COMMON FIXED POINTS FOR GENERALIZED CONTRACTION AND ZAMFIRESCU PAIR OF MAPS IN CONE B-METRIC SPACES

BY

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DECLARATION

I, undersigned declare that this research is original and it has not been submitted to any institution elsewhere for the award of any academic degree or like, where other sources of information have been used, they have been acknowledged.

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Abstract

This research dealt with common fixed points for generalized contraction and Zamfirescu pair of maps in cone b-metric spaces. In 2010, Babu et al. [1] established the existence of unique common fixed points for generalized contraction and zamfirescu pair of maps in complete cone metric spaces. Recently, Haung and Xu [9] have proved some fixed point theorems of contraction mappings without the assumption of normality condition in complete cone b-metric space. Motivated by the work of Babu et al. [1], in this work we have extended the main results of them [1 and 9] to cone b-metric spaces and examples in support of our main findings are also provided.

1. Introduction

1.1 Background of the study

Fixed point theory has fascinated hundreds of researchers since 1922 with the celebrated Banach's fixed point theorem, which is stated as follows.

"Let (X, d) be a complete metric space and $T : (X, d) \to (X, d)$ be a self- map. If there exists a constant $k \in (0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y)$$

for every $x, y \in X$, then T has a unique fixed point $x \in X$ such that $Tx = x$."

A mapping T for which the inequality mentioned above holds is called a **contraction**. Since its first appearance, the Banach contraction mapping principle has become the main tool to study contractions as they appear abundantly in a wide array of quantitative sciences. This theorem provides a technique for solving a variety of applied problems in mathematical sciences and engineering. Its most well-known application is in ordinary differential equations, particularly, in the proof of the Picard-Lindelöf theorem which guarantees the existence and uniqueness of solutions of first-order initial value problems. It is worth emphasizing that remarkable strength of the Banach contraction principle originates from the constructive processes it provides to identify the fixed point. This notable strength further attracted the attention of not only many prominent mathematicians studying in many branches of mathematics related to nonlinear analysis, but also many researchers who are interested in iterative methods to examine the quantitative problems involving certain mappings and space structures required in their work in various areas such as Social sciences, Biology, Economics, and Computer sciences.

There are great numbers of generalizations of the Banach contraction principle. In 2007, Huang and Zhuang [8] re-introduced the concept of a cone metric space where every pair of elements are assigned to an element of a Banach space equipped with a cone which induces a natural partial ordering. They proved some fixed point theorems for contraction maps in such space in the same work. In [2], Bakhtin introduced b-metric spaces as a generalization of metric spaces. He proved the contraction mapping principle in b-metric spaces that generalized the famous

Banach contraction principle in metric spaces. Since then, several papers have dealt with fixed point theory for single-valued and multi-valued operators in b-metric spaces (see [4–6] and the references therein). In recent investigations, the fixed point in non-convex analysis, especially in an ordered normed space, occupies a prominent place in many aspects (see [7–10]). In [10], Hussain and Shah introduced cone b-metric spaces as a generalization of b-metric spaces and cone metric space. They established some topological properties in such spaces. In 2010, Babu, Alemayehu and Prasad [1] established the existence of unique common fixed points for generalized and Zamfirescu pair of maps in complete cone metric spaces. Recently, Haung and Xu [9] have proved some fixed point theorems of contraction mappings without the assumption of normality condition in complete cone b-metric space. In this research the work of Babu, Alemayehu and Prasad [1] has been extended to cone b-metric spaces.

1.2 STATEMENT OF THE PROBLEM

The aim of this research is to prove the existence and uniqueness of common fixed point of generalized contraction and Zamfirescu pair of maps in cone b-metric spaces.

Motivated by the work of Babu, Alemayehu and Prasad [1], in this research work we extended the main results of [1] to cone b-metric spaces and applicable examples were provided.

This research answered the following basic questions:

i. What are cones, cone metric space, b-metric and cone b-metric space and how do we relate them?
ii. What are the topological properties of cone b-metric space?
iii. How do we prove the existence and uniqueness of common fixed point of generalized contraction pair in cone b-metric space?
iv. How do we prove the existence and uniqueness of common fixed point of Zamfirescu pair of maps in cone b-metric spaces?
v. Can we provide examples in support of the main results of this research problem?

1.3 Objective of the study

The main objective of this study was to deal with existence of common fixed points of generalized contraction and Zamfirescu pair of maps in cone b-metric spaces.

The topics of the research problem had the following specific objectives:

i. To discuss about Cones, Cone metric space, b-metric and cone b-metric on real Banach space and show their relationship.

ii. To discuss about the topological properties of con b-metric space.
iii. To prove existence and uniqueness of common fixed point of generalized contraction pair in cone b-metric space.
iv. To prove existence and uniqueness of common fixed point of Zamfirescu pair in cone b-metric space.
v. To provide examples in support of the main results.

1.4 Significance of the study

Fixed point theory is one of the interesting areas of research with a wide range of application in various fields. There are many works about fixed point of contraction maps [6, 7].We hope that the results obtained in this research will contribute to research activities in this area. The researcher also beneficial from this study since it uses to develop scientific research writing skill and scientific communication in mathematics

1.5 Delimitation of the study

This research is delimited to finding common fixed points of generalized contraction and Zamfirescu pair of maps in cone b-metric spaces which has been done under Differential Equation and Functional Analysis Streams.

2. Methodology

2.1 Study site and period

The study uses a generalization of the Banach contraction mapping principles as a base, which was stated in some of the researchers and then to prove different fixed point results in cone b-metric space. This study was conducted from November to June 2013/2014 G.C in Jimma University under Mathematics Department.

2.2 Study Design

In order to achieve the objective of this research numerical and analytical design method has been used.

2.3 Sources of information

This research mostly depends on document materials so, the available sources of information for the study are Books, Journals, different study related to the topic and internet services. The researcher collected different documents that are listed above which supports the research and discuss about the collected material and other activities with an advisor.

2.4 Procedure of the study

The procedures we followed for analysis were the standard technique used in Alemayehu and Babu [1] and Huang and Xu [9].

2.5 Instrumentation and Administration

We used secondary data for this study. So, to collect those materials by photo copy, by printing and taking by RW-CD, flash disk, hard disk and for organizing the literature pen and paper were used. The materials were collected till completion of the study to find possible extension of common fixed points in cone b-metric spaces.

2.6. Ethical issues

For this study it needed Books, Journals and other related materials. But there was a problem for collecting all the above listed materials without any permitted letters. So, the researcher took a letter of permission from Mathematics Department, Jimma University and then the researcher explained the aim of collecting those materials.

3. Discussion and results

3.1 Preliminaries

3.1.1. Cone Metric spaces and Cone b-metric spaces

Definition 3.1.2 [9]: Let E be a real Banach space and let P be a subset of E. By θ we denote the zero elements of E and by $\operatorname{int} P$ the interior of P. The subset P is called a ***cone*** if and only if

i) P is closed, non-empty, and $P \neq \{\theta\}$,

ii) $a, b \in \mathbb{R}$ $a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$, and

iii) $P \cap (-P) = \{\theta\}$

Examples 3.1.3

1) The set $E = \mathbb{R}$ of real numbers with the usual norm is a real Banach space. In this case the cone P is the set of non-negative real number.

2) The set $\mathrm{E} = \mathbb{R}^2$ of a Euclidian plane with the usual norm is a real Banach space. In this case the cone P is the first quadrant.

On these bases, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $\mathrm{x} < \mathrm{y}$ to indicate that $x \leq y$ but $x \neq y$, while $x << y$ stands for $y - x \in \operatorname{int} P$. Write $\|\cdot\|$ as the norm on E. The cone P is called ***normal*** if there is a number $k > 0$ such that $\forall x, y \in \mathrm{E},\ \theta \leq x \leq y \Rightarrow \|x\| \leq k\|y\|$. The least positive number k satisfying the above is called the normal constant of P. It is well known that $k \geq 1$.

There exists a cone which is not normal

Example 3.1.4 [5] Let $\mathrm{E} = C_R^1[0,1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and

$P = \{x \in \mathrm{E} : x(t) \geq 0 \text{ on } [0,1]\}$. This cone is not normal.

Consider for example $x_n(t) = \dfrac{1 - \sin nt}{n + 2}$ and $y_n(t) = \dfrac{1 + \sin nt}{n + 2}$

Then $0 \le x_n \le y_n + x_n$

$$\Rightarrow \|x_n\| = \|y_n\| = 1 \text{ and } \|x_n + y_n\| = \frac{2}{n+2} \to 0$$

In the following, we always suppose that E is a real Banach space, P is a cone in E with int $p \neq \phi$ and $\le$ is a partial ordering with respect to P.

Definition 3.1.5 [9]: Let X be a non-empty set. Suppose that the mapping $d : XxX \to$ E satisfies:

(d_1) $\theta < d(x,y) \forall x, y \in X$ with $x \neq y$ and $d(x,y) = \theta \Leftrightarrow x = y$.

(d_2) $d(x,y) = d(y,x) \; \forall x, y \in X$

(d_3) $d(x,y) \le d(x,z) + d(z,y) \;\; \forall x, y, z \in X$

Then d is called a ***cone metric*** on X and the pair (X, d) is called ***cone metric space***.

Note that: It is obvious that cone metric space generalizes metric space:

Remark 3.1.6 [1]: Let E be an ordered Banach space with a cone P

(1) c is an interior point of the cone P *iff* $[-c, c]$ is a neighborhood of 0.

(2) If $u \le v$ and $v << w$, then $u << w$

(3) If $u << v$ and $v \le w$, then $u << w$

(4) If $\theta \le u << c$ for each $c \in \operatorname{int} P$ then $u = \theta$

(5) If $u << v$ and $v << w$, then $u << w$

(6) If $u \le \lambda a$ where $a \in P$ and $0 < \lambda < 1$, then $a = \theta$

(7) If $a \le b + c$ for each $c \in \operatorname{int} P$ then $a \le b$

(8) If $c \in \operatorname{int} P$, $0 \le a_n$ and $a_n \to 0, \exists n_0$ such that $\forall n > n_0$ we have $a_n << c$

Properties $(2),(4)$ and (6) of this remark are often used (particularly when dealing with non-normal cones), so we give their proofs.

Proof: (2): We have to prove that $w-u \in \operatorname{int} P$ *if* $v-u \in P$ and $w-v \in \operatorname{int} P$

There exists a neighborhood V of θ in E such that $w-v+V \subset P$. Then, from $v-u \in P$ it follows that

$$w-u+V=(w-v)+V+(v-u) \subset P+P \subset P,$$

Since p is convex.

(4) Since $c-u \in \operatorname{int} P$ for each $c \in \operatorname{int} P$, it follows that $\frac{1}{n}c-u \in \operatorname{int} P$ for each $n \in \mathbb{N}$. Thus,

$$\lim_{n \to \infty}\left(\frac{1}{n}c-u\right)=\theta-u \in \overline{P}=P(\because P \text{ is closed})$$

Hence $u \in (-P) \cap P=\{\theta\}$, i.e. $u=\theta$ (by definition of cone)

(6) The condition $a \leq \lambda a$ means that $\lambda a-a \in P$, i.e., $-(1-\lambda)a \in P$.

Since $a \in P$ and $1-\lambda>0$, we have also $(1-\lambda)a \in P$. Thus we have $(1-\lambda)a \in P \cap(-P)=\{\theta\}$, and $a=\theta$

Definition 3.1.7 [5]: Let X be a non-empty set and $s \geq 1$ be given real number. A mapping $d: XxX \rightarrow \mathbb{R}_{+}$ is called a b-metric if $\forall x, y, z \in X$ the following conditions are satisfied:

(1) $d(x, y)=0 \Leftrightarrow x=y$ and $0 \leq d(x, y)$

(2) $d(x, y)=d(y, x)$

(3) $d(x, y) \leq s[d(x, z)+d(z, y)]$.

A pair (X, d) is called a ***b-metric space*** with constant s.

It is easy to see that any metric space is a b-metric space with $s=1$.

Thus, the class of b-metric space is larger than the class of metric spaces.

The following examples show that a b-metric space is a real generalization of a metric space.

Example 3.1.8 [5]: The set $\mathbb{R}$ of real numbers together with the mapping $d(x,y)=|x-y|^2 \ \forall x,y\in\mathbb{R}$ is a b-metric space with $s=2$. But d is not metric on $\mathbb{R}$.

Proof: We need to show that d satisfies Definition 3.1.7.

i) $d(x,y)=|x-y|^2>0,\ \forall x,y\in\mathbb{R}$ with $x\neq y$

Hence $d(x,y)>0$

$$d(x,y)=0 \Leftrightarrow |x-y|^2=0 \Leftrightarrow |x-y|=0 \Leftrightarrow x-y=0 \Leftrightarrow x=y.$$

Hence (1) is satisfied $\forall x,y\in\mathbb{R}$.

ii) $\forall x,y\in\mathbb{R}$,

$$d(x,y)=|x-y|^2=|-(y-x)|^2=|-1|^2\,|y-x|^2=|y-x|^2=d(y,x)$$

$$\Rightarrow d(x,y)=d(y,x), \forall x,y\in\mathbb{R}.$$

Hence (2) is satisfied.

iii) Let $x,y,z\in\mathbb{R}$, then

$$d(x,y)=|x-y|^2=|x-z+z-y|^2$$

We set $u=x-z, v=z-y$ $so, x-y=u+v$

$$\Rightarrow d(x,y)=|u+v|^2 \le 2\left(|u|^2+|v|^2\right) = 2\left(|x-z|^2+|z-y|^2\right)$$

$$\Rightarrow d(x,y)\le 2[d(x,z)+d(z,y)].$$

Hence (3) is satisfied.

Hence $(\mathbb{R}, d)$ is a b-metric space with $s = 2$. Since, $s > 1$ it is not a metric space.

Example 3.1.9 [5]: Let $X = \{0,1,2\}$ and a mapping $d : X\,x\,X \to \mathbb{R}_+$ be defined by $d(0,0) = d(1,1) = d(2,2,) = 0,\ d(0,1) = d(1,0) = d(1,2) = d(2,1) = 1$ and $d(2,0) = d(0,2) = m$ where m is a given real number such that $m \geq 2$.

Proof: It is enough to show that it satisfies Definition 3.1.7 (3).

Let $x, y, z \in X$ then

$$d(x, y) \leq \frac{m}{2}\left[d(x, z) + d(x, y)\right] \forall x, y, z \in X \,.$$

Therefore, (X, d) is a b-metric space with constant $s = \frac{m}{2}$ .However, if $m > 2,$ the ordinary triangle inequality does not hold and thus (X, d) is not metric space, for if we take $0,1,2 \in X$, then we get

$$d(2,0) = m \geq d(2,1) + d(1,0)$$

$$\Rightarrow m \geq 2 \,.$$

Hence it does not satisfy ordinary triangle inequality.

Definition 3.1.10 [9]: Let X be a non-empty set and let $s \geq 1$ be given real number. A mapping $d : X\,x\,X \to \mathrm{E}$ is said to be ***cone b-metric*** if and only if $\forall x, y, z \in X$ the following conditions are satisfied.

i) $\theta < d(x, y)$ with $x \neq y$ and $d(x, y) = \theta\, iff\ x = y$;

ii) $d(x, y) = d(y, x)$;

iii) $d(x, y) \leq s\left[d(x, z) + d(z, y)\right]$.

The pair (X, d) is called a cone b-metric space.

Remark 3.1.11 [9]: Observe that if $s=1$ then the ordinary triangle inequality in cone metric space is satisfied, however it does not hold true when $s>1$. Thus the class of cone b-metric space is larger than the class of cone metric spaces since any cone metric space must be a cone b-metric spaces with $s=1$.

The following examples show that cone b-metric spaces are more general than cone metric spaces.

Example 3.1.12 [9]: Let $\mathrm{E}=\mathbb{R}^2$, $p=\{(x,y)\in \mathrm{E},: x,y\geq 0\}\subset E, X=\mathbb{R}$ and

$d: X \, x \, X \to \mathrm{E}$ such that

$d(x,y)=\left(|x-y|^p, \alpha|x-y|^p\right)$, Where $\alpha\geq 0$ and $p>1$ are two constants.

Then, (X,d) is a cone b-metric space, but not a cone metric space.

Proof: Let $X=\mathbb{R}$ and $d: X \, x \, X \to \mathrm{E}$. We need to show that d satisfies definition 3.1.10.

i) Since $\alpha\geq 0$ and $|x-y|^p\geq 0$, $\forall x,y\in\mathbb{R}$

$\Rightarrow d(x,y)=\left(|x-y|^p, \alpha|x-y|^p\right)\geq 0$.

Hence, $d(x,y)\geq 0 \, \forall x,y\in X$

$d(x,y)=(0,0)\Leftrightarrow \left(|x-y|^p, \alpha|x-y|^p\right)=(0,0)$

$\Leftrightarrow |x-y|^p=0$ and $\alpha\,|x-y|^p=0$

$\Leftrightarrow |x-y|=0$ (Since $\alpha\geq 0$)

$\Leftrightarrow x-y=0$

$\Leftrightarrow x=y$.

Hence (i) is satisfied.

ii) $d(x,y)=\left(|x-y|^{p},\alpha|x-y|^{p}\right)=\left(|-(y-x)|^{p},\alpha|-(y-x)|^{p}\right)$

$$=\left(|-1|^{p}|y-x|^{p},\ \alpha|-1|^{p}|y-x|^{p}\right)$$

$$=\left(|y-x|^{p},\alpha|y-x|^{p}\right)$$

$$=d(y,x).$$

$\Rightarrow d(x,y)=d(y,x)$.

Hence (ii) is satisfied .

iii) $\forall x,y,z\in X$

$$d(x,y)=\left(|x-y|^{p},\alpha|x-y|^{p}\right)$$

$$=\left(|x-z+z-y|^{p},\alpha|x-z+z-y|^{p}\right)$$

$$\leq\ (|x-z|+|z-y|)^{p},\alpha(|x-z|+|z-p|)^{p}$$

$$\leq 2^{p}\quad\left(|x-z|^{p}+|z-y|^{p},\alpha|x-z|^{p}+|z-y|^{p}\right)$$

$$=\ 2^{p}\left[\ \left(|x-z|^{p},\alpha|x-z|^{p}\right)+\left(|z-y|^{p},\alpha|z-y|^{p}\right)\ \right]$$

$$=\ 2^{p}\ d(x,z)+d(z,y)$$

$\Rightarrow d(x,y)\leq 2^{p}\left[d(x,z)+d(z,y)\right]$.

Next, we let $U=x-z,V=z-y$ so $x-y=U+V$.

From the inequality

$(a+b)^{p}\leq(2\max\{a,b\})^{p}\ \leq 2^{p}(a^{p}+b^{p})\ \forall a,b\geq 0$

We have

$$|x-y|^p = |u+v|^p \leq (|u|+|v|)^p| \leq 2^p(|u|^p + |v|^p) = 2^p(|x-z|^p + |z-y|^p).$$

This implies that

$$d(x,y) \leq s[d(x,z) + d(z,y)]$$

with $s = 2^p > 1$. But,

$$|x-y|^p \leq |x-z|^p + |z-y|^p$$

is impossible for all $x > z > y$. Indeed, taking account of the inequality

$$(a+b)^p > a^p + b^p \; \forall a, b > 0$$

We arrive at

$$|x-y|^p = |u+v|^p = (u+v)^p > u^p + v^p = (x-z)^p + (z-y)^p$$

$$= |x-z|^p + |z-y|^p \, \forall x > z > y.$$

Hence,(iii) in Definition 3.1.5 is not satisfied, i.e., (X, d) is not a cone metric space.

Example 3.1.13 [9]: Let $X = \mathbb{R}, \mathrm{E} = \mathbb{R}^2$, and $p = \{(x,y) \in \mathrm{E} : x \geq 0, y \geq 0\}$. We define

$d\colon X \times X \to E$ by

$$d(x,y) = (|x-y|^2, |x-y|^2).$$

Then as it is shown below, (X, d) is a cone b-metric space with coefficient $s = 2$. But it is not a cone metric space since the triangle inequality is not satisfied.

Proof: Let $X = \mathbb{R}$ and $d\colon X \times X \to E$. We need to show d satisfies definition 3.1.10.

i) Since $|x-y|^2 \geq 0 \; \forall x, y \in \mathbb{R}$.

$$\Rightarrow (|x-y|^2, |x-y|^2) \geq \theta$$

Hence,

$$d(x,y) = |x-y|^2, |x-y|^2) \geq \theta \ \forall x, y \in X \ .$$

$$d(x,y) = (0,0) \Leftrightarrow (|x-y|^2, |x-y|^2) = (0,0)$$

$$\Leftrightarrow |x-y|^2 = 0$$

$$\Leftrightarrow x - y = 0$$

$$\Leftrightarrow x = y \ .$$

Hence, (i) is satisfied.

ii) $d(x,y) = \left(|x-y|^2, |x-y^2|\right) = \left(|-(y-x)|, |-(y-x)|\right)$

$$= \left(|-1||y-x|, |-1||y-x|\right)$$

$$= \left(|y-x|, |y-x|\right)$$

$$= d(y,x)$$

$$\Rightarrow d(x,y) = d(y,x) \ .$$

Hence, (ii) is satisfied.

iii) Let $x, y, z \in X$

$$d(x,y) = \left(|x-y|^2, |x-y|^2\right)$$

$$= \left(|x-z+z-y|^2, |x-z+z-y|^2\right)$$

$$\leq \ |x-z| + |z-y| \quad ^2, (|x-z| + |z-y|)^2$$

$$\leq 2\left(|x-z|^2 + |z-y|^2, |x-z|^2 + |z-y|^2\right)$$

$$= 2[(|x-z|^2, |x-z|^2) + (|z-y|^2, |z-y|^2)]$$

$$= 2[d(x,z) + d(z,y)]$$

$$d(x,y) \le 2[d(x,z) + d(z,y)].$$

Hence,(iii) is satisfied .

Thus, (X,d) is a cone b-metric space with $s = 2$.

Since $s > 1$, it cannot be cone metric space.

Definition 3.1.14 [9]: Let (X,d) be a cone b-metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then,

i) $\{x_n\}$ Converges to x whenever, for every $c \in E$ with $\theta << c$, there is a natural number N such that $d(x_n, x) << c$ for all $n \ge N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

ii) $\{x_n\}$ is a Cauchy sequence whenever, for every $c \in E$ with $\theta << c$, there is a natural number N such that $d(x_n, x_m) << c$ for all $n, m \ge N$

iii) (X,d) is a complete cone b-metric space if every Cauchy sequence is convergent.

3.2 Main results

In this section we state and prove common fixed points of generalized contraction and Zamfrescu pair of maps in cone b-metric spaces.

Definition 3.2.1: Let (X,d) be a cone b-metric space with $s \ge 1$ and P be a cone with non-empty interior. Let f, g: $X \to X$ be self-maps. Suppose that there exists a constant $k \in \left(0, \frac{1}{s}\right)$ and there exists

$$p(x,y) \in \quad d(x,y), d(x,fx), d(y,gy), \frac{d(x,gy)+d(y,fx)}{2} \; \},$$

such that

$$d(fx, gy) \leq k\, p(x, y) \;\; \forall x, y \text{ in } X \tag{3.2.1.1}$$

Then the pair of maps (f, g) is called a generalized contraction pair on X .

Example 3.2.2: Let $X = [0,1], E = C^1_{\mathbb{R}}[0,1]$ and $p = \{\delta \in E : \delta \geq 0\}$.

Define $d : XxX \to E$ by $d(x, y) = |x - y|^2 \delta$ where $\delta : [0,1] \to \mathbb{R}$ such that $\delta(t) = e^t$. Then (X, d) is a cone b-metric space with the coefficient s=2 but it is not a cone metric space. Consider the mappings $f, g : X \to X$ are defined by $f(x) = \begin{cases} \frac{1}{3}x, \; if \quad 0 \leq x < 5/6 \\ 1/3, \; if \; 5/6 \leq x \leq 1 \end{cases}$ and

$$g(x) = \begin{cases} 0, \; if \quad 0 \leq x < 5/6 \\ \frac{1}{3}x, \; if \; 5/6 \leq x \leq 1 \end{cases}$$

Then the pair (f, g) is a generalized contraction pair with $k = 5/6$

Solution: Let $s \geq 1$ .Then we need to show $\exists k \in (0, 1/s)$ such that

$$d(fx, gy) \leq kp(x, y) \, \forall x, y \in X$$

where

$$p(x, y) \in \{d(x, y), d(x, fx), d(y, gy), \frac{d(x, gy) + d(y, fx)}{2}\}$$

We consider the following cases

Case I: When $x, y \in [0, 5/6)$

$$d(fx, gy) = d(\frac{1}{3}x, 0) = \left|\frac{1}{3}x - 0\right|^2 \delta = \frac{1}{9}|x|^2 \delta .$$

If we take $p(x,y)=d(x,fx)$ we get

$$p(x,y)=d(x,\frac{1}{3}x)=\left|x-\frac{1}{3}x\right|^{2}\delta=\left|\frac{2}{3}x\right|^{2}\delta=\frac{4}{9}|x|^{2}\delta.$$

Thus, $d(fx,gy)=\frac{1}{9}|x|^{2}\delta\le\frac{5}{6}.\frac{4}{9}|x|^{2}\delta=\frac{20}{54}|x|^{2}\delta$

$\Rightarrow\frac{1}{9}|x|^{2}\delta\le\frac{20}{54}|x|^{2}\delta.$

This case is true.

Case II: When $x,y\in[5/6,1]$

$d(fx,gy)=d(\frac{1}{3},\frac{1}{3}y)=\left|\frac{1}{3}-\frac{1}{3}y\right|^{2}\delta=\frac{1}{9}|1-y|^{2}\delta.$

if we take $p(x,y)=d(y,gy)$ we get

$$p(x,y)=d(y,\frac{1}{3}y)=\left|y-\frac{1}{3}y\right|^{2}\delta=\left|\frac{3y-y}{3}\right|^{2}\delta=\frac{4}{9}|y|^{2}\delta.$$

Thus, $d(fx,gy)=\frac{1}{9}|1-y|^{2}\delta\le\frac{5}{6}.\frac{4}{9}|y|^{2}\delta=\frac{20}{54}|y|^{2}\delta$

$\Rightarrow\frac{1}{9}|1-y|^{2}\delta\le\frac{20}{54}|y|^{2}\delta.$

This is true .

Case III: When $x\in[0,5/6)$ and $y\in[5/6,1]$

$d(fx,gy)=d(\frac{1}{3}x,\frac{1}{3}y)=\left|\frac{1}{3}x-\frac{1}{3}y\right|^{2}\delta=\frac{1}{9}|x-y|^{2}\delta.$

If we take $p(x,y)=d(x,y)$, we get

$$p(x,y) = |x-y|^2 \delta .$$

Thus,

$$d(fx,gy) = \frac{1}{9}|x-y|^2 \delta \le \frac{5}{6}|x-y|^2 \delta .$$

Which is true.

If $p(x,y) = d(y,gy)$, we have

$$p(x,y) = d(y,\frac{1}{3}y) = \frac{4}{9}|y|^2 \delta .$$

Then, $d(fx.gy) = \frac{1}{9}|x-y|^2 \delta \le \frac{5}{6}.\frac{4}{9}|y|^2 \delta = \frac{20}{54}|y|^2 \delta .$

$$\frac{1}{9}|x-y|^2 \delta \le \frac{20}{54}|y|^2 \delta .$$

This is true .

Case IV: When $x \in [5/6, 1]$ and $y \in [0, 5/6)$

$$d(fx,gy) = d(\frac{1}{3},0) = \left|\frac{1}{3} - 0\right|^2 \delta = \frac{1}{9}\delta .$$

If $p(x,y) = d(x, fx)$, then we have

$$p(x,y) = d(x,\frac{1}{3}) = \left|x - \frac{1}{3}\right|^2 \delta = \frac{1}{9}|3x-1|^2 \delta .$$

Then, $d(fx.gy) = \frac{1}{9}\delta \le \frac{5}{6}.\frac{1}{9}|3x-1|^2 \delta = \frac{5}{54}|3x-1|^2 \delta .$

$$\Rightarrow \frac{1}{9}\delta \le \frac{5}{54}|3x-1|^2 \delta .$$

This is also true.

Then from cases I, II, III and IV we conclude that the pair (f,g) is generalized contraction.

Theorem 3.2.3: Let (X,d) be a complete cone b-metric space with $s\geq 1$. Suppose that (f,g) is a generalized contraction pair of self-maps on X. Then f and g have a unique common fixed point in X.

Proof: Let $x_0 \in X$. Since $f(X)\subset X$, $\exists x_1 \in X$ such that $x_1 = f(x_0)$.

Since $g(X)\subset X$, $\exists x_2 \in X$ such that $x_2 = g(x_1)$. By continuing this process having defined $x_n \in X$, we define $x_{n+1}\in X$ such that

$$x_{n+1} = \begin{cases} fx_n, if\ n = 0,2,4,... \\ gx_n, if\ n = 1,3,5,... \end{cases}$$

We first show that:

$$d(x_{n+1},x_n)\leq kd(x_n,x_{n-1}) \text{, for } n=1,2,3,\ldots \tag{3.2.3.1}$$

We consider two cases:

Case (i): n *is even*. Then,

$$d(x_{n+1},x_n)=d(fx_n,gx_{n-1})\leq k\,p(x_n,x_{n-1})$$

where

$$p(x_n,x_{n-1})\in \left\{d(x_n,x_{n-1}),d(x_n,fx_n),d(x_{n-1},gx_{n-1}),d\frac{(x_n,gx_{n-1})+d(x_{n-1},fx_n)}{2}\right\}$$

$$=\left\{d(x_n,x_{n-1}),d(x_n,x_{n+1}),d(x_{n-1},x_n),d\frac{(x_n,x_n)+d(x_{n-1},x_{n+1})}{2}\right\}$$

$$=\left\{d(x_n,x_{n-1}),d(x_n,x_{n+1}),\frac{1}{2}d(x_{n-1},x_{n+1})\right\}$$

Now if $p(x_n,x_{n-1})=d(x_n,x_{n-1})$, then

$$d(x_{n+1}, x_n) \leq kp(x_n, x_{n-1}) = kd(x_n, x_{n-1})$$

Clearly 3.2.3.1 holds true.

If $p(x_n, x_{n-1}) = d(x_{n+1}, x_n)$

$$\Rightarrow d(x_{n+1}, x_n) \leq kp(x_n, x_{n-1}) = kd(x_{n+1}, x_n)$$

$$\Rightarrow d(x_{n+1}, x_n) = 0 \quad \text{[by Remark 3.1.6 (6)].}$$

If $p(x_n, x_{n-1}) = \frac{1}{2} d(x_{n-1}, x_{n+1})$,

$$\Rightarrow d(x_{n+1}, x_n) \leq \frac{k}{2} p(x_n, x_{n-1}) = \frac{k}{2} d(x_{n+1}, x_{n-1})$$

$$\leq \frac{sk}{2}\left[d(x_{n+1}, x_n) + d(x_n, x_{n-1})\right]$$

$$= \frac{sk}{2} d(x_{n+1}, x_n) + \frac{sk}{2} d(x_n, x_{n-1})$$

$$\Rightarrow d(x_{n+1}, x_n) \leq \frac{sk}{2} d(x_{n+1}, x_n) + \frac{sk}{2} d(x_n, x_{n-1})$$

$$\Rightarrow \left(1 - \frac{sk}{2}\right) d(x_{n+1}, x_n) \leq \frac{sk}{2} d(x_n, x_{n-1})$$

$$\Rightarrow \left(\frac{2 - sk}{2}\right) d(x_{n+1}, x_n) \leq \frac{sk}{2} d(x_n, x_{n-1})$$

$$\Rightarrow d(x_{n+1}, x_n) \leq \frac{sk}{2 - sk} d(x_n, x_{n-1})$$

$$= hd(x_n, x_{n-1}), \text{where } h = \frac{sk}{2 - sk}.$$

$$\Rightarrow d(x_{n+1}, x_n) \leq hd(x_n, x_{n-1}).$$

Hence, 3.2.3.1 holds true.

Case (ii) n *is odd*. Then,

$$d(x_{n+1},x_n)=d(gx_n,\, fx_{n-1})=d(fx_{n-1,},gx_n)\le kp(x_{n-1},x_n),\text{ where}$$

$$p(x_{n-1},x_n)\in\left\{d(x_{n-1},x_n),d(x_{n-1},\, fx_{n-1}),d(x_n,gx_n),\frac{d(x_{n-1},gx_n)+d(x_n,\, fx_{n-1})}{2}\right\}$$

$$=\left\{d(x_n,x_{n-1}),d(x_{n-1},x_n),d(x_{n+1},x_n),\frac{d(x_{n-1},x_{n+1})+d(x_n,x_n)}{2}\right\}$$

$$=\left\{d(x_n,x_{n-1}),d(x_{n+1},x_n),\frac{1}{2}d(x_{n+1},x_{n-1})\right\}$$

Now if $p(x_{n-1},x_n)=d(x_n,x_{n-1})$, then

$$\Rightarrow d(x_{n+1},x_n)\le kp(x_{n-1},x_n)=kd(x_n,x_{n-1})$$

$$\Rightarrow d(x_{n+1},x_n)\le kd(x_n,x_{n-1})$$

Clearly 3.2.3.1 holds true.

If $p(x_{n-1},x_n)=d(x_{n+1},x_n)$

$$\Rightarrow d(x_{n+1},x_n)\le kp(x_{n+1},x_n)=kd(x_{n+1},x_n)$$

$$\Rightarrow d(x_{n+1},x_n)=0 \text{ [by Remark 3.1.6 (6)]}$$

If $p(x_{n-1},x_n)=\frac{1}{2}d(x_{n+1},x_{n-1})$, then

$$\Rightarrow d(x_{n+1},x_n)\le\frac{k}{2}p(x_n,x_{n-1})=\frac{k}{2}d(x_{n+1},x_{n-1})$$

$$\le\frac{sk}{2}\left[d(x_{n+1},x_n)+d(x_n,x_{n-1})\right]$$

$$\Rightarrow d(x_{n+1},x_n) \le \frac{sk}{2} d(x_{n+1},x_n) + \frac{sk}{2} d(x_n,x_{n-1})$$

$$\Rightarrow \left(1-\frac{sk}{2}\right) d(x_{n+1},x_n) \le \frac{sk}{2} d(x_n,x_{n-1})$$

$$\Rightarrow \left(\frac{2-sk}{2}\right) d(x_{n+1},x_n) \le \frac{sk}{2} d(x_n,x_{n-1})$$

$$\Rightarrow d(x_{n+1},x_n) \le \frac{sk}{2-sk} d(x_n,x_{n-1})$$

$$= hd(x_n,x_{n-1}), \text{where} \quad h = \frac{sk}{2-sk}$$

$$\Rightarrow d(x_{n+1},x_n) \le hd(x_n,x_{n-1}).$$

Hence, 3.2.3.1 holds true.

Hence, in both cases the inequality 3.2.3.1 holds.

By repeated application of (3.2.3.1), we get

$$d(x_{n+1},x_n) \le k^n d(x_1,x_0), n = 1,2,\ldots \qquad (3.2.3.2)$$

We now need to show that $\{x_n\}$ is a Cauchy sequence in X .

For $m > n \ge 1$, we have

$$d(x_n,x_m) \le s\left[d(x_n,x_{n+1}) + d(x_{n+1},x_m)\right]$$

$$= sd(x_n,x_{n+1}) + sd(x_{n+1},x_m)$$

$$\le sd(x_n,x_{n+1}) + s^2[d(x_{n+1},x_{n+2}) + d(x_{n+2},x_m)]$$

$$\vdots \quad \vdots$$

$$d(x_n,x_m) \le sd(x_n,x_{n+1}) + s^2 d(x_{n+1},x_{n+2}) + \ldots + s^{m-n} d(x_{m-1},x_m) \qquad (3.2.3.3)$$

Since $sk < 1$, from the inequality (3.2.3.1), we get

$$d(x_n, x_m) \le sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + ... + s^{m-n} d(x_{m-1}, x_m)$$

$$\le sk^n (x_0, x_1) + s^2 k^{n+1} d(x_0, x_1) + ... + s^{m-n} k^{m-1} d(x_0, x_1)$$

$$= [sk^n + s^2 k^{n+1} + ... + s^{m-n} k^{m-1}] d(x_0, x_1)$$

$$= sk^n \ (1 + sk + (sk)^2 + ... + (sk)^{m-n-1}) \ d(x_0, x_1)$$

$$\le \frac{sk^n}{1-sk} d(x_0, x_1) \to \theta \text{ as } n \to \infty \qquad (3.2.3.4)$$

Let $0 << c$ .From (3.2.3.4) and Remark 3.1.6 (8), there exists an integer N such that $sk^n (1-sk)^{-1} d(x_1, x_0) << c \ \forall n > N$. By Remark 3.1.6 (2), $d(x_n, x_m) << c.$

Hence by definition 3.1.14 (ii) $\{x_n\}$ is a Cauchy sequence in X .By the completeness of X, there exists z in X such that $x_n \to z$ as $n \to \infty$

We claim that $fz = z$.

Let $0 << c$. If n is odd, Then

$$\begin{aligned} d(fz, z) &\le s[(fz, gx_n) + d(gx_n, z)] \\ &\le s[kp(z, x_n) + d(x_{n+1}, z)] \end{aligned} \qquad (3.2.3.5)$$

When $p(z, x_n) \in \left\{ \ d(z, x_n), d(z, fz), d(x_n, gx_n), \frac{d(z, gx_n) + d(x_n, fz)}{2} \ \right\}$

$$= \qquad d(z, x_n), d(z, f_z), d(x_n, x_{x+1}), \frac{(z, x_{n+1}) + d(x_n, fz)}{2}$$

One of the following cases holds true for infinitely many n.

If $p(z, x_n) = d(z, x_n)$, then from (3.2.3.5) we have

$$d(fz,z) \le s[k\,d(z,x_n) + d(x_{n+1},z)] << sk\,\frac{c}{2sk} + \frac{c}{2s} = c\,.$$

If $p(z,x_n) = d(z,\ fz),$ then from (3.2.3.5) we get

$$d(fz,z) \le sk\,d(z,\ fz) + s\,d(x_{n+1},z)$$

$$\Rightarrow (1-sk)d(fz,z) \le s\,d(x_{n+1},z)$$

$$\Rightarrow d(fz,z) \le \frac{s}{1-sk}\,d(x_{n+1},z) << \frac{s}{1-sk}\ \frac{c}{\left(\dfrac{s}{1-sk}\right)} = c\,.$$

If $p(z,x_n) = d(x_n,x_{n+1}),$ then from (3.2.3.5) we get

$$d(fz,z) \le skd(x_n,x_{n+1}) + sd(x_{n+1},z)$$

$$\le s^2k\ [d(x_n,z) + d(z,x_{n+1})] + sd(x_{n+1},z)$$

$$= s^2k\ \ d(x_n,z) + s(sk+1)d(x_{n+1},z)$$

$$<< s^2k\,\frac{c}{2s^2k} + s(sk+1)\frac{c}{2s(sk+1)} = c\,.$$

If $p(z,x_n) = \dfrac{d(z,x_{n+1}) + d(x_n,\ fz)}{2}$, then from (3.2.3.5) we get

$$d(f_z,z) \le s[\frac{k}{2}(d(z,x_{n+1}) + d(x_n,\ fz))] + s\,d(x_{n+1},z)$$

$$\le\ (\frac{sk}{2}+s)d(x_{n+1},z) + \frac{s^2k}{2}\ [d(x_n,z) + d(z,\ fz)]$$

$$=\ \left(\ \frac{sk+2s}{2}\ \right)d(x_{n+1},z) + \frac{s^2k}{2}\ d(x_n,z) + \frac{s^2k}{2}\,d(z,\ fz)$$

$$\Rightarrow\ (1-\frac{s^2k}{2})d(fz,z) \le \frac{sk+2s}{2}\ d(x_{n+1},z) + \frac{s^2k}{2}\,d(x_n,z)$$

$$\Rightarrow d(fz,z) \leq \frac{sk+2s}{2-s^2k} d(x_{n+1},z) + \frac{s^2k}{2-s^2k} d(x_n,z)$$

$$<< \frac{s(k+2)}{2(2-s^2k} \frac{(2-s^2k)}{s(k+2)} c + \frac{s^2k}{2-s^2k} \frac{(2-s^2k)}{2s^2k} c = c.$$

In all cases, we obtain $d(fz,z) << c$ for each $c \in \operatorname{int} P$ using remark 3.1.6 (4) it follows that $d(fz,z) = 0$ or $fz = z$.

Next we prove that $gz = z$.

Now consider

$$d(z,gz) = d(fz,gz) \leq kp(z,z) \ , \qquad (3.2.3.6)$$

where $p(z,z) \in \{d(z,z), d(z,fz), d(z,gz), \frac{d(z,fz)+(z,gz)}{2}\}$

$$= \{0, d(z,gz), \frac{d(z,gz)}{2}\}.$$

If $p(z,z) = 0$ from (3.2.3.6), we get $gz = z$.

If either $p(z,z) = \frac{d(z,gz)}{2}$ or $p(z,z) = d(z,gz),$ then from (3.2.3.6) and remark 3.1.6 (6) we have, $d(z,gz) = 0$

Thus, $z = gz$.

Hence $fz = gz = z$.

The uniqueness of z follows from the inequality (3.2.1.1). Hence the Theorem follows.

Corollary 3.2.4 [1]: Let (X,d) be a complete cone metric space. Suppose that (f,g) is a generalized contraction pair of self-maps on X. Then f and g have a unique common fixed point in X.

Proof: Since every cone metric space is a cone b-metric space, the proof follows from Theorem 3.2.3.

Example 3.2.5: Let X, E, P, d, δ, f and g be as in example 3.2.2.The pair (f, g) is a generalized contraction pair with $k = \frac{5}{6}$; and the maps f and g satisfy all the conditions of theorem 3.2.3 and 0 is the unique common fixed point of f and g.

Definition 3.2.6: Let (X, d) be a cone b-metric space with $s \geq 1$ and P be a cone with non-empty interior. Let $f, g : X \to X$ be self-maps. Suppose that there exists a constant $k \in (0, 1/s)$ and there exists

$$p(x, y) \in \left\{ d(x, y), \frac{d(x, fx) + d(y, gy)}{2}, \frac{d(x, gy) + d(y, fx)}{2} \right\}$$

such that

$$d(fx, gy) \leq kp(x, y) \text{ for all } x, y \text{ in } X. \tag{3.2.6.1}$$

Then the pair of maps (f, g) is called a *Zamfirescu pair of maps on* X.

Example 3.2.7: Let $X = [0,1], E = C^1_{\mathbb{R}}[0,1]$ and $P = \{\delta \in E : \delta \geq 0\}$..

Define $d : XxX \to E$ by $d(x, y) = |x - y|^2 \delta$ where $\delta : [0,1] \to \mathbb{R}$ Such that $\delta(t) = e^t$. Then (X, d) is a cone b-metric space with the coefficient s=2 but it is not a cone metric space. Consider the mappings $f, g : X \to X$ are defined by

$$f(x) = \begin{cases} \frac{1}{4}x, & if \quad x \neq 1 \\ 1/5, & if x = 1 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \frac{1}{5}x, & if \quad x \neq 1 \\ \frac{1}{4}, & if x = 1 \end{cases}$$

Then the pair (f, g) is a Zamfirescu pair with $k = \frac{4}{5}$.

Now we need to show

$d(fx, gy) \le kp(x, y)$, $\forall x, y \in X$, $\exists k \in (0,1)$ where

$$p(x, y) \in \{d(x, y), \frac{d(x, fx) + d(y, gy)}{2}, \frac{d(x, gy) + d(y, fx)}{2}\}$$

Case I: When $x, y \ne 1$

$$d(fx, gy) = d(\frac{1}{4}x, \frac{1}{5}y) = \left|\frac{1}{4}x - \frac{1}{5}y\right|^2 \delta = \frac{1}{400}|x - y|^2 \delta .$$

If we take $p(x, y) = d(x, y)$ we get

$$p(x, y) = |x - y|^2 \delta$$

Thus, $d(fx, gy) = \frac{1}{400}|x - y|^2 \delta \le \frac{4}{5}|x - y|^2 \delta$.

which is true.

Case II: when $x, y = 1$

$$d(fx, gy) = d(\frac{1}{5}, \frac{1}{4}) = \left|\frac{1}{5} - \frac{1}{4}\right|^2 \delta = \frac{1}{400}\delta .$$

If we take $p(x, y) = \frac{d(x, fx) + d(y, gy)}{2}$

$$= \frac{d(x, \frac{1}{5}) + d(y, \frac{1}{4})}{2}$$

$$= \frac{\left|x - \frac{1}{5}\right|^2 \delta + \left|y - \frac{1}{4}\right|^2 \delta}{2}$$

$$= \frac{16|5x - 1|^2 \delta + 25|4y - 1|^2 \delta}{800} = \frac{16^2 \delta + (25 \times 9)\delta}{800} .$$

$$\Rightarrow p(x,y)=\frac{481}{800}\delta .$$

Then

$$d(fx,gy)=\frac{1}{400}\delta \leq \frac{4}{5}.\frac{481}{800}\delta =\frac{481}{1{,}000}\delta$$

$$\Rightarrow \frac{1}{400}\delta \leq \frac{481}{1{,}000}\delta .$$

This case is also true.

Case III: When $x=1$ and $y\neq 1$

$$d(fx,gy)=d(\frac{1}{5},\frac{1}{5}y)$$

$$=\left|\frac{1}{5}-\frac{1}{5}y\right|^{2}\delta = \frac{1}{25}|1-y|^{2}\delta = \frac{1}{25}|1-y|^{2}\delta .$$

If we take $p(x,y)=d(x,y)$

$$= |x-y|^{2}\delta = |1-y|^{2}\delta .$$

$$\Rightarrow d(fx,gy)=\frac{1}{25}|1-y|^{2}\delta \leq \frac{4}{5}|1-y|^{2}\delta$$

$$\Rightarrow \frac{1}{25}|1-y|^{2}\delta \leq \frac{4}{5}|1-y|^{2}\delta .$$

This case is true.

Case IV: when $x\neq 1$ and $y=1$

$$d(fx,gy)=d(\frac{1}{4}x,\frac{1}{4})=\left|\frac{1}{4}x-\frac{1}{4}\right|^{2}\delta =\frac{1}{16}|x-1|^{2}\delta .$$

If we take $p(x,y)=d(x,y)$, we get

$p(x, y) = |x - y|^2 \delta = |x - 1|^2 \delta$.

Then $d(fx, gy) = \frac{1}{16}|x - 1|^2 \delta \leq \frac{4}{5}|x - 1|^2 \delta$.

which is true.

From case I, II, III, and IV the pair (f, g) is a zamfirescu pair of maps on X .

Theorem 3.2.8: Let (X, d) be a complete cone b-metric space with $s \geq 1$. Suppose that (f, g) is a Zamfirescu pair on X . Then f and g have a unique common fixed point in X .

Proof: Let $x_0 \in X$ since $f(X) \subset X$ there exists $x_1 \in X$ such that $x_1 = fx_0$.

Since $g(X) \subset X$, there exists $x_2 \in X$ such that $x_2 = gx_1$. By continuing this process, having defined $x_n \in X$, we can define $x_{n+1} \in X$ such that

$$x_{n+1} = \begin{cases} fx_n, if\ n = 0,2,4,... \\ gx_n, if\ n = 1,3,5,... \end{cases}$$

We first show that

$$d(x_{n+1}, x_n) \leq kd(x_n, x_{n-1}), \text{for } \ n = 1,2,3,... \qquad (3.2.8.1)$$

We consider two cases:

Case (i) n is even. Then,

$$d(x_{n+1}, x_n) = d(fx_n, gx_{n-1}) \leq kp(x_n, x_{n-1}), \text{where}$$

$$p(x_n, x_{n-1}) \in \left\{ d(x_n, x_{n-1}), \frac{d(x_n, fx_n) + d(x_{n-1}, gx_{n-1})}{2}, \frac{d(x_n, gx_{n-1}) + d(x_{n-1}, fx_n)}{2} \right\}$$

$$= \quad d(x_n, x_{n-1}), \frac{d(x_{n+1}, x_n) + d(x_n, x_{n-1})}{2}, \frac{1}{2} d(x_{n+1}, x_{n-1}) \quad .$$

Now if $p(x_n, x_{n-1}) = d(x_n, x_{n-1})$

$$\Rightarrow d(x_{n+1}, x_n) \leq kd(x_n, x_{n-1})$$

Hence (3.2.8.1) holds

If $p(x_n, x_{n-1}) = \dfrac{d(x_{n+1}, x_n) + d(x_n, x_{n-1})}{2}$, then we have

$$\Rightarrow d(x_{n+1}, x_n) \leq \frac{k}{2}\left[\ d(x_{n+1}, x_n) + d(x_n, x_{n-1})\ \right] = \frac{k}{2} d(x_{n+1}, x_n) + \frac{k}{2} d(x_n, x_{n-1})$$

$$\Rightarrow d(x_{n+1}, x_n) \leq \frac{1}{2} d(x_{n+1}, x_n) + \frac{k}{2} d(x_n, x_{n-1})$$

$$\Rightarrow \frac{1}{2} d(x_{n+1}, x_n) \leq \frac{k}{2} d(x_n, x_{n-1})$$

$$\Rightarrow d(x_{n+1}, x_n) \leq kd(x_n, x_{n-1})$$

Hence (3.2.8.1) holds.

If $p(x_n, x_{n-1}) = \dfrac{1}{2} d(x_{n+1}, x_{n-1})$, then we have

$$d(x_{n+1}, x_n) \leq \frac{k}{2} d(x_{n+1}, x_{n-1})$$

$$\leq \frac{sk}{2}[d(x_{n+1}, x_n) + d(x_n, x_{n-1})]$$

$$= \frac{sk}{2} d(x_{n+1}, x_n) + \frac{sk}{2} d(x_n, x_{n-1})$$

$$\Rightarrow d(x_{n+1}, x_n) \leq \frac{sk}{2}\ d(x_{n+1}, x_n) + \frac{sk}{2} d(x_n, x_{n-1})$$

$$\Rightarrow \left(1 - \frac{sk}{2}\right) d(x_{n+1}, x_n) \leq \frac{sk}{2} d(x_n, x_{n-1})$$

$$\Rightarrow\left(\frac{2-sk}{2}\right)d(x_{n+1},x_n)\le\frac{sk}{2}d(x_n,x_{n-1})$$

$$\Rightarrow d(x_{n+1},x_n)\le\frac{sk}{2-sk}d(x_n,x_{n-1})$$

$$= hd(x_n,x_{n-1}),\ \text{where}\ \ h=\frac{sk}{2-sk}.$$

$$\Rightarrow d(x_{n+1},x_n)\le hd(x_n,x_{n-1}).$$

Hence (3.2.8.1) holds.

Case (ii) n is odd. Then

$$d(x_{n+1},x_n)=d(gx_n,\ fx_{n-1})=d(\ fx_{n-1},gx_n)\le kp(x_{n-1},x_n)$$

$$p(x_{n-1},x_n)\in\left\{\ d(x_{n-1},x_n),\frac{d(x_{n-1},\ fx_{n-1})+d(x_n,gx_n)}{2},\frac{d(x_{n-1},gx_n)+d(x_n,\ fx_{n-1})}{2}\ \right\}$$

$$=\left\{\ d(x_n,x_{n-1}),\frac{d(x_{n+1},x_n)+d(x_n,x_{n-1})}{2},\ \frac{1}{2}d(x_{n+1},x_{n-1})\ \right\}$$

Now if $p(x_{n-1},x_n)=d(x_n,x_{n-1}),$ then

$$d(x_{n+1},x_n)\le kp\ (x_{n-1},x_n)=kd(x_n,x_{n-1})$$

$$\Rightarrow d(x_{n+1},x_n)\le kd\ (x_n,x_{n-1})$$

Hence 3.2.8.1holds

If $p(x_{n-1},x_n)=\dfrac{d(x_{n+1},x_n)+d(x_n,x_{n-1})}{2}$, then we have

$$d(x_{n+1},x_n)\le k\left[\ \frac{d(x_{n+1},x_n)+d(x_n,x_{n-1})}{2}\ \right]=\frac{k}{2}d(x_{n+1},x_n),+\frac{k}{2}d(x_n,x_{n-1})$$

$$\le\frac{1}{2}\ d(x_{n+1},x_n)+\frac{k}{2}\ d(x_n,x_{n-1})$$

$$\Rightarrow \frac{1}{2} d(x_{n+1}, x_n) \leq \frac{k}{2} d(x_n, x_{n-1})$$

$$\Rightarrow d(x_{n+1}, x_n) \leq k d(x_n, x_{n-1}).$$

Hence 3.2.8.1 holds.

If $p(x_{n-1}, x_n) = \frac{1}{2} d(x_{n+1}, x_{n-1})$, we have

$$d(x_{n+1}, x_n) \leq \frac{k}{2} d(x_{n+1}, x_{n-1})$$

$$\leq \frac{k}{2} s[d(x_{n+1}, x_n) + d(x_n, x_{n-1})]$$

$$= \frac{ks}{2} d(x_{n+1}, x_n) + \frac{ks}{2} d(x_n, x_{n-1})$$

$$\Rightarrow d(x_{n+1}, x_n) \leq \frac{ks}{2} d(x_{n+1}, x_n) + \frac{ks}{2} d(x_n, x_{n-1})$$

$$\Rightarrow \left(1 - \frac{ks}{2}\right) d(x_{n+1}, x_n) \leq \frac{ks}{2} d(x_n, x_{n-1})$$

$$\Rightarrow \left(\frac{2 - sk}{2}\right) d(x_{n+1}, x_n) \leq \frac{ks}{2} d(x_n, x_{n-1})$$

$$\Rightarrow (2 - ks) d(x_{n+1}, x_n) \leq ks\, d(x_n, x_{n-1})$$

$$\Rightarrow d(x_{n+1}, x_n) \leq \frac{ks}{2 - ks} d(x_n, x_{n-1})$$

$$= hd(x_n, x_{n-1}), \text{ where } h = \frac{sk}{2 - sk}$$

$$\Rightarrow d(x_{n+1}, x_n) \leq hd(x_n, x_{n-1}).$$

Hence 3.2.8.1 holds true.

Hence, in both cases the inequality 3.2.8.1 holds.

By repeated application of (3.2.8.1), we get

$$d(x_{n+1},x_n)\le k^n d(x_1,x_0), n=1,2,... \qquad (3.2.8.2)$$

Next, we need to show that $\{x_n\}$ is a Cauchy sequence in X .

For $m>n\ge 1.$ it follows that

$$d(x_n,x_m)\le s[d(x_n,x_{n+1})+d(x_{n+1},x_m)]$$

$$\le sd(x_n,x_{n+1})+s[s[d(x_{n+1},x_{n+2})+d(x_{n+2},x_m)]]$$

$$=sd(x_n,x_{n+1})+s^2d(x_{n+1},x_{n+2})+s^2d(x_{n+2},x_m)$$

$$\le sd(x_n,x_{n+1})+s^2d(x_{n+1},x_{n+2})+s^3d(x_{n+2},x_{+3})+s^3d(x_{n+3},x_m)$$

$$\le sd(x_n,x_{n+1})+s^2d(x_{n+1},x_{n+2})+...+s^{m-n}d(x_{m-1},x_m).$$

Now 3.2.8.1 and $sk<1$imply that

$$d(x_n,x_m)\le sd(x_n,x_{n+1})+s^2d(x_{n+1},x_{n+2})+...+s^{m-n}d(x_{m-1},x_m)$$

$$\le sk^n d(x_0,x_1)+s^2k^{n+1}d(x_0,x_1)+...+s^{m-n}k^{m-1}d(x_0,x_1)$$

$$=(sk^n+s^2k^{n+1}+...+s^{m-n}k^{m-n1})d(x_0,x_1)$$

$$=sk^n(1+sk^n+...+(sh)^{m-n-1})d(x_0,x_1)$$

$$=sk^n(1+sk+(sk)^2+...+(sk)^{m-n-1})d(x_0,x_1)$$

$$\le\frac{sk^n}{1-sk}d(x_0,x_1)\to\theta \text{ as } n\to\infty. \qquad (3.2.8.3)$$

Let $0<<c$ from (3.2.8.3) and Remark 3.1.6 (8), there exists an integer N such that $\frac{sk^n}{1-sk}d(x_0,x_1)<<c \;\; \forall n>N.$ By Remark (3.1.6) (2), $d(x_n,x_m)<<c$.

Hence by Definition 3.1.14 (ii), $\{x_n\}$ is a Cauchy sequence in X. By the completeness of X, there exists z in x such that $x_n \to z$ as $n \to \infty$.

We claim that $fz = z$

Let $0 << c$. If n is odd. Then,

$$
\begin{aligned}
d(fz,z) &\le s[d(fz,gx_n)+d(gx_n,z)] \\
&\le s\ k\ p(z,x_n)+d(gx_n,z) \\
&= sk\ p(z,x_n)+s\,d(gx_n,z).
\end{aligned} \tag{3.2.8.4}
$$

Where $p(z,x_n) \in \left\{\ d(z,x_n), \dfrac{d(z,\,fz)+d(x_n,gx_n)}{2},\ \dfrac{d(z,gx_n)+d(x_n,\,fz)}{2}\ \right\}$

$$= \left\{\ d(z,x_n), \frac{d(z,\,fz)+d(x_n,x_{n+1})}{2},\ \frac{d(z,x_{n+1})+d(x_n,\,fz)}{2}\ \right\}.$$

One of the following cases holds true for infinitely many n

If $p(z,x_n) = d(z,x_n)$, then from (3.2.8.4) we have

$$d(fz,z) \le sk\,d(z,x_n)+s\,d(x_{n+1},z) << sk\frac{c}{2sk}+\frac{sc}{2s} = c.$$

If $p(z,x_n) = \dfrac{d(z,\,fz)+d(x_n,x_{n+1})}{2}$, then from (3.2.8.4) we get

$$
\begin{aligned}
d(fz,z) &\le sk\left(\frac{d(z,\,fz)+d(x_n,x_{n+1})}{2}\right)+s\,d(x_{n+1},z) \\
&\le \frac{sk}{2}\ d(z,\,fz)+d(x_n,x_{n+1})\ +s\,d(x_{n+1},z) \\
&\le \frac{sk}{2}d(z,\,fz)+\frac{s^2k}{2}d(x_n,z)+\frac{s^2k}{2}d(z,x_{n+1})+sd(x_{n+1},z)
\end{aligned}
$$

$$\Rightarrow \left(\frac{2-sk}{2}\right)d(fz,z) \le \frac{s^2k}{2}d(x_n,z)+\frac{s}{2}(sk+2)d(z,x_{n+1})$$

$$\Rightarrow d(fz,z) \le \frac{s^2k}{2-sk} d(x_n,z) + s(sk+2)d(z,x_{n+1})$$

$$<< \frac{s^2k}{2-sk}\frac{c(2-sk)}{2s^2k} + s(sk+2)\frac{c}{2s(sk+2} = c\,.$$

If $p(z,x_n) = \dfrac{d(z,x_{n+1}) + d(x_n,\, fz)}{2}$, then from (3.2.8.4) we get

$$d(fz,z) \le \frac{sk}{2}\left[d(z,x_{n+1}) + d(x_n,\, fz)\right] + sd(x_{n+1},z)$$

$$\le \left(\frac{sk}{2} + s\right) d(z,x_{n+1}) + \frac{s^2k}{2} d(x_n,z) + \frac{s^2k}{2} d(z,\, fz)$$

$$\Rightarrow \left(1 - \frac{s^2k}{2}\right) d(fz,z) \le \left(\frac{sk+2s}{2}\right) d(z,x_{n+1}) + \frac{s^2k}{2} d(x_n,z)$$

$$\Rightarrow \frac{2-s^2k}{2} d(fz,z) \le \frac{sk+2s}{2} d(z,x_{n+1}) + \frac{s^2k}{2} d(x_n,z)$$

$$\Rightarrow d(fz,z) \le \frac{sk+2s}{2-s^2k} d(z,x_{n+1}) + \frac{s^2k}{2-s^2k} d(x_n,z)$$

$$<< \frac{s(k+2)}{2(2-s^2k)} \quad \frac{(2-s^2k)}{s(k+2)} c + \frac{s^2k}{2(2-s^2k)} \quad \frac{(2-s^2k)}{s^2k} c = c.$$

In all cases, we obtain $d(fz,z) << c$ for each $c \in \operatorname{int} p$ .Using Remark 3.1.6 (4), it follows that $d(fz,z) = 0$ or $fz = z$.

Next we prove that gz =z. To do so consider

$$d(z,gz) = d(fz,gz) \le kp\,(z,z) \quad , \tag{3.2.8.5}$$

where $p(z,z) \in \left\{ d(z,z), \dfrac{d(z,gz) + d(z,\, fz)}{2},\ \dfrac{d(z,\, fz) + d(z,gz)}{2} \right\}$

$$= \left\{0, \frac{d(z,gz)}{2}\right\}.$$

Now if $p(z,z) = 0$, from (3.2.8.5) trivially we get $gz = z$

If $p(z,z)=\dfrac{d(z,gz)}{2}$, then from (3.2.8.5) and remark 3.1.6 (6), we have $d(z,gz)=0$

i.e. $z=gz$.

Hence, $fz=gz=z$.

The uniqueness of z follows from inequality the (3.2.6.1). Hence the theorem follows.

Corollary 3.2.9 [1]: Let (X,d) be a complete cone metric space. Suppose that (f,g) is a Zamfirescu pair on X. Then f and g have a unique common fixed point in X.

Proof: Since every cone metric space is a cone b-metric space, the proof follows from Theorem 3.2.8.

Example 3.2.10: Let X, E, P, d, δ, f and g be as in Example 3.2.7. The pair (f,g) is a Zamfirescu pair with $k=\dfrac{4}{5}$; and the maps f and g satisfy all the conditions of Theorem 3.2.8 and 0 is the unique common fixed point of f and g.

4. Conclusions and Future scopes

4.1 Conclusions

In 2010 Babu, Alemayehu and Prasad [1] established the existence of common fixed points for generalized contraction and Zamfirescu pair of maps in complete cone metric spaces. Recently Haung and Xu [9] have proved some fixed point theorems of contraction maps in complete cone b-metric spaces. In this research the work of Babu, Alemayehu and Prasad [1] is extended to cone b-metric spaces by proving

- ❖ The existence of common fixed point for generalized contraction pair in cone b-metric spaces.
- ❖ The existence of common fixed point of Zamfirescu pair of maps in cone b-metric spaces.

And we have also provided examples to substantiate the aforementioned main results.

- ➢ Since Corollary 3.3, Corollary 3.4 and Corollary 3.5 in Babu, Alemayehu and Prasad [1] are corollaries to Corollary 3.2.9; Theorem 2.3, Theorem 2.4 and Theorem 2.5 of Razapour and Hamlbarani [12] are generalized by Theorem 3.2.8. As a result some of the main results in Huang and Zhang [8] are also generalized by Theorem 3.2.8.

4.2 Future scopes

Common fixed points of two or more operators defined on cone b-metric space is new area of study. Recently there are a number of published research papers related to this area of study. So the student researcher recommend the upcoming Post graduate students of the department to have interest to do their research work in this area of study.

References

1. Babu, G. V. R. , Alemayehu, G. N. and Prasad, K.N.V.V., Common fixed point theorems of generalized contraction and Zamfirescu pair of maps in cone metric spaces, Albanian J. Math, 4 (1) (2010), 19-29.
2. Bakhtin, I. A., The contraction mapping principle in almost metric spaces, Funct. Anal., Gos. Ped. Inst. Unianowsk, 30 (1989), 26-37.
3. Banach, S., Sur les operations dans les ensembles abstraitetleur application aux equations, integrals, Fundam. Math., 3 (1922), 133-181.
4. Boriceanu, M., Bota, M., Petrusel, A., Multivalued fractals in b-metric spaces, Cent. Eur. J. Math. 8(2) (2010), 367-377.
5. Bota, M., Molnar, A., Csaba, V., On Ekeland's variational principle in b-metric spaces. Fixed Point Theory, 12 (2011), 21-28.
6. Czerwik, S., Nonlinear set-valued contraction mappings in b-metric spaces, Atti Semin. Mat. Fis. Univ. Modena, 46 (1998), 263-276.
7. Deimling, K., Nonlinear Functional Analysis. Springer, Berlin (1985).
8. Huang, L-G., Zhang, X., Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl., 332 (2) (2007), 1468-1476
9. Huang, H. and Xu, S., Fixed point theorems of contractive mappings in cone b-metric spaces, Fixed Point Theory and Applications, vol. 2013, 2013:112, 10 pages.
10. Hussian, N. and Shah, M.H., KKM mappings in cone b-metric spaces. Comput. Math. Appl. 62(2011), 1677-1684.
11. Jankovi´c, S., Kadelburg, Z., Radenovi´c, S., on cone metric spaces: a survey, Nonlinear Anal. 4(7) (2011), 2591-2601.
12. Rezapour, S., Hamlbarani, R., Some notes on the paper 'Cone metric spaces and fixed point theorems of contractive mappings', J. Math. Anal. Appl. 345(2008), 719-724.