COMMON FIXED POINT OF NONCOMMUTING ALMOST CONTRACTION MAPPING IN CONE b-METRIC SPACES

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Abstract

Vasile Berinde [9] obtained the existence and uniqueness of coincidence and common fixed points of non-commuting almost contractions in cone metric spaces. Inspired and motivated by the main result of Berinde [9], in this research we have studied the existence and uniqueness of coincidence points and common fixed points of a class of almost contraction maps in complete cone b-metric space. In this research, we followed Numerical and Analytical design. The secondary data were collected from relevant source of information and the techniques of Huang and Xu [15], Shi and Xu [24] and Berinde [9] were used to achieve the objective of the study. Finally, we have also provided examples in support of our main results.

CHAPTER ONE

1. **INTRODUTION**

1.1 BACKGROUND OF THE STUDY

Fixed point theory is one of the famous theories in mathematics and has a broad set of application in many branches of mathematics such as the theory of differential and integral equations. In1922, Stefan Banach [6], a Polish mathematician, established a very important result regarding existence of fixed points for contraction mapping on metric spaces. A mapping $T: X \to X$, where (X, d) is a metric space, is said to be a contraction if there exists $k \in [0,1)$ such that for all $x, y \in X$

$$d(Tx, Ty) \leq kd(x, y) \tag{1.1}$$

If the metric space (X, d) is complete, then a mapping satisfying (1.1) has a unique fixed point. Inequality (1.1) implies continuity of T. A natural question is that whether we can find a contractive condition which will imply existence of fixed point in a complete metric space but will not imply continuity. In 1969, Kannan in[19] established the following results in which the above question has been answered in the affirmative. If $T: X \to X$, where (X, d) is a complete space, satisfies the inequality

$$d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)] \tag{1.2}$$

where $\beta \in \left[0, \frac{1}{2}\right)$and $x, y \in X$, then Thas a unique fixed point. A similar contractive condition has been introduced by Chatterjee [11] as follows: If $T: X \to X$, where (X, d) a complete metric space, satisfies the inequality

$$d(Tx, Ty) \leq \gamma[d(x, Ty) + d(y, Tx)] \tag{1.3}$$

where $\gamma \in [0, \frac{1}{2})$ and $x, y \in X$,then T has a unique fixed point. The mapping satisfying (1.3) are called Chatterjee type mapping.

In 1972, Zamfirescu [28] obtained a generalization of Banach's,Kannan's and Chatterjee's fixed point theorems. One of the most general contraction condition for satisfying the following condition has been obtained by Ciric [12] in 1974. If $T: X \to X$ where (X, d) is a complete metric space satisfying the inequality

$$d(Tx, Ty) \leq h. max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \tag{1.4}$$

where $h \in [0,1)$ and for all $x, y \in X$. A mapping satisfying (1.5) is commonly called quasi contraction. In 2004, Berinde [7] defined the notion of weak contraction mapping which is more general than a contraction mapping. In [10] Berinde renamed it as an almost contraction mapping. The Zamfirescu fixed point theorem has been further extended to almost contractions [10], a class of contractive type mappings which exhibits totally different features than the ones of the particular results incorporated.i.e., an almost contraction generally does not have a unique fixed point. [See Example 1 in [7]. Moreover, he proved that any strict contraction, the Kannan [19] and Zamfirescu [28] mapping as well as a large class of quasi-contractions are all almost contractions.

In [14], Huang and Zhang initiated cone metric spaces, which is a generalization of metric spaces, by substituting the real numbers withorder Banach spaces. They have considered convergence in cone metric spaces, introduced completeness of cone metric spaces, and proved a Banach contraction mapping theorem, and some other fixed point theorem involving contractive type mappings in cone metric spaces using normality condition.

Abbas and Jungck [2] used this setting as ambient space in order to formulate and prove several fixed point theorems that extends well known fixed point theorems for contractive type mapping from the case of usual metric spaces. Indirect relation to this result, in [21] the author pointed out that all the fixed point theorems, established in [14] for the case a cone metric space ordered by normal cone p with normal constant K, could be formulated and proved in a more general case of a cone metric space.

On the other hand Sessa [23] introduce the notion of weakly commuting maps in metric spaces which are the generalization of commuting maps. Jungck [18] enlarged this concept of weakly commutativity by introducing compatible maps.

In [9], Vasile Berinde obtained coincidence and common fixed point theorems, similar to the one in [2], but for more general class of almost contraction, by restricting the ambient space to the class of usual metric spaces.

In [5], Bakhtin introduced b-metric space as a generalization of metric spaces and proved a contraction mapping principle in b-metric space that generalized the famous Banach contraction principle in metric spaces. In 2011, Hussain and Shah [16] introduced cone b-metric spaces as a generalization of b-metric spaces and cone metric spaces. Recently, Huang and Xu [15] have proved some fixed point theorems of contraction mapping without the assumption of normality condition in complete cone b-metric space.

Inspired and motivated by a result mentioned on [9] and using the notion introduced on [24] and [15], the purpose of the research is to study common fixed point results for a large class of almost contraction in complete cone b-metric space.

1.2 STATEMENT OF THE PROBLEM

The aim of research is to study common fixed points of non-commuting self-maps for a large class of almost contraction in cone b-metric space. There are various generalizations of contraction mapping principles in the literature which has been established either by relaxing the contractive condition or by imposing some additional conditions on complete metric spaces. Being attracted by these results, the researcher initiated to work in this line.

This research answers the following question:

1) What are the relationship between the existence of coincidence point and common fixed point of non-commuting almost contraction on cone b-metric space?
2) What are the sufficient condition for the existence of common fixed point of non-commuting almost contraction on cone b-metric space?
3) What are the generalized common fixed point result of non-commuting almost contraction on cone b-metric space?
4) Can we provide supporting example for the main result?

1.3 OBJECTIVE

1.3.1 General Objective

The general objective of this research is to study common fixed point of non-commuting self-maps for a large class of almost contraction in cone b-metric space.

1.3.2 Specific objectives

The specific objective of this research was:

- To compare and contrast generalized common fixed point of non-commuting almost contraction on cone b-metric space.
- To identify the relation between coincidence point and common fixed point of non-commuting almost contraction on cone b-metric space.
- To prove the existence of common fixed points of non–commuting almost contraction on cone b-metric space.
- To provide examples in support of our main results.

1.4 Significance of the study

Currently Fixed point theory is one of active areas of research with wide range of application in various fields (see [13, 15, 21, 24]). I hope that the result obtained in this study contributes to a large extent to research activities in this area. Moreover, it provides some back ground information for other researchers who need to conduct further research on the study area.

1.5 Delimitation of the study

This study was delimited to the existence of common fixed point of non-commuting almost contraction mapping on cone b-metric space.

CHAPTER TWO

2. Methodology

This chapter addressed study design, description of the research methodology, data collection procedures and data analysis process.

2.1 Study Site

The study was conducted in Jimma University under mathematics department.

2.2 Study Design

The study design was Analytical and Numerical designs.

2.3 Source of Information

In this study secondary data was used. So, the available source of information for the study was books, journals, different study related to the topic and internet services.

2.4 Instrumentation and Administration

In this study data was collected using documentary analysis from internet, journals, and published research and mathematics reference books. Besides it, the researcher took training for 7 days on cone b-metric space for better understanding of the study area from expertise at Addis Ababa University, Addis Ababa.

2.5 Procedure of the study

This study intended to establish common fixed point for a large class of almost contraction in cone b-metric spaces by using the standard techniques similar to that of Huang and Xu[15] , Shi and Xu[24] and Berinde[9].

2.6 Ethical consideration

The researcher took care of ethical considerations. To make the study legal, the researcher took a permission letter from Research Review and Ethical Committee of college of Natural Science,Jimma University.

CHAPTER THREE

3. RESULT AND DISCUSSION

3.1 PRELIMENARIES

Definition3.1.1 [9]: Let E be a real Banach space and P be a subset of$E$.The subset P is called a cone if and only if:

i. P is non-empty, closed and $P \neq \{0\}$

ii. $a, b \in R, \quad a, b \geq 0$ and $x, y \in P \Rightarrow ax + by \in P$.

iii. $P \cap -P = \{0\}$

On this basis, we define a partial ordering$\leq$with respect to P by$x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$. Write $\|.\|$ as the norm on$E$.The cone P is called **normal** if there is a number $k > 0$ such that $0 \leq x \leq y$ implies$\|x\| \leq k\|y\|$ for all $x, y \in E$.

The least positive number $k$satisfying the above condition is called the normal constant of$P$.

Definition 3.1.2 [9]: Let Xbe a nonempty set. Suppose that the mapping $d: X \times X \to E$satisfies:

i. $0 \leq d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $d(x, y) = 0$ if and only if $x = y$

ii. $d(x, y) = d(y, x)$ for all $x, y \in X$

iii. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$

Then d is called a cone metric on$X$ and $(X, d)$is called a **cone metric space**.

The concept of a cone metric space is more general than that of a metric space.

Example 3.1.1 [27]: Let $E = \mathbb{R}^2, P = \{(x, y) \in E |\, x, y \geq 0\}$, $X = R$ and $d : X \times X \to E$ be such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 3.1.3 [20]: Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d: X \times X \to \mathbb{R}$ is called a **b-metric** provided that, for all $x, y, z \in X$

i. $d(x,y) = 0$ if and only if $x = y$
ii. $d(x,y) = d(y,x)$
iii. $d(x,y) \leq s[d(x,z) + d(z,y)]$for all $x, y, z \in X$.

In this case the pair (X, d) is called a ***b-metric space***.

It is clear that the definition of b-metric space is an extension of metric space. Also, if we consider s = 1 in Definition 3.1.3, then we obtain definition of metric space.

Remark 3.1.1: Note that a metric space is evidently a b-metric space. However, b-metric on X need not be a metric on X.

Example 3.1.2 [3]: Let (X, d) be a metric space and $\rho(x,y) = (d(x,y))^p$ where$p > 1$ is a real number. Then ρ is a b-metric with$S = 2^{p-1}$. However, (X, ρ) is not necessarily a metric space.

Example 3.1.3 [3]: Let X be a set of real numbers and let $d(x,y) = |x - y|$ be the usual Euclidean metric. Then $\rho(x,y) = (x-y)^2$ is a b-metric on $\mathbb{R}$ with $s = 2$, but it is not a metric on $\mathbb{R}$.

Definition 3.1.4 [15]: Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d: X \times X \to E$ is said to be ***cone b-metric*** if and only if, for all $x, y, z \in X$ the following conditions are satisfied:

i) $0 \leq d(x,y)$ with $x \neq y$ and $d(x,y) = 0$ if and only if $x = y$
ii) $d(x,y) = d(y,x)$
iii) $d(x,y) \leq s[d(x,z) + d(z,y)]$

In this case the pair (X, d) is called a ***cone b-metric space***.

Remark 3.1.2: The class of cone b-metric spaces is larger than the class of cone metric spaces. Since any cone metric space must be a cone b- metric space. Therefore, it is obvious that cone b-metric spaces generalize b-metric spaces and cone metric spaces.

Example 3.1.4 [15]: Let $X = [1,2,3,4], E = \mathbb{R}^2, P = \{(x,y) \in E: x \geq 0, y \geq 0\}$.

Defined by, $d(x,y) = \begin{cases} (|x-y|^{-1}, |x-y|^{-1}), & \text{if } x \neq y \\ (0,0), & \text{if } x = y \end{cases}$

Then, (X,d) is a cone b-metric with coefficient $s = \frac{6}{5}$. But it is not cone metric space, since the triangular inequality is not satisfied. Indeed,

$$d(1,2) > d(1,4) + d(4,2), \;\; d(3,4) > d(3,1) + d(1,4).$$

Observe that if $s = 1$, then the ordinary triangle inequality in a cone metric space is satisfied, however it does not hold true when $s > 1$. Thus, the class of cone b-metric spaces is effectively larger than that of the ordinary cone metric spaces. That is, every cone metric space is a cone b-metric space, but the converse need not be true. The following examples illustrate the above remarks.

Example 3.1.5 [22]: Let $X = \mathbb{R}, E = \mathbb{R}^2$ and $P = \{(x,y) \in E: x \geq 0, y \geq 0\}$. Define $d: X \times X \to E$ by

$$d(x,y) = (|x-y|^2, |x-y|^2).$$

Then, (X,d) is a cone b-metric space with coefficient $s = 2$. But it is not a cone metric space, since the triangular inequality is not satisfied.

Definition 3.1.5 [15]: Let (X,d) be a cone b-metric space, $x \in X$ and $\{x_n\}_{n\geq 1}$ a sequence in X then:

i. $\{x_n\}_{n\geq 1}$ Converges to x whenever, for every $c \in E$ with $0 \ll c$, there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n\to\infty} x_n = x$, or $x_n \to x$ (as $n \to \infty$).

ii. $\{x_n\}_{n\geq 1}$ is a Cauchy sequence whenever for every $c \in X$ with $0 \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

iii. (X,d) is a complete cone b-metric space if every Cauchy sequence is convergent.

Lemma 3.1.1: Let $\{x_n\}$ be a sequence in a metric space (X, d) with $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$.Then $\{x_n\}$ is a Cauchy sequence.

Remark 3.1.3 [25] It follows from above definitions that if $\{x_{n_k}\}$ is a subsequence of a Cauchy sequence $\{x_n\}$ in a cone metric space (X, d) and $x_{n_k} \to z$ as $k \to \infty$ then $x_n \to z$.

Proposition 3.1.1 [15] Let (X, d) be a cone b-metric space the following properties are often used while dealing with cone b-metric space in which P is not necessarily normal.

a) If $u << v$ and $v << w$, then $u << w$
b) If $0 \leq u \ll c$ for each $c \in intP$, then $u = 0$.
c) If $a \leq b + c$ for each$c \in intP$, then $a \leq b$.
d) If 0≤ $d(x_n, x) \leq b_n$, and $b_n \to 0$, then $x_n \to x$.
e) If $a \leq \lambda a$,where $a \in P$,$0 < \lambda < 1$, then $a = 0$.
f) If $c \in intP$, $0 \leq a_n$ and $a_n \to 0$,then there exists $n_0 \in N$ such that$a_n \ll c$ for all $n > n_0$.

Definition 3.1.6 [26]: Let (X, d) be metric space. A map $T : X \to X$ is called an almost contraction with respect to a mapping $S : X \to X$ if there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(Sx, Sy) + Ld(Sy, Tx), \text{ for all } x, y \in X.$$

If we choose $S = I$, I is the identity map on X, we obtain the definition of almost contraction, the concept introduced by Berinde [10].

Definition 3.1.7 [26]: Let E be a subset of a metric space(X, d). Let S and T be two self- maps of a metric space(X, d), T is called ***S-contraction*** if there exists $k \in [0,1)$ such that

$$d(Tx, Ty) \leq kd(Sx, Sy), \text{ for all } x, y \in E.$$

Definition 3.1.8 [26]: Let (X, d) be a metric space. A map $T : X \to X$ is said to satisfy condition (B)' if there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that
$d(Tx, Ty) \leq \delta d(x, y) + L min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$, for all $x, y \in X$.

Definition 3.1.9 [9]: Two self-mappings$T$ and S on X is said to be ***weakly compatible*** if S and T commute at their coincidence point (i.e., $STx = TSx, x \in X$ whenever $Sx = Tx$.) A point $y \in X$is called a ***point of coincidence*** of two self-mappings $S$and T on Xif there exists a point $x \in X$ such that $y = Tx = Sx$.

Definition 3.1.10 [1]: Let (X, d) be a metric space, S and T be self-mappings on X, with $T(X) \subseteq S(X)$ and $x_0 \in X$. Choose a point x_1 in Xsuch that $Sx_1 = Tx_0$. Thiscan be done since $T(X) \subseteq S(X)$. Continuing this process, for x_n in Xwe can find x_{n+1} in Xsuch that

$$Sx_{k+1} = Tx_k\,;\ k = 0, 1, 2, \ldots$$

The sequence $\{Sx_n\}$is called a ***T-sequence***with initial point x_0.

Lemma 3.1.2 [14]: Let X be a non-empty set and the mappings $S, T, : X \to X$ have a unique point of coincidence v in X.If (S, T) are weakly compatible, then S, T have a unique common fixed point.

Proof: Let v be the point of coincidence of$S$, and T. Then$v = Su = Tu$ for some $u \in X$. By weakly compatibility of (S, T) we have,$Sv = STu = TSu = Tv$. It implies that $Sv = Tv = w$(say). Thus, w is a point of coincidence of S,and T.Therefore, $v = w$

In 2010, Berinde proved the following existence and uniqueness theorems of common fixed points of a pair of self-maps which generalizes and extends so many existing related results in [10, 14, 2, 21].

Theorem 3.1.1 [9]: Let (X, d) be a cone b-metric space and let $T, S: X \to X$ be mappings for which there exists a constant $\delta \in (0,1)$ and some $L \geq 0$such that

$$d(Tx, Ty) \leq \delta d(Sx, Sy) + Ld(Sy, Tx), \text{ for all } x, y \in X \tag{3.1.1}$$

If the range of S contains the range of $T$and $S(X)$ is complete subspace of X, then T and S have a coincidence point in X. Moreover, for any $x_0 \in X$, the iteration $\{Sx_n\}$ converges to some coincidence point x^* of T and S.

Theorem 3.1.2 [9]: Let (X, d) be a cone b-metric space and let $T, S: X \to X$ be mappings satisfying (3.1.1) for which there exists a constant $\theta \in (0,1)$ and some $L_1 \geq 0$ such that

$$d(Tx, Ty) \leq \theta d(Sx, Sy) + L_1 d(Sx, Tx), \text{ for all } x, y \in X$$

If the range of S contains the range of $T$and $S(X)$ is complete subspace of X, then T and S have a coincidence point in X. Moreover, for any $x_0 \in X$, the iteration $\{Sx_n\}$ converges to some coincidence point x^* of T and S.

We now establish the main results of this research work.

3.2. MAIN RESULT

We start this section by presenting a coincidence point theorem.

Theorem 3.2.1: Let (X,d) be a cone b-metric space with coefficient $s \geq 1$ and let $T,S: X \to X$ be mappings for which there exists a constant $k \in [0,\frac{1}{s})$ and some $L \geq 0$ such that

$$d(Tx,Ty) \leq kd(Sx,Sy) + Ld(Sy,Tx), \quad \text{for all } x,y \in X \tag{3.2.1}$$

If $T(X) \subseteq S(X)$and $S(X)$ is complete subspace of X, then T and S have a coincidence point in X. Moreover, for any $x_0 \in X$, the iteration $\{Sx_n\}$ converges to some coincidence point x^* of T and S.

Proof: Let x_0 be an arbitrary point in X. Since $T(X) \subseteq S(X)$ we can choose $x_1 \in X$ suchthat $Tx_0 = Sx_1$. Also since $T(X) \subseteq S(X)$, $Tx_1 = Sx_2$ for some $x_2 \in X$. Continuing in this way, for x_n in X, we can find $x_{n+1} \in X$ such that

$$Sx_{n+1} = Tx_{n,} \quad \text{for} \quad n = 0,1,2,\ldots \tag{3.2.2}$$

If $x \coloneqq x_n$ and $y \coloneqq x_{n-1}$ are two successive terms of the sequence defined by (3.2.2), then by (3.2.1), we have

$$d(Sx_n, Sx_{n+1}) = d(Tx_{n-1}, Tx_n) \leq kd(Sx_{n-1}, Sx_n) + Ld(Sx_n, Tx_{n-1})$$

Now, we consider two cases.

Case i) Suppose $Sx_n = Sx_{n+1}$ for some $n \in \mathbb{N}$, then by using inequality (3.2.1),

We have,

$$d(Sx_{n+1}, Sx_{n+2}) = d(Tx_n, Tx_{n+1}) \leq kd(Sx_n, Sx_{n+1}) + Ld(Sx_{n+1}, Tx_n).$$

This implies that

$$d(Sx_{n+1}, Sx_{n+2}) \leq kd(Sx_n, Sx_{n+1}).$$

This yield

$$d(Sx_{n+1}, Sx_{n+2}) = 0$$

$$\Rightarrow Sx_{n+1} = Sx_{n+2}$$

$\Rightarrow Sx_n = Sx_{n+2}$.

Continuing on this process, inductively, it follows that $Sx_n = Sx_m$ for all $m \geq n$.

So, that $\{Sx_n\} m \geq n$ is a constant sequence and hence it is a Cauchy sequence.

Case ii) Suppose$Sx_n \neq Sx_{n+1}$ for all $n \in \mathbb{N}$,then we have

$$d(Sx_n, Sx_{n+1}) = d(Tx_{n-1}, Tx_n)$$
$$\leq kd(Sx_{n-1}, Sx_n) + Ld(Sx_n, Tx_{n-1})$$
$$\leq kd(Sx_{n-1}, Sx_n) + Ld(Sx_n, Sx_n)$$

This implies that

$$d(Sx_n, Sx_{n+1}) \leq kd(Sx_{n-1}, Sx_n), \text{ for all } n = 1,2,3, \ldots \qquad (3.2.3)$$

Thus, for each $n = 1,2,3, \ldots$, we have

$$d(Sx_{n+1}, Sx_n) \leq kd(Sx_n, Sx_{n-1}) \leq k^2 d(Sx_{n-1}, Sx_{n-2}) \leq \cdots \leq k^n d(Sx_1, Sx_0) \qquad (3.2.4)$$

Then, for all $p \geq 1$, we have

$$d\big(Sx_n, Sx_{n+p}\big) \leq sd(Sx_n, Sx_{n+1}) + sd(Sx_{n+1}, Sx_{n+p})$$
$$\leq sd(Sx_n, Sx_{n+1}) + s^2 d(Sx_{n+1}, Sx_{n+2}) + s^2 d(Sx_{n+2}, Sx_{n+p})$$
$$\leq sd(Sx_n, Sx_{n+1}) + s^2 d(Sx_{n+1}, Sx_{n+2}) + \cdots + s^p d(Sx_0, Sx_1).$$

Now, by (3.2.4) and $sk < 1$ imply that

$$d(Sx_n, Sx_{n+p}) \leq sk^n d(Sx_0, Sx_1) + s^2 k^{n+1} d(Sx_0, Sx_1) \ldots s^p k^{n+p-1} d(Sx_0, Sx_1)$$
$$\leq (sk^n + s^2 k^{n+1} + \cdots + s^{p-1} k^{n+p-1}) d(Sx_0, Sx_1)$$
$$\leq sk^n (1 + sk + s^2 k^2 + \cdots s^{m-n-1} k^{m-n-1}) d(Sx_0, Sx_1)$$
$$\leq \frac{sk^n}{1-sk} d(Sx_0, Sx_1).$$

Since $k \in [0, \frac{1}{s})$, we notice that $\frac{sk^n}{1-sk} d(Sx_0, Sx_1) \to 0$ as $n \to \infty$.

Thus, for each $0 \ll \epsilon$, there exists $N \in \mathbb{N}$ such that $d(Sx_n, Sx_{n+p}) \ll \epsilon$ for all $n > N$ and $p \geq 1$.

Therefore,$\{Sx_n\}$ is a Cauchy sequence in $S(X)$.

Since $S(X)$ is complete, there exists x^* in $S(X)$ suchthat

$$\lim_{n\to\infty} Sx_{n+1} = x^* \qquad (3.2.5)$$

We can find $p \in X$ such that $Sp = x^*$.Then by (3.2.3) and (3.2.4) we further have,

$$d(Sx_n, Tp) = d(Tx_{n-1}, Tp) \leq kd(Sx_{n-1}, Sp) \leq \frac{k^n}{1-k} d(Sx_1, Sx_0).$$

This shows that

$$\lim_{n\to\infty} Sx_n = Tp \quad (3.2.6)$$

By (3.2.5), (3.2.6) and Remark (3.1.3), it results now that $Tp = Sp$. That is P is a coincidence point of T and S(or x^* is a point of $T$and S).

Remark 3.1.4: Almost contraction need not have a unique fixed point. By Theorem 3.2.1 above coincidence point is not generally unique (see Example 1 in [10]). In order to obtain a common fixed point theorem from Theorem3.2.1, we need the uniqueness of the coincidence point, which could be obtained by imposing additional contractive condition, quite similar to (3.2.1).

Theorem 3.2.2: Let (X, d) be a cone b-metric space with coefficient $S \geq 1$ and let $T, S: X \times X \to X$ mapping satisfies (3.2.1) for which there exists a constant $k \in [0, \frac{1}{s})$ and some $L_1 \geq 0$such that

$$d(Tx, Ty) \leq kd(Sx, Sy) + L_1 d(Sx, Tx), \text{ for all } x, y \in X \quad (3.2.7)$$

If $T(X) \subseteq S(X)$and $S(X)$ is a complete subspace of X, then T and S have a unique coincidence point in X. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point in X. In both cases, for $x_0 \in X$, the iteration$\{Sx_n\}$defined by (3.2.2) converges to the unique common fixed point(coincidence point) $x^*$of S and T.

Proof: By the proof of Theorem 3.3.1, we have that T and S have at least a point of coincidence, say $x^* = Tp = Sp,\ p \in X$.

Now, let us show that T and S have a unique point of coincidence.

Assume, there exists $q \in X$ such that $Tq = Sq$.

Then, by inequality (3.2.7), we get

$$\begin{aligned} d(Sq, Sp) &= d(Tq, Tp) \\ &\leq kd(Sq, Tp) + L_1 d(Sq, Tq). \end{aligned}$$

This implies that,

$$d(Sq, Sp) \leq kd(Sq, Sp)$$

Which yields,

$$(1-k)d(Sq, Sp) \leq 0.$$

By definition, $0 \leq d(Sq, Sp)$, that is, $d(Sq, Sp) \in P$ and by proposition 3.1.1(e),

$$d(Sq, Sp) = 0$$

This shows that $Sq = Sp = x^*$.

That is, T and$S$ has a unique point of coincidence x^*.

Now, if T and S are weakly compatible, by Lemma [3.2.2] it follows that x^* is their unique common fixed point.

The following corollaries are also obtained from our main results.

Corollary 3.2.3: Let (X,d) be a cone b-metric space with coefficient $s \geq 1$ and let $T,S: X \to X$ be two mappings for which there exist $sk \in [0,\frac{1}{2})$ such that for all $x, y \in X$,

$$d(Tx,Ty) \leq k[d(Sx,Tx) + d(Sy,Ty)] \quad (3.2.8)$$

If $T(X) \subseteq S(X)$ and $S(X)$ is a complete subspace of X, then T and S have a unique coincidence point in X. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point in X. In both cases, the iteration $\{Sx_n\}$ defined by (3.2.2) converges to the unique(coincidence) common fixed point x^* of S and T, for any $x_0 \in X$.

Proof: Let x_0 be an arbitrary point in X. since$T(X) \subseteq S(X)$ we can choose $x_1 \in X$such that $Tx_0 = Sx_1$. Also since $T(X) \subseteq S(X)$, $Tx_1 = Sx_2$ for some $x_2 \in X$. Continuing in this way, for x_n in X,we can find $x_{n+1} \in X$ such that

$$Sx_{n+1} = Tx_n \text{ for } n = 0,1,2,\ldots$$

Without loss of generality assume that $Sx_n \neq Sx_{n+1}$ for all $n = 1,2,3,\ldots$

Then, we have

$$\begin{aligned} d(Sx_{n+1},Sx_n) = d(Tx_n,Tx_{n-1}) &\leq k[d(Sx_n,Tx_n) + d(Sx_{n-1},Tx_{n-1})] \\ &\leq k[d(Sx_n,Sx_{n+1}) + d(Sx_{n-1},Sx_n)] \\ &\leq kd(Sx_n,Sx_{n+1}) + kd(Sx_{n-1},Sx_n) \end{aligned}$$

This implies that

$$\begin{aligned} d(Sx_{n+1},Sx_n) &\leq \frac{k}{1-k} d(Sx_{n-1},Sx_n) \leq \left(\frac{k}{1-k}\right)^n d(Sx_0,Sx_1) \\ &\leq h^n d(Sx_0,Sx_1) \quad \text{Where } h = \tfrac{k}{1-k} \in [0,1) \end{aligned} \quad (3.2.9)$$

Then, for all $p \geq 1$, we get

$$\begin{aligned} d(Sx_n,Sx_{n+p}) &\leq s[d(Sx_n,Sx_{n+1}) + d(Sx_{n+1},Sx_{n+p}) \\ &\leq sd(Sx_n,Sx_{n+1}) + sd(Sx_{n+1},Sx_{n+p}) \end{aligned}$$

$$\leq sd(Sx_n, Sx_{n+1}) + s^2 d(Sx_{n+1}, Sx_{n+2}) + s^2 d(Sx_{n+2}, Sx_{n+p})$$

$$\leq sd(Sx_n, Sx_{n+1}) + s^2 d(Sx_{n+1}, Sx_{n+2}) + \cdots + s^p d(Sx_{n+p-1}, Sx_{n+p})$$

By using (3.2.9), we have

$$d(Sx_n, Sx_{n+p}) \leq sh^n d(Sx_0, Sx_1) + s^2 h^{n+1} d(Sx_0, Sx_1) + \cdots + s^p h^{n+p-1} d(Sx_0, Sx_1)$$

$$\leq (sh^n + s^2 h^{n+1} + \cdots + s^p h^{n+p-1}) d(Sx_0, Sx_1)$$

$$\leq \frac{sh^n}{1-sh} d(Sx_0, Sx_1) \tag{3.2.10}$$

Since $h \in [0,1)$, $\frac{sh^n}{1-sh} d(Sx_0, Sx_1) \to 0$ as $n \to \infty$.

The rest of the proof follows as in case of Theorem 3.2.2.

Corollary 3.2.4: Let (X, d) be a cone b-metric space with coefficient $s \geq 1$ and let $T, S: X \to X$ be two mappings for which there exist $s\lambda \in [0, \frac{1}{2})$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq \lambda[d(Sx, Ty) + d(Sy, Tx) \tag{3.2.11}$$

If $T(X) \subseteq S(X)$ and $S(X)$ is a complete subspace of X, then T and S have a unique coincidence point in X. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point in X. In both cases, the iteration $\{Sx_n\}$ defined by (3.2.2) converges to the unique (coincidence) common fixed point $x^*$of S and T, for any $x_0 \in X$.

Proof: Let x_0 be an arbitrary point in X. since $T(X) \subseteq S(X)$ we can choose $x_1 \in X$such that $Tx_0 = Sx_1$. Also since $T(X) \subseteq S(X)$, $Tx_1 = Sx_2$ for some $x_2 \in X$. Continuing in this way, for x_n in X,we can find $x_{n+1} \in X$ such that

$$Sx_{n+1} = Tx_n \text{ for } n = 0,1,2, \ldots$$

Without loss of generality, assume that $Sx_n \neq Sx_{n+1}$ for all $n = 1,2,3, \ldots$

Then, we obtain

$$d(Sx_n, Sx_{n+1}) = d(Tx_{n-1}, Tx_n)$$

$$\leq \lambda[d(Sx_{n-1}, Tx_n) + d(Sx_n, Tx_{n-1})]$$

$$= \lambda[d(Sx_{n-1}, Sx_{n+1}) + d(Sx_n, Sx_n)]$$

$$\leq s\lambda[d(Sx_{n-1}, Sx_n) + d(Sx_n, Sx_{n+1})]$$

Thus, we have

$$d(Sx_n, Sx_{n+1}) \leq \frac{s\lambda}{1-s\lambda} d(Sx_{n-1}, Sx_n)$$

$$\leq \left(\frac{s\lambda}{1-s\lambda}\right)^n d(Sx_0, Sx_1)$$

$$\leq v^n d(Sx_0, Sx_1), \qquad (3.2.12)$$

where $v = \frac{s\lambda}{1-s\lambda}$. Note that since $s\lambda \in [0, \frac{1}{2})$, we have $\frac{s\lambda}{1-s\lambda} \in [0,1)$.

Thus, for all $p \geq 1$, we have

$$d(Sx_n, Sx_{n+p}) \leq s[d(Sx_n, Sx_{n+1}) + d(Sx_{n+1}, Sx_{n+p})$$

$$\leq sd(Sx_n, Sx_{n+1}) + sd(Sx_{n+1}, Sx_{n+p})$$

$$\leq sd(Sx_n, Sx_{n+1}) + s^2d(Sx_{n+1}, Sx_{n+2}) + s^2d(Sx_{n+2}, Sx_{n+p})$$

$$\leq sd(Sx_n, Sx_{n+1}) + s^2d(Sx_{n+1}, Sx_{n+2}) + \cdots + s^pd(Sx_{n+p-1}, Sx_{n+p})$$

By using (3.2.12), we have

$$d\big(Sx_n, Sx_{n+p}\big) \leq sv^n d(Sx_0, Sx_1) + s^2v^{n+1}d(Sx_0, Sx_1) + \cdots + s^pv^{n+p-1}d(Sx_0, Sx_1)$$

$$\leq (sv^n + s^2v^{n+1} + \cdots + s^pv^{n+p-1})d(Sx_0, Sx_1)$$

$$\leq \frac{sv^n}{1-sv} d(Sx_0, Sx_1) \qquad (3.2.13)$$

Since, $v \in [0,1)$, $\frac{sv^n}{1-sv} d(Sx_0, Sx_1) \to 0$ as $n \to \infty$.

The rest of the proof follows as in case of Theorem 3.2.2.

Corollary 3.2.5: Let (X, d) be a cone b-metric space with coefficient $s \geq 1$ and let $T, S: X \to X$ be two mappings for which there exists $a \in \left[0, \frac{1}{s}\right)$, $sb, sc \in [0, \frac{1}{2})$ such that for all $x, y \in X$, at least one of the following conditions is true:

(z_1) $d(Tx, Ty) \leq ad(Sx, Sy)$,

(z_2) $d(Tx, Ty) \leq b[d(Sx, Tx) + d(Sy, Ty)]$ (3.2.14)

(z_3) $d(Tx, Ty) \leq c[d(Sx, Ty) + d(Sy, Tx)$.

If $T(X) \subseteq S(X)$ and $S(X)$ is a complete subspace of X, then T and S have a unique coincidence point in X. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point in X. In both cases, the iteration $\{Sx_n\}$ defined by (3.2.2) converges to the unique (coincidence) common fixed point $x^*$of S and T, for any $x_0 \in X$.

Proof: From the proof of Theorem 3.2.1, Corollary 3.2.3 and Corollary 3.2.4, the conclusion of the Corollary follows.

3.3 EXAMPLE

Let $E = \mathbb{R}^2$ be Euclidean plane, and $P = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$ be a positive cone of E.

Let $X = \{(x, 0) \in \mathbb{R}^2 : 0 \leq x \leq 1\}$ and define $d: X \times X \to P$ by

$$d\big((x, 0), (y, 0)\big) = (|x - y|^2, |x - y|^2) \quad \forall (x, 0), (y, 0) \in X,$$

then (X, d) be complete cone b- metric space. Let $T, S: X \to X$ be defined by

$$T(x, 0) = \begin{cases} (0,0), & 0 \leq x \leq \frac{1}{4} \\ \left(\frac{1}{5}, 0\right), & \frac{1}{4} < x \leq 1 \end{cases} \text{ and } S(x, 0) = \begin{cases} (x, 0), & 0 \leq x < \frac{1}{4} \\ (1,0), & \frac{1}{4} \leq x \leq 1 \end{cases} \text{ respectively.}$$

We have, $T(X) = \left\{(0,0), (\frac{1}{5}, 0)\right\} \subseteq \left\{(x, 0): 0 \leq x < \frac{1}{4}\right\} \cup \{(1,0)\} = S(X)$.

Moreover, $(0,0)$, is the unique coincidence point of S and T, and since obviously T and S commute at $(0,0)$, then S and Tare weakly compatible.

In order to show that S and T do satisfy the contractive condition of (3.2.7) in Theorem 3.2.2.

Let us denote

$$M_1 = \left[0, \frac{1}{4}\right) \times [0, \frac{1}{4}) \qquad M_2 = \left[0, \frac{1}{4}\right) \times (\frac{1}{4}, 1]$$

$$M_3 = [0, \frac{1}{4}) \times \left\{\frac{1}{4}\right\} \qquad M_4 = \left[\frac{1}{4}, 1\right] \times [\frac{1}{4}, 1]$$

Clearly,

$$[0,1] \times [0,1] = M_1 \cup M_2 \cup M_3 \cup M_4.$$

Case i) for $(x, y) \in M_1$

$$T(x, 0) = (0,0), \quad T(y, 0) = (0,0), \qquad S(x, 0) = (x, 0) \text{ and } S(y, 0) = (y, 0)$$

In this case S and T satisfy contractive condition (3.2.7) of Theorem 3.2.2.

Indeed by (3.2.7), we get

$$(0,0) \leq k(|x - y|^2, |x - y|^2) + L(|x|^2, |x|^2)$$

This holds for all $x, y \in \left[0, \frac{1}{4}\right)$ and any constant $L \geq 0$.

Caseii) for $(x, y) \in M_2$

$T(x, 0) = (0,0), T(y, 0) = \left(\frac{1}{5}, 0\right), S(x, 0) = (x, 0)$ and $S(y, 0) = (1,0)$

Again in this case S and T satisfy the contractive condition (3.2.7).

Indeed by (3.2.7), we have

$$\left(\left|\frac{-1}{5}\right|^2, \left|\frac{-1}{5}\right|^2\right) \leq \mathrm{k}(|\mathrm{x}-1|^2, |\mathrm{x}-1|^2) + \mathrm{L}(|\mathrm{x}|^2, |\mathrm{x}|^2)$$

This holds for $\mathrm{x} \in \left[0, \frac{1}{4}\right)$, $\mathrm{y} \in \left(\frac{1}{4}, 1\right]$ and $\mathrm{L} \geq 0$.

Case iii) for $x, y \in M_3$

$$\mathrm{T}(\mathrm{x}, 0) = (0,0), \mathrm{T}(\mathrm{y}, 0) = (0,0), \mathrm{S}(\mathrm{x}, 0) = (\mathrm{x}, 0) \text{and } \mathrm{S}(\mathrm{y}, 0) = (1,0).$$

By the contractive condition (3.2.7),

We get

$$(0,0) \leq \mathrm{k}(|\mathrm{x}-1|^2, |\mathrm{x}-1|^2) + \mathrm{L}(|\mathrm{x}|^2, |\mathrm{x}|^2)$$

This holds for $\mathrm{x} \in \left[0, \frac{1}{4}\right)$ and $\mathrm{L} \geq 0$.

Case iv) for $x, y \in M_4$.

$$\mathrm{T}(\mathrm{x}, 0) = \left(\frac{1}{5}, 0\right), \mathrm{T}(\mathrm{y}, 0) = \left(\frac{1}{5}, 0\right), \mathrm{S}(\mathrm{x}, 0) = (1,0) \text{and } \mathrm{S}(\mathrm{y}, 0) = (1,0).$$

By the contractive condition (3.2.7),

We have,

$$(0,0) \leq \mathrm{k}(, 0,0) + \mathrm{L}(\left|1-\frac{1}{5}\right|^2, \left|1-\frac{1}{5}\right|^2)$$

This holds for $\mathrm{x}, \mathrm{y} \in \left[\frac{1}{4}, 1\right]$ and $\mathrm{L} \geq 0$.

By summarizing, we conclude that S and T satisfy the contractive condition of (3.2.7) in Theorem 3.2.2 with $\mathrm{k} < \frac{1}{2}$ and $\mathrm{L} \geq 0$.

Hence, $(0,0)$ is a unique fixed point of S and T.

CHAPTER FOUR

4. Conclusion and Future scope

4.1. Conclusion

In [10] the author obtained coincidence and common fixed point theorems for more general class of almost contraction and also in [9] proved the existence of coincidence points and common fixed points for a large class of almost contraction in cone metric spaces. The main aim of this study is to extend the results obtained in [9] to cone b-metric spaces. We can also obtain the following particular cases from our main result.

1) If $s = 1$ in Theorem 3.2.1, then we obtain Theorem 2 in [9].
2) If in (3.2.1), we have$L = 0$, then by Theorem 3.2.1, we obtain a generalization of Theorem 2.1 in [2]. If the cone b-metric reduces to a usual metric space, then by Theorem 3.2.1 we obtain Theorem 2 in [10], which, in turn, generalizes the Jungck common fixed point [17].
3) If in Theorem 3.2.1, the cone$P = \mathbb{R}^+$, the nonnegative real semi-axis, and $s = 1,$ then by Theorem 3.2.1 we obtain the main result (Theorem 3) in [10].
4) Also we observe that by Theorem 3.2.1, if $s = 1$, we obtain a significant generalization of Theorem 2.8 in [21], which has been obtained there by imposing for the contractive inequality (3.2.1) the very restrictive condition $\delta + L < 1$.

4.2. Future scope

In this study, we have focused on the existence and uniqueness of coincidence points and common fixed points of two non-commuting almost contraction self-mapping defined on cone b-metric space, which is an active area of research in Mathematics. So we recommend some post graduate students of the department to conduct their research work for the partial fulfillment of their MSc degree in the area of existence and uniqueness of common fixed points of two or more operators defined on cone b-metric spaces.

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