CHEBYSCHEV ITERATION TECHNIQUE FOR SOLVING SECOND ORDER SINGULARLY PERTURBED 1D REACTION – DIFFUSION EQUATION

A Thesis Submitted to Jimma University, Department of Mathematics, in Partial Fulfillment for the Requirements of the Degree of Masters of Science in Mathematics

(Numerical Analysis)

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Declaration

I hereby declare that the work which is being presented in this thesis entitled **"Chebyschev Iteration Method for Solving Second Order Singularly Perturbed 1D Reaction – Diffusion Equation"** in partial fulfillment of the requirement for the degree of Masters of Science in Mathematics, submitted to Jimma University, department of Mathematics is my original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

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Abstract

In this thesis, Chebyschev iteration technique has been presented to solve second order singularly perturbed 1D reaction – diffusion equation for a very small perturbation parameter, ε *with both variable and constant coefficient of reaction term. The given problem of interest is discretized and the derivative of the given differential equation is replaced by finite central difference approximation to obtain system of algebraic equation. Chebyschev three – level scheme was developed from the two – level scheme to solve the obtained algebraic equation. To investigate the convergence of the proposed method, three examples were taken and compared with other methods listed in the literature and exact solution. The relationship between number of iteration number and the condition number is analyzed and found to be: the larger the condition number the slower is the rate of convergence. Finally, pointwise and maximum absolute error for each example was shown both by table and numerical approximation and exact solution is demonstrated on the same graph with different iteration number.*

Table of Content

Declaration i
Acknowledgement ii
Abstract iii
Table of Content iv
Chapter One 1
Introduction 1
1.1Background of the study 1
1.2. Objectives of the Study 4
1.2.1. General objective of the study 4
1.2.2. Specific objectives 5
1.3. Significance of the Study 5
1.4. Delimitation of the Study 5
Chapter Two 6
Review of Related Literature 6
2.1. Singularly Perturbed Problems 6
2.2 Reaction – Diffusion Boundary Value Problems 7
2.3 Iteration Methods 9
Chapter Three 11
Methodology 11
3.1. Study Area and Period 11
3.2. Mathematical Design of the Study 11
3.3. Source of Information 11
3.4. Mathematical Procedure 11
Chapter Four 12
Description of The Method, Results and Discussion 12
4.1 Description of the Method 12
4.2 Chebyschev Iteration Technique 14
4.3 An estimate for the norm of the error 17
4.4 Numerical Examples and Results 27

4.5 Discussion 33
Chapter Five 34
Conclusion and Scope of Future Work 34
5.1 Conclusion 34
5.2 Scope of Future Work 34
References 35

Chapter One

Introduction

1.1Background of the study

Numerical analysis is a branch of mathematics that deals with the computational methods which helps to find approximate solutions for difficult problems such as finding roots of non- linear equations, integrations involving complex expressions and solving differential equations for which analytic solution does not exist (Terefe *et al*., 2016). It is both a science and an art in that, as a science, numerical analysis is concerned with the process by which mathematical problems can be solved by the operations of arithmetic. On the other hand, as an art numerical analysis is concerned with choosing that procedure (and suitably applying it) which is "best" suited to the solution of particular problem (Anthony and Philip, 1978).

According to James (1990), numerical analysis is the study of the methods and procedures used to obtain approximate solutions to mathematical problems. While this definition is broad, it does pinpoint some of the key issues in numerical analysis, namely, approximate solution (there is usually no reasonable hope of obtaining the exact solution); mathematical problems; the study of methods and procedures.

An equation involving derivatives of one or more dependent variables to one or more independent variable is called differential equation. A differential equation involving ordinary derivatives of one or more dependent variables with respect to one independent variable is called ordinary differential equation while a differential equation involving partial derivatives of one or more dependent variables with respect to more than one independent variable is called a partial differential equation (Shepley, 1984).

Any differential equation in which the highest order derivative is multiplied by a small positive parameter $\varepsilon\left(0<\varepsilon \ll 1\right)$is called a singularly perturbed problem and the parameter ε is called perturbation parameter (Fasika *et al*., (2016), Gayatri and Jugal, (2012), Phaneendra *et al*., (2015)). Singularly perturbed second order two – point boundary value problem occur very frequently in fluid motion, chemical reactor theory, elasticity, diffusion in polymer, reaction – diffusion equation, stability, control of chaotic system and so on and have received a significant amount of attention in past and recent year. A boundary layer is a narrow region in which solution of the problem changes rapidly. In these problems, there are thin transition layer where the solution varies rapidly or jumps abruptly, while away from the layer the solution behaves regularly and varies slowly. The solution of singularly perturbed problems exhibits boundary layers. For these problems the existing numerical methods give good results when $h<<\varepsilon$, where h is the mesh size and ε is perturbation parameter. But this is costly and time consuming process. If we take $h\geq\varepsilon$, the existing numerical methods produce oscillatory solution and pollute the solution in the entire interval, because of boundary layer behavior. Thus, numerical treatment of such problems is not trivial because of the boundary layer behavior of their solutions (Gemechis *et al*., (2017), Phaneendra *et al*., (2015))

Depending on the solution behavior of the problem in the limiting case when perturbation parameter goes to zero, such type of problems are classified into two classes, namely, regularly perturbed and singularly perturbed problems. If the solution of the original problem tends to the solution of the reduced problem (i.e., the problem which is obtained by putting $\varepsilon = 0$ in the original problem) as the perturbation parameter tends to zero, the problem is known as regularly perturbed otherwise, it is known as singularly perturbed (Phaneendra *et al*., 2014). As Gulsemay and Mustafa, (2017) stated if the order of the higher order is reduced by one the problem becomes convection – diffusion type and if the order is reduced by two it is reaction – diffusion type.

Neumann *et al*., (2000), have suggested on booster method for solving singularly perturbed one – dimensional reaction – diffusion problems that improves numerical solution which were obtained by exponentially fitted difference, exponentially fitted spline difference and upwind scheme. In particular for a small ε , there is a significant reduction on the error for a fixed mesh size h. For numerical solutions, various finite difference schemes have been proposed to guarantee the stability of the schemes for all values of perturbation parameters. Careful examination of the numerical results from such schemes on uniform grid shows that, for fixed (small) value of perturbation parameter, the maximum pointwise error usually increases as the mesh is refined, because of the presence of boundary layers or interior layers, until the mesh diameter is comparable in size to the parameter. This behavior is clearly unsatisfactory. Therefore, a separate treatment is necessary to deal with such problems.

Phaneendra *et al*., (2015) have proposed an exponentially fitted arithmetic average finite difference method for solving singularly perturbed two – point boundary value problem with boundary layers at both (left and right) end points. They introduced a fitting factor in three – point arithmetic average finite difference scheme to take care of the rapid changes that occur in the boundary layer. Pankaj *et al*., (2016) have considered orthogonal spline collocation for a class of singularly perturbed reaction – diffusion problems in one – dimension with mixed boundary conditions assuming $h \geq \varepsilon$. But, analysis part of the proposed scheme was left and different examples were taken not on the specified boundary conditions. Ivanka and Lubin, (2016) have proposed two – grid algorithms for the finite difference solutions of singularly perturbed problems. In these two – grid algorithms, the solution of the fully nonlinear coarse problem is used in a single – step linear fine mesh problem. Gulsemay and Mustafa, (2017) have examined an efficient method to yield solutions of higher order singularly perturbed problems numerically. The Chebyschev based differential quadrature method was applied to perturbation problems to obtain approximated results. They pointed out that both ordinary differential equation and partial differential equations can be solved using Chebyschev polynomial. However, the accuracy of the method was presented only by table and the graphical comparison of the approximate solution with exact solution was left.

Fasika *et al*., (2017) have developed a fourth-, sixth-, and a tenth order compact finite difference method for solving a singularly perturbed one – dimension reaction – diffusion two point boundary value problem. To demonstrate the efficiency of the method, they implemented numerical examples by taking different values for perturbation parameter ε , and the mesh size h. i.e., of the type $h \geq \varepsilon$. (Feyisa and Gemechis, 2017) have presented a higher order finite difference method, eighth order compact difference method for solving singularly perturbed one – dimension reaction – diffusion equation.

Yet, the methods are appropriate when the coefficient of the reaction term is constant and the size of the perturbation parameter ε is comparable with the mesh size h.

(Rajashekhar, 2016) has presented numerical solution to linear singular perturbation two – point boundary value problems using B – Spline collocation method. In the paper, the B – Spline basis functions was defined recursively and the B – Spline collocation method was described and formulated. The efficiency of the method is demonstrated using second order singular differential equation with Neumann's boundary conditions. The proposed method was tested for two numerical examples taking a very small perturbation parameter ε.

Terefe *et al*., (2016) have presented fourth order stable central difference for solving self –adjoint singularly perturbed two – point boundary value problem. Their numerical solutions are in a very good agreement with the exact solution for a small value of ε (i.e. $h \geq \varepsilon$) for which most classical numerical methods do not give good result.

Prasad and Reddy, (2014) have presented a fitted second order finite difference method for solving singularly perturbed problems exhibiting dual layers at both (left and right) end points. By introducing a fitting factor, they obtained its value from a singular perturbation theory. The efficiency of the method was shown by taking different numerical examples with constant and variable coefficient of the reaction term. They presented the result for $h \geq \varepsilon$ and comparison of graphical representation of the approximate solution with exact solution was not shown.

Thus, in this study we are interested to present Chebyschev iteration techniques for both constant and variable coefficient of reaction term of singularly perturbed 1D reaction - diffusion equation for a very small perturbation parameter in which most methods listed in the literature do not give good result. Further, we try to present a more accurate numerical method for solving singularly perturbed 1D reaction – diffusion two – point boundary value problems using Chebyschev iteration technique.

1.2. Objectives of the Study

1.2.1. General objective of the study

To apply the Chebyschev iteration technique for solving second order singularly perturbed one – dimension reaction – diffusion equation.

1.2.2. Specific objectives

The specific objectives of study are;

1. Describe Chebyschev iteration technique for singularly perturbed 1D reaction – diffusion equation.
2. To establish the convergence of the proposed method.

1.3. Significance of the Study

The importance of this study is to show how to apply Chebyschev iteration technique in solving second order singularly perturbed 1D reaction – diffusion boundary value problems and to obtain a more accurate solution with a fast rate of convergence. It also helps other scholars who want to work on this area.

1.4. Delimitation of the Study

Numerical treatment of singularly perturbed problems has received significant attention in recent year. Thus, this study is delimited to Chebyschev iteration technique for solving second order singularly perturbed 1D reaction – diffusion equation of the form: $-\varepsilon y''(x)+a(x)y(x)=f(x), a<x<b$ with the boundary conditions $y(a)=\alpha$ and $y(b)=\beta$ where ε is a small positive parameter (diffusion coefficient) such that $(0<\varepsilon \ll 1)$ and α, β are given constants and $a(x)$, $f(x)$ are assumed to be sufficiently continuously differentiable function in the given domain.

Chapter Two

Review of Related Literature

2.1. Singularly Perturbed Problems

Numerical treatment of singularly perturbed problems has received significant attention in recent year. These problems arise frequently in fluid dynamics, elasticity, quantum mechanics, chemical reactor theory, reaction – diffusion equation and other applied areas. A few notable examples are boundary layer problems. The presence of a small parameter in these problems prevents us from obtaining satisfactory numerical and asymptotic solutions by direct or classical methods. For numerical solutions, various finite – difference schemes have been proposed in the numerical literature to guarantee the stability of the scheme for all values of the perturbation parameter (Neumann *et al*., 2000). Since the mid-1960s, singular perturbation problem have flourished, the subject is now commonly a part of graduate students training in applied mathematics and in many fields of engineering (Gayatri and Jugal, 2012)

The study of many theoretical and applied problems in science and technology leads to boundary value problems for singularly perturbed differential equations that have a multi-scale character. However, most of the problems cannot be completely solved by analytic techniques. Consequently, numerical simulations are of fundamental importance in gaining some useful insights on the solutions of the singularly perturbed differential equations. (Gemechis *et al*., 2017) have presented numerical method for solving singularly perturbed delay reaction – diffusion equation with layer or oscillatory behavior via fourth order finite difference method. To demonstrate the efficiency of the method, four examples without their exact solutions have been considered.

According to Miller *et al.*, (2012) boundary layer is a region of independent variable over which the dependent variable changes rapidly. The dimension of the boundary layer, in particular its width, have to be defined with some care. This suggests that it is most appropriate to say that the boundary layer in $e^{-x/\varepsilon}$ is of width ε and in $e^{-x/\sqrt{\varepsilon}}$ is of width $\sqrt{\varepsilon}$.

Moving to two – point boundary value problem for second order singularly perturbed reaction – diffusion equation, the exact solution is a linear combination of the exponential function $\left\{ e^{\frac{-x}{\sqrt{\varepsilon}}}, e^{\frac{-(1-x)}{\sqrt{\varepsilon}}} \right\}$. Because the exponential $e^{\frac{-x}{\sqrt{\varepsilon}}}$ in the solution has the argument $x/\sqrt{\varepsilon}$, the solution changes rapidly in $\left(0, \sqrt{\varepsilon}\right)$ but not in $\left(\sqrt{\varepsilon}, 1\right)$. From this we have at most two boundary layers, each of width $\sqrt{\varepsilon}$. Here the boundary layers are much thicker than before when $(0 < \varepsilon \ll 1)$ so numerical difficulties arise only if ε is much smaller than 1.

According to Chandru and Shanti, (2014), classical numerical methods fail to produce good approximations for singularly perturbed problems. Hence, one has to go for non – classical methods. A number of articles have been appearing in the past three decades on non – classical methods which covers mostly second order equations. Singular perturbation problems are classified on the basis how the order of the original differential equation is affected. We say that singular perturbation problem is of convection – diffusion type if the order of the differential equation is reduced by one, where as it is called reaction – diffusion type if the order is reduced by two. Kadalbajoo and Patidra, (2001) have described a numerical method for self – adjoint singularly perturbed problems using cubic spline with exponentially fitting factor. They considered where the coefficient of the reaction term $a(x)$ is different from constant and analyzed the convergence of the method by four numerical examples.

2.2 Reaction – Diffusion Boundary Value Problems

According to Miller *et al.*, (2012) the simplest example of a singular perturbation problem is the initial value problem on the unit interval $\left(0,1\right)$ of the form $\varepsilon u'(x) + u(x) = 0$, $u(0) = u_0$. This singular perturbation initial value problem can arise in the model of chemical reactions, if there is a fast reaction rate. Christina, (2011), a reaction –diffusion equation comprises of a reaction term and a diffusion term, i.e. the typical form is as $-\varepsilon y''(x) + a(x)y(x) = f(x)$ with the boundary conditions $y(0) = \alpha$ and $y(1) = \beta$ where ε is a small positive parameter (diffusion coefficient) such that

$(0 < \varepsilon \ll 1)$. So, the first term on the left hand side $(y''(x))$ describes the "Diffusion" term and the second term $y(x)$ describes a "Reaction" term.

In general, the reaction – diffusion equations allow for much more complex behaviors than a scalar reaction – diffusion equation does. Especially interacting reaction terms are of interest and leads to interesting behavior. So, e.g. oscillating phenomena can evolve as these oscillations can spread in a space via diffusion and instability may develop spatial phenomena like pattern formation can be observed.

Typical examples

- Population dynamics: this reaction – diffusion is used to describe the spread of population in a space. So, we need some basics about population dynamics possibly stationary and their stabilities are of interest in 1D and 2D ordinary differential equations.
- Prey- Predator, Competition, Symbiosis, Chemical reactions…

Valanarasu and Ramanujam, (2007) have developed an asymptotic numerical method for singularly perturbed reaction – diffusion type of third order ordinary differential equation with discontinuous source term subject to a particular type of boundary conditions.

Mark, (2001), many problems in engineering and science can be formulated in terms of differential equations. A differential equation is an equation involving a relation between an unknown function and one or more of its derivatives. Equations involving derivatives of only one independent variable are called Ordinary Differential Equation and may be classified as either Initial – Value Problem or Boundary – Value Problem. The distinction between the two classifications lies in the location where extra conditions are specified. For an Initial – Value Problem, the condition are given at the same value of *x*, whereas in the case of Boundary – Value Problem, they are prescribed at two different values of *x*. There are different types of boundary conditions; Neumann boundary conditions, Robin boundary conditions, mixed boundary conditions and Dirichlet boundary conditions of which Dirichlet boundary condition is our interest.

2.3 Iteration Methods

There are two types of methods that can be used to find the roots of the equation:

1) Direct method: these methods give the exact value of the roots (in the absence of round off errors) in a finite number of steps. These methods determine all the roots at the same time (Gauss elimination, Gauss – Jordan, Triangularization, Cholesky,…) methods.
2) Iterative methods: these methods are based on the idea of successive approximations. Starting with one or more initial approximations to the root, we obtain a sequence of iterative $\{x_k\}$ which in the limit converges to the root (Jain *et al.*, 2007).

To solve complex and non – linear difference problems, iterative methods are the most commonly used. The essence of the iterative methods consists in constructing a sequence of approximations converging to the solution, starting with some initial guess. After a finite number of steps, the approximate solution is taken to be the solution of the problem. Iterative methods are more universal in that they allow us to solve not one concrete problem, but a class of problems possessing definite properties. Since in the majority of iterative methods the concrete structure of the equation is not used, the theory of iterative methods can be constructed from a single point of view, taking as our goal the investigation of first kind equation; $Au = f$ where A is operator, f is given, u is the desired of some space H.

In any iterative method the solution of the above equation is found from some initial approximation $y_0 \in H$ and a sequence of approximate solutions, $y_1, y_2, ..., y_k, y_{k+1}, ...$ is defined where k is iterative number. The approximation y_{k+1} is expressed in theory of the already known preceding approximations using a recurrence formula.

Iterative methods are characterized by the structure of the iterative scheme, by the space H in which the convergence of the method is studied, by the termination condition for the iterative process.

Thus, a few of them are; Jacobi iteration method, Gauss-Seidel iteration method, Successive over Relaxation method and Chebyschev iteration technique.

The two – level method is only valid after the completion of all the iterations. However, for the three – level methods, the estimation is also valid at any intermediate iteration. Unlike the two – level method, in the three – level method the norm of the error decreases monotonically at intermediate iterations, and this guarantees the computational stability of the three – level method (Aleksandr and Evengii, 1989).

Chapter Three

Methodology

3.1. Study Area and Period

This study will be conducted at Jimma University under mathematics department from August 2017 to November 2018.

3.2. Mathematical Design of the Study

Document review and numerical experimentation design will be employed.

3.3. Source of Information

Relevant information will be obtained from books, published journals and internets.

3.4. Mathematical Procedure

To achieve the stated objective, the study will follow the following procedures.

1) Defining problem.
2) Discretizing the solution domain
3) Replacing the differential equation by finite central difference which gives a system of algebraic equation
4) Applying the Chebyschev iteration method to solve the algebraic equation
5) Investigate the convergence of the method
6) Writing MATLAB code for the method to solve the obtained algebraic equation
7) Validating the method by numerical examples and displaying the numerical results in tabulate form as well as graphically.

Chapter Four

Description of the Method, Results and Discussion

4.1 Description of the Method

Consider the singularly perturbed boundary value problem of the form:

$$-\varepsilon y''(x)+a(x)y(x)=f(x), \quad a<x<b \tag{4.1}$$

with boundary conditions

$$y(a)=\alpha \text{ and } y(b)=\beta \tag{4.2}$$

where ε is a small positive parameter (diffusion coefficient) such that $0<\varepsilon<<1$ and α, β are given constants and $a(x), f(x)$ are assumed to be sufficiently continuously differentiable functions for every $x\in[a,b]$.

To describe the method, we divide the interval $[a,b]$ into N equal subintervals of mesh size h.

Let $a=x_0, x_1,..., x_N=b$ be the nodal points. Then, we have $x_i=a+ih,\ i=0,1,...,N$.

For convenience, let $a(x_i)=a_i$, $f(x_i)=f_i$, $y(x_i)=y_i$, $y'(x_i)=y_i'$, ..., $y^{(n)}(x_i)=y_i^{(n)}$

Assume that $y(x)$ has continuous higher order derivatives on $[a,b]$.

Thus, using Taylor series expansion we have,

$$y_{i+1}=y_i+hy_i'+\frac{h^2}{2!}y_i''+\frac{h^3}{3!}y_i'''+\frac{h^4}{4!}y_i^{(4)}+O(h^5) \tag{4.3}$$

$$y_{i-1}=y_i-hy_i'+\frac{h^2}{2!}y_i''-\frac{h^3}{3!}y_i'''+\frac{h^4}{4!}y_i^{(4)}-O(h^5) \tag{4.4}$$

Adding Eq. (3) and Eq. (4) we obtain the second order finite difference approximation for the second order derivative of y_i as:

$$y_i''=\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}+TE_2 \tag{4.5}$$

where $TE_2 = -\frac{h^2}{12} y_i^{(4)}$

Writing Eq. (1) at discretized nodal points, we get:

$$-\varepsilon y_i'' + a_i y_i = f_i \tag{4.6}$$

Substituting the values of y_i'' from Eq. (5) into Eq. (6) we obtain:

$$\Rightarrow -\varepsilon\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) + a_i y_i = f_i$$

$$\Rightarrow -\varepsilon\left(y_{i+1} - 2y_i + y_{i-1}\right) + h^2 a_i y_i = h^2 f_i$$

Thus, $-\varepsilon y_{i-1} + \left(2\varepsilon + h^2 a_i\right) y_i - \varepsilon y_{i+1} = h^2 f_i,\ i = 1, 2, ..., N-1$ (4.7)

From Eq. (7) we obtain the equivalent three – term recurrence relation given by:

$$E_i y_{i-1} + F_i y_i + G_i y_{i+1} = H_i, \quad \text{for } i = 1, 2, ..., N-1 \tag{4.8}$$

where, $E_i = -\varepsilon = G_i$, $F_i = 2\varepsilon + h^2 a_i$, $H_i = h^2 f_i$

Now, Eq. (8) can be written in matrix form as:

$$AY = B \tag{4.9}$$

where $A = \begin{bmatrix} F_1 & G & 0 & \cdots & 0 \\ E & F_2 & G & 0 & \vdots \\ 0 & E & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & F_{N-2} & G \\ 0 & \cdots & 0 & E & F_{N-1} \end{bmatrix}$ $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-2} \\ y_{N-1} \end{bmatrix}$ and $B = \begin{bmatrix} H_1 + \varepsilon y_0 \\ H_1 \\ \vdots \\ H_{N-2} \\ H_{N-1} + \varepsilon y_N \end{bmatrix}$

A matrix M *is* said to be tri-diagonal if $a_{ij} = 0$ for $|i - j| > 1$ and diagonally dominant if $|a_{ii}| \geq \sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}|,\ i = 1(1)n$. If $\det(M) \neq 0$ then, the matrix is non-singular. Otherwise, it is singular (Jain *et al.*, 2007)

4.2 Chebyschev Iteration Technique

To find an approximate solution of Eq. (9), with a non – singular matrix A, it is important to apply the three – level iterative schemes.

The three – level iterative scheme for Eq. (9), is linked with three iterative approximations $Y^{(p+1)}, Y^{(p)}, Y^{(p-1)}$ so that $Y^{(p+1)}$ is defined in terms of $Y^{(p)}$ and $Y^{(p-1)}$ where p is the iteration number.

In order to realize the three – level scheme, it is necessary to give two initial approximations $Y^{(0)}$ and $Y^{(1)}$. Usually $Y^{(0)}$ is arbitrary and $Y^{(1)}$ is found from the two – level scheme.

Now, according to (Aleksandr and Evengii, 1989), in order to find a new approximation $Y^{(p+1)}$, from the two – level scheme, Eq. (9) can be written as:

$$I\,\frac{Y^{(p+1)}-Y^{(p)}}{\tau}+AY^{(p)}=B, \qquad p=0,1,2,... \tag{4.10}$$

where τ is positive parameter to be determined and I is the identity matrix with the same size to the matrix A.

From Eq. (10) we have:

$$IY^{(p+1)}-IY^{(p)}+\tau AY^{(p)}=\tau B$$

$$\Rightarrow IY^{(p+1)}+\left(\tau A-I\right)Y^{(p)}=\tau B$$

$$\Rightarrow Y^{(p+1)}=\left(I-\tau A\right)Y^{(p)}+\tau B\,, \quad p=0,1,2,... \tag{4.11}$$

$$\Rightarrow Y^{(1)}=\left(I-\tau A\right)Y^{(0)}+\tau B, \text{at } p=0\,,\ Y^{(0)} \text{ is an initial guess}$$

From Eq. (11), denote $F=\left(I-\tau A\right)$

Where F is called iteration matrix

The above Eq. (11) is called the two – level iterative scheme.

To develop the three –level scheme, subtract $Y^{(p-1)}$ from both sides of Eq. (11) to get:

$$Y^{(p+1)} - Y^{(p-1)} = FY^{(p)} + \tau B - Y^{(p-1)} \qquad (4.12)$$

Multiplying the right hand side of Eq. (12) by α_{p+1} , where α_{p+1} is a parameter to be determined from the condition that the resolving matrix *A* has minimal norm and is used to control errors produced in the computation, it gives:

$$Y^{(p+1)} - Y^{(p-1)} = \alpha_{p+1}\left\{FY^{(p)} + \tau B - Y^{(p-1)}\right\}$$

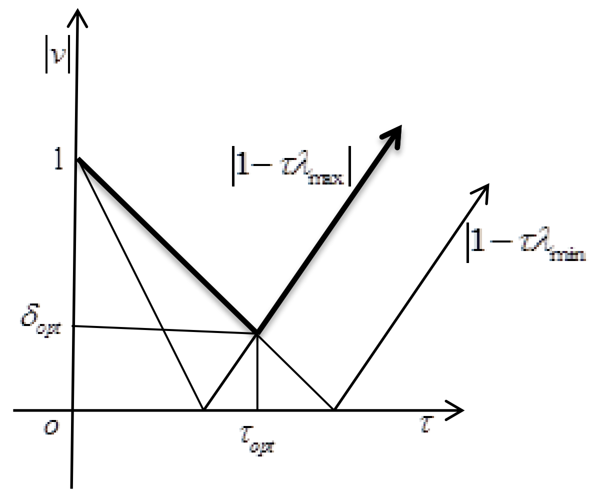
$$Y^{(p+1)} = \alpha_{p+1}FY^{(p)} + \alpha_{p+1}\tau B - \alpha_{p+1}Y^{(p-1)} + Y^{(p-1)}$$

$$Y^{(p+1)} = \alpha_{p+1}FY^{(p)} + \left(1-\alpha_{p+1}\right)Y^{(p-1)} + \alpha_{p+1}\tau B \qquad (4.13)$$
$$p = 1, 2, \ldots$$

Eq. (13) above is called Chebyschev three – level iterative scheme.

Consider Eq. (9), where *Y, B* $\in$ *L* and *L* is an n – dimensional Euclidean space (*e.g.* $L \in \mathbb{R}^n$) Suppose that λ_j , $j = 1, 2, \ldots, n$ are the eigenvalues of the matrix *A* arranged in the ascending order: $0 < \lambda_{\min} = \lambda_1 \le \lambda_2 \le \ldots \le \lambda_n = \lambda_{\max}$.

Now we analyze the behavior of quantities $|v_1| = |1-\tau\lambda_{\min}|$ and $|v_n| = |1-\tau\lambda_{\max}|$ as function of τ . From the following figure, (Victor and Semyon, 2007), we determine that for the smaller value of τ the quantity $|v_1|$ dominates whereas for larger value of τ the quantity $|v_n|$ dominates. The value of $\delta = \delta(\tau) = \max\left\{|1-\tau\lambda_{\min}|, |1-\tau\lambda_{\max}|\right\}$ is shown by a **bold polygonal line**. It coincides with $|1-\tau\lambda_{\min}|$ before the intersection point and after this point it coincides with $|1-\tau\lambda_{\max}|$.



Consequently, the minimum value of $\delta = \delta(\tau)$ is achieved precisely at the intersection point. This condition reads:

$$1-\tau\lambda_{\min} = \tau\lambda_{\max} - 1 \Rightarrow \tau\lambda_{\min} + \tau\lambda_{\max} = 2$$

Therefore, $\tau_{opt} = \tau = \dfrac{2}{\lambda_{\min}+\lambda_{\max}}$ (4.14)

Consequently,

$$\delta_{opt} = \rho(\tau_{opt}) = 1-\tau_{opt}\lambda_{\min} = 1-\left(\frac{2}{\lambda_{\min}+\lambda_{\max}}\right)\lambda_{\min}$$

After rearrangement, we get;

$$\Rightarrow \delta_{opt} = \frac{\lambda_{\max}-\lambda_{\min}}{\lambda_{\max}+\lambda_{\min}} = \frac{1-\dfrac{\lambda_{\min}}{\lambda_{\max}}}{1+\dfrac{\lambda_{\min}}{\lambda_{\max}}}$$

Therefore, $\delta_0 = \dfrac{1-\xi}{1+\xi}$, (4.15)

where $\xi = \dfrac{\lambda_{\min}}{\lambda_{\max}}$ and $\xi \in (0,1)$

According to Victor and Semyon , (2007), the condition number, $\mu(A)$, of a matrix A acted on a normed space L is defined as:

$$\mu(A) = \det(A).\det(A^{-1}). \tag{4.16}$$

where, $\det(A) \geq \rho(A) = \lambda_{\max}$ and $\det(A^{-1}) \geq \rho(A^{-1}) = \frac{1}{\lambda_{\min}}$

$\rho(A)$ and $\rho(A^{-1})$ are the spectral radius of A and A^{-1} respectively.

The quantity of Eq. (16) can be referred to as the condition number of a linear system of Eq. (9). Therefore, we can write condition number in terms of eigenvalues as:

$$\mu(A) = \det(A).\det(A^{-1}) = \frac{\lambda_{\max}}{\lambda_{\min}} = \frac{1}{\xi}$$

Therefore, $\mu(A) = \frac{1}{\xi}$ (4.17)

James (1990), Condition number is related to how sensitive the solution of a problem is to change in its data. In any norm, $\mu(A) \geq 1$, and for "large" value of $\mu(A)$ the matrix A is said to be ill conditioned. Here "large" must be interpreted in somewhat subjective term. If $\mu(A)$=1, then A is said to be perfectly conditioned

4.3 An estimate for the norm of the error

Let $\varepsilon^{(p)}$ be the error and $Y^{(p)}$ be the numerical solution at p iterate and Y be the exact solution of Eq. (9). Multiplying both sides of Eq. (9) by τ and adding Y on both sides, it gives:

$$Y = FY + \tau B \tag{4.18a}$$

Then, from Eq. (18a) a family of iterative scheme can be generated as:

$$Y^{(p+1)} = FY^{(p)} + \tau B \text{ , } p = 0, 1, 2, \ldots \tag{4.18b}$$

Subtracting Eq. (18a) from Eq. (18b),

$$
- \begin{cases} Y^{(p+1)} = FY^{(p)} + \tau B \\ Y = FY + \tau B \end{cases}
$$

$$Y^{(p+1)} - Y = F\left(Y^{(p)} - Y\right) \tag{4.19}$$

Then, Eq. (19) can be rearranged as:

$$Y^{(p)} = \varepsilon^{(p)} + Y \quad \text{and} \quad Y^{(p+1)} = \varepsilon^{(p+1)} + Y \tag{4.20}$$

Substituting Eq. (20) into Eq. (11), it gives the equation of the error for the two – level scheme $\varepsilon^{(p+1)}$ as:

$$Y^{(p+1)} = FY^{(p)} + \tau B\,, \quad p = 0, 1, 2, \ldots$$

$$\varepsilon^{(p+1)} + Y = F\left(\varepsilon^{(p)} + Y\right) + \tau B$$

After rearrangement,

$$
\begin{aligned}
&\varepsilon^{(p+1)} = F\varepsilon^{(p)}\,,\ p = 0, 1, 2, \ldots \\
&\varepsilon^{(1)} = F\varepsilon^{(0)}\,, \quad \text{where} \quad \varepsilon^{(0)} = Y^{(0)} - Y
\end{aligned} \tag{4.21}
$$

Also substituting Eq. (20) into Eq. (13) the following equation of the error for the three – level scheme $\varepsilon^{(p+1)}$ can be expressed as:

$$
\begin{aligned}
&Y^{(p+1)} = \alpha_{p+1} FY^{(p)} + \left(1 - \alpha_{p+1}\right) Y^{(p-1)} + \alpha_{p+1} \tau B \\
&p = 1, 2, \ldots
\end{aligned}
$$

$$\varepsilon^{(p+1)} + Y = \alpha_{p+1} F\,(\varepsilon^{(p)} + Y) + (1 - \alpha_{p+1})(\varepsilon^{(p-1)} + Y)$$

$$\varepsilon^{(p+1)} = \alpha_{p+1} F\,\varepsilon^{(p)} + (1 - \alpha_{p+1})\,\varepsilon^{(p-1)}\ ,\ p = 1, 2, \ldots \tag{4.22}$$

Theorem: Let L be an n – dimensional normed vector spaces $\left(\text{say } \mathbb{R}^n \text{ or } \mathbb{C}^n\right)$ and assume that the induced operator norm of the iteration matrix F of Eq. (13) satisfies $\|F\| = q < 1$ where q is the spectral radius of $F, \rho(F)$. Then, the system of Eq. (13) has a unique solution $Y \in L$.

Moreover, the iteration sequence of Eq. (13) **converges** to the solution Y for an arbitrary initial guess $Y^{(0)}$. The error at p iteration is defined by Eq. (13) satisfies the estimate:

$$\left\|\varepsilon^{(p)}\right\| = \left\|Y^{(p)} - Y\right\| \leq q^p \left\|Y^{(0)} - Y\right\| = q^p \left\|\varepsilon^{(0)}\right\|$$

Proof: For $p = 0$ we have $\varepsilon^{(0)} = Y - Y^{(0)}$ and $\varepsilon^{(p+1)} = F\varepsilon^{(p)} \Rightarrow \varepsilon^{(1)} = F\varepsilon^{(0)}$

$$\Rightarrow \left\|\varepsilon^{(1)}\right\| \leq \|F\| \left\|\varepsilon^{(0)}\right\| = q \left\|\varepsilon^{(0)}\right\|$$

For $p = 1$ we have $\varepsilon^{(1)} = Y - Y^{(1)}$ and $\varepsilon^{(p+1)} = F\varepsilon^{(p)} \Rightarrow \varepsilon^{(2)} = F\varepsilon^{(1)}$

$$\Rightarrow \left\|\varepsilon^{(2)}\right\| \leq \|F\| \left\|\varepsilon^{(1)}\right\| = q^2 \left\|\varepsilon^{(0)}\right\|$$

For $p = 2$ we have $\varepsilon^{(2)} = Y - Y^{(2)}$ and $\varepsilon^{(p+1)} = F\varepsilon^{(p)} \Rightarrow \varepsilon^{(3)} = F\varepsilon^{(2)}$

$$\Rightarrow \left\|\varepsilon^{(3)}\right\| \leq \|F\| \left\|\varepsilon^{(2)}\right\| = q^3 \left\|\varepsilon^{(0)}\right\|$$

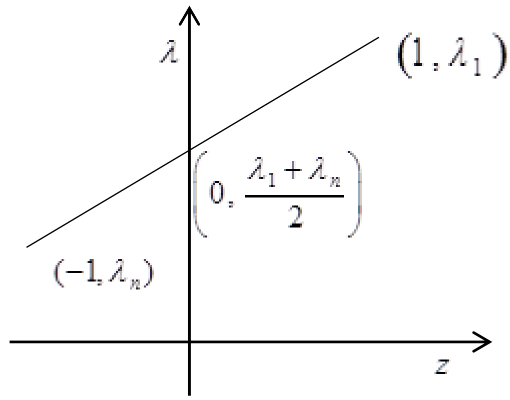
Inductively, for $p \geq 3$, $\left\|\varepsilon^{(p)}\right\| = \left\|Y - Y^{(p)}\right\|$ and

$$\left\|\varepsilon^{(p+1)}\right\| = \left\|F\varepsilon^{(p)}\right\| \Rightarrow \left\|\varepsilon^{(p)}\right\| \leq \left\|F^{(p)}\right\| \left\|\varepsilon^{(0)}\right\| = q^p \left\|\varepsilon^{(0)}\right\|$$

$$\therefore \left\|\varepsilon^{(p)}\right\| \leq q^p \left\|\varepsilon^{(0)}\right\| \tag{4.23}$$

Now, let us transform the interval $\left[\lambda_{\min}, \lambda_{\max}\right]$ into the interval $[-1, 1]$. To do this we make a linear change of variables, mapping the intervals $\lambda_{\min} = \lambda_1 < \lambda < \lambda_n = \lambda_{\max}$ on the interval $-1 < z < 1$ and point λ_1 to the point 1.

It can be shown on $z\lambda$ plane as:



Slope $m = \dfrac{\lambda_1 - \lambda_n}{2}$ and $m = \dfrac{\lambda_1 - \lambda_n}{2} = \dfrac{\Delta\lambda}{\Delta z} = \dfrac{\lambda - \dfrac{\lambda_1 + \lambda_n}{2}}{z - 0}$ (4.24)

$$\Rightarrow \frac{\lambda_1 - \lambda_n}{2} = \frac{\lambda - \dfrac{\lambda_1 + \lambda_n}{2}}{z} \Rightarrow z\left(\frac{\lambda_1 - \lambda_n}{2}\right) = \lambda - \frac{\lambda_1 + \lambda_n}{2}$$

$$\Rightarrow \lambda = \frac{1}{\tau} + z\left(\frac{\lambda_1 - \lambda_n}{2}\right) \Rightarrow \lambda\tau = 1 + z\left(\frac{\lambda_1 - \lambda_n}{2}\right)\tau$$

$$\Rightarrow \lambda\tau = 1 + z\left(\frac{\lambda_1 - \lambda_n}{2}\right)\frac{2}{\lambda_1 + \lambda_n} = 1 + z\frac{\lambda_1 - \lambda_n}{\lambda_1 + \lambda_n}$$

$$\Rightarrow \lambda\tau = 1 - z\frac{\lambda_n - \lambda_1}{\lambda_1 + \lambda_n} = 1 - z\frac{1-\xi}{1+\xi}$$

where $\xi = \dfrac{\lambda_1}{\lambda_n}$

$$\Rightarrow \lambda\tau = 1 - z\delta_0$$

where $\delta_0 = \dfrac{1-\xi}{1+\xi}$

$$\Rightarrow \lambda = \frac{1 - z\,\delta_0}{\tau} \quad \text{and} \quad z = \frac{1-\lambda\,\tau}{\delta_0} \tag{4.25}$$

Using this transformation, the point $\lambda \to 0$ corresponds to:

$$0 = \frac{1 - z\,\delta_0}{\tau} \;\Rightarrow z = \frac{1}{\delta_0} > 1 \tag{4.26}$$

The desired polynomial transformation and the solution of this problem was obtained by the Russian mathematician V.A. Markov in 1982, has the form:

$$\begin{aligned} &Q^{(p)}(z) = q^p T^{(p)}\left(\frac{1-\tau\lambda}{\delta_0}\right), \\ &q^p = \frac{1}{T^{(p)}\left(\dfrac{1}{\delta_0}\right)}, \;\; p = 0,1,2,\ldots \end{aligned} \tag{4.27}$$

$T^{(p)}(z)$ is the Chebyschev polynomial of first kind with degree $p = 0,1,2,\ldots$ such that

$$T^{(p)}(z) = \cos(p\cos^{-1} z), \;\; |z| \le 1 \tag{4.28}$$

$$T^{(p)}(z) = \frac{1}{2}\left(z + \sqrt{z^2 - 1}\right)^p + \frac{1}{2}\left(z - \sqrt{z^2 - 1}\right)^p$$

$$T^{(p)}\left(\frac{1}{\delta_0}\right) = \frac{1}{2}\left(\frac{1}{\delta_0} + \sqrt{\left(\frac{1}{\delta_0}\right)^2 - 1}\right)^p + \frac{1}{2}\left(\frac{1}{\delta_0} - \sqrt{\left(\frac{1}{\delta_0}\right)^2 - 1}\right)^p$$

When rearranged,

$$T^{(p)}\left(\frac{1}{\delta_0}\right) = \frac{1}{2}\left\{\left(\frac{1}{\delta_0} + \frac{1}{\delta_0}\sqrt{1-\delta_0{}^2}\right)^p + \frac{1}{2}\left(\frac{1}{\delta_0} - \frac{1}{\delta_0}\sqrt{1-\delta_0{}^2}\right)^p\right\}$$

$$T^{(p)}\left(\frac{1}{\delta_0}\right)=\frac{1}{2\delta_0{}^p}\left\{\left(1+\sqrt{1-\delta_0{}^2}\right)^p+\left(1-\sqrt{1-\delta_0{}^2}\right)^p\right\}$$

From Eq. (27), it can be observed that: $q^p=\dfrac{2\delta_0{}^p}{\left(1+\sqrt{1-\delta_0{}^2}\right)^p+\left(1-\sqrt{1-\delta_0{}^2}\right)^p}$

Substituting Eq. (15) into the above:

$$q^p=\frac{2\left(\frac{1-\xi}{1+\xi}\right)^p}{\left(1+\sqrt{1-\left(\frac{1-\xi}{1+\xi}\right)^2}\right)^p+\left(1-\sqrt{1-\left(\frac{1-\xi}{1+\xi}\right)^2}\right)^p}$$

$$q^p=\frac{2\left(\frac{1-\xi}{1+\xi}\right)^p}{\left(1+\sqrt{1-\left(\frac{1-\xi}{1+\xi}\right)^2}\right)^p+\left(1-\sqrt{1-\left(\frac{1-\xi}{1+\xi}\right)^2}\right)^p}$$

$$q^p=\frac{2(1-\xi)^p}{\left((1+\xi)+\sqrt{4\xi}\right)^p+\left((1+\xi)-\sqrt{4\xi}\right)^p}=\frac{2(1-\xi)^p}{\left((1+\xi)+2\sqrt{\xi}\right)^p+\left((1+\xi)-2\sqrt{\xi}\right)^p}$$

After rearrangement,

$$q^p=\frac{2\left(\frac{1-\sqrt{\xi}}{1+\sqrt{\xi}}\right)^p}{1+\left(\frac{1-\sqrt{\xi}}{1+\sqrt{\xi}}\right)^{2p}}=\frac{2(\delta_1)^p}{1+(\delta_1)^{2p}} \tag{4.29}$$

where $\delta_1=\dfrac{1-\sqrt{\xi}}{1+\sqrt{\xi}}$

According to, Victor and Semyon, (2007), for any z: $|z| \le 1$ the Chebyschev polynomial of the first kind $T^{(p)}(z)$ satisfy the following recurrence relations:

$$\begin{aligned} &T^{(p+1)}(z) = 2zT^{(p)}(z) - T^{(p-1)}(z), \quad p = 1, 2, \dots \\ &T^{(1)}(z) = z, \ \ T^{(0)}(z) = 1 \end{aligned} \tag{4.30}$$

Using Eq. (27) and Eq. (30), we obtain:

$$\frac{Q^{(p+1)}(z)}{q^{p+1}} = 2\left(\frac{1-\tau\lambda}{\delta_0}\right)\frac{Q^{(p)}(z)}{q^p} - \frac{Q^{(p-1)}(z)}{q^{p-1}} \tag{4.31}$$

$$\frac{Q^{(1)}(z)}{q^1} = \left(\frac{1-\tau\lambda}{\delta_0}\right), \quad \frac{Q^{(0)}(z)}{q^0} = 1$$

The last equations of Eq. (31) are obtained from Eq. (30) and Eq. (27).

From Eq. (27), we have the following values as:

$$T^0(z) = 1, \ T^1(z) = z = \left(\frac{1-\tau\lambda}{\delta_0}\right), T^2(z) = 2z^2 - 1 = 2\left(\frac{1-\tau\lambda}{\delta_0}\right)^2 - 1$$

$$\Rightarrow T^2(z) = \frac{2(1-\tau\lambda)^2 - \delta_0^2}{\delta_0^2}, T^3(z) = \frac{4(1-\tau\lambda)^3 - 3\delta_0^2(1-\tau\lambda)}{\delta_0^3}$$

$$q^0 = 1, \ q^1 = \delta_0, \ q^2 = \frac{\delta_0^2}{2-\delta_0^2}, \ \ q^3 = \frac{\delta_0^3}{4-3\delta_0^2}$$

Using Eq. (31), we have:

$$Q^{(0)}(z) = 1, \ Q^{(1)}(z) = 1-\tau\lambda, Q^{(2)}(z) = \frac{2(1-\tau\lambda)^2 - \delta_0^2}{2-\delta_0^2}, Q^{(3)}(z) = \frac{4(1-\tau\lambda)^2 - 3\delta_0^2(1-\tau\lambda)}{4-3\delta_0^2}$$

Then, from the above equations when λ approaches zero, we get:

$$\frac{1}{q^{p+1}} = \frac{2}{\delta_0 q^p} - \frac{1}{q^{p-1}}, \quad p = 1, 2, \dots \tag{4.32}$$

Multiplying both sides of Eq. (32) by q^{p+1} we get:

$$\frac{q^{p+1}}{q^{p-1}} = \frac{2q^{p+1}}{\delta_0 q^p} - 1, \quad p = 1, 2, \dots \tag{4.33}$$

Substituting Eq. (33) into Eq. (31), we obtain a recurrence formula for the polynomial $Q^{(p+1)}(z)$:

$$Q^{(p+1)}(z)=\frac{2q^{p+1}}{\delta_0 q^p}\left(1-\tau\lambda\right)Q^{(p)}(z)+\left(1-\frac{2q^{p+1}}{\delta_0 q^p}\right)Q^{(p-1)}(z) \qquad (4.34)$$
$$p=1,2,\ldots$$

From Eq. (34), we obtain recurrence relations for $Y^{(p+1)}$ as:

$$Y^{(p+1)}=\frac{2q^{p+1}}{\delta_0 q^p}\left(1-\tau\lambda\right)Y^{(p)}+\left(1-\frac{2q^{p+1}}{\delta_0 q^p}\right)Y^{(p-1)} \qquad (4.35)$$
$$p=1,2,\ldots$$

Comparing Eq. (35) with Eq. (13), we get:

$$\alpha_{p+1} = \frac{2q^{p+1}}{\delta_0 q^p},\ \ p=0,1,2,\ldots \qquad (4.36)$$

Using Eq. (36) we can write α_{p+1} in terms of α_p.

i. Put $p=0$ into Eq. (36) to get $\alpha_1 = 2$

ii. Put $p=1$ into Eq. (36) to get $\alpha_2=\frac{2}{2-\delta_0^2}$, multiplying the numerator and denominator of α_2 by α_1, the result will be $\alpha_2=\frac{4}{4-\delta_0^2\alpha_1}$

iii. Put $p=2$ into Eq. (36) to get $\alpha_3 = \frac{2q^{(3)}}{\delta_0 q^{(2)}}=\frac{2\delta_0^3(2-\delta_0^2)}{\delta_0^3\left(4-3\delta_0^2\right)}=\frac{4-2\delta_0^2}{4-3\delta_0^2}$

$$\alpha_3=\frac{4}{4-\delta_0^2\left(\frac{2}{2-\delta_0^2}\right)}=\frac{4}{4-\delta_0^2\alpha_2}$$

$$\Rightarrow \alpha_3=\frac{4}{4-\delta_0^2\alpha_2}$$

Hence, by induction for $p\geq 1$,

$$\alpha_{p+1} = \frac{4}{4-\delta_0^2\alpha_p} \qquad (4.37)$$

where, $\alpha_{p+1}\in\left(1,\,2\right]$, $p=0,1,2,\ldots$

Hence, from Eq. (23) and Eq. (29), we have; $\left\|\varepsilon^{(p)}\right\| \le q^p \left\|\varepsilon^{(0)}\right\| = \left(\frac{2\delta_1^{\,p}}{1+\delta_1^{\,2p}}\right)\left\|\varepsilon^{(0)}\right\|$

But we have,

$$\frac{\delta_1^{\,p}}{1+\delta_1^{\,2p}} \le \frac{\delta_1^{\,p}}{\delta_1^{\,2p}} = \frac{1}{\delta_1^{\,p}} \tag{4.38}$$

$$\Rightarrow \ln\left(\frac{\delta_1^{\,p}}{1+\delta_1^{\,2p}}\right) \le \ln\left(\frac{1}{\delta_1^{\,p}}\right) = \ln(1) - \ln\left(\delta_1^{\,p}\right) = -p\ln(\delta_1)$$

Therefore, $$\ln\left(\frac{\delta_1^{\,p}}{1+\delta_1^{\,2p}}\right) \le -p\ln(\delta_1) \tag{4.39}$$

Let, $\frac{2\delta_1^{\,p}}{1+\delta_1^{\,2p}} \le \sigma \Rightarrow \frac{\delta_1^{\,p}}{1+\delta_1^{\,2p}} \le \frac{\sigma}{2} \Rightarrow \ln\left(\frac{\delta_1^{\,p}}{1+\delta_1^{\,2p}}\right) \le \ln\left(\frac{\sigma}{2}\right)$

$$\Rightarrow \ln\left(\frac{\delta_1^{\,p}}{1+\delta_1^{\,2p}}\right) \le -p\ln(\delta_1) \le \ln\left(\frac{\sigma}{2}\right)$$

$$\Rightarrow -p\ln(\delta_1) \le \ln\left(\frac{\sigma}{2}\right)$$

$$\Rightarrow p \ge -\frac{\ln\left(\frac{\sigma}{2}\right)}{\ln(\delta_1)} \tag{4.40}$$

Putting $\delta_1 = \frac{1-\sqrt{\xi}}{1+\sqrt{\xi}}$ into Eq. (40) and using Taylor series for the function $f(\delta_1) = \ln(\delta_1)$ at the point $\delta_1 = 1$, we get;

$$\ln(\delta_1) = f(\delta_1) = f(1) + \frac{f'(1)}{1!}(\delta_1) + \frac{f''(1)}{2!}(\delta_1)^2 + \frac{f'''(1)}{3!}(\delta_1)^3 + \frac{f^{(4)}(1)}{4!}(\delta_1)^4 + \ldots$$

$$= 0 + \delta_1 - \frac{1}{2}(\delta_1)^2 + \frac{1}{3}(\delta_1)^3 - \frac{1}{4}(\delta_1)^4 + \ldots$$

$$\ln(\delta_1) = \delta_1\left(1 - \frac{1}{2}(\delta_1) + \frac{1}{3}(\delta_1)^2 - \frac{1}{4}(\delta_1)^3 + \ldots\right) \tag{4.41}$$

By absolutely alternating series test,

$$\ln\left(\delta_1\right) = \delta_1\left(\sum_{k=0}^{\infty}\left(\frac{\left(-1\right)^k}{k+1}\right)\left(\delta_1\right)^k\right) = \delta_1\left(\frac{1}{1-\delta_1}\right) = \frac{\delta_1}{1-\delta_1}$$

Therefore, $\ln\left(\delta_1\right)=\frac{1-\sqrt{\xi}}{2\sqrt{\xi}}$ (4.42)

Substituting Eq. (42) into Eq. (40), we find;

$$p \geq -\frac{\ln\left(\frac{\sigma}{2}\right)}{\ln\left(\delta_1\right)} = -\frac{1-\sqrt{\xi}}{2\sqrt{\xi}}\ln\left(\frac{\sigma}{2}\right) = -\frac{1}{2}\left(\frac{1}{\sqrt{\xi}}-1\right)\ln\left(\frac{\sigma}{2}\right)$$

$$p \geq -\frac{1}{2}\left(\frac{1}{\sqrt{\xi}}-1\right)\ln\left(\frac{\sigma}{2}\right) \geq -\frac{1}{2}\ln\left(\frac{\sigma}{2}\right)\frac{1}{\sqrt{\xi}}$$

So it is sufficient to choose the number of iteration that:

$$p \geq -\frac{1}{2}\ln\left(\frac{\sigma}{2}\right)\sqrt{\mu\left(A\right)} \tag{4.43}$$

The number of iteration p is proportional to the square root of the condition number, $\mu\left(A\right)$. Therefore, the larger the condition number, the more substantial is the relative economy offered by the method and it is also important to mention that the scheme is computationally stable.

4.4 Numerical Examples and Results

Example 1: Consider the singularly perturbed boundary value problem given by:

$$-\varepsilon y''(x)+y(x)=-\left(\cos(\pi x)\right)^2-2\varepsilon\pi^2\cos(2\pi x),\quad 0\le x\le 1,$$

with the boundary conditions: $y(0)=0=y(1)$

The exact solution for this problem is given by: $y(x)=\dfrac{e^{-(\frac{1-x}{\sqrt{\varepsilon}})}+e^{-\frac{x}{\sqrt{\varepsilon}}}}{1+e^{-\frac{1}{\sqrt{\varepsilon}}}}-\left(\cos(\pi x)\right)^2$

Table 1: Pointwise and maximum absolute errors for Example 1 at $N=8$ and different small values of perturbation parameters such that $\varepsilon << h$, with different number of iteration p.

x_i	$\varepsilon=10^{-3}$ $p=3$	$\varepsilon=10^{-4}$ $p=2$	$\varepsilon=10^{-5}$ $p=2$	$\varepsilon=10^{-7}$ $p=1$	$\varepsilon=10^{-9}$ $p=1$
0.125	3.7113e-02	6.2282e-03	6.3216e-04	6.3296e-06	6.3297e-08
0.25	2.8967e-03	3.9942e-05	4.0856e-07	4.0959e-11	4.6629e-15
0.375	8.9291e-04	7.0285e-05	7.0266e-06	7.0289e-08	7.0290e-10
0.5	1.0113e-03	9.9041e-05	9.9367e-06	9.9445e-08	9.9405e-10
0.625	8.9291e-04	7.0285e-05	7.0266e-06	7.0289e-08	7.0290e-10
0.75	2.8967e-03	3.9942e-05	4.0856e-07	4.0959e-11	4.6629e-15
0.875	3.7113e-02	6.2462e-03	6.3216e-04	6.3296e-06	6.3297e-08
Max.	3.7113e-02	6.2462e-03	6.3216e-04	6.3296e-06	6.3297e-08
Phaneendra *et. al.*, 2015	1.23e-02	2.35e-02	2.52e-02	---	---

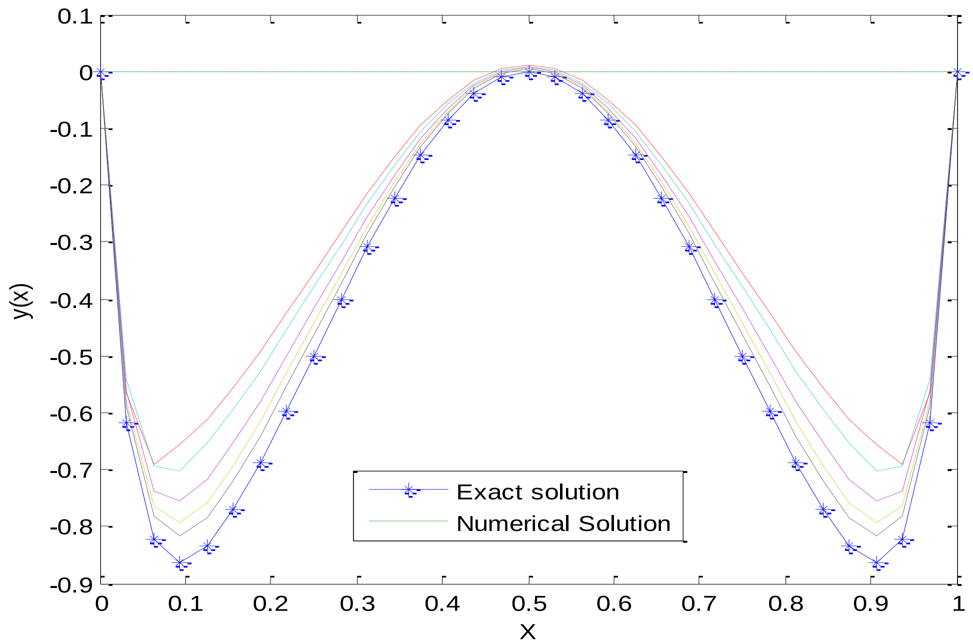


Figure 1: Numerical and Exact solutions of Example 1 at $N = 32$ and $\varepsilon = 10^{-3}$ with $p = 6$ and condition number $\mu(A) = 5.0365$

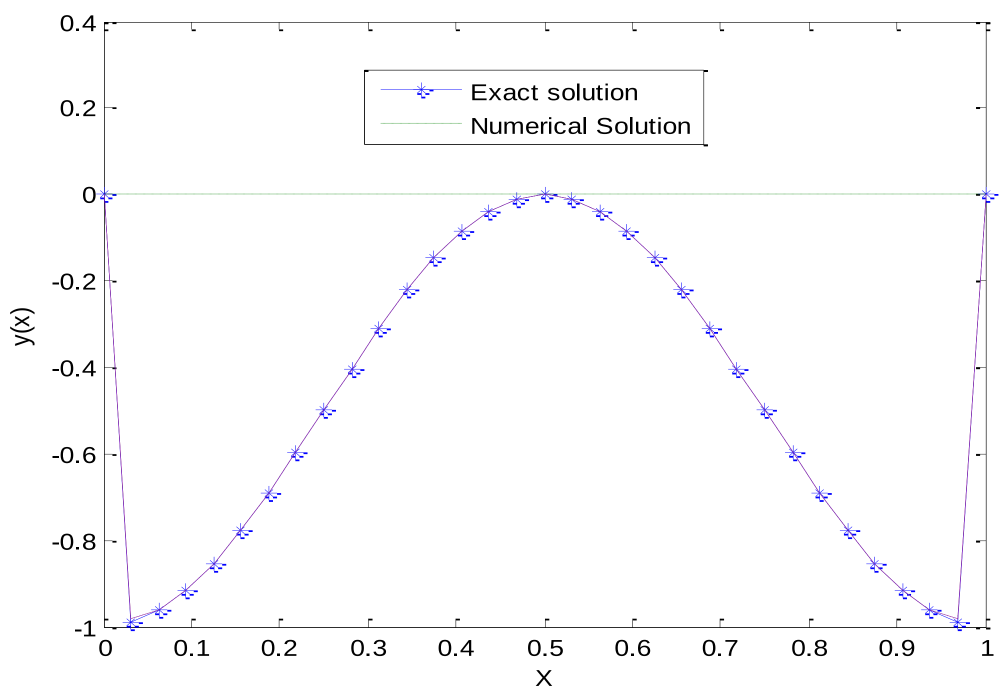


Figure 2: Numerical and Exact solutions of Example 1 at $N = 32$ and $\varepsilon = 10^{-5}$ with $p = 3$ and condition number $\mu(A) = 1.0408$

Example 2: Consider the singularly perturbed boundary value problem given by (Phaneendra *et. al*., 2015):

$$-\varepsilon y''(x)+(2-x^2)y(x)=1, \quad -1<x<1$$

with the boundary conditions: $y(-1)=0=y(1)$

The exact solution is $y(x)=\dfrac{1}{2-x^2}-e^{\frac{-(1+x)}{\sqrt{\varepsilon}}}-e^{\frac{-(1-x)}{\sqrt{\varepsilon}}}$

Table 2: Pointwise and maximum absolute errors for Example 2 at $N=16$ and different small values of perturbation parameters such that $\varepsilon << h$, with number of iteration $p=6$

x_i	**Exact Sol.**	**Pointwise Absolute errors**		
	$\varepsilon=10^{-5}$	$\varepsilon=10^{-3}$	$\varepsilon=10^{-4}$	$\varepsilon=10^{-5}$
-1.0	0	3.4104e-28	1.3839e-87	2.1284e-275
-0.75	6.9565e-01	2.6094e-04	1.5603e-04	1.7607e-05
-0.5	5.7143e-01	6.0129e-04	5.9841e-05	5.8282e-06
-0.25	5.1613e-01	3.1540e-04	9.1770e-06	6.9714e-06
0	5.0000e-01	2.5301e-04	3.6423e-05	1.7836e-05
0.25	5.1613e-01	3.1540e-04	9.1770e-06	6.9714e-06
0.5	5.7143e-01	6.0129e-04	5.9841e-05	5.8282e-06
0.75	6.9565e-01	2.6094e-04	1.5603e-04	1.7607e-05
1.0	0	3.4104e-28	1.3839e-87	2.1284e-275
Max. Absolute error		2.4279e-02	4.7656e-03	5.0435e-04
Phaneendra *et. al*., 2015		2.46e-02	2.16e-02	2.16e-02

Table 3: Pointwise and maximum absolute errors for Example 2 at $N=10$ and different small values of perturbation parameters such that $\varepsilon << h$, with number of iteration $p=3$

$x_i \downarrow$	**Exact Sol.**	Pointwise absolute errors			
	$\varepsilon=10^{-4}$	$\varepsilon=10^{-4}$	$\varepsilon=10^{-6}$	$\varepsilon=10^{-8}$	$\varepsilon=10^{-10}$
-1.0	0	1.3839e-87	0	0	0
- 0.8	7.3529e-01	2.3032e-03	7.4612e-04	1.3928e-04	1.3912e-04
- 0.6	6.0976e-01	8.4182e-05	3.8385e-06	8.0236e-08	7.1416e-08
- 0.4	5.4348e-01	5.0642e-05	9.2496e-07	3.5316e-08	3.9664e-08
- 0.2	5.1020e-01	2.1310e-04	2.6584e-04	4.4399e-05	4.4397e-05
0	5.0000e-01	4.3497e-04	4.9602e-04	9.4601e-05	9.4600e-05
0.2	5.1020e-01	2.1310e-04	2.6584e-04	4.4399e-05	4.4397e-05
0.4	5.4348e-01	5.0642e-05	9.2496e-07	3.5316e-08	3.9664e-08
0.6	6.0976e-01	8.4182e-05	3.8385e-06	8.0236e-08	7.1416e-08
0.8	7.3529e-01	2.3032e-03	7.4612e-04	1.3928e-04	1.3912e-04
1.0	0	1.3839e-87	0	0	0
Max. Absolute error		1.5765e-03	2.0870e-05	1.6489e-07	8.2263e-09

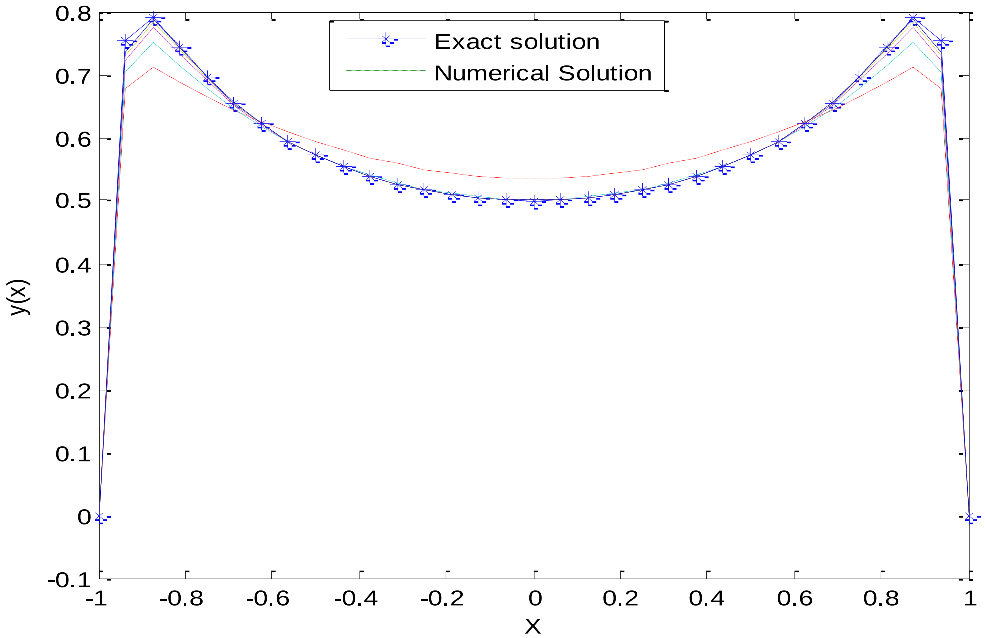


Figure 3: Numerical and Exact solutions of Example 2 at $N = 32$ and $\varepsilon = 10^{-3}$ with $p = 6$ and condition number $\mu(A) = 2.2258$

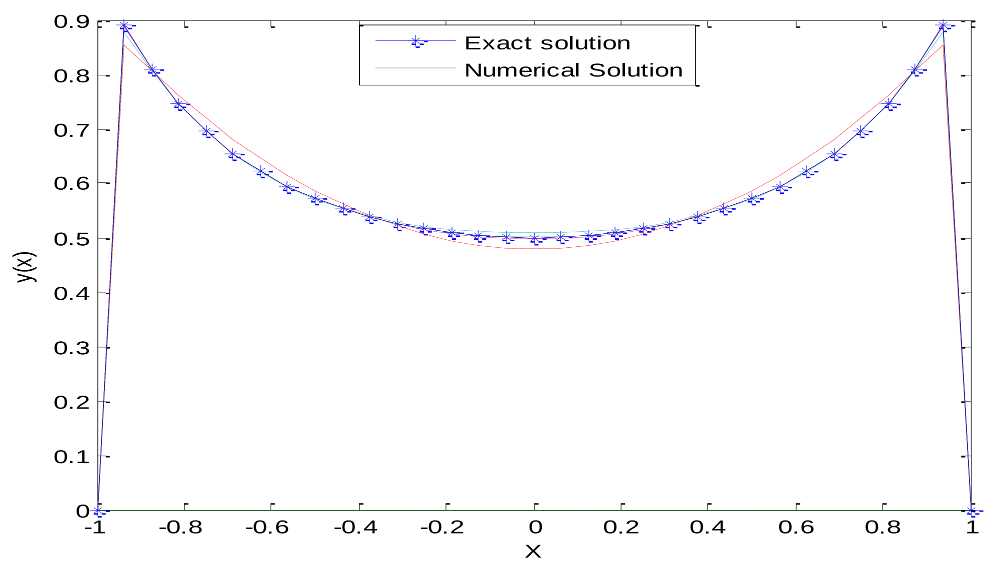


Figure 4: Numerical and Exact solutions of Example 2 at $N = 32$ and $\varepsilon = 10^{-5}$ with $p = 6$ and condition number $\mu(A) = 1.7825$

Example 3: Consider the singularly perturbed boundary value problem given by;

$$-\varepsilon y''(x)+(1+x-x2)y(x)=f(x),\ 0<x<1$$

with boundary conditions: $y(0)=0=y(1)$

where $f(x)=1+x-x2+\left(2\sqrt{\varepsilon}-x2+x3\right)e^{\left(-\frac{(1-x)}{\sqrt{\varepsilon}}\right)}+\left(2\sqrt{\varepsilon}-x\left(1-x\right)^2\right)e^{\left(\frac{-x}{\sqrt{\varepsilon}}\right)}$

The exact solution is given by: $y(x)=1+\left(x-1\right)e^{\left(\frac{-x}{\sqrt{\varepsilon}}\right)}-xe^{\left(-\frac{(1-x)}{\sqrt{\varepsilon}}\right)}$

Table 4: Pointwise and maximum absolute errors for Example 3, $N=10$ and different small values of perturbation parameters such that $\varepsilon << h$, with different condition numbers at iteration number $p=9$

x_i	Exact Solution $\varepsilon=10^{-3}$	Exact Solution $\varepsilon=10^{-6}$	Pointwise absolute errors $\varepsilon=10^{-3}$ $\mu(A)=1.3568$	Pointwise absolute errors $\varepsilon=10^{-6}$ $\mu(A)=1.1468$
0.1	9.6190e-01	1	4.0462e-02	9.1726e-05
0.2	9.9857e-01	1	4.4595e-03	7.9061e-09
0.3	9.9995e-01	1	3.7148e-04	6.5192e-13
0.4	1.0000e+00	1	2.8025e-05	3.8136e-13
0.5	1.0000e+00	1	4.0086e-06	1.6481e-12
0.6	1.0000e+00	1	2.8025e-05	3.8136e-13
0.7	9.9995e-01	1	3.7148e-04	6.5192e-13
0.8	9.9857e-01	1	4.4595e-03	7.9061e-09
0.9	9.6190e-01	1	4.0462e-02	9.1726e-05
Max. Absolute error			4.0462e-02	9.1726e-05

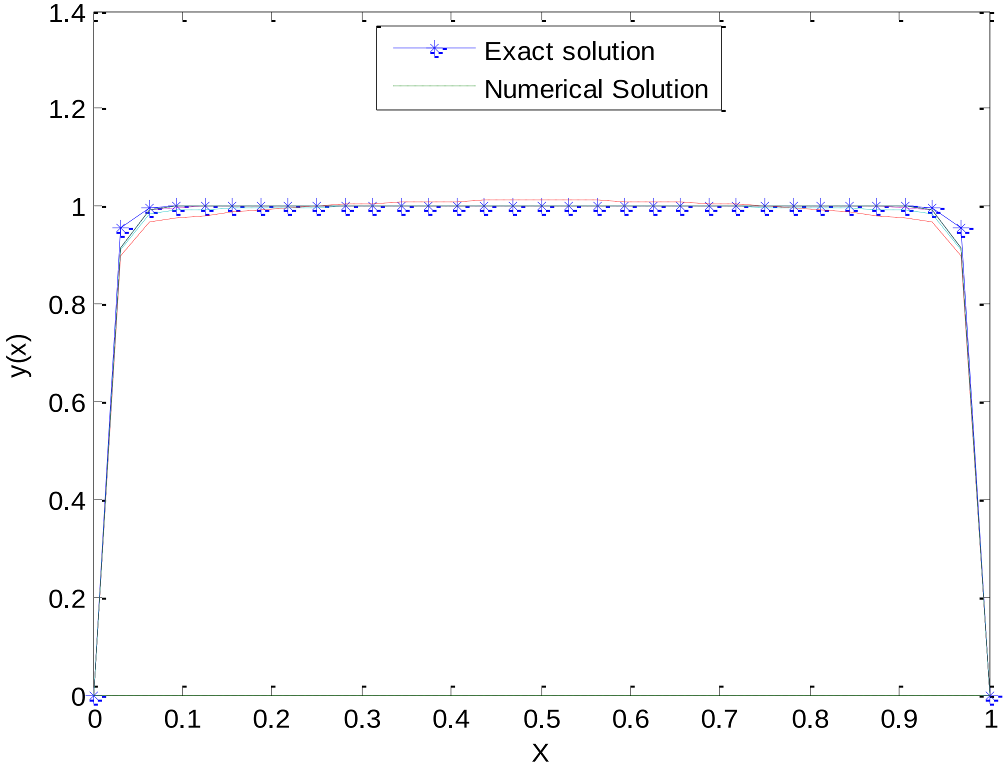


Figure 5: Numerical and Exact solutions of Example 3 at $N = 32$ and $\varepsilon = 10^{-4}$ with $p = 6$ and condition number $\mu(A) = 1.4993$

4.5 Discussion

In this thesis, Chebyschev iteration technique has been presented to solve second order singularly perturbed 1D reaction – diffusion equation for a very small perturbation parameter, ε and both variable and constant coefficient of reaction term. The given problem of interest is discretized and the derivative of the given differential equation is replaced by finite central difference to obtain system of algebraic equation. Chebyschev three – level scheme was developed from a two – level scheme to solve the obtained algebraic equation. To investigate the convergence and stability of the proposed method, three examples were taken and compared with exact solution for a very small value of perturbation parameter, ε and larger step size than step sizes in the literature. Constant and variable coefficient of the reaction term was treated to ensure that the present method approximates the exact solution very well when compared with schemes listed in the literature. Finally, a pointwise and maximum absolute error for each example was shown both by table and graph with different iteration number.

Table 1 indicates that as ε decreases maximum absolute error also decreases. From figure 1 and figure 2 we can observe that when ε is decreasing, $\mu(A)$ approaches one and the rate of convergence is fast. Table 2 and Table 3 also depicts that maximum absolute error decreases when ε is decreasing and iteration number is increasing. Figure 3 and figure 4 read as; at the same number of iteration and when ε is decreasing, the rate of convergence is faster when $\mu(A)$ approaches one.

Lastly, to apply the Chebyschev iteration method there are different parameters to be determined like eigenvalues, condition number and so on needed to speed up the rate of convergence and we can observe that the $\mu(A)$ of a matrix determine how sensitive the solution of the corresponding linear system be to the perturbation of the input data. In addition, the $\mu(A)$ also determines the rate of convergence of the iteration method. Indeed, it is clear that the closer the $\mu(A)$ to one, the faster is the decay of the error.

Chapter Five

Conclusion and Scope of Future Work

5.1 Conclusion

Chebyschev iteration technique has been presented to solve second order singularly perturbed 1D reaction – diffusion equation. Using different parameters obtained in computation, MATLAB code have been carried out on three numerical examples taking both constant and variable coefficient of the reaction term with $\varepsilon \ll h$. The result was shown by table and graph and found to approximate the exact solution very well than methods presented in the literature. The convergence of the method were established well.

5.2 Scope of Future Work

In this thesis, Chebyschev iteration technique has been presented to solve second order singularly perturbed 1D reaction – diffusion equation. Therefore, the scheme proposed in this thesis can also be treated by other iteration techniques and can be extended to higher order of singularly perturbed 1D reaction – diffusion equation.

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