COMMON FIXED POINT RESULTS IN B-RECTANGULAR PARTIALLY ORDERED METRIC SPACES.

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS

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DECLARATION

I, the undersigned declare that, the proposal entitled "**Commo**n **Fixed point results in b-rectangular partially ordered metric spaces"** is original and it has not been submitted to any institution elsewhere for the award of any academic degree or like, where other sources of information that have been used, they have been acknowledged .

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ABSTRACT

In this study, we established some common fixed point results in b-rectangular partially ordered metric space in the framework of metric spaces endowed with a partial order by extending the works of Roshan et al,. Our results extend some results of Roshan *et al*. The researcher followed analytical design in this research work. Secondary source of data such as journal articles and books which are found in different libraries and internet were used for the study. The mathematical procedures we employed for this study were the techniques used by Roshan *et al*. and Arab and we provided examples in support of our main results.

CHAPTER ONE

INTRODUCTION

1.1 Background of the study

Let X be a non-empty and $T: X \to X$, then we call T a self map of X. A point $x \in X$ is said to be fixed point of T if $Tx = x$.

Let (X, d) be a metric space. A self-map $T: X \to X$ is said to be a contraction if there is a real number $k \in [0, 1)$ such that:

$$d(Tx, Ty) \leq kd(x, y), \text{ for all } x, y \in X. \quad (1.1)$$

In this case k is called a contraction constant.

A theory of fixed point is one of the most powerful and popular tools of modern mathematics. Its use is not only confined to pure and applied mathematics but also it serves as a bridge between analysis and topology. It is a fruitful area of interaction between analysis and topology and also to examine the quantitative problems involving certain mappings and space structures required in various areas such as: economics, chemistry, biology, computer science, engineering, and others. For more details one can refer, (Banach, 1922, Roshan et.al. 2016 Czerwik, 1993).

The first most significant result of metric fixed point theory was given by the Polish mathematician Stefan Banach, in 1922, which is known as Banach contraction principle. Banach contraction principle states that if (X, d) is a complete metric space and $T: X \to X$ is a contraction, then T has unique fixed point x in X. It also helps to show that the Picard iteration $x_{n+1} = Tx_n,\ x_0 \in X$, converges to the fixed point. For the reason that the contraction mapping is continuous, many researchers established fixed point theorems on various classes of operators which satisfy conditions that are weaker than the contractive condition in Banach Contraction Principle but are not continuous.

Kannan, (1968) gave a fixed point theorem for a self-map $T: (X, d) \to (X, d)$ which need not be continuous and satisfying the contraction condition

$$d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X, \text{ where } 0 \leq \alpha < \tfrac{1}{2}. \quad (1.2)$$

A lot of authors were devoted to obtain fixed point theorem for various classes of contractive type conditions that do not require the continuity of T on X. One of them is actually a sort of dual Kannan fixed point theorem.

Chatterjea , (1972) gave the dual of Kannan fixed point theorem as follows:

$$d(Tx, Ty) \leq \beta[d(x, Ty) + d(y, Tx)] \text{ for all } x, y \in X, \text{ where } 0 \leq \beta < \tfrac{1}{2}. \quad (1.3)$$

Banach contractions (1.1), Kannan mappings (1.2) and Chatterjea's mappings (1.3) are independent, (Rhoades, 1977).

Zamfirescu, (1972) proved the following fixed point theorem by combining (1.1), (1.2) and (1.3) as follows.

Theorem1.1: Let (X, d) be a complete metric space and $T: X \to X$ be a map for which there exist the real numbers a, b and c satisfying $0 \leq a < 1$, $0 \leq b <$ ½, $0 \leq c <$ ½, such that for each pair $x, y \in X$ at least one of the following holds.

$$(Z_1) \ d(Tx, Ty) \leq ad(x, y);$$

$$(Z_2) \ d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)];$$

$$(Z_3) \ d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)].$$

Then T has a unique fixed point. Zamfirescu's Theorem, (1972) is a generalization of (Z_1) Banach's Theorem, (1922), (Z_2) Kannan's Theorem, (1968) and (Z_3) Chatterjea's Theorem, (1972).

Definition 1.1 A partially ordered set is a set X and a binary relation $\preccurlyeq$, denoted by $(X, \preccurlyeq)$ such that for all $a, b, c \in X$

i. $a \preccurlyeq a$. (Reflexivity),
ii. $a \preccurlyeq b$ and $b \preccurlyeq a \Rightarrow a = b$. (Anti-symmetry),
iii. $a \preccurlyeq b$ and $b \preccurlyeq c \Rightarrow a \preccurlyeq c$. (Transitivity).

Example 1.1:

(1) If X any non-empty set $(p(X), \subseteq)$ is a partially ordered set. Where $p(X)$ = the power set of X and " $\subseteq$ "is to mean subset of ".

(2) On the set of natural number N, define $m \preccurlyeq n$ if m divides n then $(N, \preccurlyeq)$ is a partially ordered set.

The concept of b-metric space was introduced by Czerwik,(1993). Since then, several papers have been published on the fixed point theory of various classes of single-valued and multi-valued operators in b-metric spaces (Singh and Prasad,2008 and Boriceanu,2009). On the other hand, George et al., (2015) introduced the concept of rectangular b-metric space. Branciari, (2000) introduced the notation of a generalized (rectangular) metric space where the triangle inequality of a metric space was replaced by another inequality, the so called rectangular inequality.

Definition 1.2: Branciari, (2000). Let X be a nonempty set and let d: $X \times X \rightarrow [0, +\infty)$ be a mapping such that for all $x, y \in X$ and for all distinct points $u, v \in X$, each of them different from x and y, one has

(RM_1) $d(x,y) = 0 \quad iff\ x = y$;

(RM_2) $d(x,y) = d(y,x)$;

(RM_3) $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$ (the rectangular inequality).

Then, the map d is called rectangular metric space. The pair (X, d) is called a rectangular metric space.

Definition 1.3: Roshan et al., (2016) Let X be a nonempty set, $s \geq 1$ be a given real number, and let $d: X \times X \rightarrow [0, \infty)$ be a mapping such that for all $x, y \in X$ and all distinct points $u, v \in X$, each distinct from x and y:

$(br1)$ $d(x,y) = 0\ iff\ x = y$;

$(br2) d(x,y) = d(y,x)$;

$(br3) d(x,y) \leq s\,[d(x,u) + d(u,v) + d(v,y)]$. (b-rectangular inequality).

Then (X, d) is called b-rectangular metric space $(b - r.m.s)$.

Example 1.2: George *et.al.*, (2015) let $X = A \cup B$, where $A = \{\frac{1}{n} : n \in N\}$ and B is the set of all positive integers define $d: X \times X \rightarrow [0, \infty)$ such that $d(x,y) = d(y,x)$ for all $x, y \in X$ and

$$d(x,y) = \begin{cases} 0, & if\ x = y \\ 2\alpha, & if\ x, y \in A \\ \frac{\alpha}{2n}, if\ x \in A\ and\ y \in \{2,3\}; \\ \alpha, & otherwise \end{cases}$$

where $\alpha > 0$ is a constant. Then (X, d) is a b-rectangular metric space with coefficient $s = 2 > 1$.

Roshan *et al.*, (2016) Proved fixed point results in ordered b-rectangular partially ordered metric spaces.

Notation:

1. Fs denotes the class of all functions $\beta: [0, \infty) \rightarrow [0, \frac{1}{s})$ satisfying the following condition:

   $\lim_{n \rightarrow \infty} \beta t_n = \frac{1}{s}$ implies $\lim_{n \rightarrow \infty} t_n = 0$, where $s > 1$.
2. $C\ (f, g)$ is the set of coincidence point of f and g.

Theorem 1.2: Roshan *et al*., (2016) Let $(X, \preccurlyeq, d)$ be a complete ordered $b - r.m.s.$ with parameter $s > 1$. Let $f : X \to X$ be an increasing mapping with respect to $\preccurlyeq$ such that there exist an element $x_0 \in X$ with $x_0 \preccurlyeq f\, x_0$ suppose that

$$d(fx, fy) \leq \beta(d(x, y))M(x, y)$$

For some $\beta \epsilon Fs$ and all comparable elements, $x, y \in X$ where

$M = \max\left\{d(x, y), \frac{d(x,fx)d(y,fy)}{1+d(fx,fy)}, \frac{d(x,fx)d(y,fy)}{1+d(x,y)}, \frac{d(x,fx)d(x,fy)}{1+d(x,fy)+d(y,fx)}\right\}$, if f is continuous, then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Theorem 1.3: Roshan *et al*., (2016) Under the hypotheses of Theorem 1.2, without the continuity assumption on f, assume that whenever $\{x_n\}$ is a non decreasing sequence in X such that $x_n \longrightarrow u$, one has $x_n \preccurlyeq u$ for all $n \in N$. Then f has a fixed point.

Inspired and motivated by the work of the Roshan *et al*., (2016) the researcher has extended this work to common fixed point results for a pair of self- maps.

Therefore, the purpose of this study is to establish some common fixed point results and proving existence and uniqueness of common fixed point results in b-rectangular partially ordered metric spaces by extending the works of Roshan *et al.,* (2016).

1.3 Objectives of the study

1.3.1 General objective

1. The general objective of this study is to establish common fixed point results in b-rectangular partially ordered metric spaces by extending the works of (Roshan et al.,2016).

1.3.2 Specific objective

The specific objectives of this study are:

1. To prove the existence of common fixed point results for a pair of self-mapping in b-rectangular partially ordered metric spaces.
2. To establish additional conditions required to obtain uniqueness of common fixed Point in b-rectangular partially ordered metric spaces.
3. To verify the applicability of the results obtained using specific examples.

1.4 Significance of the study

The study may have the following importance:

- The results obtained in this study may contribute to research activities in this area.
- The researcher may develop scientific research writing skills and scientific communication in Mathematics.
- The results obtained in this study may help to solve existence and uniqueness of solution involving differential equation.

1.5 Delimitation of the study

This study was delimited to prove the existence and uniqueness of common fixed point result for a pair of self-mapping in b-rectangular partially ordered metric spaces which was done under the stream of Functional Analysis.

CHAPTER TWO
LITERATURE REVIEW

The Banach Contraction Principle is a very popular tool which is used to solve existence problems in many branches of Mathematical Analysis and its applications. It is no surprise that there are a great number of generalizations of this fundamental theorem. They go in several directions-modifying the basic contractive condition or changing the ambient space. The Banach's contraction mapping principle is one of the cornerstones in the development of fixed point theory. In particular, this principle is used to demonstrate the existence and uniqueness of a solution of differential equations, integral equations, functional equations, partial differential equations and others. Due to their importance generalizations of Banach's contraction mapping principle have been investigated heavily by many authors , (Sintunavarat and Kumam, 2013).

Stefan Banach, (1922) Stated his celebrated theorem on the existence and uniqueness of fixed point of contraction of self- maps defined on complete metric spaces for the first time, which is known as the Banach contraction mapping principle. Since then many researchers have obtained fixed point, common fixed point and other fixed point results in metric spaces, cone- metric spaces, partially ordered metric spaces and other spaces.

Geraghty, (1973) proved a fixed point result, generalizing the Banach contraction principle. Several authors proved later various results using Geraghty-type conditions.

Fixed point results of this kind in b-metric spaces were obtained by Đuki´c *et al*., (2011).

In recent times, fixed point theory has developed rapidly in partially ordered metric spaces, that is, metric spaces endowed with a partial ordering. The triple $(X, d, \preccurlyeq)$ is called partially ordered metric spaces if $(X, \preccurlyeq)$ is a partially ordered set and (X, d) is a metric space, (Das and Dey, 2007).

The study of common fixed points of mappings satisfying certain contractive conditions can be employed to establish existence of solutions of certain operator equations such as differential and integral equations. Beg and Abbas, (2006) obtained common fixed points by extending a weak contractive condition to two maps. Abbas and Dori´c, (2009) obtained a common fixed point theorem for four maps.

Definition 2.1: Ciric *et al.*, (2008) Let $(X, \preccurlyeq)$ be a partially ordered set and $f, g: X \to X$. One says f is g-non-decreasing if for all $x, y \in X$, we have

$gx \preccurlyeq gy \Rightarrow fx \preccurlyeq fy.$

Theorem 2.1: Ding *et al.*, (2015) Let (X, d) be a b-rectangular space with $s > 1$, and let $f, g: X \to X$ be two self-maps such that $f(X) \subseteq g(X)$, one of these two subsets of X being complete. If for some real numbers $a, b \geq 0$ with $a + 2b < \frac{1}{s}$, $d(fx, fy) \leq ad(gx, gy) + b[d(gx, fx) + d(gy, fy)]$ holds for all $x, y \in X$, then f and g have a unique point of coincidence w. Moreover, for each, a corresponding Jungck sequence $\{y_n\}$ can be chosen such that $lim_{n\to\infty} y_n = w$. Moreover,if f and g are weakly compatible, then they have a unique common fixed point**.**

CHAPTER THREE
METHODOLOGY

3.1 Study Site and Period

This study is conducted from September 2016 to October 2017 in Jimma University at Mathematics Department.

3.2 Study Design

The design of the study is analytical.

3.3 Source of Information

This study mostly depended on document materials, so the available source of information for the study were Books related to the area of study, Journals, different study results related to the topic and internet services.

3.4 Mathematical Procedure

The mathematical procedure employed in the researcher work was the analysis techniques used by Sumit Roshan *et al.*, (2016) and (Arab, 2016).

CHAPTER 4

4 DISCUSSION AND RESULT

4.1 PRELIMINARIES

Definition 4.1: Arshad et al., (2013) Let f and g be self-mappings of a non-empty set X.

(i) A point $x \in X$ is said to be a common fixed point of f and g if $x = fx = gx$.

(ii) A point $x \in X$ is called a coincidence point of f and g if $fx = gx$. And if $w = fx = gx$, then w is said to be a point of coincidence of f and g.

(iii) The mappings $f, g: X \to X$ are said to be weakly compatible if they commute at their coincidence point that is, $fgx = gfx$ whenever $gx = fx$.

Definition 4.2: Let (X, d) be a metric space. Then the pair (f, g) is said to be compatible if and only if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Definition 4.3: Al-Thagafi and Shahzad, (2009) Let f and g be self-maps of a metric space *(X, d)*. The pair (f, g) is called:
i. commuting if $fgx = gfx$, for all $x \in$ X.
ii. Weakly commuting if $d(fgx, gfx) \leq d(fx, gx)$, for all $x \in X$.

Example 4.1: Let $X = [0, \infty)$. Define $f, g: X \to X$ by $fx = \frac{x}{8} - \frac{x^2}{64}$ and $gx = \frac{x}{2}$ for all $x \in X$. Then

$$d(fgx, gfx) = \frac{x^2}{256} \leq \frac{24x + x^2}{64} = d(gx, fx).$$

which implies $d(fgx, gfx) \leq d(fx, gx)$.

Therefore, f and g are weakly commuting. But

$fgx = \frac{x}{16} - \frac{x^2}{256} \neq \frac{x}{16} - \frac{x^2}{128} = gfx$. This shows that f and g are not commuting.

Definition 4.4: Beg and Butt, (2013) Let $(X, \preccurlyeq)$ be a partially ordered set and $x, y \in X$, then x and y are said to be comparable elements of X if $x \preccurlyeq y \text{ or } y \preccurlyeq x$.

4.2 MAIN RESULT.

Theorem 4.1: Let $(X, d, \preccurlyeq)$ be a partially ordered complete b-rectangular metric space with $s > 1$. Suppose $f, g: X \to X$ are such that f is a g-non decreasing and for every two comparable gx and gy, we have

$$d(fx, fy) \leq \beta\big(d(gx, gy)\big)M(x, y) \tag{4.1}$$

Where

$$\mathrm{M(x, y)} = \max\left\{d(gx, gy), \frac{d(gx,fx)d(gy,fy)}{1+d(fx,fy)}, \frac{d(gx,fx)d(gy,fy)}{1+d(gx,gy)}, \frac{d(gx,fx)d(gx,fy)}{1+d(gx,fy)+d(gy,fx)}\right\} \tag{4.2}$$

Suppose that

i) $fX \subseteq gX$;

ii) f is g-non- decreasing;

iii) f and g are continuous;

iv) There exists $x_0 \in X$ such that $gx_0 \preccurlyeq fx_0$. If the pair of (f, g) is compatible, then f and g have a coincidence point. Furthermore if f and g are weakly compatible maps, then f and g have a common fixed point, in X under the assumption that there exists $u \in X$ such that fu is comparable to fx and to fy. The common fixed point is unique.

Proof Let x_0 be an arbitrary point of X such that $g(x_0) \preccurlyeq f(x_0)$. Since $fX \subseteq gX$ we can choose $x_1 \in X$ so that $g(x_1) = f(x_0)$. Again from $fX \subseteq gX$ we can choose $x_2 \in X$ so that $g(x_2) = f(x_1)$. Since $g(x_0) \preccurlyeq f(x_0) = g(x_1)$ and f is g-non decreasing, we have $f(x_0) \preccurlyeq f(x_1)$ continuing this process we can choose a sequence$\{x_n\}$ in X such that $g(x_{n+1}) = f(x_n)$ with

$f(x_0) \preccurlyeq f(x_1) \preccurlyeq f(x_2) \preccurlyeq f(x_3) \preccurlyeq \cdots f(x_n) \preccurlyeq f(x_{n+1}) \preccurlyeq \cdots$.

Therefore,

$$g(x_1) \preccurlyeq g(x_2) \preccurlyeq g(x_3) \preccurlyeq \cdots g(x_n) \preccurlyeq g(x_{n+1}) \preccurlyeq \cdots . \tag{4.3}$$

If there exists $n_0 \in X$ such that $d(gx_{n_0}, gx_{n_0+1}) = 0$, then we have $gx_{n_0} = gx_{n_0+1} = fx_{n_0}$. Hence x_{n_0} is a coincidence point of f and g. so we assume that $d(gx_n, gx_{n+1}) > 0$, for all $n \in N$. we will show that

$$d(gx_{n+1}, gx_n) \leq d(gx_n, gx_{n-1}), \text{for all } n \geq 1. \tag{4.4}$$

From (4.1) and (4.3) with $x = x_n$ and $y = x_{n+1}$, we have

$$d(gx_{n+1}, gx_n) = d\big(fx_n, fx_{n-1}\big) \leq \beta\Big(d\big(gx_n, gx_{n-1}\big)\Big)M\big(x_n, x_{n-1}\big), \tag{4.5}$$

Now,

$$M(x_n, x_{n-1}) = \max\ \{d((gx_n, gx_{n-1}), \frac{d\big(gx_n, fx_n\big)d\big(gx_{n-1}, fx_{n-1}\big)}{1 + d\big(fx_n, fx_{n-1}\big)},$$

$$\frac{d\big(gx_n, fx_n\big)d\big(gx_{n-1}, fx_{n-1}\big)}{1+d\big(gx_n, gx_{n-1}\big)}, \frac{d\big(gx_n, fx_n\big)d\big(gx_n, fx_{n-1}\big)}{1+d\big(gx_n, fx_{n-1}\big)+d\big(gx_{n-1}, fx_n\big)}\ \}.$$

$$= \max\{d((gx_n, gx_{n-1}), \frac{d\big(gx_n, gx_{n+1}\big)d\big(gx_{n-1}, gx_n\big)}{1+d\big(gx_{n+1}, gx_n\big)},$$

$$\frac{d(gx_{n\,,}\,gx_{n+1})d(gx_{n-1\,,}\,gx_n)}{1+d(gx_{n,}\,gx_{n-1})}, \frac{d(gx_{n\,,}\,gx_{n+1})d(gx_{n,}\,gx_n)}{1+d(gx_{n\,,}\,gx_n)+d(gx_{n-1\,,}\,gx_{n+1})}\ \}$$

$$\leq \max\left\{d\left(gx_{n,}\,gx_{n-1}\right), d\left(gx_{n,}\,gx_{n+1}\right)\right\}.$$

If $\max\left\{d\left(gx_{n\,,}\,gx_{n-1}\right), d\left(gx_{n\,,}\,gx_{n+1}\right)\right\} = d\left(gx_{n\,,}\,gx_{n+1}\right)$ from (4.5) we have

$$d(gx_{n+1},\ gx_n) \leq \beta\left(d\left(gx_{n\,,}\,gx_{n-1}\right)\right) M(x_n\,,\ x_{n-1}),$$

$$d(gx_{n+1,}\,gx_n) \leq \beta\left(d\left(gx_{n\,,}\,gx_{n-1}\right)\right)\ d\left(gx_{n\,,}\,gx_{n+1}\right) < \frac{1}{s} d\left(gx_{n\,,}\,gx_{n+1}\right)$$

$$< d\left(gx_{n\,,}\,gx_{n+1}\right).$$

Which is a Contradiction.

Therefore, we proved that (4.4) holds. Then, the sequence $\{d\left(gx_{n\,,}\,gx_{n+1}\right)\}$ is decreasing sequence of non-negative real numbers.

Hence, there exists a real number $\delta \geq 0$ such that,

$\lim_{n\to\infty} d\left(gx_{n\,,}gx_{n+1}\right) = \delta.$

Claim; $\delta = 0$. suppose to contrary, that is, $\delta > 0$. Then letting $n \to \infty$ in (4.4) we have

$\delta \leq \frac{\delta}{s} < \delta$, which is impossible (since $s > 1$). Hence $\delta = 0$.

That is, $\lim_{n\to\infty} d\left(gx_{n\,,}\,gx_{n+1}\right) = 0.$ (4.6)

Suppose $gx_n = gx_m$ for some $n > m$, so, we have $gx_{n+1} = fx_n = fx_m = gx_{m+1}$, by continuing this process $gx_{n+k} = gx_{m+k}$ for $k \in N$.

Then (4.1) imply that

$$d(gx_{m\,,}\,gx_{m+1}) = d\left(gx_{n\,,}\,gx_{n+1}\right) \leq \beta\left(d\left(gx_{n,}\,gx_{n-1}\right)\right) M\ d\left(gx_{n,}\,gx_{n-1}\right)$$

$$\leq \beta\left(d\left(gx_{n\,,}\,gx_{n-1}\right)\right) max\{d\left(gx_{n\,,}\,gx_{n-1}\right),$$

$$d\left(gx_{n\,,}gx_{n+1}\right)\}.$$

If $\max\ \{d\left(gx_{n,}\ \ gx_{n-1}\right), d\left(gx_{n\,,}\,gx_{n+1}\right)\} =\ d\left(gx_{n\,,}\,gx_{n+1}\right)$, then we have

$$d\left(gx_{m\,,}\,gx_{m+1}\right) \leq \beta\left(d\left(gx_{n\,,}\ \ gx_{n-1}\right)\right)\ d\left(gx_{n\,,}\,gx_{n+1}\right) <\ d\left(gx_{n,}\ \ gx_{n+1}\right)$$

$$= d(gx_{m,}\ \ gx_{m+1).}$$

Which is a contradiction.

If $\max\ \{d\left(gx_{n,}\ \ gx_{n-1}\right), d\left(gx_{n,}\ \ gx_{n+1}\right)\} =\ d\left(gx_{n,}\ \ gx_{n-1}\right)$, then we have

$$d(gx_{m,}\ \ gx_{m+1)} < d\left(gx_{n,}\ \ gx_{n-1}\right) \leq \beta\left(d\left(gx_{n-2,}\ \ gx_{n-1}\right)\right) M\ d\left(gx_{n-2,}\ \ gx_{n-1}\right)$$

$$\leq \beta\left(d\left(gx_{n-2,}\ gx_{n-1}\right)\right)\max\{\, d\left(gx_{n-2\,,}gx_{n-1}\right),$$

$$d\left(gx_{n-1,}\ gx_n\right)\}$$

$$<\ d\left(gx_{n-2,}\ gx_{n-1}\right) < \cdots \ldots\ldots < d\left(gx_{m,}\ gx_{m+1}\right).$$

Which is a contradiction.

Thus, we can assume that $gx_n \neq gx_m$ for $n \neq m$. Then, we can prove that the sequence $\{gx_n\}$ is Cauchy sequence in $b - rms$.

Using b-rectangular inequality and by (4.1), we have

$$d(gx_{n\,,}\, gx_m) \leq sd\left(gx_{n\,,}\, gx_{n+1}\right) + sd\left(gx_{n+1\,,}\, gx_{m+1}\right) + sd(gx_{m+1\,,}\, gx_m)$$

$$\leq sd\left(gx_{n\,,}\, gx_{n+1}\right) + s\beta d\left(gx_{n\,,}\, gx_m\right)Md\left(gx_{n\,,}\, gx_m\right) + d(gx_{m\,,}\, gx_{m+1}).$$

(4.7)

Here $d(gx_{n\,,}\, gx_m) \leq M\left(gx_{n\,,}\, gx_m\right)$

$$= \max\{d((gx_n,\ gx_m), \frac{d(gx_{n\,,}\, fx_n)d(gx_{m\,,}\, fx_m)}{1+d(fx_n,\ fx_m)},$$

$$\frac{d(gx_{n\,,}\, fx_n)d(gx_m,\ fx_m)}{1+d(gx_n,\ gx_m)}, \frac{d(gx_{n\,,}\, fx_n)d(gx_{n\,,}\, fx_m)}{1+d(gx_{n\,,}\, fx_m)+d(gx_{n\,,}\, fx_n)}\}.$$

Taking the upper limits as $\mathrm{m,n} \to \infty$ in the above inequality and using (4.6) we get

$$\lim_{\mathrm{m,n}\to\infty} sup\ M\left(gx_{n,}\, gx_m\right) = \lim_{\mathrm{m,n}\to\infty} sup\ d\left(gx_{n,}\, gx_m\right).$$

Hence, letting $\mathrm{m,n} \to \infty$ in (4.7), we obtain

$$\lim_{\mathrm{m,n}\to\infty} supd\left(gx_{n\,,}gx_m\right) \leq$$

$$\lim_{\mathrm{m,n}\to\infty} sups\beta d\left(gx_{n,}\, gx_m\right)\ \lim_{\mathrm{m,n}\to\infty} supd\left(gx_{n\,,}gx_m\right).$$

Now we claim that $\lim_{\mathrm{m,n}\to\infty} d\left(gx_{n\,,}gx_m\right) = 0$. If on the contrary,

$\lim_{\mathrm{m,n}\to\infty} d\left(gx_{n,}\, gx_m\right) \neq 0$, then we get

$$\frac{1}{s} \leq \lim_{\mathrm{m,n}\to\infty} sup\beta\left(d\left(gx_{n\,,}gx_m\right)\right).$$

Since $\beta \in F_S$. we deduce that

$$\lim_{\mathrm{m,n}\to\infty} d\left(gx_{n\,,}gx_m\right) = 0.$$

Which is a contradiction.

Consequently, $\{\, gx_{n\,,}\}$ is a Cauchy sequence in $b - rms\ \ X$.

Since X is complete $b -\ rms$, there exists x in X such that

$$\lim_{\mathrm{n}\to\infty} gx_{n+1} = \lim_{\mathrm{n}\to\infty} fx_n = x.$$

Now we show that $gx = fx$.

From the b-rectangular inequality, we have

$$d(fx, gx) \leq sd(fx, fgx_n) + sd(fgx_n, gfx_n) + sd(gfx_n, gx) . \qquad (4.8)$$

Also, from continuity of f and g,

$$\lim_{n\to\infty} f\big(g(x_n)\big) = f(\lim_{n\to\infty} g(x_n)) = fx \text{ and}$$

$$\lim_{n\to\infty} g\big(f(x_n)\big) = g(\lim_{n\to\infty} f(x_n)) = gx. \qquad (4.9)$$

Letting $n \to \infty$ in (4.8) and using (4.9) and compatibility of f and g, we get $d(fx, gx) = 0$, implies $fx = gx$, that is x is a coincidence point of f and g.
Hence the set of coincidence point of f and g is non-empty or $C(g, f) \neq \emptyset$.
From this there exists at least a coincidence point. Suppose that x and y are coincidence points of f and g, that is, $fx = gx$ and $fy = gy$. We shall show that $gx = gy$. By assumptions, there exists $u \in X$ such that fu is comparable to fx and to fy. Without any restriction of the generality, we can assume that $fx \preccurlyeq fu$ and $fy \preccurlyeq fu$.
Put $u_0 = u$ and choose $u_1 \in X$ such that $gu_1 = fu_0$. for $n \geq 1$, continuing this process we can construct sequence $\{gx_n\}$ such that $gu_{n+1} = fu_n$ for all n.
Further, set $x_0 = x$ and $u_0 = u$ and on the same way define sequences $\{gx_n\}$ and $\{gu_n\}$.
Since $gx = fx = gx_1$ and $fu = gu_1$ are comparable, $gx \preccurlyeq gu$. One can show, by induction

$$gx_n = gu_n \text{ for all } n.$$

Thus from (4.1), we have

$$d(gx, gu_{n+1}) = d(fx, fu_n)$$

$$\leq \beta(d(gx, gu_n)M(x, u_n).$$

Where $M(x, u_n) = \max\{d(gx, gu_n), \frac{d(gx,fx)d(gu_n,fu_n)}{1+d(fx,fu_n)}$

$$, \frac{d(gx,\ fx)d(gu_n, fu_n)}{1+d(gx,\ gu_n)}, \frac{d(gx, fx)d(gx, fu_n)}{1+d(gx, fu_n)+d(gu_n, fx)}\}$$

$$= d\big(gx, gu_n\big).$$

Hence, $$d\big(gx, gu_{n+1}\big) \leq \beta\left(d\big(gx, gu_n\big)\right) d\big(gx, gu_n\big) < d\big(gx, gu_n\big). \qquad (4.10)$$

This implies that $d\big(gx, gu_n\big)$ is non increasing sequence. Hence there exists $\delta \geq 0$, such that

$\lim_{n\to\infty} d(gx, gu_n) = \delta$, now we show $\delta = 0$. Suppose $\delta > 0$.

Passing to the upper limit in (4.10) as $n \to \infty$, we obtain

$$\lim_{n\to\infty} d(gx, gu_{n+1}) \leq \beta \lim_{n\to\infty} d(gx, gu_n)\, d(gx, gu_n)$$

$$< \frac{1}{s} \lim_{n\to\infty} d(gx, gu_n) < \lim_{n\to\infty} d(gx, gu_n).$$

That is $\delta < \frac{1}{s}\delta < \delta$, which is a contradiction.

Thus, $\delta = 0$. Therefore,

$$\lim_{n\to\infty} d(gx, gu_n) = 0. \qquad (4.11)$$

Similarly, one can prove that

$$\lim_{n\to\infty} d(gy, gu_n) = 0. \qquad (4.12)$$

From (4.11) and (4.12), we have $gx = fx$. Since $gx = fx$ and $gy = fy$,

by weakly compatible of f and g, we have $g(gx) = g(fx) = f(gx)$. (4.13)

Denote $gx = v$, then from (4.13)

$$gv = fv. \qquad (4.14)$$

Thus, v is a coincidence point of f and g.

It follows that $gv = gx$, that is

$$gv = v. \qquad (4.15)$$

From (4.14) and (4.15) ,

$$v = gv = fv.$$

Therefore v is a common fixed point of f and g.

To prove the uniqueness of the common fixed point of f and g, we assume that u is another common fixed point of f and g.

Then we prove $u = gu = fu$. Since u is a coincidence point of f and g,

we have $gu = gx = v$. Thus,

$u = gu = gv = v$. Hence the result.

Now we give an example in support of Theorem 4.1.

Example 4.2: Let $X = \{1,2,3,4,\}$ define $d: X \times X \to [0, \infty]$ such that $d(x, y) = d(y, x)$ for all $x, y \in X$.

$d(1,1) = d(2,2) = d(3,3) = d(4,4) = 0.$

$\mathrm{d}(1,2) = \mathrm{d}(2,3) = 2. \qquad \mathrm{d}(1,3) = 20. \qquad \mathrm{d}(1,4) = \mathrm{d}(2,4) = \mathrm{d}(3,4) = 4.$

Then (X, d) is a b-rectangular metric space with $s = 2$ but (X, d) is not rectangular metric space, since $\mathrm{d}(1,3) = 20 > 10 = \mathrm{d}(1,2) + \mathrm{d}(2,4) + \mathrm{d}(4,3)$

We define a partial order $\preccurlyeq$ on X by

$\preccurlyeq = \{(1,1), (2,2), (3,3), (4,4), (1,2), (1,4), (2,4)\}.$

Define $f, g: X \to X$ by

$gx = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix}$ $\qquad fx = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 \end{pmatrix}$. Define $\beta: [0, \infty) \to [0, \frac{1}{2})$ by $\beta(t) = \frac{2t}{4t+1}$, $t \in [0, \infty)$.

We take $x_0 = 1$ such that $gx_0 \preccurlyeq fx_0$

Since

$1 = g1 \preccurlyeq g1 = 1 \implies 1 = f1 \preccurlyeq f1 = 1$

$1 = g1 \preccurlyeq g2 = 1 \implies 1 = f1 \preccurlyeq f2 = 1$

$1 = g1 \preccurlyeq g3 = 2 \implies 1 = f1 \preccurlyeq f3 = 1$

$1 = g2 \preccurlyeq g1 = 1 \implies 1 = f2 \preccurlyeq f1 = 1$

$1 = g2 \preccurlyeq g2 = 1 \implies 1 = f2 \preccurlyeq f2 = 1$

$1 = g2 \preccurlyeq g3 = 2 \implies 1 = f2 \preccurlyeq f3 = 1$

$2 = g3 \preccurlyeq g3 = 2 \implies 1 = f3 \preccurlyeq f3 = 1$

$3 = g4 \preccurlyeq g4 = 3 \implies 2 = f4 \preccurlyeq f4 = 2.$

This shows that f is g-non decreasing, we observe $g1 \preccurlyeq f1$.

Now we show that f and g satisfy all the conditions of the theorem 4.1 with $\beta(t) = \frac{2t}{4t+1}$, $t \in [0, \infty)$.

$$d(fx, fy) \leq \beta\big(d(gx, gy)\big) M(x, y).$$

Where $M(x, y) = \max\left\{d(gx, gy), \frac{d(gx,fx)d(gy,fy)}{1+d(fx,fy)}, \frac{d(gx,fx)d(gy,fy)}{1+d(gx,gy)}, \frac{d(gx,fx)d(gx,fy)}{1+d(gx,fy)+d(gy,fx)}\right\}.$

Case (1): $(1,2) \Rightarrow d(fx, fy) = d(f1, f2) = d(1,1)$

$$M(1,2) = \max\{d(g1, g2), \frac{d(g1,1)d(g2, f2)}{1 + d(f1, f2)},$$

$$\frac{d(g1,f1)d(g2,f2)}{1+d(g1,g2)}, \frac{d(g1,f1)d(g1,f2)}{1+d(g1,f2)+d(g2,f1)}\}$$

$$= \max\{\mathrm{d}(1,1), \frac{\mathrm{d}(1,1)\mathrm{d}(1,1)}{1+\mathrm{d}(1,1)} \frac{\mathrm{d}(1,1)\mathrm{d}(1,1)}{1+\mathrm{d}(1,1)}, \frac{\mathrm{d}(1,1)\mathrm{d}(1,1)}{1+\mathrm{d}(1,1)\mathrm{d}(1,1)+\mathrm{d}(1,1))}\}$$

$$= \max\ \{\mathrm{d}\ (1,1), 0, 0, 0\}$$

$$= \max\{0, 0, 0, 0\} = 0.$$

So, $d(fx, fy) \leq \beta\big(d(gx, gy)\big)M(x, y)$

$$d(1,1) \leq \beta\big(d(g1, g2)\big)M(1,2)$$

$$\mathrm{d}(1,1) \leq \beta\big(\mathrm{d}(1,1)\big)\mathrm{M}(1,2)$$

$$0 \leq \beta(0)(0) = 0.$$

Case (2): $(x, y) = (1,4) \Longrightarrow d(fx, fy) = d(f1, f4) = (1,2)$

$$\mathrm{M}(1,4) = \max\left\{d(g1, g4), \frac{d(g1,f1)d(g4,f4)}{1+d(f1,f4)}, \frac{d(g1,f1)d(g4,f4)}{1+d(g1,g4)}, \frac{g1,f1)d(g1,f2)}{1+d(g1,f2)+d(g1,f1)}\right\}$$

$$= \max\left\{d(1,3), \frac{d(1,1)d(3,2)}{1+d(1,2)}, \frac{d(1,1)d(3,2)}{1+d(1,3)}, \frac{d(1,1)d(1,1)}{1+d(1.1)+d(1,1)}\right\}$$

$$= \max\ \{20,0,0,0\} = 20.$$

So, $\mathrm{d}(fx, fy) \leq \beta\big(d(gx, gy)\big)M(x, y)$.

$$d(1,2) \leq \beta\big(d(g1, g4)\big)M(1,4)$$

$$2 \leq \beta\big(\mathrm{d}(1,3)\big) \times 20$$

$$= \beta(20) \times 20$$

$$= 9.88$$

$$2 \leq 9.88.$$

Case (3): $(x, y) = (2,4) \Longrightarrow d(fx, fy) = d(f2, f4) = d(1,2)$

$$\mathrm{M}(2,4) = \max\left\{\mathrm{d}(\mathrm{g}2, \mathrm{g}4), \frac{\mathrm{d}(\mathrm{g}2,\mathrm{f}2)\mathrm{d}(\mathrm{g}4,\mathrm{f}4)}{1+\mathrm{d}(\mathrm{f}2,\mathrm{f}4)}, \frac{\mathrm{d}(\mathrm{g}2,\mathrm{f}2)\mathrm{d}(\mathrm{g}4,\mathrm{f}4)}{1+\mathrm{d}(\mathrm{g}2,\mathrm{g}4)}, \frac{\mathrm{d}(\mathrm{g}2,\mathrm{f}2)\mathrm{d}(\mathrm{g}2,\mathrm{f}4)}{1+\mathrm{d}(\mathrm{g}2,\mathrm{f}4)+\mathrm{d}(\mathrm{g}4,\mathrm{f}2)}\right\}$$

$$= \max\left\{d(1,3), \frac{d(4,4)d(3,2)}{1+d(4,2)}, \frac{d(4,4)d(3,2)}{1+d(4,3)}, \frac{d(4,4)d(4,2)}{1+d(4,2)+d(3,4)}\right\}$$

$$= \max\{20,0,0,0\} = 20.$$

So, $d(fx, fy) \leq \beta\big(d(gx, gy)\big)M(x, y)$

$$d(1,2) \leq \beta\big(d(g2, g4)\big)M(2,4)$$

$$2 \leq \beta\big(d(1,3)\big) \times 20$$

$$= \beta(20) \times 20$$

$$= \frac{40}{81} \times 20$$

$$= 9.88$$

$$2 \leq 9.88.$$

Case (4): $(x, y) = (1,1) \Rightarrow d(fx, fy) = d(f1, f1) = d(1,1)$

$$M(1,1) = \max\left\{d(g1, g1), \frac{d(g1,f1)d(g1,f1)}{1+d(f1,f1)}, \frac{d(g1,f1)d(g1,f1)}{1+d(g1,g1)}, \frac{d(g1,f1)d(g1,f1)}{1+d(g1,f1)+d(g1,f1)}\right\}$$
$$= \max\left\{d(1,1), \frac{d(g1,f1)d(g1,f1)}{1+d(f1,f1)}, \frac{d(g1,f1)d(g1,f1)}{1+d(g1,g1)}, \frac{d(g1,f1)d(g1,f1)}{1+d(g1,f1)+d(g1,f1)}\right\}$$
$$= \max\{0,0,0,0\} = 0.$$

So, $d(fx, fy) \leq \beta\big(d(gx, gy)\big)M(x, y)$

$$d(1,1) \leq \beta\big(d(g1, g1)\big)M(1,1)$$

$$0 \leq \beta\big(d(1,1)\big) \times 0$$

$$0 \leq \beta(0) \times 0$$

$$0 \leq 0.$$

Case (5): $(x, y) = (2,2) \Rightarrow d(fx, fy) = d(f2, f2) = d(1,1)$

$$M(2,2) = \max\left\{d(g2, g2), \frac{d(g2,f2)d(g2,f2)}{1+d(f4,f4)}, \frac{d(g2,f2)d(g2,f2)}{1+d(g2,g2)}, \frac{d(g2,f2)d(g2,f2)}{1+d(g2,f2)+d(g2,f2)}\right\}$$

$$= \max\left\{d(1,1), \frac{d(1,1)d(1,1)}{1+d(1,1)}, \frac{d(1,1)d(1,1)}{1+d(1,1)}, \frac{d(1,1)d(1,1)}{1+d(1,1)+d(1,1)}\right\}$$

$$= \max\{0,0,0,0\} = 0.$$

So, $d(fx, fy) \leq \beta\big(d(gx, gy)\big)M(x, y)$

$$d(4,4) \leq \beta\big(d(1,1)\big)M(2,2)$$

$$0 \leq \beta(0)0$$

$$0 \leq 0.$$

Case (6): $(x, y) = (4,4) \Rightarrow d(fx, fy) = d(f4, f4) = d(2,2)$

$$M(4,4) = \max\left\{d(g4, g4), \frac{d(g4,f4)d(g4,f4)}{1+d(f4,f4)}, \frac{d(g4,f4)d(g4,f4)}{1+d(g4,g4)}, \frac{d(g4,f4)d(g4,f4)}{1+d(g4,f4)+d(g4,f4)}\right\}$$

$$= \max\left\{d(3,3), \frac{d(3,2)d(3,2)}{1+d(2,2)}, \frac{d(3,2)d(3,2)}{1+d(3,3)}, \frac{d(3,2)d(3,2)}{1+d(3,2)+d(3,2)}\right\}$$

$$= \max\left\{0, 200, 200, \frac{400}{41}\right\} = 400.$$

So, $d(fx, fy) \leq \beta\big(d(gx, gy)\big)M(x, y)$

$$d(2,2) \leq \beta\big(d(3,3)\big)M(4,4)$$

$$= \beta(0) \times 400$$

$$= 0 \times 400$$

$$0 \leq 0.$$

Case (7): $(x, y) = (3,3) \Longrightarrow d(fx, fy) = d(f3, f3) = d(2,2)$

$$M(3,3) = \max\left\{d(g3, g3), \frac{d(g3,f3)d(g3,f3)}{1+d(f3,f3)}, \frac{d(g3,f3)d(g3,f3)}{1+d(g3,g3)}, \frac{d(g3,f3)d(g3,f3)}{1+d(g3,f3)+d(g3,f3)}\right\}$$

$$= \max\left\{d(2,2), \frac{d(2,2)d(2,2)}{1+d(2,2)}, \frac{d(2,2)d(2,2)}{1+d(2,2)}, \frac{d(2,2)d(2,2)}{1+d(2,2)+d(2,2)}\right\}$$

$$= \max\ \{0,0,0,0\} = 0.$$

So, $d(fx, fy) \leq \beta\big(d(gx, gy)\big)M(x, y)$.

$$d(2,2) \leq \beta\big(d(2,2)\big)M(3,3);$$

$$0 \leq \beta(0)0 = 0.$$

Thus, the pair of mappings f and g satisfies all conditions of Theorem 4.1 and 1 is the unique common fixed point of f and g.

The following is also an example in support of Theorem 4.1.

Example 4.3: Let $X = \{5,6,7,8,\}$. Define $d: X \times X \to X$ such that $d(x, y) = d(y, x)$, for all $x, y \in X$ $\quad d(x, y) = 0$ iff $x = y$.

$d(5,7) = 10$, $d(5,6) = d(7,8) = d(6,7) = 1$, $d(5,8) = d(6,8) = 2$.

Then, (X,d) is a b-rectangular metric space with $s=2$ but (X,d) is neither metric space nor rectangular metric space.

$d(5,7)=10>5=d(5,8)+d(8,6)+d(6,7)$

We define a partial order $\preccurlyeq$ on X by

$\preccurlyeq=\{(5,5),(6,6),(7,7),(8,8),(9,9),(6,7),(6,8),(7,8)\}$.

Define $f,g\colon X\to X$ by

$gx=\begin{pmatrix}5 & 6 & 7 & 8\\ 5 & 6 & 7 & 7\end{pmatrix}$, $fx=\begin{pmatrix}5 & 6 & 7 & 8\\ 5 & 6 & 6 & 6\end{pmatrix}$. Define $\beta\colon[0,\infty)\to[0,\frac{1}{2})$ by $\beta(t)=\frac{t}{1+2t}$, $t\in[0,\infty)$.

$5=g5\preccurlyeq g5=5 \;\Rightarrow\; 5=f5\preccurlyeq f5=5$

$6=g6\preccurlyeq g6=6 \;\Rightarrow\; 6=f6\preccurlyeq f6=6$

$6=g6\preccurlyeq g7=7 \;\Rightarrow\; 6=f6\preccurlyeq f7=6$

$6=g6\preccurlyeq g8=7 \;\Rightarrow\; 6=f6\preccurlyeq f8=6$

$7=g7\preccurlyeq g7=7 \;\Rightarrow\; 6=f7\preccurlyeq f7=6$

$7=g7\preccurlyeq g8=7 \;\Rightarrow\; 6=f7\preccurlyeq f8=6$

$7=g8\preccurlyeq g8=7 \;\Rightarrow\; 6=f8\preccurlyeq f8=6$

$7=g8\preccurlyeq g7=7 \;\Rightarrow\; 6=f8\preccurlyeq f7=6$.

This shows that f is g-non decreasing. Now we show that f and g satisfy the condition of theorem 4.1 with $\beta(t)=\frac{t}{1+2t}$, $t\in[0,\infty)$.

$$d(fx,fy)\leq\beta\big(d(gx,gy)\big)M(x,y).$$

Where $M(x,y)=\max\left\{d(gx,gy),\frac{d(gx,fx)d(gy,fy)}{1+d(fx,fy)},\frac{d(gx,fx)d(gy,fy)}{1+d(gx,gy)},\frac{d(gx,fx)d(gx,fy)}{1+d(gx,fy)+d(gy,fx)}\right\}$

Case (1): $(x,y)=(6,7)\Rightarrow d(f6,f7)=d(6,6)$

$$\begin{aligned}M(6,7)&=\max\left\{d(g6,g7),\frac{d(g6,f6)d(g7,f7)}{1+d(f6,f7)},\frac{d(g6,f6)d(g7,f7)}{1+d(g6,g7)},\frac{d(g6,f6)d(g7,f7)}{1+d(g6,f7)+d(g7,f6)}\right\}\\
&=\max\left\{d(6,7),\frac{d(6,6)d(7,6)}{1+d(6,6)},\frac{d(6,6)d(7,6)}{1+d(6,7)},\frac{d(6,6)d(7,6)}{1+d(6,6)+d(7,6)}\right\}\\
&=\max\{1,0,0,0\}=1.\end{aligned}$$

So, $d(fx, fy) \leq \beta\big(d(gx, gy)\big)M(x, y)$

$$d(6,6) \leq \beta\big(d(6,7)\big)M(6,7)$$

$$0 \leq \beta(1)(1)$$

$$= \frac{1}{1+2(1)}(1)$$

$$= \frac{1}{3}(1)$$

$$0 \leq \frac{1}{3}.$$

<u>Case (2):</u> $(x, y) = (7,8) \Longrightarrow d(fx, fy) = d(f7, f8) = d(6, 6)$

$$M(7,8) = \max\{d(g7, g8), \frac{d(g7,f7)d(g8,f8)}{1+d(f7,f8)} \frac{d(g7,f7)d(g8,f8)}{1+d(g7,g8)}, \frac{d(g7,f7)d(g7,f8)}{1+d(g7,f8)+d(g8,f7)}\}$$

$$= \max\left\{d(7,7), \frac{d(7,6)d(7,6)}{1 + d(6,6)}, \frac{d(7,6)d(7,6)}{1 + d(7,7)}, \frac{d(7,6)d(7,6)}{1 + d(7,6) + d(7,6)}\right\}$$

$$= \max\left\{0,1,1,\frac{1}{3}\right\} = 1.$$

So, $d(fx, fy) \leq \beta\big(d(gx, gy)\big)M(x, y)$

$$d(6,6) \leq \beta\big(d(g7, g8)\big)M(7,8)$$

$$d(6,6) \leq \beta\big(d(7,7)\big)M(7,8)$$

$$0 \leq \beta(0)1$$

$$0 \leq 0.$$

<u>Case (3):</u> $(x, y) = (6,8) \Longrightarrow d(f6, f8) = d(6,6)$

$$M(6,8) = \max\{d(g6, g8), \frac{d(g6,f6)d(g8,f8)}{1+d(f6,f8)}, \frac{d(g6,f6)d(g8,f8)}{1+d(g6,g8)}, \frac{d(g6,f6)d(g6,f8)}{1+d(g6,f8)+d(g8,f6)}\}$$

$$= \max\left\{d(6,7), \frac{d(6,6)d(g8,f8)}{1+d(6,6)}, \frac{d(6,6)d(6,6)}{1+d(6,7)}, \frac{d(6,6)d(6,6)}{1+d(6,6)+d(7,6)}\right\}$$

$$= \max\ \{1,0,0,0\} = 1.$$

So, $d(fx, fy) \leq \beta\big(d(gx, gy)\big)M(x, y)$

$$d(6,6) \leq \beta\big(d(g6, g8)\big)M(6,8)$$

$0 \leq \beta\big(d(6,7)\big) \times 1$

$= \beta(1) \times 1$

$= \frac{1}{1+2(1)}$

$0 \leq \frac{1}{3}.1$

Case (4): $(x, y) = (5,5) \Rightarrow d(f5, f5) = d(5,5)$

$$\mathrm{M}(5,5) = \max\{\mathrm{d}(g5, g5), \frac{d(g5, f5)d(g5, f5)}{1 + d(f5, f5)},$$

$$\frac{d(g5, f5)d(g5, f5)}{1 + d(g5, g5)}, \frac{d(g5, f5)d(g5, f5)}{1 + d(g5, f5) + d(g5, f5)}\}$$

$$= \max\left\{d(5,5), \frac{d(5,5)d(5,5)}{1+d(5,5)}, \frac{d(5,5)d(5,5)}{1+d(5,5)}, \frac{d(5,5)d(5,5)}{1+d(5,5)+d(5,5)}\right\}$$

$= \max\ \{0, 0,0\ , 0\} = 0.$

So, $d(fx, fy) \leq \beta\big(d(gx, gy)\big)M(x, y)$

$d(5,5) \leq \beta\big(d(g5, g5)\big)M(5,5)$

$= \beta\big(d(5,5)\big)0$

$0 \leq 0$.

Case (5): $(x, y) = (6,6) \Longrightarrow d(f6, f6) = (6,6)$

$$\mathrm{M}(6,6) = \max\ \{d(g6, g6)\ , \frac{d(g6, f6)d(g6, f6)}{1 + d(f6, f6)},$$

$$\frac{d(g6, f6)d(g6, f6)}{1 + d(g6, g6)}, \frac{d(g6, f6)d(g6, f6)}{1 + d(g6, f6) + d(g6, f6)}\}$$

$$= \max\left\{d(6,6), \frac{d(6,6)d(6,6)}{1+d(6,6)}, \frac{d(6,6)d(6,6)}{1+d(6,6)}, \frac{d(6,6)d(6,6)}{1+d(6,6)+d(6,6)}\right\}$$

$= \max\ \{0,0,0,0\} = 0.$

So, $d(fx, fy) \leq \beta\big(d(gx, gy)\big)M(x, y)$

$d(6,6) \leq \beta\big(d(g6, g6)\big)M(6,6)$

$$0 \leq \beta\big(d(6,6)\big)0$$

$$0 \leq 0.$$

Case (6): $(x,y) = (7,7) \Longrightarrow d(fx,fy) = d(f7,f7) = d(6,6)$

$$M(7,7) = \max d(g7,g7)\,, \frac{d(g7,f7)d(g7,f7)}{1+d(f7,f7)},$$

$$\frac{d(g7,f7)d(g7,f7)}{1+d(g7,g7)}, \frac{d(g7,f7)d(g7,f7)}{1+d(g7,f7)+d(g7,f7)}\}$$

$$= \max\left\{d(7,7), \frac{d(7,6)d(7,6)}{1+d(7,7)}, \frac{d(7,6)d(7,6)}{1+d(7,7)}, \frac{d(7,6)d(7,6)}{1+d(7,6)+d(7,6)}\right\}$$

$$= \max\left\{0\,, \frac{1}{1+0}, \frac{1}{1+0}, \frac{1}{1+1+1}\right\}$$

$$= \max\left\{0,1,1,\tfrac{1}{3}\right\} = 1.$$

So, $d(fx,fy) \leq \beta\big(d(gx,gy)\big)M(x,y)$

$$d(6,6) \leq \beta\big(d(g7,g7)\big)M(7,7)$$

$$0 \leq \beta\big(d(7,7)\big)1$$

$$= \beta(0)1$$

$$0 \leq 0.$$

Case (7): $(x,y) = (8,8) \Longrightarrow d(fx,fy) = d(f8,f8) = d(6,6)$

$$M(8,8) = \max\{d(g8,g8), \frac{d(g8,f8)d(g8,f8)}{1+d(f8\,,f8)},$$

$$\frac{d(g8,f8)d(g8,f8)}{1+d(g8,g8)}, \frac{d(g8,f8)d(g8,f8)}{1+d(g8,f8)+d(g8,f8)}\}$$

$$= \max\left\{d(7,7), \frac{d(7,6)d(7,6)}{1+d(6,6)}, \frac{d(7,6)d(7,6)}{1+d(7,7)}, \frac{d(7,6)d(7,6)}{1+d(7,6)+d(7,6)}\right\}$$

$$= \max\left\{0, \frac{1}{1+0}, \frac{1}{1+0}, \frac{1}{1+1+1}\right\}$$

$$= \max\left\{0,1,1,\tfrac{1}{3}\right\} = 1.$$

So, $d(fx,fy) \leq \beta\big(d(gx,gy)\big)M(x,y)$

$$d(8,8) \leq \beta\big(d(9,9)\big)M(8,8)$$

$$0 \leq \beta(0)(1)$$

$$0 \leq 0.$$

Since $C(f,g) \neq \emptyset$, f and g are weakly compatible maps. Moreover, 5 and 7 are common fixed points of f and g.

Theorem 4.2: Let $(X, d, \leq)$ be a partially ordered complete b-rectangular metric space with $s > 1$.suppose $f, g: X \to X$ are such that f is a g-non decreasing and for every two comparable gx and gy , we have

$$d(fx, fy) \leq \beta(d(gx, gy)M(x, y)$$

where,

$$M(x, y) = \max\left\{d(gx, gy), \frac{d(gx, fx)d(gy, fy)}{1 + d(fx, fy)}, \frac{d(gx, fx)d(gy, fy)}{1 + d(gx, gy)}, \frac{d(gx, fx)d(gx, fy)}{1 + d(gx, fy) + d(gy, fx)}\right\}.$$

Suppose that

i) $fX \subseteq gX$;

ii) gX is closed;

iii) f is g-non decreasing;

iv) there exists $x_0 \in X$ such that $gx_0 \preccurlyeq fx_0$;

v) $\{gx_n\} \subseteq X$ is a non-decreasing sequence with $gx_n \to gz = sup gx_n$ in gX, then $gx_n \to gz$ for all n.

Then f and g have a coincidence point. Further more if f and g are weakly compatible maps then f and g have common fixed point.

Proof: Since $\{fx_n\} \subseteq gX$ and gX is closed, then there exist $z \in X$. Also $\{gx_n\}$ is a non-decreasing sequence and $gx_n \to gz = x$ by (v), we have $x = gz = sup(gx_n)$. Particularly, $gx_n \preccurlyeq gz$ for all *n*, now we claim that *z* is a coincidence point of f and g.

Which implies $d(gz, fz) \leq d(gz, gx_{n+1}) + d(gx_{n+1}, fz)$.

Taking $n \to \infty$ in the above in equality, we have

$$d(gz, fz) \leq \lim_{n\to\infty} sup(d(fx_n, fz))$$

$$\leq \lim_{n\to\infty} sup\beta(d(gx_n, gz)M(x_n, z).$$

Where,

$$\lim_{n\to\infty} M(x_n, z) = \lim_{n\to\infty} \max\{d(gx_n, gz), \frac{d(gx_n, fx_n)d(gz, fz)}{1 + d(fx_n, fz)},$$
$$\frac{d(gx_n, fx_n)d(gz, fz)}{1 + d(gx_n, gz)}, \frac{d(gx_n, fx_n)d(gx_n, fz)}{1 + d\big(gx_n, fz\big) + d(gz, gx_{n+1})}\}$$

$$= \lim_{n\to\infty} \max\{d(gx_n, gz), \frac{d(gx_n,gx_{n+1})d(gz,fz)}{1+d(gx_{n+1},fz)},$$

$$\frac{d(gx_n,gx_{n+1})d(gz,fz)}{1+d(gx_n,gz)}, \frac{d(gx_n,gx_{n+1})d(gx_n,fz)}{1+d(gx_n,fz)+d(gz,gx_{n+1})}\}$$

$= 0.$

Which is possible only when $d(gz, fz) = 0$.

This implies $fz = gz$, *i.e.* z is a coincidence point, which implies that $C(g,f) \neq \emptyset$. Since f and g are weakly compatible pair of self- maps, f and g commute at some point $p \in C(g,f)$.

Now set $w = fp = gp$. Since f and g are weakly compatible, which gives

$$fw = f(fp) = f(gp) = g(fP) = gw.$$

Now we claim that w is a common fixed point of f and g.

Suppose $fw \neq w$, since p is coincidence point, $gp \preccurlyeq g(gp) = gw$, we have

$d(fw,w) = d(fw,fp) \leq \beta\big(d(gw,gp)\big)M(gw,gp) < d(fw,w).$

This is contradiction. Hence $fw = w$.

Therefore, $fw = gw = w$. Hence w is the a common fixed point of f and g.

Now we present an example in support of Theorem 4.2.

Example 4.4: Let $X = [0,2]$ and define $d: X \times X \to [0,\infty)$ such that $d(x,y) = d(y,x)$, for all $x \in X$, and

$d\left(\frac{1}{2},\frac{1}{3}\right) = d\left(\frac{1}{4},\frac{1}{5}\right) = 0.03; d\left(\frac{1}{2},\frac{1}{5}\right) = d\left(\frac{1}{3},\frac{1}{4}\right) = 0.02;$ $d\left(\frac{1}{2},\frac{1}{4}\right) = d\left(\frac{1}{5},\frac{1}{3}\right) = 0.06$;

$d(x,y) = (x-y)^2$ otherwise.

Then (X, d) is a b-rectangular metric space with $s = 3$. (X,d) is neither metric space nor rectangular metric space.

i.e., $d\left(2,\frac{1}{2}\right) = \mathbf{\frac{9}{4}} > \frac{3}{4} = d\left(2,\frac{3}{2}\right) + d\left(\frac{3}{2},1\right) + d\left(1,\frac{1}{2}\right) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$

We define the partial order "$\preccurlyeq$" on X by

$\preccurlyeq\ = \{(x,y);\ x,y \in [0.1]: x = y\} \cup \{(x,y): x,y \in (1,2], x \leq y\}.$

Define $f, g: X \to X$ *by*

$$fx = \begin{cases} \frac{x}{6}, & if\ 0 \leq x < 1, \\ \frac{1}{5}, & if\ x = 1, \\ \frac{1}{4}, & if\ 1 < x \leq 2. \end{cases} \quad \text{and} \quad gx = \begin{cases} \frac{x}{5}, & if\ 0 \leq x < 1, \\ \frac{1}{5}, & if\ x = 1, \\ \frac{1}{4}, & if\ 1 < x \leq 2. \end{cases}$$

also define $\beta: [0,\infty) \to \left[0,\frac{1}{3}\right)$ by $\beta(t) = \begin{cases} 0, if\ t = 0, \\ \frac{e^{-t}}{3}, if\ t > 0. \end{cases}$

$fX = \left[0, \frac{1}{6}\right) \cup \left\{\frac{1}{5}, \frac{1}{4}\right\} \subset \left[0, \frac{1}{5}\right] \cup \{\frac{1}{4}\} = gX$ and $\left[0, \frac{1}{5}\right] \cup \{\frac{1}{4}\}$ *is closed in X.*

Clearly

$gx \preccurlyeq gy \Longrightarrow fx \preccurlyeq fy$
$gx_0 \preccurlyeq fx_0$
Clearly there exist $x_0 = 0 \in [0,2]$ such that $fx_0 = gx_0$.
$0 = g0 \preccurlyeq g0 = 0 \Longrightarrow 0 = f0 \preccurlyeq f0 = 0$.
This shows that f is g-non decreasing.
Now we show that f and g satisfy the condition of theorem 4.2 with

$$\beta(t) = \begin{cases} 0, if\ t = 0\ , \\ \frac{e^{-t}}{3}, if\ t > 0. \end{cases}$$

Case (1): let $x, y \in [0,1)$ then $d(fx, fy) = 0$.

We take $d(fx, fy) = d(\frac{x}{6}, \frac{x}{6}) = 0$ since $x, y \in [0,1)$.

$$\text{M}(x, y) = \max\{d(gx, gy), \frac{d(gx,fx)d(gy,fy)}{1+d(fx,fy)} + \frac{d(gx,fx)d(gy,fy)}{1+d(gx,gy)}, \frac{d(gx,fx)d(gx,fy)}{1+d(gx,fy)+d(gy,fx)}\}$$

$$= \max\{d\left(\frac{x}{5}, \frac{y}{5}\right), \frac{d\left(\frac{x}{5},\frac{x}{6}\right)d\left(\frac{y}{5},\frac{y}{6}\right)}{1+d\left(\frac{x}{6},\frac{y}{6}\right)}, \frac{d\left(\frac{x}{5},\frac{x}{6}\right)d\left(\frac{y}{5},\frac{y}{6}\right)}{1+d\left(\frac{x}{5},\frac{y}{5}\right)}, \frac{d\left(\frac{x}{5},\frac{x}{6}\right)d\left(\frac{x}{5},\frac{y}{6}\right)}{1+d\left(\frac{x}{5},\frac{y}{6}\right)+d\left(\frac{y}{5},\frac{x}{6}\right)}\}$$

$$= \max\{d\left(\frac{x}{5}, \frac{x}{5}\right), \frac{d\left(\frac{x}{5},\frac{x}{6}\right)d\left(\frac{x}{5},\frac{x}{6}\right)}{1+d\left(\frac{x}{6},\frac{x}{6}\right)} + \frac{d\left(\frac{x}{5},\frac{x}{6}\right)d\left(\frac{x}{5},\frac{x}{6}\right)}{1+d\left(\frac{x}{5},\frac{x}{5}\right)}, \frac{d\left(\frac{x}{5},\frac{x}{6}\right)d\left(\frac{x}{5},\frac{x}{6}\right)}{1+d\left(\frac{x}{5},\frac{x}{6}\right)+d\left(\frac{x}{5},\frac{x}{6}\right)}\}$$

$$= \left\{0, \frac{x^4}{810000}, \frac{x^4}{810000}, \frac{x^4}{900(2x^2+900)}\right\} = \frac{x^4}{810000}.$$

So, $d(fx, fy) \leq \beta\big(d(gx, gy)\big)M(x, y)$

$$d\left(\frac{x}{6}, \frac{x}{6}\right) \leq \beta(0)\frac{x^4}{810000} = 0$$

$$0 \leq 0.$$

Case (2): $(x, y) = (1,1) \Longrightarrow \text{d}(f1, f1) = \text{d}\left(\frac{1}{5}, \frac{1}{5}\right)$

$$\text{M}(x, y) = \max\{d(gx, gy), \frac{d(gx,fx)d(gy,fy)}{1+d(fx,fy)} + \frac{d(gx,fx)d(gy,fy)}{1+d(gx,gy)}, \frac{d(gx,fx)d(gx,fy)}{1+d(gx,fy)+d(gy,fx)}\ \}$$

$$\text{M}(1,1) = \max\{d(g1, g1), \frac{d(g1,f1)d(g1.f1)}{1+d(f1,f1)} + \frac{d(g1,f1)d(g1.f1)}{1+d(g1,g1)}, \frac{d(g1,f1)d(g1.f1)}{1+d(g1,f1)+d(g1,f1)}\ \}$$

$$= \max\{\text{d}\left(\frac{1}{5}, \frac{1}{5}\right), \frac{\text{d}\left(\frac{1}{5},\frac{1}{5}\right)\text{d}\left(\frac{1}{5}\ ,\frac{1}{5}\right)}{1+\text{d}\left(\frac{1}{5},\frac{1}{5}\right)} + \frac{\text{d}\left(\frac{1}{5},\frac{1}{5}\right)\text{d}\left(\frac{1}{5},\frac{1}{5}\right)}{1+\text{d}\left(\frac{1}{5},\frac{1}{5}\right)}, \frac{\text{d}\left(\frac{1}{5},\frac{1}{5}\right)\text{d}\left(\frac{1}{5},\frac{1}{5}\right)}{1+\text{d}\left(\frac{1}{5},\frac{1}{5}\right)+\text{d}\left(\frac{1}{5},\frac{1}{5}\right)}\ \}$$

$= \max\{0, 0,0,0\} = 0.$

So, $d(fx, fy) = \beta\big(d(gx, gy)\big)M(x, y)$

$$\mathrm{d}\left(\frac{1}{5}, \frac{1}{5}\right) \leq \beta(0)0 = 0 \times 0 = 0$$

$$0 \leq 0.$$

<u>Case (3):</u> $x, y \in (1,2] \Longrightarrow fx = fy = \frac{1}{4}$

$$\mathrm{M}(x, y) = \max\{d(gx, gy), \frac{d(gx,fx)d(gy,fy)}{1+d(fx,fy)} + \frac{d(gx,fx)d(gy,fy)}{1+d(gx,gy)}, \frac{d(gx,fx)d(gx,fy)}{1+d(gx,fy)+d(gy,fx)}\}$$

$$= \max\{\mathrm{d}\left(\frac{1}{4}, \frac{1}{4}\right), \frac{\mathrm{d}\left(\frac{1}{4}, \frac{1}{4}\right)\mathrm{d}\left(\frac{1}{4}, \frac{1}{4}\right)}{1+\mathrm{d}\left(\frac{1}{4}, \frac{1}{4}\right)} + \frac{\mathrm{d}\left(\frac{1}{4}, \frac{1}{4}\right)\mathrm{d}\left(\frac{1}{4}, \frac{1}{4}\right)}{1+\mathrm{d}\left(\frac{1}{4}, \frac{1}{4}\right)}, \frac{\mathrm{d}\left(\frac{1}{4}, \frac{1}{4}\right)\mathrm{d}\left(\frac{1}{4}, \frac{1}{4}\right)}{1+\mathrm{d}\left(\frac{1}{4}, \frac{1}{4}\right)+\mathrm{d}\left(\frac{1}{4}, \frac{1}{4}\right)}\}$$

$$= \max\ \{0,0,0,0\} = 0.$$

So, $d(fx, fy) \leq \beta\big(d(gx, gy)\big)M(x, y)$

$$0 \leq \beta\left(\mathrm{d}\left(\frac{1}{4}, \frac{1}{4}\right)\right)\mathrm{M}(x, y)$$

$$0 \leq \beta(0) \times 0 = 0.$$

Thus, f and g satisfy all the condition of Theorem 4.2. Moreover, 0 is a unique common fixed point of f and g.

Remark 1: If we choose $g = I_X =$ Identity map on X, is Theorem 1.2 we get theorem 4.1.Hence Theorem 1.2 follows as a corollary to Theorem 4.1.

Remark 2: If we choose $g = I_X =$ Identity map on X, is Theorem 1.3 we get Theorem 4.2. Hence Theorem 1.3 follows as a corollary to Theorem 4.2.

CHAPTER FIVE

5. CONCLUSION AND FUTURE SCOPE

5.1.CONCLUSION

In this research work, we constructed two common fixed point theorems for a pair of continuous maps and proved the existence and uniqueness of common fixed point results in the setting of complete b-rectangular metric spaces. By replacing the continuity assumption by sequential condition and considering closeness of one of the range spaces we also proved the existence of common fixed point results in b-rectangular metric space endowed with partial order metric spaces. We also provided examples in support of our main results. Our results extend some of the works of Roshal et.al. , (2016) to a pair of self-maps.

5.2 FUTURE SCOPE

Fixed point theory is one of active and desirable area of research in pure and applied mathematics. The existence of common fixed points of b-rectangular in partially ordered metric spaces were the research papers recently published. There are a number of published research papers related to this area of study. So we recommend the upcoming Post Graduate students and any other interested researchers of the department to do their research work in this area of study.

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