# COMMON FIXED POINT RESULTS OF $(\phi, \psi)$-CONTRACTIONS IN PARTIALLY ORDERED METRIC SPACES INVOLVING RATIONAL EXPRESSIONS

**A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS, JIMMA UNIVERSITY IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS**

**BY**


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## DECLARATION

I, the undersigned declare that, the thesis entitled "**Common fixed point results of $(\boldsymbol{\phi}, \boldsymbol{\psi})$ -contractions in partially ordered metric spaces involving rational expressions"** is original and it has not been submitted to any institution elsewhere for the award of any academic degree or like and where other sources of information have been used, they have been acknowledged .

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## Abstract

This research dealt with common fixed point results of $(\phi, \psi)$-contractions involving rational expressions in partial order metric spaces by extending the works of Chandok *et al.,*. Our results extend the results of Chandok *et al.,* to a pair of maps. In this work analytical design was employed. Secondary source of data such as published articles, books related to the research area and related works from internet were used. The procedures employed were the techniques used by Chandok *et al.* and Arab. We provided examples in support of our main findings.

# CHAPTER ONE

# 1. INTRODUCTION

## 1.1 Background of the Study

Fixed point theory is one of the most important topics in the development of nonlinear functional analysis. Fixed point theorems in metric spaces play an essential role to construct methods to solve problems in Mathematics and Sciences. In this area the first and significant result was proved by Banach, (1922) for contraction mapping in metric spaces. Let $X$ be a nonempty set and $T: X \to X$ a map. Then $T$ is said to be a self-map of $X$. A point $x$ in $X$ is said to be a fixed point of $T$ if $Tx = x$.

**Definition 1.1:** Let $(X, d)$ be a metric space. A self-map $T: X \to X$ is said to be contraction map if there exists a constant $k \in [0,1)$ such that,

$$d(Tx, Ty) \leq kd(x, y), \text{ for all } x, y \in X. \tag{1.1}$$

In this case k is called a contraction constant.

**Theorem 1.1 (Banach Contraction Principle)** Any contraction mapping on a complete metric space has a unique fixed point.

For the reason that the contraction mapping is continuous, many researchers established fixed point theorems on various classes of operators which satisfy conditions that are weaker than the contractive condition in Banach Contraction Principle but are not continuous.

Kannan, (1968) established the idea of contractive type mapping which imply existence of fixed point in complete metric space in which the map $T$ is not continuous, but continuous at fixed point.

**Definition 1.2:** A self-map $T$ in a metric space $(X, d)$ is called Kannan mapping if there exists $h \in \left[0, \frac{1}{2}\right)$ such that,

$$d(Tx, Ty) \leq h[d(Tx, x) + d(Ty, y)], \text{ for all } x, y \in X. \tag{1.2}$$

**Theorem 1.2 (Kannan Contraction Principle)** Let $(X, d)$ be a non-empty complete metric space and a self-map $T$ satisfies inequality (1.2). Then $T$ has a unique fixed point. Chatterjea, (1972) established a similar contractive condition which is dual of Kannan type which gave a new direction to the study of fixed point theory.

**Definition 1.3:** A self-map $T$ in a metric space $(X, d)$ is called Chatterjea mapping if there exists $h \in [0, \frac{1}{2})$ such that,

$$d(Tx, Ty) \leq h[d(Tx, y) + d(Ty, x)] \text{ , for all } x, y \in X. \tag{1.3}$$

**Theorem 1.3: (Chatterjea mapping)** Let $(X, d)$ be a non-empty complete metric space and a self-map $T$ satisfies inequality (1.3). Then $T$ has a unique fixed point.

Rhoades, (1977) compared various definitions of contractive mapping on a complete metric space which were used to generalize the contraction mapping principle.

Banach mappings, Kannan mappings and Chatterjea's mappings are independent (Rhoades, 1977).

Zamfirescu, (1972) proved the following fixed point theorem by combining (1.1), (1.2) and (1.3) as follows.

**Theorem1.4: (Zamfrescu mapping)** Let $(X, d)$ be a complete metric space and $T: X \to X$ be a map for which there exist real numbers $a, b$ and $c$ satisfying.

$0 \leq a < 1, 0 \leq b, c < \frac{1}{2}$ such that for each pair $x, y \in X$ at least one of the following holds.

($Z_1$) $d(Tx, Ty) \leq ad(x, y)$

($Z_2$) $d(Tx, Ty) \leq b[d(Tx, x) + d(Ty, y)]$

($Z_3$) $d(Tx, Ty) \leq c[d(Tx, y) + d(Ty, x)]$.

Then $T$ has a unique fixed point. Zamfirescu's Theorem is a generalization of ($Z_1$) Banach's Theorem, ($Z_2$) Kannan's Theorem and ($Z_3$) Chatterjea's Theorem.

Notation: Throughout this paper we denote:

$\mathbb{R}^+ = [0, \infty)$. (The set of non-negative real numbers) and

$\Phi = \{\phi: \mathbb{R}^+ \to \mathbb{R}^+, \text{ such that } \phi \text{ is continuous, nondecreasing and } \phi(t) = 0 \text{ if and only if } t = 0\}$.

$\Psi = \{\psi: \mathbb{R}^+ \to \mathbb{R}^+, \text{such that for any sequence } \{x_n\} \text{ in } \mathbb{R}^+ \text{ with } x_n \to t, (t > 0), \psi \underline{\lim}(x_n) > 0\}$.

$C(T, g) \neq \emptyset)$. (The set of coincidence points of $T$ and $g$ is non-empty).

One of the generalizations of Banach contraction principle is through the method of altering distances between the points with the help of a continuous control function.

Khan *et al*., (1984) proved fixed point theorems by altering distance between the points.

Since the early days of metric fixed point theory, numerous authors attempted to vary the contraction conditions by improving the existing contraction conditions and replacing with various types of the general conditions.

The first result in existence of fixed point in partially ordered sets was given by (Turinici, 1986), where he extended the Banach contraction principle in partially ordered sets.

Ran and Reurings, (2004) extended the Banach contraction principle in partially ordered sets with some applications to linear and nonlinear matrix equations.

Dass and Gupta, (1975) extended Banach's contraction principle through rational expression as follows.

**Theorem 1.5:** Let $(X, d)$ be a complete metric space and $T: X \to X$ be a mapping such that there exist $\alpha, \beta \geq 0$with$\alpha + \beta < 1$ satisfying;

$d(Tx, Ty) \leq \alpha \frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)} + \beta d(x, y)$, then $T$ has a unique fixed point.

Arshad *et al*., (2013) proved some unique fixed point theorems for rational type contractions in partially ordered metric spaces.

Chandok *et al*., (2015) proved fixed point results in ordered metric spaces with rational type expressions using some auxiliary functions.

**Theorem 1.6:** Chandok, (2015) Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric $d \; on \; X$ such that $(X, d)$ is a complete metric space. Let $T: X \to X$ be a continuous and non-decreasing mapping. Suppose that there exists $\phi \in \Phi$ and $\psi \in \Psi$ such that,

$$\phi(d(Tx, Ty) \leq \phi(M(x, y) - \psi\big(N(x, y)\big),$$

for all $x, y \in X$ with $x \preccurlyeq y$ where,

$$M(x, y) = \max\left\{\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, \frac{d(y,Tx)[1+d(x,Ty)]}{1+d(x,y)}, d(x, y)\right\} \text{ and}$$

$$N(x, y) = \max\left\{\frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}, d(x, y)\right\}.$$

If there exists $x_0 \in X$ with $x_0 \preccurlyeq Tx_0$ , then $T$ has a fixed point in $X$.

**Theorem 1.7:** Chandok, (2015) Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Assume that if $\{x_n\}$ a non-decreasing sequence in $X$ such that $x_n \to x$, then $x_n \preccurlyeq x$, for all $n \in N$. Let $T: X \to X$ be a non-decreasing mapping. Suppose that there exists $\phi \in \Phi$ and $\psi \in \Psi$ such that,

$$\phi(d(Tx, Ty) \leq \phi(M(x, y) - \psi\big(N(x, y)\big),$$

for all $x, y \in X$ with $x \preccurlyeq y$ where

$$M(x, y) = \max\left\{\frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}, \frac{d(y, Tx)[1 + d(x, Ty)]}{1 + d(x, y)}, d(x, y)\right\}$$

$$N(x, y) = \max\left\{\frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}, d(x, y)\right\}.$$

If there exists $x_0 \in X$ with $x_0 \preccurlyeq Tx_0$, then $T$ has a fixed point in $X$.

Inspired and motivated by the work of Chandok *et al*., (2015) the researcher has extended this work to common fixed point results for a pair of self-maps.

Therefore, the purpose of this study is to establish common fixed point results and proving the existence and uniqueness of common fixed point results of $(\phi, \psi)$-contractions in partially ordered metric spaces involving rational expressions for a pair of self-maps by extending the works of (Chandok *et al*., 2015). We provided illustrative examples which support the main result of the study.

## 1.3 Objectives of the Study

### 1.3.1 General Objective of the Study

The main objective of this study is to establish the existence of common fixed point results of $(\phi, \psi)$-contractions in partially ordered metric spaces involving rational expressions by extending the works of Chandok *et al*.

### 1.3.2 Specific Objectives

The specific objectives of this study are:

1. To prove the existence of common fixed point results of $(\phi, \psi)$-contractions involving rational expressions in partially ordered metric spaces.
2. To identify additional conditions required to obtain unique common fixed point results of $(\phi, \psi)$-contractions in partially ordered metric spaces involving rational expressions.
3. To verify the applicability of the result using specific examples.

## 1.4 Significance of the Study

The study may have the following importance:

1. The results obtained in this study may contribute to research activities in this area.
2. It may help the researcher to develop scientific research writing skills and scientific communication in Mathematics.
3. It may help to solve problem involving differential equation.

## 1.5 Delimitation of the Study

This study was delimited to prove the existence and uniqueness of common fixed point results for a pair of self- maps satisfying $(\phi, \psi)$-contractive conditions in partially ordered metric spaces with rational expressions under the stream of Functional Analysis.

# CHAPTER TWO

## 2. Literature Review

In 1922, the Polish mathematician Stefan Banach established a remarkable fixed point theorem known as the "Banach Contraction Principle" which is one of the most important results of analysis and considered as the main source of metric fixed point theory. A theory of fixed point is one of the most powerful and popular tools of modern mathematics and it is the most widely applied fixed point result in many branches of mathematics because it requires the structure of complete metric space with contractive condition on the map which is easy to test in this setting.

Banach's contraction principle, which gives an answer on the existence and uniqueness of a solution of an operator equation, $Tx = x$ is the most widely used fixed point theorem in all of analysis.

There have been a number of generalizations of metric spaces such as, cone-metric spaces, cone-b metric spaces, partially ordered metric spaces and other spaces.

Alber and Guerre-Delbariere, (1997) introduced the concept of weakly contractive maps in a complete Hilbert spaces as a generalization of weakly contractive maps. Rhoades, (2001) extended this concept to Banach spaces and proved the existence of fixed points of weakly contractive maps in the setting of metric space. Harjani and Sadarangani, (2009) established some fixed point theorems for weak contractions and generalized contraction in partially ordered metric spaces by using the altering distance function.

Jaggi and Dass, (1980) proved the following fixed point theorem. Let $T$ be a continuous self-map defined on a complete metric space $(X, d)$. Suppose that $T$ satisfies the following contraction condition:

$d(Tx, Ty) \leq \alpha \frac{d(x,Tx)[1+d(y,Ty)]}{d(x,y)+d(x,Ty)+d(y,Tx)} + \beta d(x, y)$ for all $x, y \in X,\ x \ngeq y$ and for some $\alpha, \beta \in (0,1)$ with $\alpha + \beta < 1$, then $T$ has a unique fixed point.

So the rational type contractions have been improved by many researchers in various ways.

Recently, many researchers have obtained fixed point, common fixed point results in partially ordered metric spaces.

**Definition 2.1:** Doric, (2009) Let $(X, d)$ be a metric space and $f$ and $T$ be two self-mappings on $(X, d)$. A point $x \in X$ is said to be a common fixed point of $f$ and $T$ if $fx = Tx = x$.

Jungck and Rhoades, (1998) weakened the notion of compatibility by giving the concept of weakly compatibility.

**Definition 2.2:** Let $(X, d)$ be a metric space. Then the pair $(f, g)$ is said to be compatible if and only if $\lim_{n\to\infty} d(fgx_n,\ gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in $X$ such that, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

**Definition 2.3:** Jungck and Rhoades, (1998) Two mappings $f, g\colon \mathrm{X} \to \mathrm{X}$ are said to be weakly compatible if, they commute at their coincidence point, that is;

$fgx = gfx$ holds, whenever $fx = gx,\ x \in X$.

**Definition 2.4:** Jungck, (1986) Let $f$ and $g$ be self-maps of a set $X$. If $\mathrm{w} = fx = gx$ for some $x$ in X then $x$ is called a coincidence point of $f$ and $g$ and w is called a point of coincidence of $f$ and $g$.

Chandok, (2015) proved fixed point results in ordered metric spaces with rational type expressions using some auxiliary functions.

In recent times, fixed point theory has developed rapidly in partially ordered metric spaces, that is, metric spaces endowed with a partial ordering.

**Definition 2.5:** Ciri'c *et al.*, (2008) Let $(X, \preccurlyeq)$ be a partially ordered set and $T, g\colon X \to X$. One says $T$ is $g$-non-decreasing if for all $x, y \in X,\ gx \preccurlyeq gy \implies Tx \preccurlyeq Ty$.

Choudhury, (2005) proved a common fixed point theorem using altering distances for three variables and Doric , (2009) proved some common fixed point results in partially ordered metric spaces for generalized rational type contraction mappings.

**Theorem 2.3:** Doric, (2009) Let $(X, d)$ be a complete metric space and let $T, S\colon X \to X$ be two self-maps. Suppose that there exists $\phi \in \Phi$ and $\psi \in \Psi$ such that,

$$\psi(\mathrm{d}(Tx, Sx)) \leq \psi(M(x, y) - \phi(M(x, y)),$$

for all $x, y \in X$, where

$$M(x, y) = \max\left\{d(x, y), d(Tx, x), d(Sy, y), \frac{1}{2}[d(y, Tx) + d(x, Sy)]\right\}$$

Then there exists a unique $u \in X$ such that $Tu = Su = u$.

# CHAPTER THREE

## 3. Methodology

### 3.1 Study Site and Period

This study was conducted from September 2016 G.C to October 2017 G.C, in Jimma University under Mathematics Department.

### 3.2 Study Design

In order to achieve the objectives of the study, Analytical design method was used.

### 3.3 Source of Information

This study mostly depended on document materials, so the available source of information for the study were Books, Journals, different study related to the topic and internet services. So, the researcher collected different documents that were listed which support the study and discuss about the collected materials and other activities with advisor.

### 3.4 Mathematical Procedure

The mathematical procedures employed in this research work were the standard technique used by Chandok *et al.*, (2015) and Arab, (2016).

# CHAPTER FOUR

## 4. Discussion and Result

### 4.1 Preliminaries

**Definition 4.1:** Let $X$ be a non-empty set and $d: X \times X \to \mathbb{R}^+$ be a mapping satisfying the following conditions for all $x, y, z \in X$;

i. $d(x, y) = 0$ if and only if $x = y$.
ii. $d(x, y) = d(y, x)$. (Symmetry)
iii. $d(x, z) \leq d(x, y) + d(y, z)$. (Triangular inequality)

Then $d$ is called a metric on $X$ and the pair $(X, d)$ is called a metric space.

**Example 4.1:** Let $X = R$ (the set of real numbers). Define $d: X \times X \to \mathbb{R}^+$ by

$d(x, y) = |x - y|$, for all $x, y \in X$ then clearly the pair $(X, d)$ is a metric space.

**Definition 4.2:** Al-Thagafi and Shahazad, (2009) Let $T$ and $g$ be two self-maps of a metric space $(X, d)$. The pair $(T, g)$ is called;

i. Commuting if $Tgx = gTx$, for all $x \in X$.

ii. Weakly commuting if $d(Tgx, gTx) \leq d(Tx, gx)$, for all $x \in X$.

**Example 4.2:** Let $X = \mathbb{R}^+$. Define $T, g: X \to X$ by $Tx = \frac{x}{8} - \frac{x^2}{64}$ and $gx = \frac{x}{2}$, for all $x \in X$.

Then $d(Tgx, gTx) = \frac{x^2}{256} \leq \frac{24x + x^2}{64} = d(Tx, gx)$,

which implies $d(Tgx, gTx) \leq d(Tx, gx)$.

Therefore, $T$ and $g$ are weakly commuting. But

$Tgx = \frac{x}{16} - \frac{x^2}{256} \neq \frac{x}{16} - \frac{x^2}{128} = gTx$ .

Which show that $T$ and $g$ are not commuting.

**Definition 4.3:** Su, (2014) A partially ordered set (poset) is a system $(X, \preccurlyeq)$ , where $X$ is a non-empty set and $\preccurlyeq$ is a binary relation of $X$ satisfying for all $x, y, z \in X$;

i. $x \preccurlyeq x$ (Reflexivity)

ii. if $x \preccurlyeq y$ and $y \preccurlyeq x$ then $x = y$ (Anti symmetry)

iii. if $x \preccurlyeq y$ and $y \preccurlyeq z$ then $x \preccurlyeq z$. (Transitivity)

**Example 4.3:**

(i) If $X$ any non-empty set $(\mathrm{P}(X), \subseteq)$ is a partially ordered set. Where $\mathrm{P}(X)$ = the power set of $X$ and " $\subseteq$ " is to mean a subset of.

(ii) On set of natural numbers $N$, define m $\preccurlyeq$ n if m divides n then $(N, \preccurlyeq)$ is a partially ordered set.

**Definition 4.4:** Let $X$ be a non-empty set. Then $(X, d, \preccurlyeq)$ is called partially ordered metric space if:

i. $(X, d)$ is a metric space and

ii. $(X, \preccurlyeq)$ is a partially ordered set.

**Definition 4.5:** Su, (2014) **A** function $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ is called an altering distance function if:

i. $\phi$ is non-decreasing, continuous and

ii. $\phi(t) = 0$ if and only if $t = 0$.

**Example 4.5:**

$\phi(t) = \begin{cases} t, & 0 \le t < 1 \\ at^2, & t \ge 1 \end{cases}$, where $a \ge 1$. Then $\phi$ is an altering distance function.

## 4.2 Main Result

**Theorem: 4.1** Let $(X, d, \preccurlyeq)$ be a partially ordered complete metric space. Suppose $T, g: X \to X$ be two maps and for every two comparable $gx$ and $gy$. Suppose that there exists $\phi \in \Phi$ and $\psi \in \Psi$ such that;

$$\phi(d(Tx, Ty)) \leq \phi(M(x, y) - \psi(N(x, y)) \text{ , where} \tag{4.1}$$

$$M(x, y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, \frac{d(gy,Tx)[1+d(gx,Ty)]}{1+d(gx,gy)}, d(gx, gy)\right\}$$

$$N(x, y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, d(gx, gy)\right\} \text{ and} \tag{4.2}$$

satisfying the following conditions:

i. $TX \subseteq gX$;
ii. $T$ is a $g$-non-decreasing;
iii. $T$ and $g$ are continuous;
iv. There exists an $x_0 \in X$ with $g(x_0) \preccurlyeq T(x_0)$.

If $T$ and $g$ are compatible, then $T$ and $g$ have a coincidence point. Moreover, if $T$ and $g$ are weakly compatible, then $T$ and $g$ have a common fixed point, under the assumption there exists $u \in X$ such that $Tu \preccurlyeq Tx$ and $Tu \preccurlyeq Ty$, the common fixed point is unique.

**Proof:** Let $x_0$ be an arbitrary point of $X$ such that $g(x_0) \preccurlyeq T(x_0)$. Since $T(X) \subseteq g(X)$, we can choose $x_1 \in X$ so that $g(x_1) = T(x_0)$. Again from $T(X) \subseteq g(X)$, we can choose $x_2 \in X$ so that $g(x_2) = T(x_1)$, since $g(x_0) \preccurlyeq T(x_0) = g(x_1)$ and $T$ is $g$-non-decreasing, we have

$T(x_0) \preccurlyeq T(x_1)$. Continuing this process we can choose a sequence $\{x_n\}$ in $X$ such that,

$$gx_{n+1=}Tx_n, \text{ with} \tag{4.3}$$

$T(x_0) \preccurlyeq T(x_1) \preccurlyeq T(x_2) \preccurlyeq, \dots, \preccurlyeq T(x_n) \preccurlyeq, \dots$ , therefore

$$g(x_1) \preccurlyeq g(x_2) \preccurlyeq g(x_3) \preccurlyeq, \dots, \preccurlyeq g(x_{n+1}) \preccurlyeq, \dots \tag{4.4}$$

If there exists $n_0 \in N$ such that, $d(gx_{n_0}, gx_{n_{0+1}}) = 0$, then we have;

$$gx_{n_0} = gx_{n_{0+1}} = Tx_{n_0}, \text{ hence } x_{n_0} \text{ is a coincidence point of } T \text{and } g. \tag{4.5}$$

Assume that $d(gx_{n,}\ gx_{n+1}) > 0$, for all $n \in N$. We will show that,

$$d(gx_{n,}\ gx_{n+1}) \leq d(gx_{n-1,}\ gx_n), \text{ for all n} \geq 1, \tag{4.6}$$

From (4.1) and (4.4), with $x = x_{n-1}$ and $y = x_n$, we have;

$$\phi\left(d\left(gx_{n,}\ gx_{n+1}\right)\right) = \phi\left(d\left(Tx_{n-1,}\ Tx_n\right)\right) \leq \phi\left(M\left(x_{n-1,}x_n\right)\right) - \psi\left(N\left(x_{n-1,}x_n\right)\right). \qquad (4.7)$$

Where, $M\left(x_{n-1,}x_n\right) = \max\left\{\frac{d(gx_{n,}\ Tx_n)[1+d(gx_{n-1,}\ Tx_{n-1})]}{1+d(gx_{n-1,}\ gx_n)}, \frac{d(gx_{n,}\ Tx_{n-1})[1+d(gx_{n-1,}\ Tx_n)]}{1+d(gx_{n-1,}\ gx_n)}, d(gx_{n-1,}\ gx_n)\right\}$

$$= \max\left\{\frac{d(gx_{n,}\ gx_{n+1})[1+d(gx_{n-1,}\ gx_n)]}{1+d(gx_{n-1,}\ gx_n)}, \frac{d(gx_{n,}\ gx_n)[1+d(gx_{n-1,}\ gx_{n+1})}{1+d(gx_{n-1,}\ gx_n)}, d(gx_{n-1,}\ gx_n)\right\}$$

$$= \max\{d(gx_{n,}\ gx_{n+1}),\ \ 0,\ \ d(gx_{n-1,}\ gx_n)\}$$

$$= \max\{d(gx_{n,}\ gx_{n+1}), d(gx_{n-1,}\ gx_n)\}.$$

$$N(x_{n-1,}x_n) = \max\left\{\frac{d(gx_n, Tx_n)[1 + d(gx_{n-1},\ Tx_{n-1})}{1 + d(gx_{n-1,}\ gx_n)}, d(gx_{n-1,}\ gx_n)\right\}$$

$$= \max\left\{\frac{d(gx_{n,}\ gx_{n+1})[1+d(gx_{n-1,}\ gx_n)]}{1+d(gx_{n-1,}\ gx_n)}, d(gx_{n-1,}\ gx_n)\right\}$$

$$= \max\{d(gx_{n,}\ gx_{n+1}), d(gx_{n-1,}\ gx_n)\}.$$

Suppose that $d\left(gx_{n,}\ gx_{n+1}\right) > d(gx_{n-1,}\ gx_n)$, from (4.7), we have;

$$\phi\left(d\left(gx_{n,}\ gx_{n+1}\right)\right) \leq \phi\left(d\left(gx_{n,}\ gx_{n+1}\right)\right) - \psi\left(d\left(gx_{n,}\ gx_{n+1}\right)\right). \qquad (4.8)$$

Which implies $\psi\left(d\left(gx_{n,}\ gx_{n+1}\right)\right) < 0$, which is a contradiction. Therefore, we prove that,

$d(gx_{n,}\ \ gx_{n+1}) \leq d(gx_{n-1,}\ \ gx_n)$, for all $n \geq 1$. Hence (4.6) holds. Thus the sequence $\left\{d\left(gx_{n,}\ gx_{n+1}\right)\right\}$ is a decreasing sequence of non-negative real numbers and consequently there exists $r \geq 0$, such that $\lim_{n\to\infty} d\left(gx_{n,}\ gx_{n+1}\right) = r.$ (4.9)

We shall show that $r = 0$. Suppose to the contrary, that $r > 0$ .Taking the upper limit as $n \to \infty$ in (4.8) and using the properties of $\phi$ and $\psi$ we get,

$$\phi(r) \leq \phi(r) + \overline{\lim}\ (-\psi\left(d\left(gx_{n,}\ gx_{n+1}\right)\right))$$

$$= \phi(r) - \underline{\lim}(\psi\left(d\left(gx_{n,}\ gx_{n+1}\right)\right))$$

$\leq \phi(r) - \psi(r) < \phi(r)$ which is a contradiction. Hence $r = 0$.

Therefore $\lim_{n\to\infty} d\left(gx_{n,}\ gx_{n+1}\right) = 0.$ (4.10)

Now we claim that $\{gx_n\}$ is a Cauchy sequence. Suppose that $\{gx_n\}$ is not a Cauchy sequence, then there exists a positive real number $\varepsilon$ such that for any given $N$ there exists $m, n \in N$ such that $m > n > N$ and $d(gx_m, gx_n) \geq \varepsilon$. Since $\{d(gx_{n,}\ gx_{n+1})\}$, converges to zero, it follows that there exists strictly increasing sequences, $\{n_k\}$and $\{m_k\}$, of positive integers $1 < n_k < m_{k,} k \geq 1$ such that

$d(gx_{m_k}, gx_{n_k}) \geq \varepsilon$ and $d(gx_{m_{k-1}}, gx_{n_k}) < \varepsilon$. (4.11)

Using the triangular inequality and the condition (4.11), we have;

$$\varepsilon \leq d\big(gx_{m_k}, gx_{n_k}\big) \leq d\big(gx_{m_k}, gx_{m_{k-1}}\big) + d\big(gx_{m_{k-1}}, gx_{n_k}\big) \leq d\big(gx_{m_k}, gx_{m_{k-1}}\big) + \varepsilon$$

Letting $k \to \infty$ and using (4.10), we get $\lim_{k\to\infty} d(gx_{m_k},\ gx_{n_k}) = \varepsilon$. (4.12)

Again $d\big(gx_{m_{k-1}}, gx_{n_{k-1}}\big) \leq d\big(gx_{m_{k-1}}, gx_{m_k}\big) +\ d\big(gx_{m_k}, gx_{n_k}\big) + d\big(\ gx_{n_k}, gx_{n_{k-1}}\big)$ (4.13)

$d\big(gx_{m_k}, gx_{n_k}\big) \leq d\big(gx_{m_k}, gx_{m_{k-1}}\big) + d\big(gx_{m_{k-1}}, gx_{n_{k-1}}\big) + d(\ gx_{n_{k-1}}, gx_{n_k})$ (4.14)

Letting $k \to \infty$ on (4.13), (4.14) and using (4.12), we have;

$$\lim_{k\to\infty} d\big(gx_{m_{k-1}}, gx_{n_{k-1}}\big) = \varepsilon. \tag{4.15}$$

Since $\phi$ is non-decreasing on $\mathbb{R}^+$ and from (4.11), we have $\phi(\varepsilon) \leq \phi(d\big(gx_{m_k}, gx_{n_k}\big))$ for all $k \geq 1$ and as $m_k > n_k$ , by (4.4) we have $gx_{m_{k-1}}$and $gx_{n_{k-1}}$ are comparable and taking upper limit as $k \to \infty$ on (4.8) and using (4.15) and properties of $\phi$ and $\psi$, we have;

$$\phi(\varepsilon) \leq \phi(\varepsilon) + \overline{\lim}\ (-\psi\left(d\big(gx_{mk,}\ gx_{nk}\big)\right)$$

$$= \ \phi(\varepsilon) + \overline{\lim}\ (-\psi\left(d\big(Tx_{m_{k-1}}, Tx_{n_{k-1}}\big)\right))$$

$$= \phi(\varepsilon) - \underline{\lim}\ (\psi\left(d\big(Tx_{m_{k-1}}, Tx_{n_{k-1}}\big)\right))$$

$\leq \phi(\varepsilon) - \psi(\varepsilon) < \phi(\varepsilon)$. Which is a contradiction, since $\varepsilon > 0$.

Thus the sequence $\{gx_n\}$ is a Cauchy sequence in $X$.

Hence $\{gx_n\}$ is convergent in the complete metric space $(X, d)$. So there exists $x \in X$ such that

$$\lim_{n\to\infty} g_{x_{n+1}} = \lim_{n\to\infty} T_{x_n} = x.$$

Suppose that the continuity of $T$ and $g$ holds. We show that $gx = Tx$.

From the triangular inequality we have;

$$d(Tx, gx) \leq d(Tx, Tgx_n) + d(Tgx_n, gTx_n) + d(gTx_n, gx). \quad (4.16)$$

Also from the continuity of $T$ and $g$ we have;

$$\lim_{n\to\infty} T(g(x_n)) = T(\lim_{n\to\infty} g(x_n)) = Tx. \quad (4.17)$$

$$\lim_{n\to\infty} g(T(x_n)) = g(\lim_{n\to\infty} T(x_n)) = gx.$$

Letting $n \to \infty$ in (4.16) and using (4.17) and compatibility of $T$ and $g$ we get, $d(Tx, gx) = 0$. Which implies $Tx = gx$, that is $x$ is a coincidence point of $T$ and $g$. (4.18)

Hence the set of coincidence points of $T$ and $g$ is non-empty or $C(T, g) \neq \emptyset$). Suppose that $x$ and $y$ are coincidence points of $T$and $g$, that is $Tx = gx$ and $Ty = gy$. We shall show that $gx = gy$. By assumption there exists $u \in X$ such that $Tu$ is comparable to $Tx$ and $Ty$. Without any restriction of generality, we can assume that;

$$Tx \preccurlyeq Tu \text{ and } Ty \preccurlyeq Tu. \quad (4.19)$$

Put $u_0 = u$ and choose $u_1 \in X$ such that $gu_1 = Tu_0$. For $n \geq 1$, continuing this process we can construct sequence $\{gu_n\}$ such that $gu_{n+1} = Tu_n$ for all n. (4.20)

Further, set $x_0 = x$ and $y_0 = y$ and on the same way define sequences $\{gx_n\}$ and $\{gy_n\}$. Since $gx = Tx = gx_1$ and $Tu = gu_1$ are comparable, $gx \preccurlyeq gu$. One can show by induction that $gx_n \preccurlyeq gu_n$ for all $n$. (4.21)

Thus from (4.1), we have;

$$\phi(d(gx, gu_{n+1})) = \phi(d(Tx, Tu_n)$$

$$\leq \phi\big(M(x, u_n)\big) - \psi\big(N(x, u_n)\big). \quad (4.22)$$

Where $\quad M(x, u_n) = \max\left\{\frac{d(gu_n,Tu_n)[1+d(gx,Tx)]}{1+d(gx,gu_n)}, \frac{d(gu_n,Tx)[1+d(gx,Tu_n)]}{1+d(gx,gu_n)}, d(gx, gu_n)\right\}$

$$= \max\{d(gu_n, gx), d(gx, gu_n)\}$$

$$= d(gx, gu_n).$$

$$N(x, u_n) = \max\left\{\frac{d(gu_n,Tu_n)[1+d(gx,Tx)]}{1+d(gx,gu_n)}, d(gx, gu_n)\right\}$$

$$= \max\{0, d(gx, gu_n)\}$$

$$= d(gx, gu_n).$$

Hence $\phi(d(gx, gu_{n+1})) \leq \phi\big(d(gx, gu_n)\big) - \psi(d(gx, gu_n))$
$$\leq \phi\big(d(gx, gu_n)\big). \qquad (4.23)$$

Using the non-decreasing property of $\phi$ we get,

$d(gx, gu_{n+1}) \leq d(gx, gu_n)$, implies that $\{d(gx, gu_n)\}$ is a non-increasing sequence. Hence, there exists $s \geq 0$ such that,

$$\lim_{n\to\infty} d(gx, gu_n) = s.$$

Taking the upper limit in (4.23) as $n \to \infty$ we obtain, $\phi(s) \leq \phi(s) - \psi(s)$ which implies that $\psi(s) = 0$ and hence $s = 0$. We deduce that,

$$\lim_{n\to\infty} d(gx, gu_n) = 0. \qquad (4.24)$$

Similarly, one can prove that,

$$\lim_{n\to\infty} d(gy, gu_n) = 0. \qquad (4.25)$$

Thus from (4.24) and (4.25), we have $gx = gy$. Since $gx = Tx$ and $gy = Ty$, by weakly compatible of $T$ and $g$, we have;

$$g(gx) = g(Tx) = T(gx). \qquad (4.26)$$

Denote $gx = a$, then from (4.26), we have;

$$ga = Ta. \qquad (4.27)$$

Thus, $a$ is a coincidence point of $T$and $g$, it follows that $ga = gx = a$. (4.28)

So from (4.27) and (4.28), we have $a = ga = Ta$. (4.29)

Therefore, $a$ is a common fixed point of $T$and $g$. To prove the uniqueness of the common fixed point of $T$ and $g$ we assume that $b$ is another common fixed point of $T$and $g$. Then we have

$$b = gb = Tb. \qquad (4.30)$$

Since $b$ is a coincidence point of $T$ and $g$, we have $gb = gx = a$ .

Thus $b = gb = ga = a$, which is the desired result.

**Remark 1:** By choosing $g =$ Ix =Identity map on $X$ in Theorem 4.1 we get Theorem 1.6. Hence Theorem 1.6 follows as a corollary to Theorem 4.1.

The following is an example in support of Theorem 4.1.

**Example 4.6** Let $X = \{1,2,3,4,5\}$, with $d(x,y) = |x - y|$ and define $\preccurlyeq := \{(1,1),(2,2),(3,3),(4,4),(5,5),(2,3),(3,4),(2,4)\}$, then $(X, \preccurlyeq)$ is a partially ordered set.

Define $T, g: X \to X$ by $TX = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 2 & 2 & 2 \end{pmatrix}$ and $gX = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 2 & 3 & 4 \end{pmatrix}$.

Then $TX = \{1,2\}$ and $gX = \{1,2,3,4\}$ and hence $TX = \{1,2\} \subset \{1,2,3,4\} = gX$. $T$ and $g$ are continuous.

Next we show that $T$ is $g$-non-decreasing.

$$1 = g(1) \preccurlyeq g(1) = 1 \Rightarrow 1 = T(1) \preccurlyeq T(1) = 1$$

$$2 = g(2) \preccurlyeq g(2) = 2 \Rightarrow 2 = T(2) \preccurlyeq T(2) = 2$$

$$2 = g(2) \preccurlyeq g(3) = 2 \Rightarrow 2 = T(2) \preccurlyeq T(3) = 2$$

$$2 = g(3) \preccurlyeq g(2) = 2 \Rightarrow 2 = T(3) \preccurlyeq T(2) = 2$$

$$2 = g(2) \preccurlyeq g(4) = 3 \Rightarrow 2 = T(2) \preccurlyeq T(4) = 2$$

$$2 = g(2) \preccurlyeq g(5) = 4 \Rightarrow 2 = T(2) \preccurlyeq T(5) = 2$$

$$2 = g(3) \preccurlyeq g(3) = 2 \Rightarrow 2 = T(3) \preccurlyeq T(3) = 2$$

$$2 = g(3) \preccurlyeq g(4) = 3 \Rightarrow 2 = T(3) \preccurlyeq T(4) = 2$$

$$2 = g(3) \preccurlyeq g(5) = 4 \Rightarrow 2 = T(3) \preccurlyeq T(5) = 2$$

$$3 = g(4) \preccurlyeq g(4) = 3 \Rightarrow 2 = T(4) \preccurlyeq T(4) = 2$$

$$3 = g(4) \preccurlyeq g(5) = 4 \Rightarrow 2 = T(4) \preccurlyeq T(5) = 2$$

$$4 = g(5) \preccurlyeq g(5) = 4 \Rightarrow 2 = T(5) \preccurlyeq T(5) = 2.$$

From the above steps $T$ is $g$-non-decreasing and we also observe that $x_0 = 1 \in X$ such that $gx_0 \preccurlyeq Tx_0$.

Define $\phi, \psi: \mathbb{R}^+ \to \mathbb{R}^+$, by $\phi(t) = t$ and $\psi(t) = \frac{1}{2}t$ and we also consider the following cases to verify the inequality (4.1).

**<u>Case 1</u>:** Let $x = 1$ and $y = 1$. Then we have;

1. $d(Tx, Ty) = d(T(1), T(1)) = d(1,1) = 0.$
2. $M(x, y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, \frac{d(gy,Tx)[1+d(gx,Ty)]}{1+d(gx,gy)}, d(gx, gy)\right\}$

$$= \max\left\{\frac{d(g(1),T(1))[1+d(g(1),T(1))]}{1+d(g(1),g(1))}, \frac{d(g(1),T(1))[1+d(g(1),T(1))]}{1+d(g(1),g(1))}, d(g(1), g(1))\right\}$$

$$= \max\left\{\frac{d(1,1)[1+d(1,1)]}{1+d(1,1)}, \frac{d(1,1)[1+d(1,1)]}{1+d(1,1)}, d(1,1)\right\}$$

$$= \max\{0,\ 0, 0\} = 0.$$

3. $N(x, y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, d(gx, gy)\right\}$

$$= \max\left\{\frac{d(g(1),T(1))[1+d(g(1),T(1))]}{1+d(g(1),g(1))}, d(g(1), g(1))\right\}$$

$$= \max\left\{\frac{d(1,1)[1+d(1,1)]}{1+d(1,1)}, d(1,1)\right\}$$

$$= \max\{0,\ 0\} = 0.$$

Thus from inequality (4.1), we have;

$$\phi(d(Tx, Ty) \leq \phi(M(x, y)) - \psi(N(x, y))$$

$$\phi(0) \leq \phi(0) - \psi(0)$$

$$0 \leq 0.$$

**<u>Case 2</u>:** Let $x = 2$ and $y = 2$. Then we have;

1. $d(Tx, Ty) = d(T(2), T(2)) = d(2,2) = 0.$

2. $M(x,y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, \frac{d(gy,Tx)[1+d(gx,Ty)]}{1+d(gx,gy)}, d(gx,gy)\right\}$

$$= \max\left\{\frac{d(g(2),T(2))[1+d(g(2),T(2))]}{1+d(g(2),g(2))}, \frac{d(g(2),T(2))[1+d(g(2),T(2))]}{1+d(g(2),g(2))}, d(g(2),g(2))\right\}$$

$$= \max\left\{\frac{d(2,2)[1+d(2,2)]}{1+d(2,2)}, \frac{d(2,2)[1+d(2,2)]}{1+d(2,2)}, d(2,2)\right\}$$

$$= \max\{0,\ 0, 0\} = 0.$$

3. $N(x,y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, d(gx,gy)\right\}$

$$= \max\left\{\frac{d(g(2),T(2))[1+d(g(2),T(2))]}{1+d(g(2),g(2))}, d(g(2),g(2))\right\}$$

$$= \max\left\{\frac{d(2,2)[1+d(2,2)]}{1+d(2,2)}, d(2,2)\right\}$$

$$= \max\{0,\ 0\} = 0.$$

Thus from inequality (4.1), we have;

$$\phi(d(Tx,Ty) \le \phi(M(x,y)) - \psi(N(x,y))$$

$$\phi(0) \le \phi(0) - \psi(0)$$

$$0 \le 0.$$

**<u>Case 3</u>:** Let $x = 3$ and $y = 3$. We have;

1. $d(Tx,Ty) = d(T(3),T(3)) = d(2,2) = 0.$

2. $M(x,y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, \frac{d(gy,Tx)[1+d(gx,Ty)]}{1+d(gx,gy)}, d(gx,gy)\right\}$

$$= \max\left\{\frac{d(g(3),T(3))[1+d(g(3),T(3))]}{1+d(g(3),g(3))}, \frac{d(g(3),T(3))[1+d(g(3),T(3))]}{1+d(g(3),g(3))}, d(g(3),g(3))\right\}$$

$$= \max\left\{\frac{d(2,2)[1+d(2,2)]}{1+d(2,2)}, \frac{d(2,2)[1+d(2,2)]}{1+d(2,2)}, d(2,2)\right\}$$

$$= \max\{0,\ 0,\ 0\} = 0.$$

3. $N(x,y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, d(gx,gy)\right\}$

$$= \max\left\{\frac{d(g(3),T(3))[1+d(g(3),T(3))]}{1+d(g(3),g(3))}, d(g(3),g(3))\right\}$$

$$= \max\left\{\frac{d(2,2)[1+d(2,2)]}{1+d(2,2)}, d(2,2)\right\}$$

$$= \max\{0,\ 0\} = 0.$$

Thus from inequality (4.1), we have;

$$\phi(d(Tx,Ty) \leq \phi(M(x,y)) - \psi(N(x,y))$$

$$\phi(0) \leq \phi(0) - \psi(0)$$

$$0 \leq 0.$$

**<u>Case 4</u>:** Let $x = 4$ and $y = 4$. Then we have;

1. $d(Tx,Ty) = d(T(4),T(4)) = d(2,2) = 0.$
2. $M(x,y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, \frac{d(gy,Tx)[1+d(gx,Ty)]}{1+d(gx,gy)}, d(gx,gy)\right\}$

$$= \max\left\{\frac{d(g(4),T(4))[1+d(g(4),T(4))]}{1+d(g(4),g(4))}, \frac{d(g(4),T(4))[1+d(g(4),T(4))]}{1+d(g(4),g(4))}, d(g(4),g(4))\right\}$$

$$= \max\left\{\frac{d(3,2)[1+d(3,2)]}{1+d(3,3)}, \frac{d(3,2)[1+d(3,2)]}{1+d(3,3)}, d(3,3)\right\}$$

$$= \max\{(1)(2),(1)(2),0\}$$

$$= \max\{2,2,0\} = 2.$$

3. $N(x,y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, d(gx,gy)\right\}$

$$= \max\left\{\frac{d(g(4),T(4))[1+d(g(4),T(4))]}{1+d(g(4),g(4))}, d(g(4),g(4))\right\}$$

$$= \max\left\{\frac{d(3,2)[1+d(3,2)]}{1+d(3,3)}, d(3,3)\right\}$$

$$= \max\{(1)(2),0\}$$

$$= \max\{2, 0\} = 2.$$

Thus from inequality (4.1), we have;

$$\phi(d(Tx, Ty) \leq \phi\big(M(x, y)\big) - \psi(N(x, y))$$

$$\phi(0) \leq \phi(2) - \psi(2)$$

$$0 \leq 2 - \tfrac{1}{2}(2)$$

$$0 \leq 1.$$

**<u>Case 5</u>:** Let $x = 5$ and $y = 5$. Then we have;

1. $d(Tx, Ty) = d(T(5), T(5)) = d(2,2) = 0.$

2. $M(x, y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, \frac{d(gy,Tx)[1+d(gx,Ty)]}{1+d(gx,gy)}, d(gx, gy)\right\}$

$$= \max\left\{\frac{d\big(g(5),T(5)\big)\big[1+d\big(g(5),T(5)\big)\big]}{1+d\big(g(5),g(5)\big)}, \frac{d\big(g(5),T(5)\big)\big[1+d\big(g(5),T(5)\big)\big]}{1+d\big(g(5),g(5)\big)}, d(g(5), g(5))\right\}$$

$$= \max\left\{\frac{d(4,2)[1+d(4,2)]}{1+d(4,4)}, \frac{d(4,2)[1+d(4,2)]}{1+d(4,4)}, d(4,4)\right\}$$

$$= \max\{(2)(3), (2)(3), 0\}$$

$$= \max\{6, 6, 0\} = 6.$$

3. $N(x, y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, d(gx, gy)\right\}$

$$= \max\left\{\frac{d\big(g(5),T(5)\big)\big[1+d\big(g(5),T(5)\big)\big]}{1+d\big(g(5),g(5)\big)}, d(g(5), g(5))\right\}$$

$$= \max\left\{\frac{d(4,2)[1+d(4,2)]}{1+d(4,4)}, d(4,4)\right\}$$

$= \max\{(2)(3),\ 0\}$

$= \max\{6,\ 0\} = 6.$

Thus from inequality (4.1), we have;

$$\phi(d(Tx, Ty) \leq \phi\big(M(x, y)\big) - \psi(N(x, y))$$

$$\phi(0) \leq \phi(6) - \psi(6)$$

$$0 \leq 6 - \tfrac{1}{2}(6).$$

$$0 \leq 3.$$

**<u>Case 6</u>:** Let $x = 2$ and $y = 3$. Then we have;

1. $d(Tx, Ty) = d\big(T(2), T(3)\big) = d(2{,}2) = 0.$

2. $M(x, y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, \frac{d(gy,Tx)[1+d(gx,Ty)]}{1+d(gx,gy)}, d(gx, gy)\right\}$

$$= \max\left\{\frac{d\big(g(3),T(3)\big)\big[1+d\big(g(2),T(2)\big)\big]}{1+d\big(g(2),g(3)\big)}, \frac{d\big(g(3),T(2)\big)\big[1+d\big(g(2),T(3)\big)\big]}{1+d\big(g(2),g(3)\big)}, d(g(2), g(3))\right\}$$

$$= \max\left\{\frac{d(2{,}2)[1+d(2{,}2)]}{1+d(2{,}2)}, \frac{d(2{,}2)[1+d(2{,}2)]}{1+d(2{,}2)}, d(2{,}2)\right\}$$

$$= \max\{0,\ 0,\ 0\} = 0.$$

3. $N(x, y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, d(gx, gy)\right\}$

$$= \max\left\{\frac{d\big(g(3),T(3)\big)\big[1+d\big(g(2),T(2)\big)\big]}{1+d\big(g(2),g(3)\big)}, d(g(2), g(3))\right\}$$

$$= \max\left\{\frac{d(2{,}2)[1+d(2{,}2)]}{1+d(2{,}2)}, d(2{,}2)\right\}$$

$$= \max\{0, 0\} = 0.$$

Thus from inequality (4.1), we have;

$$\phi(d(Tx,Ty) \leq \phi\big(M(x,y)\big) - \psi(N(x,y))$$

$$\phi(0) \leq \phi(0) - \psi(0)$$

$$0 \leq 0.$$

**<u>Case 7</u>:** Let $x = 3$ and $y = 4$. Then we have;

1. $d(Tx,Ty) = d\big(T(3),T(4)\big) = d(2,2) = 0.$

2. $M(x,y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, \frac{d(gy,Tx)[1+d(gx,Ty)]}{1+d(gx,gy)}, d(gx,gy)\right\}$

$$= \max\left\{\frac{d\big(g(4),T(4)\big)\big[1+d\big(g(3),T(3)\big)\big]}{1+d\big(g(3),g(4)\big)}, \frac{d\big(g(4),T(3)\big)\big[1+d\big(g(3),T(4)\big)\big]}{1+d\big(g(3),g(4)\big)}, d(g(3),g(4))\right\}$$

$$= \max\left\{\frac{d(3,2)[1+d(2,2)]}{1+d(2,3)}, \frac{d(3,2)[1+d(2,2)]}{1+d(2,3)}, d(2,3)\right\}$$

$$= \max\left\{\frac{1}{2}, \frac{1}{2}, 1\right\} = 1.$$

3. $N(x,y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, d(gx,gy)\right\}$

$$= \max\left\{\frac{d\big(g(4),T(4)\big)\big[1+d\big(g(3),T(3)\big)\big]}{1+d\big(g(3),g(4)\big)}, d(g(3),g(4))\right\}$$

$$= \max\left\{\frac{d(3,2)[1+d(2,2)]}{1+d(2,3)}, d(2,3)\right\}$$

$$= \max\left\{\frac{1}{2}, 1\right\} = 1.$$

Thus from inequality (4.1), we have;

$$\phi(d(Tx,Ty) \leq \phi\big(M(x,y)\big) - \psi\big(N(x,y)\big).$$

$$\phi(0) \leq \phi(1) - \psi(1)$$

$$0 \leq 1 - \frac{1}{2}$$

$$0 \leq \frac{1}{2}.$$

**<u>Case 8</u>:** Let $x = 2$ and $y = 4$. Then we have;

1. $d(Tx, Ty) = d\big(T(3), T(4)\big) = d(2,2) = 0.$

2. $M(x, y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, \frac{d(gy,Tx)[1+d(gx,Ty)]}{1+d(gx,gy)}, d(gx, gy)\right\}$

$$= \max\left\{\frac{d\big(g(4),T(4)\big)\big[1+d\big(g(2),T(2)\big)\big]}{1+d\big(g(2),g(4)\big)}, \frac{d\big(g(4),T(2)\big)\big[1+d\big(g(2),T(4)\big)\big]}{1+d\big(g(2),g(4)\big)}, d(g(2), g(4))\right\}$$

$$= \max\left\{\frac{d(3,2)[1+d(2,2)]}{1+d(2,3)}, \frac{d(3,2)[1+d(2,2)]}{1+d(2,3)}, d(2,3)\right\}$$

$$= \max\left\{\frac{1}{2}, \frac{1}{2}, 1\right\} = 1.$$

3. $N(x, y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, d(gx, gy)\right\}$

$$= \max\left\{\frac{d\big(g(4), T(4)\big)\big[1 + d\big(g(2), T(2)\big)\big]}{1 + d\big(g(2), g(4)\big)}, d(g(2), g(4))\right\}$$

$$= \max\left\{\frac{d(3,2)[1+d(2,2)]}{1+d(2,3)}, d(2,3)\right\}$$

$$= \max\left\{\frac{1}{2}, 1\right\} = 1.$$

Thus from inequality (4.1), we have;

$$\phi(d(Tx, Ty) \leq \phi\big(M(x, y)\big) - \psi\big(N(x, y)\big).$$

$$\phi(0) \leq \phi(1) - \psi(1)$$

$$0 \leq 1 - \frac{1}{2}$$

$$0 \leq \frac{1}{2}.$$

Therefore, from all the cases considered above the pair $(T, g)$, satisfies all hypothesis of Theorem (4.1) for $\phi$ and $\psi$ chosen in example (4.6) and moreover, 1 and 2 are common fixed points of $T$ and $g$.

The following is also an example in support of Theorem 4.1.

**Example 4.7** Let $X = \{2, 3, 4, 5, 6\}$, with $d(x, y) = |x - y|$ and define

$\preccurlyeq := \{(2,2), (2,3), (3,3), (4,4), (5,5), (6,6), (4,5), (4,6), (5,6)\}$ , then $(X, \preccurlyeq)$ is a partially ordered set.

Define $T, g: X \to X$ by $T = \begin{pmatrix} 2 & 3 & 4 & 5 & 6 \\ 2 & 2 & 3 & 3 & 3 \end{pmatrix}$ and $g = \begin{pmatrix} 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 4 & 4 \end{pmatrix}$.

Then $TX = \{2, 3\}$ and $gX = \{2, 3, 4\}$ and hence $TX = \{2, 3\} \subset \{2, 3, 4\} = gX$ and $T$ and $g$ are continuous.

Next we show that $T$ is $g$-non-decreasing.

$$2 = g(2) \preccurlyeq g(2) = 2 \Rightarrow 2 = T(2) \preccurlyeq T(2) = 2$$

$$3 = g(3) \preccurlyeq g(3) = 3 \Rightarrow 2 = T(3) \preccurlyeq T(3) = 2$$

$$2 = g(2) \preccurlyeq g(3) = 3 \Rightarrow 2 = T(2) \preccurlyeq T(3) = 2$$

$$4 = g(4) \preccurlyeq g(4) = 4 \Rightarrow 3 = T(4) \preccurlyeq T(4) = 3$$

$$4 = g(5) \preccurlyeq g(5) = 4 \Rightarrow 3 = T(5) \preccurlyeq T(5) = 3$$

$$4 = g(6) \preccurlyeq g(6) = 4 \Rightarrow 3 = T(6) \preccurlyeq T(6) = 3$$

$$4 = g(4) \preccurlyeq g(5) = 4 \Rightarrow 3 = T(4) \preccurlyeq T(5) = 3$$

$$4 = g(5) \preccurlyeq g(4) = 4 \Rightarrow 3 = T(5) \preccurlyeq T(4) = 3$$

$$4 = g(4) \preccurlyeq g(6) = 4 \Rightarrow \ 3 = T(4) \preccurlyeq T(6) = 3$$

$$4 = g(6) \preccurlyeq g(4) = 4 \Rightarrow \ 3 = T(6) \preccurlyeq T(4) = 3$$

$$4 = g(5) \preccurlyeq g(6) = 4 \Rightarrow \ 3 = T(5) \preccurlyeq T(6) = 3$$

$$4 = g(6) \preccurlyeq g(5) = 4 \Rightarrow \ 3 = T(6) \preccurlyeq T(5) = 3.$$

From the above steps $T$ is $g$-non-decreasing and we also observe that $x_0 = 2 \in X$ such that $gx_0 \preccurlyeq Tx_0$.

Define $\phi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\phi(t) = \frac{1}{4}t$ and $\psi(t) = \frac{1}{8}t$ and we also consider the following cases to verify the inequality (4.1).

**<u>Case 1</u>:** Let $x = 2$ and $y = 2$. Then we have;

1. $d(Tx, Ty) = d\big(T(2), T(2)\big) = d(2,2) = 0.$

2. $M(x, y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, \frac{d(gy,Tx)[1+d(gx,Ty)]}{1+d(gx,gy)}, d(gx, gy)\right\}$

$$= \max\left\{\frac{d\big(g(2),T(2)\big)[1+d\big(g(2),T(2)\big)]}{1+d\big(g(2),g(2)\big)}, \frac{d\big(g(2),T(2)\big)[1+d\big(g(2),T(2)\big)]}{1+d\big(g(2),g(2)\big)}, d\big(g(2), g(2)\big)\right\}$$

$$= \max\left\{\frac{d(2,2)[1 + d(2,2)]}{1 + d(2,2)}, \frac{d(2,2)[1 + d(2,2)]}{1 + d(2,2)}, d(2,2)\right\}$$

$$= \max\{0, 0, 0\} = 0.$$

3. $N(x, y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, d(gx, gy)\right\}$

$$= \max\left\{\frac{d\big(g(2),T(2)\big)[1+d\big(g(2),T(2)\big)]}{1+d\big(g(2),g(2)\big)}, d(g(2), g(2))\right\}$$

$$= \max\left\{\frac{d(2,2)[1+d(2,2)]}{1+d(2,2)}, d(2,2)\right\}$$

$$= \max\{0, 0\} = 0.$$

Thus from inequality (4.1), we have;

$$\phi(d(Tx,Ty) \leq \phi\big(M(x,y)\big) - \psi\big(N(x,y)\big).$$

$$\phi(0) \leq \tfrac{1}{4}(0) - \tfrac{1}{8}(0)$$

$$0 \leq 0.$$

**<u>Case 2</u>:** Let $x = 3$ and $y = 3$. Then we have;

1. $d(Tx,Ty) = d\big(T(3),T(3)\big) = d(2,2) = 0.$
2. $M(x,y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, \frac{d(gy,Tx)[1+d(gx,Ty)]}{1+d(gx,gy)}, d(gx,gy)\right\}$

$$= \max\left\{\frac{d\big(g(3),T(3)\big)[1+d\big(g(3),T(3)\big)]}{1+d\big(g(3),g(3)\big)}, \frac{d\big(g(3),T(3)\big)[1+d\big(g(3),T(3)\big)]}{1+d\big(g(3),g(3)\big)}, d(g(3),g(3))\right\}$$

$$= \max\left\{\frac{d(3,2)[1+d(3,2)]}{1+d(3,3)}, \frac{d(3,2)[1+d(3,2)]}{1+d(3,3)}, d(3,3)\right\}$$

$$= \max\{(1)(2),(1)(2),0\}$$

$$= \max\{2,2,0\} = 2.$$

3. $N(x,y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, d(gx,gy)\right\}$

$$= \max\left\{\frac{d\big(g(3),T(3)\big)[1+d\big(g(3),T(3)\big)]}{1+d\big(g(3),g(3)\big)}, d(g(3),g(3))\right\}$$

$$= \max\left\{\frac{d(3,2)[1+d(3,2)]}{1+d(3,3)}, d(3,3)\right\}$$

$$= \max\{(1)(2),(1)(2),0\}$$

$$= \max\{2,0\} = 2.$$

Thus from inequality (4.1), we have;

$$\phi(d(Tx,Ty) \leq \phi\big(M(x,y)\big) - \psi\big(N(x,y)\big).$$

$$\phi(0) \leq \frac{1}{4}(2) - \frac{1}{8}(2)$$

$$0 \leq \frac{1}{2} - \frac{1}{4}$$

$$0 \leq \frac{1}{4}.$$

**<u>Case 3</u>:** Let $x = 4$ and $y = 4$. Then we have;

1. $(Tx, Ty) = d\big(T(4), T(4)\big) = d(3,3) = 0.$
2. $M(x, y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, \frac{d(gy,Tx)[1+d(gx,Ty)]}{1+d(gx,gy)}, d(gx, gy)\right\}$

$$= \max\left\{\frac{d\big(g(4),T(4)\big)\big[1+d\big(g(4),T(4)\big)\big]}{1+d\big(g(4),g(4)\big)}, \frac{d\big(g(4),T(4)\big)\big[1+d\big(g(4),T(4)\big)\big]}{1+d\big(g(4),g(4)\big)}, d(g(4), g(4))\right\}$$

$$= \max\left\{\frac{d(4,3)[1+d(4,3)]}{1+d(4,4)}, \frac{d(4,3)[1+d(4,3)]}{1+d(4,4)}, d(4,4)\right\}$$

$$= \max\{(1)(2), (1)(2), 0\}$$

$$= \max\{2, 2, 0\} = 2.$$

3. $N(x, y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, d(gx, gy)\right\}$

$$= \max\left\{\frac{d\big(g(4),T(4)\big)\big[1+d\big(g(4),T(4)\big)\big]}{1+d\big(g(4),g(4)\big)}, d(g(4), g(4))\right\}$$

$$= \max\left\{\frac{d(4,3)[1+d(4,3)]}{1+d(4,4)}, d(4,4)\right\}$$

$$= \max\{(1)(2), 0\}$$

$$= \max\{2, 0\} = 2.$$

Thus from inequality (4.1), we have;

$$\phi(d(Tx, Ty) \leq \phi\big(M(x, y)\big) - \psi\big(N(x, y)\big).$$

$$\phi(0) \leq \frac{1}{4}(2) - \frac{1}{8}(2)$$

$$0 \leq \frac{1}{2} - \frac{1}{4}$$

$$0 \leq \frac{1}{4}.$$

**<u>Case 4</u>:** Let $x = 5$ and $y = 5$. Then we have;

1. $(Tx, Ty) = d\big(T(5), T(5)\big) = d(3,3) = 0.$

2. $M(x, y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, \frac{d(gy,Tx)[1+d(gx,Ty)]}{1+d(gx,gy)}, d(gx, gy)\right\}$

$$= \max\left\{\frac{d\big(g(5),T(5)\big)\big[1+d\big(g(5),T(5)\big)\big]}{1+d\big(g(5),g(5)\big)}, \frac{d\big(g(5),T(5)\big)\big[1+d\big(g(5),T(5)\big)\big]}{1+d\big(g(5),g(5)\big)}, d(g(5), g(5))\right\}$$

$$= \max\left\{\frac{d(4,3)[1+d(4,3)]}{1+d(4,4)}, \frac{d(4,3)[1+d(4,3)]}{1+d(4,4)}, d(4,4)\right\}$$

$$= \max\{(1)(2), (1)(2), 0\}$$

$$= \max\{2, 2, 0\} = 2.$$

3. $N(x, y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, d(gx, gy)\right\}$

$$= \max\left\{\frac{d\big(g(5), T(5)\big)\big[1 + d\big(g(5), T(5)\big)\big]}{1 + d\big(g(5), g(5)\big)}, d(g(5), g(5))\right\}$$

$$= \max\left\{\frac{d(4,3)[1+d(4,3)]}{1+d(4,4)}, d(4,4)\right\}$$

$$= \max\{(1)(2), 0\}$$

$$= \max\{2, 0\} = 2.$$

Thus from inequality (4.1), we have;

$$\phi(d(Tx, Ty) \leq \phi\big(M(x, y)\big) - \psi\big(N(x, y)\big).$$

$$\phi(0) \leq \frac{1}{4}(2) - \frac{1}{8}(2)$$

$$0 \leq \frac{1}{2} - \frac{1}{4}$$

$$0 \leq \frac{1}{4}.$$

**<u>Case 5</u>:** Let $x = 6$ and $y = 6$. Then we have;

1. $(Tx, Ty) = d\big(T(6), T(6)\big) = d(3{,}3) = 0.$
2. $M(x, y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, \frac{d(gy,Tx)[1+d(gx,Ty)]}{1+d(gx,gy)}, d(gx, gy)\right\}$

$$= \max\left\{\frac{d\big(g(6),T(6)\big)\big[1+d\big(g(6),T(6)\big)\big]}{1+d\big(g(6),g(6)\big)}, \frac{d\big(g(6),T(6)\big)\big[1+d\big(g(6),T(6)\big)\big]}{1+d\big(g(6),g(6)\big)}, d\big(g(6), g(6)\big)\right\}$$

$$= \max\left\{\frac{d(4{,}3)[1+d(4{,}3)]}{1+d(4{,}4)}, \frac{d(4{,}3)[1+d(4{,}3)]}{1+d(4{,}4)}, d(4{,}4)\right\}$$

$$= \max\{(1)(2), (1)(2), 0\}$$

$$= \max\{2, 2, 0\} = 2.$$

3. $N(x, y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, d(gx, gy)\right\}$

$$= \max\left\{\frac{d\big(g(6),T(6)\big)\big[1+d\big(g(6),T(6)\big)\big]}{1+d\big(g(6),g(6)\big)}, d(g(6), g(6))\right\}$$

$$= \max\left\{\frac{d(4{,}3)[1+d(4{,}3)]}{1+d(4{,}4)}, d(4{,}4)\right\}$$

$$= \max\{(1)(2), 0\}$$

$$= \max\{2, 0\} = 2.$$

Thus from inequality (4.1), we have;

$$\phi(d(Tx, Ty) \leq \phi\big(M(x, y)\big) - \psi\big(N(x, y)\big).$$

$$\phi(0) \leq \frac{1}{4}(2) - \frac{1}{8}(2)$$

$$0 \leq \frac{1}{2} - \frac{1}{4}$$

$$0 \leq \frac{1}{4}.$$

**<u>Case 6</u>:** Let $x = 4$ and $y = 5$. Then we have;

1. $(Tx, Ty) = d\big(T(4), T(5)\big) = d(3,3) = 0.$
2. $M(x,y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, \frac{d(gy,Tx)[1+d(gx,Ty)]}{1+d(gx,gy)}, d(gx,gy)\right\}$

$$= \max\left\{\frac{d\big(g(5),T(5)\big)\big[1+d\big(g(4),T(4)\big)\big]}{1+d\big(g(4),g(5)\big)}, \frac{d\big(g(5),T(4)\big)\big[1+d\big(g(4),T(5)\big)\big]}{1+d\big(g(4),g(5)\big)}, d(g(4),g(5))\right\}$$

$$= \max\left\{\frac{d(4,3)[1+d(4,3)]}{1+d(4,4)}, \frac{d(4,3)[1+d(4,3)]}{1+d(4,4)}, d(4,4)\right\}$$

$$= \max\{(1)(2),(1)(2),0\}$$

$$= \max\{2,2,0\} = 2.$$

3. $N(x,y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, d(gx,gy)\right\}$

$$= \max\left\{\frac{d\big(g(5),T(5)\big)\big[1+d\big(g(4),T(4)\big)\big]}{1+d\big(g(4),g(5)\big)}, d(g(4),g(5))\right\}$$

$$= \max\left\{\frac{d(4,3)[1+d(4,3)]}{1+d(4,4)}, d(4,4)\right\}$$

$$= \max\{(1)(2),0\}$$

$$= \max\{2,0\} = 2.$$

Thus from inequality (4.1), we have;

$$\phi\big(d(Tx,Ty)\big) \leq \phi\big(M(x,y)\big) - \psi\big(N(x,y)\big).$$

$$\phi(0) \leq \frac{1}{4}(2) - \frac{1}{8}(2)$$

$$0 \leq \frac{1}{2} - \frac{1}{4}$$

$$0 \leq \frac{1}{4}.$$

**<u>Case 7</u>:** Let $x = 5$ and $y = 6$. Then we have;

1. $(Tx, Ty) = d\big(T(5), T(6)\big) = d(3,3) = 0.$
2. $M(x,y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, \frac{d(gy,Tx)[1+d(gx,Ty)]}{1+d(gx,gy)}, d(gx,gy)\right\}$

$$= \max\left\{\frac{d\big(g(6),T(6)\big)\big[1+d\big(g(5),T(5)\big)\big]}{1+d\big(g(5),g(6)\big)}, \frac{d\big(g(6),T(5)\big)\big[1+d\big(g(5),T(6)\big)\big]}{1+d\big(g(5),g(6)\big)}, d(g(5),g(6))\right\}$$

$$= \max\left\{\frac{d(4,3)[1+d(4,3)]}{1+d(4,4)}, \frac{d(4,3)[1+d(4,3)]}{1+d(4,4)}, d(4,4)\right\}$$

$$= \max\{(1)(2),(1)(2),0\}$$

$$= \max\{2,2,0\} = 2.$$

3. $N(x,y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, d(gx,gy)\right\}$

$$= \max\left\{\frac{d\big(g(6),T(6)\big)\big[1+d\big(g(5),T(5)\big)\big]}{1+d\big(g(5),g(6)\big)}, d(g(5),g(6))\right\}$$

$$= \max\left\{\frac{d(4,3)[1+d(4,3)]}{1+d(4,4)}, d(4,4)\right\}$$

$$= \max\{(1)(2),0\}$$

$$= \max\{2,0\} = 2.$$

Thus from inequality (4.1), we have;

$$\phi(d(Tx,Ty)) \le \phi\big(M(x,y)\big) - \psi(N(x,y))$$

$$\phi(0) \le \frac{1}{4}(2) - \frac{1}{8}(2)$$

$$0 \le \frac{1}{2} - \frac{1}{4}$$

$$0 \le \frac{1}{4}.$$

**<u>Case 8</u>:** Let $x = 4$ and $y = 6$. Then we have;

1. $(Tx, Ty) = d\big(T(4), T(6)\big) = d(3,3) = 0.$

2. $M(x, y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, \frac{d(gy,Tx)[1+d(gx,Ty)]}{1+d(gx,gy)}, d(gx, gy)\right\}$

$$= \max\left\{\frac{d\big(g(6),T(6)\big)\big[1+d\big(g(4),T(4)\big)\big]}{1+d\big(g(4),g(6)\big)}, \frac{d\big(g(6),T(4)\big)\big[1+d\big(g(4),T(6)\big)\big]}{1+d\big(g(4),g(6)\big)}, d(g(4), g(6))\right\}$$

$$= \max\left\{\frac{d(4,3)[1+d(4,3)]}{1+d(4,4)}, \frac{d(4,3)[1+d(4,3)]}{1+d(4,4)}, d(4,4)\right\}$$

$$= \max\{(1)(2), (1)(2), 0\}$$

$$= \max\{2, 2, 0\} = 2.$$

3. $N(x, y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, d(gx, gy)\right\}$

$$= \max\left\{\frac{d\big(g(6),T(6)\big)\big[1+d\big(g(4),T(4)\big)\big]}{1+d\big(g(4),g(6)\big)}, d(g(4), g(6))\right\}$$

$$= \max\left\{\frac{d(4,3)[1+d(4,3)]}{1+d(4,4)}, d(4,4)\right\}$$

$$= \max\{(1)(2), 0\}$$

$$= \max\{2, 0\} = 2.$$

Thus from inequality (4.1), we have;

$$\phi(d(Tx, Ty)) \leq \phi\big(M(x, y)\big) - \psi(N(x, y))$$

$$\phi(0) \leq \frac{1}{4}(2) - \frac{1}{8}(2)$$

$$0 \leq \frac{1}{2} - \frac{1}{4}$$

$$0 \leq \frac{1}{4}.$$

**Case 9:** Let x = 2 and y = 3. Then we have;

1. $d(Tx,Ty) = d\big(T(2),T(3)\big) = d(2,2) = 0.$
2. $M(x,y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, \frac{d(gy,Tx)[1+d(gx,Ty)]}{1+d(gx,gy)}, d(gx,gy)\right\}$

$$= \max\left\{\frac{d\big(g(3),T(3)\big)\big[1+d\big(g(2),T(2)\big)\big]}{1+d\big(g(2),g(3)\big)}, \frac{d\big(g(3),T(2)\big)\big[1+d\big(g(2),T(3)\big)\big]}{1+d\big(g(2),g(3)\big)}, d(g(2),g(3))\right\}$$

$$= \max\left\{\frac{d(3,2)[1+d(2,2)]}{1+d(2,3)}, \frac{d(3,2)[1+d(2,2)]}{1+d(2,3)}, d(2,3)\right\}$$

$$= \max\left\{\frac{1}{2}, \frac{1}{2}, 1\right\} = 1.$$

3. $N(x,y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, d(gx,gy)\right\}$

$$= \max\left\{\frac{d\big(g(3),T(3)\big)\big[1+d\big(g(2),T(2)\big)\big]}{1+d\big(g(2),g(3)\big)}, d(g(2),g(3))\right\}$$

$$= \max\left\{\frac{d(3,2)[1+d(2,2)]}{1+d(2,3)}, d(2,3)\right\}$$

$$= \max\left\{\frac{1}{2}, 1\right\} = 1.$$

Thus from inequality (4.1), we have;

$$\phi(d(Tx,Ty) \leq \phi\big(M(x,y)\big) - \psi\big(N(x,y)\big).$$

$$\phi(0) \leq \phi(1) - \psi(1)$$

$$0 \leq \frac{1}{4} - \frac{1}{8}$$

$$0 \leq \frac{1}{8}.$$

Therefore from all the cases considered above the pair $(T,g)$ satisfies all the hypothesis of Theorem (4.1) for $\phi$ and $\psi$ chosen in example (4.7). Moreover $x_0 = 2$ is a common fixed point of $T$and $g$.

In the following, we prove common fixed point results by relaxing the continuity assumption in Theorem 4.1

**Theorem 4.2:** Let $(X, d, \preccurlyeq)$ be a partially ordered complete metric space. Suppose $T, g: X \to X$ be two maps and for every two comparable $gx$ and $gy$. Suppose that there exists $\phi \in \Phi$ and $\psi \in \Psi$ such that;

$$\phi(d(Tx, Ty)) \leq \phi(M(x, y) - \psi(N(x, y)) \text{ , where} \tag{4.31}$$

$$M(x, y) = \max\left\{\frac{d(gy, Ty)[1 + d(gx, Tx)]}{1 + d(gx, gy)}, \frac{d(gy, Tx)[1 + d(gx, Ty)]}{1 + d(gx, gy)}, d(gx, gy)\right\}$$

$N(x, y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, d(gx, gy)\right\}$ and satisfying the following conditions:

(i) $TX \subseteq gX$

(ii) $gX$ is closed

(iii) $T$ is $g$-non-decreasing mapping

(iv) if $\{gx_n\} \subseteq X$ is a non-decreasing sequence with $gx_n \to gz$ in $gX$, then $gx_n \preccurlyeq gz$ for all $n$.

If there exists an $x_0 \in X$ with $gx_0 \preccurlyeq Tx_0$, then $T$ and $g$ have a coincidence point. Moreover if $gz \preccurlyeq ggz$ and $T$ and $g$ are weakly compatible then $T$ and $g$ have a common fixed point in $X$.

**Proof:** Suppose that (ii) holds. Since $Tx_n = gx_{n+1} \subseteq gX$ and $gX$ is closed, there exists $z \in X$, for which $x = gz$ and

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} Tx_n = gz. \tag{4.32}$$

Now we claim that z is a coincidence point of $T$and $g$.

From condition (iv), we have $\{gx_n\} \subseteq X$ is a non-decreasing sequence with $gx_n \to gz$ in $gX$, then $gx_n \preccurlyeq gz$ for all $n$. (4.33)

Taking $x = x_n$ and $y = z$ in (4.31) and by (4.32), we get,

$$\phi\big(d(Tx_n, Tz)\big) \leq \phi(M(gx_n, gz)) - \psi(N(gx_n, gz))$$

$$\phi\big(d(Tx_n, Tz)\big) \leq \phi(\max\left\{\frac{d(gz, Tz)[1 + d(gx_n, Tx_n)]}{1 + d(gx_n, gz)}, \frac{d(gz, Tx_n)[1 + d(gx_n, Tz)]}{1 + d(gx_n, gz)}, d(gx_n, gz)\right\}$$

$$-\psi(\max\left\{\frac{d(gz,Tz)[1+d(gx_n,Tx_n)]}{1+d(gx_n,gz)}, d(gx_n, gz)\right\}.$$

Taking $n \to \infty$ in the above inequality, we get,

$$\phi\big(d(gz,Tz)\big) \leq \phi\big(d(gz,Tz)\big) - \psi\big(d(gz,Tz)\big).$$

Which is possible only when $d(gz,Tz) = 0$ and which in turn,

$Tz = gz$, that is z is a coincidence point of $T$ and $g$. (4.34)

From the condition that $gz \preccurlyeq ggz$ and by (4.32), that is a non-decreasing sequence $\{gx_n\}$ converging to $gz$. Since $T$ and $g$ are weakly compatible, by (4.34), we have that $Tgz = gTz$. We set,

$$w = gz = Tz. \tag{4.35}$$

Therefore we have $gz \preccurlyeq ggz = gw$. (4.36)

Also $Tw = Tgz = gTz = gw$. (4.37)

If $z = w$, then z is a common fixed point of $T$ and $g$. If $z \neq w$, then by (4.31), we have;

$\phi\big(d(gz, gw)\big) = \phi\big(d(Tz, Tw)\big)$.

$\leq \phi\big(d(gz, gw)\big) - \psi\big(d(gz, gw)\big)$, which is possible only when $d(gz, gw) = 0$, implies $gz = gw$. Then by (4.35) and (4.37), we have;

$$w = gw = Tw.$$

Therefore $w$ is a common fixed point of $T$ and $g$.

**Remark 2:** By choosing $g =$ Ix =Identity map on $X$ in Theorem 4.2 we get Theorem 1.7. Hence Theorem 1.7 follows as a corollary to Theorem 4.2.

The following is an example in support of Theorem 4.2.

Example 4.8: Let $\mathrm{X} = [0,2]$ with the usual metric and we define the partial order as follows;

$\preccurlyeq := \{(x,y): x,y \in [0,1], x = y\} \cup \{(x,y): x,y \in (1,2], x \leq y\}$. Where $\leq$ is the usual less than or equal sign, then $(X, \preccurlyeq)$ is a partially ordered set.

We define $T, g: X \to X$ by $TX = \begin{cases} \frac{x}{3}, if\ 0 \leq x < 1. \\ \frac{1}{2},\ \ if\ \ x = 1. \\ \frac{2}{3}, if\ 1 < x \leq 2. \end{cases}$ and $gX = \begin{cases} \frac{x}{2}, if\ 0 \leq x < 1. \\ \frac{1}{2},\ \ if\ \ x = 1. \\ \frac{2}{3}, if\ 1 < x \leq 2. \end{cases}$

Also we define $\phi, \psi: \mathbb{R}^+ \to \mathbb{R}^+$, by $\phi(t) = t^2$ and $\psi(t) = \frac{1}{2}t^2$, then

$TX = \left[0, \frac{1}{3}\right) \cup \left\{\frac{1}{2}, \frac{2}{3}\right\} \subset \left[0, \frac{1}{2}\right] \cup \left\{\frac{2}{3}\right\} = gX$ and $gX$ closed there exists $x_0 = 0 \in X$, such that, $gx_0 \preccurlyeq Tx_0$ and clearly $T$ is a $g$-non-decreasing.

Now we verify the inequality (4.31).

**<u>Case 1:</u>** Let $x = y$ in [0,1] so, $Tx = \frac{x}{3} = Ty$. Then we have;

1. $d(Tx, Ty) = d\left(\frac{x}{3}, \frac{x}{3}\right) = \left|\frac{x}{3} - \frac{x}{3}\right| = 0$, since $x = y$.
2. $M(x,y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, \frac{d(gy,Tx)[1+d(gx,Ty)]}{1+d(gx,gy)}, d(gx,gy)\right\}$

$$= \max\left\{\frac{d\left(\frac{y}{2},\frac{y}{3}\right)\left[1+d\left(\frac{x}{2},\frac{x}{3}\right)\right]}{1+d\left(\frac{x}{2},\frac{y}{2}\right)}, \frac{d\left(\frac{y}{2},\frac{x}{3}\right)\left[1+d\left(\frac{x}{2},\frac{y}{3}\right)\right]}{1+d\left(\frac{x}{2},\frac{y}{2}\right)}, d\left(\frac{x}{2},\frac{y}{2}\right)\right\}$$

$$= \max\left\{\frac{\left|\frac{x}{2}-\frac{x}{3}\right|\left[1+\left|\frac{x}{2}-\frac{x}{3}\right|\right]}{1+\left|\frac{x}{2}-\frac{x}{2}\right|}, \frac{\left|\frac{x}{2}-\frac{x}{3}\right|\left[1+\left|\frac{x}{2}-\frac{x}{3}\right|\right]}{1+\left|\frac{x}{2}-\frac{x}{2}\right|}, \left|\frac{x}{2}-\frac{x}{2}\right|\right\}, \text{ since } x = y.$$

$$= \max\left\{\frac{x}{6}\left(1+\frac{x}{6}\right), \ \frac{x}{6}\left(1+\frac{x}{6}\right), \ 0\right\}$$

$$= \frac{x}{6}\left(1+\frac{x}{6}\right).$$

3. $N(x,y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, d(gx,gy)\right\}$

$$= \max\left\{\frac{d\left(\frac{y}{2},\frac{y}{3}\right)\left[1+d\left(\frac{x}{2},\frac{x}{3}\right)\right]}{1+d\left(\frac{x}{2},\frac{y}{2}\right)}, d\left(\frac{x}{2},\frac{y}{2}\right)\right\}$$

$$= \max\left\{\frac{\left|\frac{x}{2}-\frac{x}{3}\right|\left[1+\left|\frac{x}{2}-\frac{x}{3}\right|\right]}{1+\left|\frac{x}{2}-\frac{x}{2}\right|}, \left|\frac{x}{2}-\frac{x}{2}\right|\right\}, \text{ since } x = y.$$

$$= \max\left\{\frac{x}{6}\left(1+\frac{x}{6}\right),\ 0\right\}$$

$$= \frac{x}{6}\left(1+\frac{x}{6}\right).$$

Thus from inequality (4.31), we have;

$$\phi\big(d(Tx,Ty)\big) \le \phi\big(M(x,y)\big) - \psi\big(N(x,y)\big).$$

$$\phi(0) \le \frac{x^2}{36}\left(1+\frac{2x}{6}+\frac{x^2}{36}\right) - \frac{\frac{x^2}{36}\left(1+\frac{2x}{6}+\frac{x^2}{36}\right)}{2}$$

$0 \le \frac{\frac{x^2}{36}\left(1+\frac{2x}{6}+\frac{x^2}{36}\right)}{2}$, which is true.

**Case 2:** Let $x = 1, y = 1$so, $Tx = Ty = \frac{1}{2}$. Then we have;

1. $d(Tx,Ty) = d(T(1),T(1)) = d\left(\frac{1}{2},\frac{1}{2}\right) = \left|\frac{1}{2}-\frac{1}{2}\right| = 0.$`
2. $M(x,y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, \frac{d(gy,Tx)[1+d(gx,Ty)]}{1+d(gx,gy)}, d(gx,gy)\right\}$

$$= \max\left\{\frac{d\big(g(1),T(1)\big)\big[1+d\big(g(1),T(1)\big)\big]}{1+d\big(g(1),g(1)\big)}, \frac{d\big(g(1),T(1)\big)\big[1+d\big(g(1),T(1)\big)\big]}{1+d\big(g(1),g(1)\big)}, d\big(g(1),g(1)\big)\right\}$$

$$= \max x\left\{\frac{d\left(\frac{1}{2},\ \frac{1}{2}\right)\left[1+d\left(\frac{1}{2},\ \frac{1}{2}\right)\right]}{1+d\left(\frac{1}{2},\ \frac{1}{2}\right)}, \frac{d\left(\frac{1}{2},\ \frac{1}{2}\right)\left[1+d\left(\frac{1}{2},\ \frac{1}{2}\right)\right]}{1+d\left(\frac{1}{2},\ \frac{1}{2}\right)}, d\left(\frac{1}{2},\frac{1}{2}\right)\right\}$$

$$= \max\{0,0,0\} = 0.$$

3. $N(x,y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, d(gx,gy)\right\}$

$$= \max\left\{\frac{d\big(g(1),T(1)\big)\big[1+d\big(g(1),T(1)\big)\big]}{1+d\big(g(1),g(1)\big)}, d\big(g(1),g(1)\big)\right\}$$

$$= \max\left\{\frac{d\left(\frac{1}{2},\ \frac{1}{2}\right)\left[1+d\left(\frac{1}{2},\ \frac{1}{2}\right)\right]}{1+d\left(\frac{1}{2},\ \frac{1}{2}\right)}, d\left(\frac{1}{2},\frac{1}{2}\right)\right\}$$

$$= \max\{0,0\} = 0.$$

Thus from inequality (4.31), we have;

$$\phi\big(d(Tx,Ty)\big) \le \phi\big(M(x,y)\big) - \psi\big(N(x,y)\big).$$

$$\phi(0) \le \phi(0) - \psi(0)$$

$$0 \le 0.$$

**Case 3:** Let $x, y \in (1,2]$, then $Tx = \frac{2}{3} = Ty$.

1. $d(Tx, Ty) = d(\frac{2}{3}, \frac{2}{3}) = 0.$
2. $M(x, y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, \frac{d(gy,Tx)[1+d(gx,Ty)]}{1+d(gx,gy)}, d(gx, gy)\right\}$

$$= \max\left\{\frac{d(\frac{2}{3},\frac{2}{3})\left[1+d(\frac{2}{3},\frac{2}{3})\right]}{1+d(\frac{2}{3},\frac{2}{3})}, \frac{d(\frac{2}{3},\frac{2}{3})\left[1+d(\frac{2}{3},\frac{2}{3})\right]}{1+d(\frac{2}{3},\frac{2}{3})}, d(\frac{2}{3}, \frac{2}{3})\right\}$$

$$= \max\{0, 0, 0\} = 0.$$

3. $N(x, y) = \max\left\{\frac{d(gy,Ty)[1+d(gx,Tx)]}{1+d(gx,gy)}, d(gx, gy)\right\}$

$$= \max\left\{\frac{d(\frac{2}{3},\frac{2}{3})\left[1+d(\frac{2}{3},\frac{2}{3})\right]}{1+d(\frac{2}{3},\frac{2}{3})}, d(\frac{2}{3}, \frac{2}{3})\right\}$$

$$= \max\{0, 0\} = 0.$$

Thus from inequality (4.31), we have;

$$\phi\big(d(Tx, Ty)\big) \leq \phi\big(M(x, y)\big) - \psi\big(N(x, y)\big).$$

$$\phi(0) \leq \phi(0) - \phi(0)$$

$$0 \leq 0.$$

Therefore from the cases considered above $T$ and $g$ satisfy all the conditions of Theorem (4.2) for $\phi$ and $\psi$ chosen in example (4.8). Moreover, 0 is a common fixed point of $T$ and $g$.

# CHAPTER FIVE

# 5. CONCLUSION AND FUTURE SCOPE

## 5.1 Conclusion

In this research, we constructed two common fixed point theorems for a pair of continuous maps and proved the existence and uniqueness of common fixed point results for $(\phi, \psi)$-contractions involving rational expressions in partially ordered metric spaces. By replacing the continuity assumption by sequential condition and considering closeness of one of the range space we also proved the existence of common fixed point results for $(\phi, \psi)$-contractions involving rational expressions in partially ordered metric spaces. We provided examples in support of our main results. Our results extend the works of Chandok *et al.*, to a pair of maps.

## 5.2 Future scope

The existence of common fixed point results of $(\phi, \psi)$-contractions involving rational expressions in partially ordered metric space is active area of study. Recently there are a number of published research papers related to this area of study. So the researcher recommends the upcoming post graduate students of the department and other researchers to do their research work in this area of study.

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