e-002**	**7.96762e-003**	**7.53930e-006**
10^{-4}	**2.55682e-003**	**1.07245e-003**	**7.53555e-007**
10^{-5}	**2.59680e-004**	**1.10428e-004**	**7.53551e-009**
10^{-6}	**2.60086e-005**	**1.10751e-005**	**7.52905e-011**

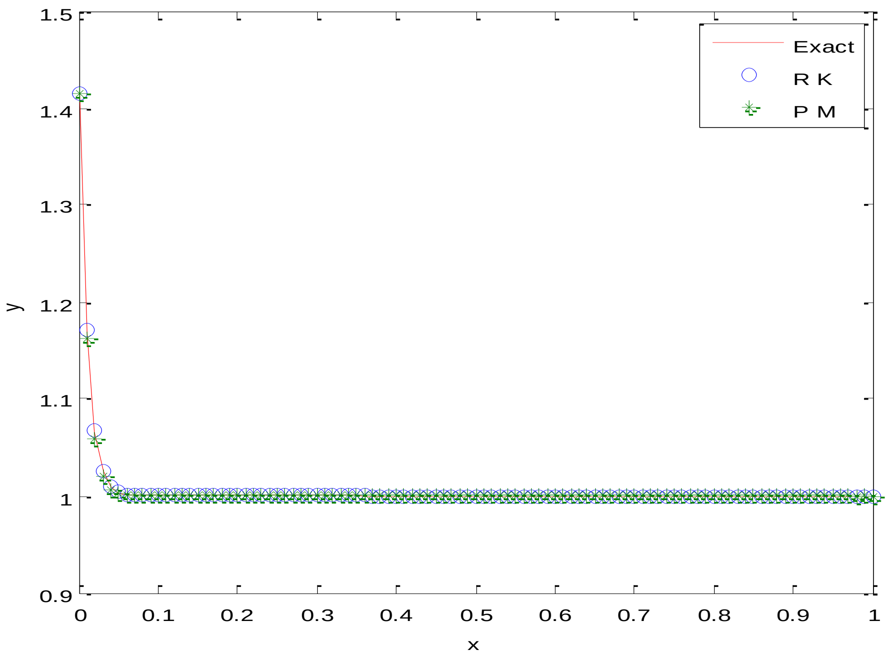

Figure4.4: Graph of Exact, RK, PM solution curves for problem 3 when h=0.01.

The results from the tables4.5 and 4.6 above show that for all the two problems solved, the derived proposed method is better in terms of accuracy than the two methods 2BBDF and 2BEBEDF. And also from the graphs in figures 3 and 4 we deduced that both methods RK and PM agree with the exact solution graph.

Problem4.4: $y'(x) = -2100(y - \cos(x)) - \sin(x), \quad y(0) = 1, \quad 0 \le x \le 1$.

The exact solution is: $y(x) = \cos(x)$

[See, (Randall, 2004)]

Table4.7: Comparison of maximum absolute errors for RK and PM

H	Method	
	RK	PM
10^{-1}	1.22516e+024	1.12538e-005
10^{-2}	2.41053e+304	9.67880e-008
10^{-3}	1.53563e-007	6.46041e-013
10^{-4}	5.09304e-012	3.33844e-013
10^{-5}	1.22125e-015	4.10783e-015

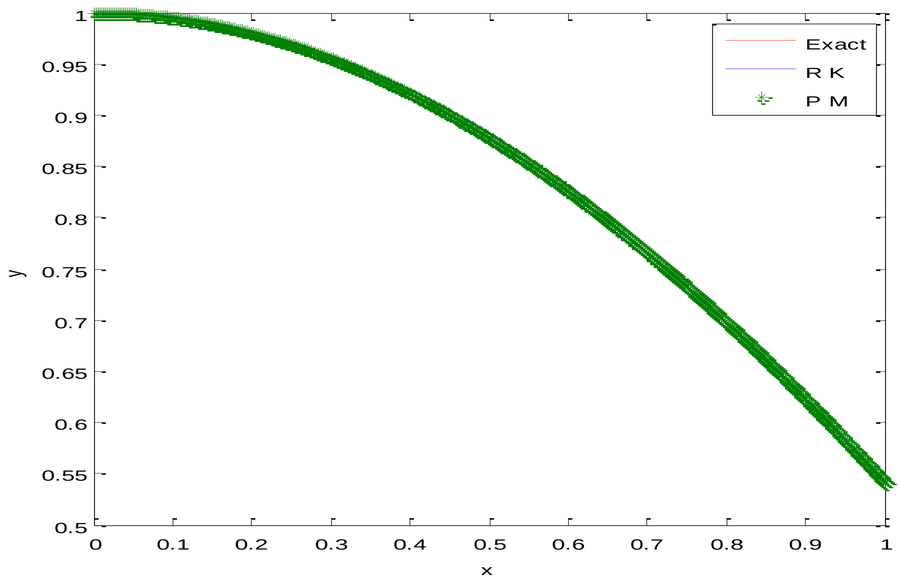

Figure4.5: Graph of Exact, RK and PM for problem 4 when h=0.001.

In table 4.7 above the maximum absolute errors show that the method RK diverges for step sizes $h = 0.1$ and $h = 0.01$, which means the maximum absolute errors are out of ranges, but the proposed method PM approximates with maximum errors of $1.13X10^{-5}$ and $9.68X10^{-8}$ respectively. Therefore, for high stiff problems our method is preferable when the step size relatively large. Note that the exponential growth of errors does not contradict zero-stability or convergence of the method in any way. Thus as $h \to 0$ both RK and PM converges to the exact solutions even if RK diverges when $h \ge 0.01$.

CHAPTER FIVE

Conclusion and Future scope

5.1. Conclusion

This study presented a block procedure with implicit linear multistep method based on Legendre polynomials for solving first order IVPs in ODEs. A perturbed collocation approach along with interpolation at some grid points which produces a family of maximal order six multi-derivative schemes has been proposed for the numerical solution of stiff problems in ODEs. The properties of the Legendre polynomials are used to introduce the proposed problems to system of equations which are solved by a suitable method. The desirable property of a numerical solution is to behave like the theoretical solution of the problem which can be seen in the above experimental results. The method is tested and found to be consistent, zero stable and convergent. We implement the method on four numerical examples and the numerical evidences shows that the method is accurate and effective for stiff problems and therefore effective for wide range of stiff IVPs in ODEs.

5.2 Future scope

In this study, a collocation approach which produces a block procedure with implicit linear multistep method based on Legendre polynomials for solving stiff first order initial value problems in ODEs with non-uniform orders has been proposed. The method has been applied to find the numerical solutions of stiff IVPs. Hence further research should be performed to enhance the accuracy of the method by extending the step number k and taking into consideration the off grid points to produce a method of uniform orders.

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