Common Fixed Points of f-Contraction Mapping With Generalized Altering Distance Function In Partially Ordered Metric Spaces

A Research Submitted To The Department Of Mathematics, Jimma University In Partial Fulfillment Of The Requirements For The Degree Of Masters Of Sciences In Mathematics (Differential Equation)

By

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DECLARATION

I, the undersigned declare that, the research entitled "**Common Fixed Points of f-Contraction Mapping With Generalized Altering Distance Function In Partially Ordered Metric Spaces"** is original and it has not been submitted to any institution elsewhere for the award of any academic degree or like, where other sources of information that have been used, they have been acknowledged .

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ABSTRACT

The aim of this research project was to extend the result of Yongfu Su [26] and to obtain some new fixed point theorems for f-contraction mapping in a complete metric space endowed with a partial order by using generalized altering distance function. This study was mostly depended on Secondary source of data such as journals and books which are found in different libraries and internet service were used for the study. The researcher followed analytical and numerical design in this research work. The procedures employed for the analyses of this study were the standard iterative techniques and procedures used in Su [26]. We have stated some examples showing that our results are effective. This study was conducted in Jimma University under Mathematics Department from November 2014 to June 2015 G.C.

Key words*:* f-*Contraction mapping, partially ordered metric spaces, common fixed point, generalized altering distance function.*

1. INTRODUCTION

1.1. Background of the study

Fixed point theory contains many different fields of mathematics, such as nonlinear functional analysis, mathematical analysis, operator theory and general topology. The fixed point theory is divided into two major areas: One is the fixed point theory on contraction or contraction type mappings on complete metric spaces and the second is the fixed point theory on continuous operators on compact and convex subsets of a normed space. The beginning of fixed point theory in normed space is attributed to the work of Brouwer in 1910, who proved that any continuous self-map of the closed unit ball of $\mathbb{R}^n$ has a fixed point. The beginning of fixed point theory on complete metric space is related to the work of Polish mathematician Stefan Banach [4], Banach Contraction Principle, published in 1922. Let (X, d) be a complete metric space and $T: X \rightarrow X$ be a mapping. Then T is said to be a contraction mapping if there exists a constant $k \in [0,1)$, called a contraction factor, such that

$$d(Tx, Ty) \leq kd(x, y) \text{ for all } x, y \in X.$$

Banach Contraction Principle says that any contraction self-mappings on a complete metric space has a unique fixed point. This principle is one of a very power test for existence and uniqueness of the solution of considerable problems arising in mathematics. Because of its importance for mathematical theory, Banach Contraction Principle has been extended and generalized in many directions.

Definition 1.1.1 Let X be non-empty set and $T: X \rightarrow X$ a self map. The point $x \in X$ is said to be fixed point of T if $Tx = x$.

Definition 1.1.2[18] A mapping$T: X \rightarrow X$, where (X, d) is a metric space, is said to be weakly contractive if

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y))$$

where$x, y \in X$and $\varphi: [0, \infty) \rightarrow [0, \infty)$ is a continuous and non-decreasing function such that $\varphi(t) = 0$if and only if$t = 0$.

Theorem 1.1.3[18] If $T: X \rightarrow X$ is a weakly contractive mapping, where (X, d) is a complete metric space, then T has a unique fixed point.

Now the weak contractions are generalization of the Banach contraction mapping, which have been studied by several authors. In [1, 2, 3, 5, 6, 7, 8], the authors proved some types of weak contractions in complete metric spaces, respectively. In particular the existence of a fixed point for weak contraction was extended to partial ordered metric spaces in [2, 8, 16]. Among them some involve altering distance functions. Such functions become famous by Khan et al in [18], where they present some fixed point theorems with the help of such functions. Since its first appearance, the Banach contraction mapping principle has become the main tool to study contractions as they appear abundantly in a wide array of quantitative sciences. Its well known application is in Ordinary Differential Equations, particularly in the proof of the Picard-Lindlof theorem which guarantees the existence and uniqueness of solutions of first order initial value problems. It is worth emphasizing that the remarkable strength of the contraction principle originates from the constructive process it provides to identify the fixed point.

This notable strength further attracted the attention of not only many prominent mathematicians studying in many branches of Mathematics related to non-linear analysis, but also many researchers who are interested in iterative methods to examine the quantitative problems involving certain mappings and space structures required in their work in various areas such as social sciences, Biology, Economics, and Computer Sciences.

In 2012, Yan et al. [29] established a new contraction mapping principle in partially ordered metric spaces. In his research paper published on 03 Nov 2014, Su [26] has proved some fixed point theorems of generalized contraction mappings in a complete metric space endowed with a partial order by using generalized altering distance functions.

Inspired and motivated by the results mentioned on [29] and [26], the purpose of this project was to extend the main theorem of [26] to f-contraction mapping in a complete metric space endowed with a partial order by using generalized altering distance functions and application examples were provided.

1.2. Statement of the problem

This study focused on proving the existence of common fixed points of f-contraction mapping defined on complete metric spaces endowed with a partial order by using generalized altering distance functions. This study tried to answer the following questions:

i. How can we prove the existence of common fixed points of f-contraction mappings defined on complete metric spaces endowed with a partial order by using generalized altering distance functions?

ii. If such common fixed points exist, how can we verify their uniqueness?

iii. How can we support the main result by providing application examples?

1.3. Objective of the study

1.3.1. General Objective

The general objective of this research was to establish common fixed point theorem for f-contraction mapping on complete metric spaces endowed with a partial order by using generalized altering distance function in partially ordered metric spaces.

1.3.2. Specific Objectives

i) To prove common fixed points of f-contraction mapping defined on complete metric spaces endowed with a partial order by using generalized altering distance function in partially ordered metric spaces.

ii) To verify the uniqueness of the common fixed point if it exists.

iii) To provide supporting and application examples in support of the result.

1.4. Significance of the study

Fixed point theory is an interesting area of research with a wide range of application in various fields. This concept is recently becoming a topic of considerable research interest. There are many works about fixed points of contraction mappings [2, 4, 10, 16, 17, 22, 27, 29]. We hope that the result obtained in this study will contribute to research activities in this area. The researcher also benefited from this study since he used to develop scientific research writing skill and scientific communication in Mathematics.

1.5. Delimitation of the study

This study was delimited to finding the common fixed points of f-contraction mappings defined on complete metric spaces endowed with partial order by using generalized altering distance function which will be done under Differential Equations and Functional Analysis streams.

2. REVIEW LITERATURE

Fixed point theory is one of the famous theories in mathematics and has broad applications. The applications of fixed point theory are very important in different disciplines of mathematics. The Banach contraction mapping principle is one of the pivotal results of analysis. It is widely considered as the source of metric fixed point theory and its significance lies in its vast applicability in a number of branches of mathematics. There are a lot of generalizations of the Banach contraction mapping principle in the literature. One of the most interesting of them is the result of Khan et al. [18]. They addressed a new category of fixed point problems for a single self-map with the help of a control function which they called an altering distance function.

Definition 2.1[18] A function $\psi\colon [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:

a. ψ is continuous and monotonically non-decreasing.
b. $\psi(t) = 0$ if and only if $t = 0$.

Example The following function is an altering distance function:

$$\psi(t) = \begin{cases} 0, & t = 0 \\ \beta t, & t \geq 1 \end{cases} \qquad \text{where } \beta \geq 1.$$

Khan et al. [18] tried to prove the next fixed point theorem.

Theorem 2.1 Let (X, d)be a complete metric space, let ψ be an altering distance function,and let$f\colon X \to X$ be a self-mapping which satisfies the following inequality:

$$\psi(d(fx, fy)) \leq c\psi(d(x, y)) \tag{2.1}$$

for all $x, y \in X$and for some $0 \leq c < 1$. Then $f$has a unique fixed point.

In fact Khan et al. proved a more general theorem of which theorem 2.1 is a corollary. Altering distance functions have been used in metric fixed point theory in a number of papers. Some of the works utilizing the concept of altering distance function are noted in [3, 12, 18, 23].

Another generalization of the Banach contraction was suggested by Alber and Guerre-Delabriere [28] in Hilbert Spaces. Rhoades [22] has shown that the result which Alber and Guerre-Delabriere have proved in [28] is also valid in complete metric spaces.

In fact, Alber and Guerre-Delabriere assumed an additional condition on φwhich is $\lim_{t\to\infty} \varphi(t) = \infty$. But Rhoades [22] obtained the result noted in theorem 2.1without using this particular assumption. Also, the weak contractions are closely related to maps of Boyd and Wong [6] and Reich type [21]. Namely, if φ is a lower semi-continuous function from the right, then $\psi(t) = t - \varphi(t)$ is an upper semi-continuous function from the right, and moreover, (2.1) turns into $d(Tx, Ty) \leq \psi(d(x, y))$. Therefore, the weak contraction is of Boyd and Wong type. And if we define $\beta(t) = t - \frac{\varphi(t)}{t}$ for $t > 0$ and $\beta(0) = 0$ then (2.1) is replaced by

$$d(Tx, Ty) \leq \psi(d(x, y))d(x, y) \tag{2.2}$$

Therefore, the weak contraction becomes a Reich-type one. It may be observed that though the function φ has been defined in the same way as the altering distance function, the way it has been used in [16] is completely different from the use of altering distance function. Weakly contractive mappings have been dealt with in a number of papers. Some of these works are noted in [4, 5, 10, 16, 17, 22].

Also, Zhang and Song [21] have given the following generalized version of Theorem 2.1.

Theorem 2.2 Let (X, d) be a complete metric space, and let $f, T: X \to X$ be two mappings such that for each $x, y \in X$,

$$d(Tx, fy) \leq \psi(x, y) - \phi\big(\psi(x, y)\big)$$

where $\phi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function with $\phi(t) > 0$ for $t > 0$ and $\phi(0) = 0$,

$$\psi(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, fy), \frac{1}{2}[d(y, Tx) + d(x, fy)]\right\}$$

Then, there exists a unique point $z \in X$ such that $z = Tz = fz$.

In recent years, many results appeared related to fixed point theorem in complete metric spaces endowed with a partial ordering $\preccurlyeq$ in the literature [2, 13, 20, 27, 29]. Most of them are a hybrid of two fundamental principles: Banach contraction theorem and the weakly contractive condition. Indeed, they deal with a monotone (either order-preserving or order-reversing) mapping satisfying, with some restriction, a classical contractive condition, and such that for some $x_0 \in X$ either $x_0 \preccurlyeq Tx_0$ or $Tx_0 \preccurlyeq x_0$, where T is a self-map on metric space.

Harjani and Sadarangani [16] established some fixed point theorems for weak contractions and generalized contractions in partially ordered metric spaces by using the altering distance function which improved the results in the theorems of [8,22].

Definition 2.2 [26] If $(X, \preccurlyeq)$ is a partially ordered set and$T: X \to X$, we say that $T$is monotone non-decreasing if $x, y \in X, x \preccurlyeq y \Rightarrow T(x) \preccurlyeq T(y)$.

This definition coincides with the notion of a non-decreasing function in the case where $X = \mathbb{R}$and $\preccurlyeq$ represents the usual total order in $\mathbb{R}$.

Definition 2.3 [13] we shall say that the mapping T is f-non-decreasing (resp. f-non-increasing) if $fx \preccurlyeq fy \Rightarrow Tx \preccurlyeq Ty$ (resp.,$fx \preccurlyeq fy \Rightarrow Tx \succcurlyeq Ty$) holds for each $x, y \in X$.

Theorem 2.3[16] Let$(X, \preccurlyeq)$be a partially ordered set and suppose that there exists a metric d such that (X, d) is a complete metric space. Let $f: X \to X$ be a continuous and non-decreasing mapping such that

$$d(f(x), f(y)) \leq d(x, y) - \psi(d(x, y)), \text{ for } x \geq y,$$

where $\psi: [0, \infty) \to [0, \infty)$ is continuous and non-decreasing function such that ψ is positive in $(0, \infty)$, $\psi(0) = 0$ and $\lim_{t \to \infty} \psi(t) = \infty$. If there exists $x_0 \in X$ with $x_0 \preccurlyeq f(x_0)$, then f has a fixed point.

Theorem 2.4 [16] Let$(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $f: X \to X$ be a continuous and non-decreasing mapping such that

$$\psi\Big(d\big(f(x), f(y)\big)\Big) \leq \psi\big(d(x, y)\big) - \varphi\big(d(x, y)\big), \text{for } x \geq y,$$

where ψ and φ are altering distance functions. If there exists $x_0 \in X$ with $x_0 \preccurlyeq f(x_0)$, then $f$has a fixed point.

Further, Harjani and Sadarangani [16] proved the ordered version of Theorem 2.1, Amini-Harandi and Emami [2] proved the ordered version of Rich type fixed point theorem, and Harjani and Sadarangani [16] proved ordered version of Theorem 2.2. Subsequently, Amini-Harandi and Emami proved another fixed point theorem for contraction type maps in partially ordered metric spaces in[2]. The following class of functions is used in [2]. Let $\mathcal{R}$ denote the class of those functions $\beta: [0, \infty) \to [0, \infty)$ which satisfy the condition:

$$\beta(t_n) \to 1 \Rightarrow t_n \to 0.$$

Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $f: X \to X$ be an increasing mapping such that there exists an element $x_0 \in X$ with$x_0 \preccurlyeq f(x_0)$,. Suppose that there exists $\beta \in \mathfrak{R}$ such that

$$d(f(x), f(y)) \leq \beta(d(x, y))\, d(x, y) \text{ for each } x, y \in X \text{ with } x \geq y.$$

Assume that either f is continuous or M is such that if an increasing sequence $x_n \to x \in X$, then $x_n \leq x, \forall n$, besides, if for each $x, y \in X$ there exists $z \in$ M which is comparable to x and y, then f has a unique fixed point.

In 2012, Yan et al. proved the following fixed point theorem.

Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $T: X \to X$ be a continuous and non-decreasing mapping such that

$$\psi(d(Tx, Ty)) \leq \phi(d(x, y)), \forall x \geq y,$$

where ψ is an altering distance function and $\phi : [0, \infty) \to [0, \infty)$ is a continuous function with the condition $\psi(t) > \phi(t)$ for all $t > 0$. If there exists $x_0 \in X$ with $x_0 \preccurlyeq T(x_0)$, then T has a fixed point.

In 2014, Su [26] proved the following fixed point theorem, which is the generalized type of Yan et al [29];

Let $(\mathrm{X}, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $\mathrm{T} : \mathrm{X} \to \mathrm{X}$ be a continuous and non-decreasing mapping such that

$$\psi\big(d(Tx, Ty)\big) \leq \phi\big(d(x, y)\big), \forall x \geq y,$$

where ψ is a generalized altering distance function and $\phi: [0, \infty) \to [0, \infty)$ is a right uppersemi-continuous function with the condition: $\psi(t) > \phi(t)$ for all $t > 0$. If there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$, then T has a fixed point.

In this research project we extended results of Su [26] to f-contraction mappings and obtained some new fixed point theorems for f-contraction mapping in a complete metric space endowed with a partial order by using generalized altering distance functions and common fixed point theorems obtained were proved. Examples were given to show that our results are proper extension of the existing ones.

3. METHODOLOGIES

3.1. Study site and period

This research was conducted in Jimma University under Mathematics department from November 2014 to June 2015 G.C.

3.2. Study design

The design used in this research was Analytical and numerical design.

3.3. Source of information

This study mostly depended on document materials or secondary data. So, the available sources of information for the study were Books, Journals, different study related to the topic and internet service.

3.4. Procedure of the study

The procedures we followed for analysis was the standard iterative techniques and procedures used in Su [26]. Application examples were also provided in support of the results we obtain.

3.5. Ethical consideration

Ethical clearance was obtained from Research and Post Graduate program coordinator Office of College of Natural Sciences, Jimma University and any concerned body was informed about the purpose of the study.

4. DISCUSION AND MAIN RESULT

4.1. PRELIMINARIES

Definition 4.1.1 Let f and T be self maps of a metric space (X,d). The pair (f,T) is called:

i. commuting if $fTx = Tfx$, $\forall x \in X$
ii. weakly commuting if $d(fTx, Tfx) \leq d(fx, Tx)$, $\forall x \in X$

Example 4.1.1.1

Define $f, T: X \to X$ by $fx = \frac{x}{8} - \frac{x^2}{64}$ and $Tx = \frac{x}{2} \forall x \in X$.

$$d(fTx, Tfx) = \left(\frac{x}{8} - \frac{x^2}{256} - \frac{x}{16} + \frac{x^2}{128}\right) = \frac{x^2}{256} \leq \frac{3}{8}x - \frac{x^2}{64} = \frac{x}{2} - \left(\frac{x}{8} - \frac{x^2}{64}\right) = d(Tx, fx)$$

$$d(fTx, Tfx) \leq d(Tx, fx)$$

Therefore f and T are weakly commuting. But

$$fTx = f\left(\frac{x}{2}\right) = \frac{x}{16} - \frac{x^2}{256}$$

$$Tfx = T\left(\frac{x}{8} - \frac{x^2}{64}\right) = \frac{x}{16} + \frac{x^2}{128}$$

$$\Rightarrow fTx = \frac{x}{16} - \frac{x^2}{256} \neq \frac{x}{16} - \frac{x^2}{128} = Tfx$$

$$\Rightarrow fTx \neq Tfx$$

Hence f and T are not commuting.

Therefore f and T are weakly commuting but not commuting.

Definition 4.1.2: [34] Two self maps f and T of a metric space (X,d) are said to be compatible if and only if $\lim_{n\to\infty} d(fTx_n, Tfx_n) = 0$ when $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$. Thus $d(fTx, Tfx) \to 0$ as $d(fx, Tx) \to 0 \Rightarrow f$ & T are compatible.

So, if f and T commute, then they are obviously Compatible.

Example 4.1.2.1 Let $X = \mathbb{R}$. Define $f, T: X \to X$ by $fx = 5x^3$ and $Tx = 2x^3$ $\forall x \in X$.

Then $d(fx, Tx) = |Tx - fx| = |5x^3 - 2x^3| = 3|x|^3 \to 0$ iff $x \to 0$ and

$$d(Tfx, fTx) = |fTx - Tfx|$$

$$= |40x^9 - 250x^9|$$

$$= 210|x|^9 \to 0 \ iff \ x \to 0$$

$$d(fx, Tx) \to 0 \Rightarrow d(Tfx, fTx) \to 0$$

Therefore f and T are compatible. But

$$d(fTx, Tfx) = 210|x|^9 \text{and } d(fx, Tx) = 3|x|^9$$

$$d(fTx, Tfx) = 210|x|^9 \nleq 3|x|^9 = d(fx, Tx).$$

Therefore f and T are not weakly commuting.

Hence f and T are compatible but not weakly commuting.

Definition 4.1.3 [34] Let $f$and T be self maps of a metric space (X, d). The pair (f, T) is called weakly compatible if they commute at their coincidence point. (i.e. if $f(T(x)) = T(f(x))$ for all $x \in C(f, T)$.

Definition 4.1.4 Let $f$and T be self maps of a metric space (X, d). The pair (f, T) is called occasionally weakly compatible (OWC) if there exists $x \in X$which is a coincidence point for f and T at which f and T commute (i.e. if $f(T(x)) = T(f(x))$ for some $x \in C(f, T)$).

Example 4.1.4.1 Let$X = [0, \infty)$ with the usual metric. Define $f, T: X \to X$ by $f(x) = 2x$ and $T(x) = x^2$ for all $x \in X$. Then $C(f, T) = \{0, 2\}$, $f(T(0)) = T(f(0))$and $f(T(2)) \neq T(f(2))$. Thus (f, T) is an Occasionally Weakly Compatible (OWC) pair but not weakly compatible.

Example 4.1.4.2 Let $X = [0,20]$and d is the usual metric on X. Define $f, T: X \to X$ by

$$fx = \begin{cases} 0 \ if x = 0 \\ x + 16 \ \ if \ 0 < x \leq 4 \\ x - 4 \ \ if \ 4 < x \leq 20 \end{cases} \text{ and } Tx = \begin{cases} 0 \ \ if x \in \{0\} \cup (4,20] \\ 3 \ \ if \ 0 < x \leq 4 \end{cases}$$

Show that $f$and T areweakly compatible but not compatible maps.

Solution

Let$x_n = 4 + \frac{1}{n}, n \geq 1$. Then $fx_n = x_n - 4 = 4 + \frac{1}{n} - 4 = \frac{1}{n} \to 0$ as $n \to \infty$.

$Tx_n = 0 \to 0$ as $n \to \infty$.

$f0 = 0 = T0 \Rightarrow x = 0$ is the coincidence point of f and T.

$$fT0 = 0 = Tf0$$

Therefore f and T are weakly compatible maps.

$Tfx_n = T(x_n - 4) = 3$, as $n \to \infty$

$fTx_n = f(0) = 0$, as $n \to \infty$

which implies $\lim_{n\to\infty} d(fTx_n, Tfx_n) = 3 \neq 0$.

Therefore f and T are not compatible.

Q. If $C(f,T) = \emptyset$, then f and T are weakly commuting but not occasionally weakly compatible. Give an example of such mappings.

Example 4.1.4.3 Define $f, T: X \to X$by $fx = 3x$ and $Tx = 3x + 2$.

$3x = 3x + 2 \Rightarrow 0 = 2$. Which is false. i.e. there is no $x \in X$ such that $fx = Tx$.

Therefore f and T have no coincidence point, i.e. $C(f,T) = \emptyset$.

But they are weakly commuting in the empty set. Next

$$fTx = f(3x+2) = 3(3x+2) = 9x + 6$$

$$Tfx = T(3x) = 3(3x) + 2 = 9x + 2$$

These two equations are parallel lines that do not coincide at any point in X. It shows that f and T are not occasionally weakly compatible since there does not exist $x \in X$ such that $fTx = Tfx$ for which $fx = Tx$.

Definition 4.1.5 Consider a function $f: X \to X$ and a point $x_0 \in X$. The function f is said to be upper (resp. lower) semi-continuous at the point x_0 if

$$f(x_0) \geq \lim_{x\to x_0} \sup f(x), \text{(resp. } f(x_0) \leq \lim_{x\to x_0} \inf f(x))$$

Example 4.1.5.1

$$f(x) = \begin{cases} 0, x < 0 \\ 1, x \geq 0 \end{cases} \text{Upper semi-continuous}$$

$$Tx = \begin{cases} 0, x \leq 0 \\ 1, x > 0 \end{cases} \text{ Lower semi-continuous}$$

Example 4.1.5.2 $$fx = \begin{cases} 1, x < 1 \\ 2, x = 1 \\ \frac{1}{2}, x > 1 \end{cases}$$

is upper semi-continuous at x=1 although not left or right continuous. The limit from the left is equal to 1 and the limit from the right is equal to 1/2, both of which are different from the function value of 2.

Example 4.1.5.3 $$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), x \neq 0 \\ 1, x = 0 \end{cases}$$

is upper semi-continuous at $x = 0$ while the function limits from the left or right at zero do not even exist.

Definition4.1.6 A partially ordered set is a set X and a binary relation $\preccurlyeq$, denoted by $(X, \preccurlyeq)$ such that for all $a, b, c \in X$

i. $a \preccurlyeq a$. (Reflexivity)
ii. $a \preccurlyeq b$ and $b \preccurlyeq a \Rightarrow a = b$. (Anti-symmetry).
iii. $a \preccurlyeq b$ and $b \preccurlyeq c \Rightarrow a \preccurlyeq c$. (Transitivity)

Definition 4.1.7 [26] A generalized altering distance function is a function $\psi : [0, \infty) \to [0, \infty)$ which satisfies:

(a) ψ is non-decreasing;
(b) $\psi(t) = 0$ if and only if $t = 0$.

Example 4.1.7.1 The following are some generalized altering distance functions:

$$\psi_1(t) = \begin{cases} 0, t = 0 \\ [t] + 1, t > 0 \end{cases}$$

$$\psi_2(t) = \begin{cases} 0, \ t = 0 \\ \alpha([t] + 1), t > 0 \end{cases}$$

$$\psi_3(t) = \begin{cases} t, 0 \leq t < 1 \\ \alpha t^2, t \geq 1 \end{cases}$$

where $\alpha \geq 1$ is a constant.

Definition 4.1.8 A metric space (X, d) together with a partial ordering $\preccurlyeq$ is said to be partially ordered metric space if the following conditions are satisfied:

i. (X, d) is metric space
ii. $(X, \preccurlyeq)$ is partial ordered set.

Definition 4.1.9 Let (X, d) be a metric space and $T, f: X \to X$ be two functions. A mapping T is said to be f-contraction if there exists $k \in [0,1)$ such that

$$d(fTx, fTy) \leq kd(fx, fy) \text{ for all } x, y \in X.$$

Definition 4.1.10 Let (X, d) be a partially ordered metric space and f, T be two self-mappings on (X, d). A point $z \in X$ is said to be a common fixed point of f and T if $fz = Tz = z$.

Notation:

Let $X = [0, \infty)$ and we denote:

1) The set of all right upper semi-continuous functions by

   $\Phi = \{\phi: X \to X \text{ such that } \phi(t_0) \geq \lim_{t \to t_0} \sup \phi(t) \text{ for } t_0 \in X\}$.

2) The set of all generalized altering distance functions by

   $\Psi = \{\psi: X \to X \text{ such that } \psi \text{ is non-decreasing and } \psi(t) = 0 \text{ if and only if } t = 0\}$.

Theorem 4.1.1 [26] Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d in X such that $(X, d)$is a complete metric space. Let $T : X \to X$ be a continuous and non-decreasing mapping such that

$$\psi\big(d(Tx, Ty)\big) \leq \phi\big(d(x, y)\big), \forall x \geq y,$$

where ψ is a generalized altering distance function and $\phi: [0, \infty) \to [0, \infty)$is a right upper semi-continuous function with the condition: $\psi(t) > \phi(t)$ for all $t > 0$.If there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$,then T has a fixed point.

If (x_n) *is a non-decreasing sequence in X such that* $x_n \to x$ *then* $x_n \leq x$ *for all* $n \in \mathbb{N}$. (*)

Theorem4.1.2 [26] Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Assume that X satisfies (*). Let $T : X \to X$ be a continuous and non-decreasing mapping such that

$$\psi(d(Tx, Ty)) \leq \phi(d(x, y)), \forall x \geq y,$$

where ψ is a generalized altering distance function and $\phi: [0, \infty) \to [0, \infty)$ is a right upper semi-continuous function with the condition: $\psi(t) > \phi(t)$ for all $t > 0$.If there exists $x_0 \in X$

such that $x_0 \preccurlyeq Tx_0$, then T has a fixed point.

for $x, y \in X$ *there exists* $z \in X$ *which is coparable to* x *and* y (**)

Theorem4.1.3 [26] Adding the condition (**) to the hypothesis of theorem 4.2.1 (resp. theorem 4.2.2) wee obtain the uniqueness of the fixed point of T.

4.2. Main result

We start this section with the following definitions. Consider a partially ordered set $(X, \preccurlyeq)$ and two self-maps $f, T: X \to X$ such that $T(X) \subset f(X)$.

Definition 4.2.1 A point $y \in X$ is called point of coincidence of two mappings $f, T: X \to X$ if there exists a point $x \in X$ such that $y = fx = Tx$. In this case x is called the coincidence point of f and T and the set of coincidence points of f and T is denoted by $C(f, T)$.

Definition 4.2.2 Let (X, d) be a metric space and $T, f: X \to X$ be two functions. A mapping T is said to be f-contraction if there exists $k \in [0,1)$ such that

$$d(Tx, Ty) \leq kd(fx, fy) \text{for all } x, y \in X.$$

Theorem 4.2.1: Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric $d$on Xsuch that (X, d) is a complete metric space. Let $f, T: X \to X$ be two self maps on X satisfying the following conditions:

i) $TX \subset fX$;

ii) fX is closed;

iii) T is f-non-decreasing;

iv) there exists $x_0 \in X$ such that $fx_0 \preccurlyeq Tx_0$;

v) if $z \in C(f, T)$, then $fz \preccurlyeq f(fz)$.

such that

$$\psi(d(Tx, Ty)) \leq \phi(d(fx, fy)), \ \forall x, y \in X \text{ with, } fy \preccurlyeq fx \tag{1}$$

where ψ is a generalized altering distance function and $\phi: [0, \infty) \to [0, \infty)$ is a right upper semi-continuous function with the condition$\psi(t) > \phi(t), \forall t > 0$ and $\phi(t) = 0 \Leftrightarrow t = 0$. Then f and T have a coincidence point. Furthermore if f and T are occasionally weakly compatible maps, then f and T have common fixed point, in X.

Proof From condition (iv) we have $x_0 \in X$ such that $fx_0 \preccurlyeq Tx_0$. Since $TX \subset fX$, we can choose $x_1 \in X$ such that $fx_1 = Tx_0$. Again from $TX \subset fX$, we can choose $x_2 \in X$ such that $fx_2 = Tx_1$. Continuing this process, we can choose a sequence $\{y_n\}$ which is called Jungck sequence in X such that

$$fx_{n+1} = Tx_n = y_n, \forall n \geq 0 \tag{2}$$

Since $fx_0 \preccurlyeq Tx_0$ and $fx_1 = Tx_0$, we have $fx_0 \preccurlyeq fx_1$. Then by (iii), we have

$$Tx_0 \preccurlyeq Tx_1 \tag{3}$$

Thus by (2) we obtain$fx_1 \preccurlyeq fx_2$. Again by (iii), we have

$$Tx_1 \preccurlyeq Tx_2 \tag{4}$$

That is $fx_2 \preccurlyeq fx_3$. Continuing this process we obtain

$$Tx_0 \preccurlyeq Tx_1 \preccurlyeq Tx_2 \preccurlyeq Tx_3 \preccurlyeq \cdots \preccurlyeq Tx_n \preccurlyeq Tx_{n+1} \preccurlyeq \cdots \tag{5}$$

Now considering (2) (i.e. $y_n = Tx_n = fx_{n+1}$), from (5) we note that y_n and y_{n+1} are comparable $n \geq 0$,without lose of generality we can assume that $y_n \neq y_{n+1}, \forall n \in \mathbb{N}$. Using the contractive condition (1), we get

$$\psi\big(d(y_{n+1}, y_n)\big) = \psi\big(d(Tx_{n+1}, Tx_n)\big) \leq \phi\big(d(fx_{n+1}, fx_n)\big)$$

$$= \phi(d(y_n, y_{n-1})) < \psi(d(y_n, y_{n-1}))$$

This implies that

$$\psi(d(y_{n+1}, y_n)) < \psi(d(y_n, y_{n-1})) \tag{6}$$

By the non-decreasingness ofψ, from (6) we get

$$d(y_{n+1}, y_n) < d(y_n, y_{n-1}) \tag{7}$$

Hence the sequence $\{d(y_n, y_{n+1})\}$ is decreasing sequence and consequently there exists $r > 0$ such that

$$d(y_{n+1}, y_n) \to r, \text{ as } n \to \infty.$$

Now we claim that $r = 0$. Suppose $r > 0$.

$$\psi(d(Tx_{n+1}, Tx_n)) \leq \phi(d(fx_{n+1}, fx_n)) \tag{8}$$

Considering the non-decreasingness of ψ and the upper semi-continuity ofϕ, and letting $n \to \infty$ in (8) we get

$$\psi(r) \leq \lim_{n\to\infty} \sup \psi\big(d(y_{n+1}, y_n)\big) \leq \lim_{n\to\infty} sup\phi\big(d(y_n, y_{n-1})\big) \leq \phi(r)$$

Hence, we have

$\psi(r) \leq \phi(r)$. Consequently we obtain

$\Rightarrow \psi(r) < \psi(r)$

which is impossible since $r > 0$. Thus $r = 0$.Hence

$$d(y_{n+1}, y_n) \to 0 \tag{9}$$

Now we claim that $\{y_n\}$ is a Cauchy sequence.

Suppose that $\{y_n\}$ is not Cauchy sequence. Then there exists a positive real number ε such that for a given $N \in \mathbb{N}$ there exists $m, n \in \mathbb{N}$ such that $m > n > N$ and

$$d(y_m, y_n) \geq \varepsilon.$$

Since $\{d(y_{n+1}, y_n)\}$ converges to zero, it follows that there exist strictly increasing sequences $\{n_k\}$ and$\{m_k\}$, $k \geq 1$ of positive integers such that $1 < n_k < m_k$,

$$d(y_{m_k}, y_{n_k}) \geq \varepsilon, \qquad \forall k \geq 1 \tag{10}$$

and

$$d(y_{m_k-1}, y_{n_k}) < \varepsilon \tag{11}$$

Using the triangular inequality and the conditions (10) and (11) we have

$$\varepsilon \leq d(y_{m_k}, y_{n_k}) \leq d(y_{m_k}, y_{m_k-1}) + d(y_{m_k-1}, y_{n_k}) < d(y_{m_k}, y_{m_k-1}) + \varepsilon$$

Letting $k \to \infty$ and using (7), we obtain

$$\lim_{k\to\infty} d(y_{m_k}, y_{n_k}) = \varepsilon. \tag{12}$$

Using the triangular inequality, we obtain

$$d(y_{m_k-1}, y_{n_k-1}) \leq d(y_{m_k-1}, y_{m_k}) + d(y_{m_k}, y_{n_k}) + d(y_{n_k}, y_{n_k-1})$$

and

$$d(y_{m_k}, y_{n_k}) \leq d(y_{m_k}, y_{m_k-1}) + d(y_{m_k-1}, y_{n_k-1}) + d(y_{n_k-1}, y_{n_k})$$

Now letting $k \to \infty$ in the above two inequalities and using (12), we have

$$\lim_{k\to\infty} d(y_{m_k-1}, y_{n_k-1}) = \varepsilon. \tag{13}$$

Since ψ is non-decreasing on $[0, \infty)$, from (10) we have,

$$\psi(\varepsilon) \leq \psi\left(d(y_{n_k}, y_{m_k})\right), \forall k \geq 1, \tag{14}$$

As $m_k > n_k$, by (5) $y_{m_{k-1}}$ and $y_{n_{k-1}}$ are comparable. So from the condition (1), using (5) and the upper semi-continuity of ϕ, we have

$$\psi(\varepsilon) \leq \lim_{k\to\infty} sup\psi\left(d(y_{m_k}, y_{n_k})\right) = \limsup_{k\to\infty}\psi\left(d(Tx_{m_k}, Tx_{n_k})\right)$$

$$\leq \limsup_{k\to\infty}\phi\left(d(y_{m_k-1}, y_{n_k-1})\right) \leq \phi(\varepsilon)$$

This implies

$$\psi(\varepsilon) \leq \phi(\varepsilon) < \psi(\varepsilon),$$

which is impossible since $\varepsilon > 0$.

Thus the sequence $\{y_n\}$ is a Cauchy sequence in X.

Since (X, d) is a complete metric space, there exists $y \in X$ such that $y_n \to y$ as $n \to \infty$.

By (2) $\{y_n\} \subseteq fX$, where $y_n = fx_{n+1}$, for each $n = 1,2,3, \cdots$ and fX is closed then there exists $p \in X$ such that y $= fp$.

Next we show that $Tp = y$.

Now by the continuity of f and T, we obtain

$$\begin{aligned}
\psi\big(d(Tp, y)\big) &= \psi\big(d(Tp, \lim_{n\to\infty} Tx_n)\big) \\
&= \psi\left(d\left(Tp, T(\lim_{n\to\infty} x_n)\right)\right) \\
&\leq \phi\left(d\left(fp, f(\lim_{n\to\infty} x_n)\right)\right) \\
&= \phi\left(d\left(fp, \lim_{n\to\infty} fx_n\right)\right) \\
&= \phi(d(fp, fp)) = 0
\end{aligned}$$

This implies that $\psi\big(d(Tp, y)\big) = 0$ and hence $d(Tp, y) = 0$. As a result we have

$$Tp = y = fp \tag{15}$$

Thus p is the coincidence point of f and T, which implies $C(f, T) \neq \emptyset$. Since f and T are occasionally weakly compatible pair of self maps, f and T commute at some $z \in C(f, T)$.

Now set $w = fz = Tz$. Since f and T are occasionally weakly compatible,

$$fw = f(Tz) = T(fz) = Tw,$$

which implies

$$fw = Tw. \tag{16}$$

Next we claim that $fw = Tw = w$. Suppose $Tw \neq w$. By the condition (v) of Theorem 4.2.1, we have $fz \preccurlyeq f(fz) = fw$. Then

$$\psi\big(d(Tw,w)\big) = \psi\big(d(Tw,Tz)\big) \le \phi\big(d(fw,fz)\big)$$

$$= \phi(d(Tw,w)) < \psi(d(Tw,w))$$

which implies that

$$\psi(d(Tw,w)) < \psi(d(Tw,w)),$$

a contradiction. Thus $Tw = w$. And hence by (15), we have

$$fw = Tw = w.$$

Thus, we have proved that f and T have a common fixed point.

Example 4.2.1.1 Let $X = \{-2,-1,0,1\}$ and

$$\preccurlyeq = \{(-2,-2),(-1,-1),(0,0),(1,1),(-1,0),(0,1),(-1,1),(-2,0),(-2,1)\}.$$

Let $f,T\colon X \to X$ defined by

$$f(-1) = 1, f(0) = 0, f(1) = -2, f(-2) = -1,$$

$$T(-1) = 0, T(0) = 0, T(1) = -1, T(-2) = 0.$$

$T(X) = \{-1,0\}$ and $f(X) = \{-2,-1,0,1\}$

which implies that $T(X) \subset f(X)$.

T is f-non-decreasing. Since

$$-1 = f(-2) \preccurlyeq f(-1) = 1 \Rightarrow 0 = T(-2) \preccurlyeq T(-1) = 0$$

$$-2 = f(1) \preccurlyeq f(-1) = 1 \Rightarrow -1 = T(1) \preccurlyeq T(-1) = 0$$

$$-2 = f(1) \preccurlyeq f(0) = 0 \Rightarrow -1 = T(1) \preccurlyeq T(0) = 0$$

$$-1 = f(-2) \preccurlyeq f(0) = 0 \Rightarrow 0 = T(-2) \preccurlyeq T(0) = 0$$

$$0 = f(0) \preccurlyeq f(-1) = 1 \Rightarrow 0 = T(0) \preccurlyeq T(-1) = 0$$

In all the above 5 cases, the pair of mappings f and T satisfy all conditions of Theorem 4.2.1 with $\psi(t) = \frac{1}{2}t$ and $\phi(t) = \frac{1}{3}t$. Also 0 is the common fixed point of f and T.

From the above example one can understand that T cannot satisfy the contraction condition of Su because T is f-contraction not contraction. For if $x = 0$ and $y = 1$, then

$$\psi\big(d(Tx,Ty)\big) = \psi(|Tx - Ty|) = \psi(|0+1|) = \psi(1)$$

$$\phi\big(d(x,y)\big) = \phi(|x-y|) = \phi(|0-1|) = \phi(1)$$

Then using the contraction condition of Su, we obtain

$$\psi\big(d(Tx,Ty)\big) \leq \phi\big(d(x,y)\big)$$

which implies,

$$\psi(1) \leq \phi(1)$$

Thus T does not satisfy the contraction condition of Su for any ϕ and ψ because of the condition $\psi(t) \geq \phi(t)\ \forall t > 0$.

Remark 1 From this example we conclude that theorem 4.2.1 generalizes Theorem 2.3 of Su and hence we get Theorem 4.1.1 as the corollary to theorem 4.2.1 of this paper.

In what follows, we prove that Theorem 4.2.1 is still valid for f and T being not necessarily continuous, assuming the following hypothesis in X:

$$\begin{gathered}\textit{If } \{y_n\} \textit{ is a non-decreasing sequence in } X \textit{such that } y_n \to y \\ \textit{then } y_n \preccurlyeq y \textit{ for all } n \in \mathbb{N}.\end{gathered} \tag{17}$$

Theorem 4.2.2 Let $(X, \preccurlyeq)$be a partially ordered set and suppose that there exists a metric d in X such that $(X,d)$is a complete metric space. Assume that X satisfies (17). Let $f, T: X \to X$ be two self maps satisfying the following conditions:

i) $TX \subset fX$;

ii) fX is closed;

iii) T is f-non-decreasing;

iv) there exists $x_0 \in X$ such that $fx_0 \preccurlyeq Tx_0$;

v) if $z \in C(f,T)$, then $fz \preccurlyeq f(fz)$

such that

$$\psi(d(Tx,Ty)) \leq \phi(d(fx,fy)),\ \forall x,y \in X \text{ with } fy \preccurlyeq fx$$

where ψ isan altering distance functions and $\phi: [0,\infty) \to [0,\infty)$is a right upper semi-continuous function with the condition $\psi(t) > \phi(t)$for all $t > 0$. Then f and T have a coincidence point. Furthermore if f and T are occasionally weakly compatible maps, then f and T have common fixed point.

Proof Suppose there exists$x_0 \in X$ such that $fx_0 \preccurlyeq Tx_0$. Since $TX \subset fX$, we can choose $x_1 \in X$ such that $fx_1 = Tx_0$. Again from $TX \subset fX$, we can choose $x_2 \in X$ such that $fx_2 = Tx_1$. Continuing this process, we can choose a sequence $\{y_n\}$ in X such that

$$y_n = fx_{n+1} = Tx_n,\ \forall n \geq 0. \tag{19}$$

Since $fx_0 \preccurlyeq Tx_0$ and $fx_1 = Tx_0$ we have

$fx_0 \preccurlyeq fx_1$. Then by condition (iii) $Tx_0 \preccurlyeq Tx_1$.

Thus by (18) $fx_1 \preccurlyeq fx_2$. Again by condition (iii) $Tx_1 \preccurlyeq Tx_2$. That is $fx_2 \preccurlyeq fx_3$.

Continuing this process, we obtain

$$Tx_0 \preccurlyeq Tx_1 \preccurlyeq Tx_2 \preccurlyeq Tx_3 \ldots \preccurlyeq Tx_n \preccurlyeq Tx_{n+1} \tag{20}$$

Now considering (19) (i.e. $y_n = fx_{n+1} = Tx_n$), from (18) we note that y_n and y_{n+1} are comparable for each $n \geq 0$. Without lose of generality we can assume that $y_n \neq y_{n-1} \forall n \geq 1$.

$$\psi(d(y_{n+1}, y_n)) \leq \phi(d(y_n, y_{n-1})) < \psi(d(y_n, y_{n-1}))$$

So, $\psi(d(y_{n+1}, y_n)) < \phi(d(y_n, y_{n-1}))$

Since the function ψ is non-decreasing, it follows

$d(y_{n+1}, y_n) < d(y_n, y_{n-1})$, and hence the sequence $\{d(y_{n+1}, y_n)\}$ is decreasing sequence and consequently there exists $r > 0$ such that

$$d(y_{n+1}, y_n) \to r^+, \text{ as } n \to \infty.$$

We claim that $r = 0$. Suppose $r > 0$.

$$\psi(d(Tx_{n+1}, Tx_n)) < \phi(d(fx_{n+1}, fx_n)). \tag{21}$$

Considering the property of $\psi(t) > \phi(t), \forall t > 0$, and letting $n \to \infty$ in (21) we get

$$\psi(r) \leq \lim_{n\to\infty} sup\, \psi\big(d(y_{n+1}, y_n)\big) \leq \lim_{n\to\infty} sup\, \phi\big(d(y_n, y_{n-1})\big) \leq \phi(r)$$

This implies $\psi(r) \leq \phi(r)$,

which is impossible since by our assumption $r > 0$.

Therefore $r = 0$.

Hence $$d(y_{n+1}, y_n) \to 0, \text{ as } n \to \infty \tag{22}$$

Now we claim that $\{y_n\}$ is a Cauchy sequence. Suppose that $\{y_n\}$ is not a Cauchy sequence. Then there exists a positive real number ε such that for given $N \in \mathbb{N}$, there exists $m, n \in \mathbb{N}$ and

$$d(y_m, y_n) \geq \varepsilon$$

Since $\{d(y_{n+1}, y_n)\}$ converges to zero, it follows that there exist strictly increasing sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $1 < n_k < m_k$,

$$d\left(y_{m_k}, y_{n_k}\right) \geq \varepsilon \tag{23}$$

and

$$d\left(y_{m_k-1}, y_{n_k-1}\right) < \varepsilon \tag{24}$$

Using the triangular inequality and from (23) and (24) we have

$$\varepsilon \leq d\left(y_{m_k}, y_{n_k}\right) \leq d\left(y_{m_k}, y_{m_k-1}\right) + d\left(y_{m_k-1}, y_{n_k}\right) < d\left(y_{m_k}, y_{m_k-1}\right) + \varepsilon$$

Letting $k \to \infty$ and using (22) we obtain

$$\lim_{k\to\infty} d\left(y_{m_k}, y_{n_k}\right) = \varepsilon \tag{25}$$

Again using the triangular inequality, we obtain

$$d\left(y_{m_k}, y_{n_k}\right) \leq d\left(y_{m_k}, y_{m_k-1}\right) + d\left(y_{m_k-1}, y_{n_k-1}\right) + d\left(y_{n_k-1}, y_{n_k}\right)$$

and

$$d\left(y_{m_k-1}, y_{n_k-1}\right) \leq d\left(y_{m_k-1}, y_{m_k}\right) + d\left(y_{m_k}, y_{n_k}\right) + d\left(y_{n_k}, y_{n_k-1}\right)$$

Letting $k \to \infty$ in the above two inequalities and using (25)., we have

$$\lim_{k\to\infty} d\left(y_{m_k-1}, y_{n_k-1}\right) = \varepsilon \tag{26}$$

Since ψ is non-decreasing on $[0, \infty)$, from (23) we have

$$\psi(\varepsilon) \leq \psi\left(d\left(y_{m_k}, y_{n_k}\right)\right), \quad \forall k \geq 1 \tag{27}$$

As $m_k > n_k$, by (20) y_{m_k-1} and y_{n_k-1} are comparable. So, from the condition (18) using (26),(27) and the upper semi-continuity ofϕ, we have

$$\psi(\varepsilon) \leq \limsup_{k\to\infty} \psi\left(d\left(y_{m_k}, y_{n_k}\right)\right) = \limsup_{k\to\infty} \psi\left(d\left(Tx_{m_k}, Tx_{n_k}\right)\right)$$

$$\leq \limsup_{k\to\infty} \phi\left(d\left(fx_{m_k}, fx_{n_k}\right)\right) \leq \phi(\varepsilon).$$

This implies that $\psi(\varepsilon) \leq \phi(\varepsilon)$, which is impossible since$\varepsilon > 0$. Thus the sequence $\{y_n\}$ is a Cauchy sequence in X.

Since (X, d) is a complete metric space and using (17),$\{y_n\} \subset f(X)$ where $y_n = fx_{n+1}$ for each $n \geq 1$ and $f(X)$ is closed, then there exists $p \in X$ such that$y = fp$.

Now we prove that $Tp = y$. Then, by the continuity of ψ and the upper semi-continuity of ϕ, using the condition (18), we have

$$\psi\big(d(Tp, y)\big) = \psi\left(d\left(Tp, \lim_{n\to\infty} Tx_n\right)\right) = \lim_{n\to\infty} \psi\big(d(Tp, Tx_n)\big)$$

$$\leq \lim_{n\to\infty} \phi\big(d(fp, fx_n)\big)$$

$$\leq \lim_{n\to\infty} \psi\big(d(fp, fx_n)\big)$$

$$= \psi(\lim_{n\to\infty} d(fp, fx_n))$$

$$= \psi\big(d(fp, \lim_{n\to\infty} fx_n)\big)$$

$$= \psi\big(d(fp, fp)\big) = \psi(0) = 0$$

This implies that $\psi\big(d(Tp, y)\big) = 0$ and hence $Tp = y$

Thus

$$y = Tp = fp \tag{28}$$

Thus p is the coincidence point of f and T, which implies that $C(f, T) \neq \emptyset$. Since f and T are occasionally weakly compatible pair of self maps, f and T commute at some point $z \in C(f, T)$.

Now set $w = fz = Tz$. Since f and T are occasionally weakly compatible,

$$fw = f(Tz) = T(fz) = Tw$$

$$fw = Tw \tag{29}$$

Now we claim that w is a common fixed poin of f and T.

Now if $Tw \neq w$, since by (v) of Theorem 3.2.2, $fz \preccurlyeq f(fz) = fw$, we have

$$\psi\big(d(Tw, w)\big) = \psi\big(d(Tw, Tz)\big) \leq \phi\big(d(fw, fz)\big) \leq \phi\big(d(Tw, w)\big) < \psi\big(d(Tw, w)\big)$$

which is absurd. Hence $Tw = w$.

Therefore $fw = Tw = w$.

Now we present an example where it can be appreciated that the hypothesis in Theorem 4.2.1 and Theorem 4.2.2 do not guarantee uniqueness of the common fixed point.

Example 4.2.2.1

Let $X = \{1,2,3,4,5\}$ and a metric $d: X \times X \to \mathbb{R}$ be defined by $d(x, y) = |x - y|$ for all $x, y \epsilon X$. We define a partial order "$\preccurlyeq$" on X by

$\preccurlyeq = \{(1,1), (2,2), (3,3), (4,4), (5,5), (3,4), (3,5), (4,5)\}$ and we define $\psi, \phi: [0\infty) \to [0\infty)$ by $\psi(t) = \frac{1}{3}t$ and $\phi(t) = \frac{3}{10}t$.

Then the pair $(X, \preccurlyeq)$ is partially ordered set. Consider the mappings $f, T: X \to X$ defined by

$$f(1) = 1, f(2) = 2, f(3) = 2, f(4) = 2, f(5) = 3$$

$$T(1) = 1, T(2) = 2, T(3) = 2, T(4) = 2, T(5) = 2$$

The set of all coincidence points of f and T, $C(f, T) = \{1,2\}$.

i. $T(X) = \{1,2\} \subset \{1,2,3\} = f(X)$
ii. $f(X) = \{1,2,3\}$ is closed.
iii. T is f-non-decreasing
Since $1 = f(1) \preccurlyeq 1 = f(1) \Rightarrow 1 = T(1) \preccurlyeq 1 = T(1)$
$2 = f(2) \preccurlyeq 2 = f(2) \Rightarrow 2 = T(2) \preccurlyeq 2 = T(2)$
$2 = f(3) \preccurlyeq 2 = f(3) \Rightarrow 2 = T(3) \preccurlyeq 2 = T(3)$
$2 = f(4) \preccurlyeq 2 = f(4) \Rightarrow 2 = T(4) \preccurlyeq 2 = T(4)$
$3 = f(5) \preccurlyeq 3 = f(5) \Rightarrow 2 = T(5) \preccurlyeq 2 = T(5)$
$2 = f(3) \preccurlyeq 2 = f(4) \Rightarrow 2 = T(3) \preccurlyeq 2 = T(4)$
$2 = f(3) \preccurlyeq 3 = f(5) \Rightarrow 2 = T(3) \preccurlyeq 2 = T(5)$
$2 = f(4) \preccurlyeq 3 = f(5) \Rightarrow 2 = T(4) \preccurlyeq 2 = T(5)$

iv. There is $x_0 = 1 \in X$ such that $fx_0 \preccurlyeq Tx_0$ implies $1 \preccurlyeq 1$.
v. $z = 2 \in C(f, T) = \{1,2\}$ which implies that $fz \preccurlyeq f(fz)$, implying $2 \preccurlyeq 2$.

Thus, f and T satisfy all the conditions of Theorem 4.2.1 and Theorem 4.2.2 with $\psi(t) = \frac{1}{3}t$ and $\phi(t) = \frac{3}{10}t$. Since $C(f, T) \neq \emptyset$, f and T are occasionally weakly compatible maps. Moreover, 1 and 2 are common fixed points of f and T. Hence the uniqueness of common fixed point of f and T is not guaranteed by the conditions of Theorem 4.2.1 and Theorem 4.2.2.

Remark 2 By choosing a map T to be f-non-decreasing map in theorem 4.2.2 we get theorem 4.1.2 as a corollary to theorem 4.2.2.

Lemma 4.2.1 [32] Let X be a non-empty set, f and T are occasionally weakly compatible self maps of X. If f and T have a unique point of coincidence, $w = fx = Tx$ then w is the unique common fixed point of f and T.

Proof Let z be another point of coincidence of f and T.

Now $z = Tw = Tfx = fTx = fw$

This implies $z = Tw = fw$

It follows that w is coincidence point of f and T. Since f and T have unique point of coincidence $z = w$ and hence $w = fw = Tw$. Let $y = fy = Ty$ be another coincidence point of f and T. By the hypothesis of Lemma 4.2.1 the point of coincidence of f and T is unique and hence $y = w$. Therefore f and T have unique common fixed point.

In what follows, we give sufficient condition for the uniqueness of the fixed point of f and T in Theorems 4.2.1 and 4.2.2? We try to answer this question in the following theorem.

Theorem 4.2.3 In addition to the hypothesis of Theorem 4.2.1 and Theorem 4.2.2, suppose that $f: X \to X$ is non-decreasing and for every $x, y \in X$ there exists $z \in X$ which is comparable to x and y. Then f and T have a unique common fixed point in X.

Proof By either Theorem 4.2.1 or Theorem 4.2.2, the set of common fixed points of f and T is non-empty.

Suppose that there exist $y, z \in X$ which are common fixed points of f and T. We consider two cases.

Case1. If y is comparable to z, then $y = Ty$ is comparable to $z = Tz$. So,

$$\psi\big(d(y,z)\big) = \psi\big(d(Ty,Tz)\big) \leq \phi\big(d(fy,fz)\big) = \phi\big(d(y,z)\big).$$

As the condition $\psi(t) > \phi(t)$ for $t > 0$, we obtain $d(y,z) = 0$ which in turn implies $y = z$.

Case2. If y is not comparable to z, then there exists $x_0 \in X$ which is comparable to y and z; i.e., either $x_0 \preccurlyeq y$ and $x_0 \preccurlyeq z$ or $y \preccurlyeq x_0$ and $z \preccurlyeq x_0$.

Without loss of generality let us take $y \preccurlyeq x_0$ and $z \preccurlyeq x_0$.

Now $x_0 \preccurlyeq y \Rightarrow fx_0 \preccurlyeq fy$, since f is non-decreasing on X.

$\Rightarrow Tx_0 \preccurlyeq Ty$, since T is f-non-decreasing on X.

But $TX \subset fX$. Then there exists $x_1 \in X$ such that $Tx_0 = fx_1$. It follows that

$$fx_1 \preccurlyeq y = fy.$$

Since T is f-non-decreasing on X, this implies

$$Tx_1 \preccurlyeq Ty = y.$$

Now again since $TX \subset fX$, there exists $x_2 \in X$ such that $Tx_1 = fx_2$. This implies

$$fx_2 \preccurlyeq y = fy$$

Proceeding this way, inductively we construct a sequence $\{p_n\}$ such that $\forall n \geq 0$,

$$p_n \preccurlyeq y,$$

where $p_n = Tx_n = fx_{n+1}$ for each $n = 0,1,2,\cdots$.

If there exists $N \in \mathbb{Z}_+$ such that $y = p_N$, then

$$\psi\big(d(y, p_{N+1})\big) = \psi\big(d(Ty, Tx_{N+1})\big) \leq \phi\big(d(fy, fx_{N+1})\big) = \phi\big(d(y, p_N)\big) = 0,$$

which implies that $y = p_n, \forall n \geq N$ and hence the sequence $\{p_n\} \to y$ as $n \to \infty$.

Suppose that $y \neq p_n, \forall n \geq 0$. Then

$$\psi\big(d(y, p_n)\big) = \psi\big(d(Ty, Tx_n)\big) \leq \phi\big(d(fy, fx_n)\big) = \phi\big(d(y, p_{n-1})\big) \tag{30}$$

which implies that

$$\psi\big(d(y, p_n)\big) \leq \phi\big(d(y, p_{n-1})\big) < \psi\big(d(y, p_{n-1})\big), \forall n = 1,2,3, \ldots$$

From the property of ψ, we notice that $\{d(y, p_n)\}$ is a non-decreasing sequence and hence there exists $b \geq 0$ such that

$$d(y, p_n) \to b \text{ as } n \to \infty.$$

We claim that $b = 0$.

Letting $n \to \infty$ in (30) and taking into account the property of ψ and ϕ, we obtain $\psi(b) \leq \phi(b)$. This and the condition $\psi(t) > \phi(t)$ for $t > 0$ imply $b = 0$. Hence,

$$\lim_{n\to\infty} d(y, p_n) = 0.$$

In similar line, it can be proved that

$$\lim_{n\to\infty} d(z, p_n) = 0.$$

Finally as $\lim_{n\to\infty} d(y, p_n) = \lim_{n\to\infty} d(z, p_n) = 0$, the uniqueness of limits in metric spaces gives us $y = z$.

This completes the proof.

Remark 3 By choosing T to be f-non-decreasing map in theorem 4.2.3, we get Theorem 4.1.3 is the corollary to theorem 4.2.3.

The following example is an example in support of theorem 4.2.3.

Example 4.2.3.1 Let $X = [-3,3]$ and define order relation "$\preccurlyeq$" on X by

$$x \preccurlyeq y \Leftrightarrow \{(x = y)\ or (x \in [-3,0]\ \&\ y \in [0,3])\}.$$

We observe that $(X, \preccurlyeq)$ is partially ordered set.

Define $d: X \times X \to \mathbb{R}$ by

$$d(x,y) = |x - y|\ \forall x, y \in X.$$

Consider the mapping $f, T: X \to X$ defined by $Tx = \frac{x}{3}$ and $fx = \frac{x}{2}$. Define $\psi, \phi: [0,\infty) \to [0,\infty)$ by $\psi(t) = \begin{cases} \frac{11}{60}t, 0 \le t < 1 \\ \frac{1}{5}t,\ t \ge 1 \end{cases}$ and $\phi(t) = \frac{1}{6}t$. Then ψ and ϕ satisfy the conditions of the theorem. Here we observe that

i) $T([-3,3]) = [-1,1] \subset \left[-\frac{3}{2}, \frac{3}{2}\right] = f([-3,3])$;

ii) $f([-3,3]) = \left[-\frac{3}{2}, \frac{3}{2}\right]$ is closed;

iii) T is f-non-decreasing,
Since if $x, y \in [-3,3]$ such that $fx \preccurlyeq fy$, then either $fx = fy$ or $fx \in [-3,0]$ and $fy \in [0,3]$.
$\Rightarrow x = y$ or $x \in [-3,0]$ and $y \in [0,3]$. $\left(\because fx, fy \in f([-3,3]) = \left[\frac{-3}{2}, \frac{3}{2}\right]\right)$
$\Rightarrow Tx = Ty$ or $Tx \in [-3,0]$ and $Ty \in [0,3]$
$\Rightarrow Tx \preccurlyeq Ty$

iv) Clearly there exists $x_0 = 0 \in [-3,3]$ such that $fx_0 = Tx_0$,
i.e. $fx_0 \preccurlyeq Tx_0$.

v) f is a non-decreasing, since if $x, y \in [-3,3]$ such that $x \preccurlyeq y$, then either $x = y$ or $x \in [-3,0]$ and $y \in [0,3]$
$\Rightarrow fx = fy$ or $fx = \frac{x}{2} \in [-3,0]$ and $fy = \frac{y}{2} \in [0,3]$
$\Rightarrow fx \preccurlyeq fy$

Now let $x, y \in [-3,3]$ such that$fx \preccurlyeq fy$. Then either $fx = fy$ or $fx \in [-3,0]$ and $fy \in [0,3]$.

Case (i) If $fx = fy$, we have$\frac{x}{2} = \frac{y}{2}$, which implies $Tx = Ty$ and hence obviously the inequality (1) and (18) holds.

Case (ii) If $fx \in [-3,0]$ and $fy \in [0,3]$, then $\frac{x}{2} \in [-3,0]$ and $\frac{y}{2} \in [0,3]$.

This implies that

$x \in [-3,0]$ and $y \in [0,3]$ ($\because fx, fy \in f([-3,3]) = [-\frac{3}{2}, \frac{3}{2}]$).

Now we shall consider two sub-cases

i) If $\mathbf{0} \leq \boldsymbol{y} - \boldsymbol{x} < 1$, then

$$\psi\big(d(Tx,Ty)\big) = \psi\left(\frac{1}{3}(y-x)\right)$$

$$= \frac{11}{60}\left(\frac{1}{3}(y-x)\right)$$

$$= \frac{11}{180}(y-x)$$

$$\leq \frac{1}{12}(y-x)$$

$$= \frac{1}{6}\left(\frac{1}{2}(y-x)\right)$$

$$= \phi\left(\frac{1}{3}(y-x)\right)$$

$$= \phi\big(d(fx,fy)\big)$$

ii) If $\boldsymbol{y} - \boldsymbol{x} \geq \mathbf{1}$, then

$$\psi\big(d(Tx,Ty)\big) = \psi\left(\frac{1}{3}(y-x)\right)$$

$$= \frac{1}{5}\left(\frac{1}{3}(y-x)\right)$$

$$= \frac{1}{15}(y-x)$$

$$\leq \frac{1}{12}(y-x)$$

$$= \frac{1}{6}\left(\frac{1}{2}(y-x)\right)$$

$$= \phi\left(\frac{1}{3}(y-x)\right)$$

$$= \phi\big(d(fx, fy)\big)$$

Thus

$$\psi\big(d(Tx,Ty)\big) \le \phi\big(d(fx,fy)\big)\ \forall x,y \in X \text{ with } fx \preccurlyeq fy;\text{ for}$$

$$\psi(t) = \begin{cases} \frac{11}{60}t, 0 \le t < 1 \\ \frac{1}{5}t,\ \ t \ge 1 \end{cases} \text{ and } \ \phi(t) = \frac{1}{6}t.$$

Thus, f and T satisfy all the conditions of Theorem 4.2.1 and Theorem 4.2.2.

Moreover, 0 is a unique common fixed point of f and T.

From Example 4.1.7.1 let us chose $\psi\,(t) = \psi_3(t)$

$$\psi\,(t) = \begin{cases} t, 0 \le t < 1 \\ \alpha t^2, t \ge 1 \end{cases} \text{ and}$$

$$\phi(t) = \begin{cases} \mathrm{t}^2, 0 \le t < 1 \\ \beta t,\ \ t \ge 1 \end{cases}$$

where $0 < \beta < \alpha$ is a constant. By using Theorem 4.2.1 we can get the following result.

Theorem 4.2.4 Let X be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $f, T: X \to X$ be continuous and T is f-non-decreasing map such that

$$0 \le d(Tx,Ty) < 1 \Rightarrow d(Tx,Ty) \le (d(fx,fy))^2\,,$$
$$d(Tx,Ty) \ge 1 \Rightarrow \alpha(d(Tx,Ty))^2 \le\ \beta d(fx,fy)$$

for any $x, y \in X$ for which $fx \preccurlyeq\ fy$. If there exists $x_0 \in X$ such that $fx_0 \preccurlyeq Tx_0$;
such that

$$\psi(d(Tx,Ty)) \le \phi(d(fx,fy)),\ \forall x,y \in X\ \text{ with, } \ fy \preccurlyeq fx$$

where ψ is a generalized altering distance function and $\phi: [0,\infty) \to [0,\infty)$ is a right upper semi-continuous function with the condition $\psi(t) > \phi(t), \forall t > 0$.Then f and T have a coincidence point. Furthermore if f and T are occasionally weakly compatible maps, then f and T have common fixed point, in X.

Proof:

To prove this theorem we consider two cases as follows:

Case (i) Let $0 \leq d(Tx, Ty) < 1$.

From $\psi\ (t) = \begin{cases} t, 0 \leq t < 1 \\ \alpha t^2, t \geq 1 \end{cases}$ and $\phi(t) = \begin{cases} \mathrm{t}^2, 0 \leq t < 1 \\ \beta t, \quad t \geq 1 \end{cases}$ we can obtain,

$\psi\big(d(Tx, Ty)\big) = d(Tx, Ty)$ and $\phi\big(d(fx, fy)\big) = \big(d(fx, fy)\big)^2$. Then by the contraction condition (1) we have above,

$$\psi\big(d(Tx, Ty)\big) = d(Tx, Ty) \leq \big(d(fx, fy)\big)^2 = \phi\big(d(fx, fy)\big)$$

which implies that,

$$d(Tx, Ty) \leq \big(d(fx, fy)\big)^2$$

Case (ii) Let $d(Tx, Ty) \geq 1$.

From $\psi\ (t) = \begin{cases} t, 0 \leq t < 1 \\ \alpha t^2, t \geq 1 \end{cases}$ and $\phi(t) = \begin{cases} \mathrm{t}^2, 0 \leq t < 1 \\ \beta t, \quad t \geq 1 \end{cases}$ we can obtain,

$\psi\big(d(Tx, Ty)\big) = \alpha\big(d(Tx, Ty)\big)^2$ and $\phi\big(d(fx, fy)\big) = \beta d(fx, fy)$.

Again by the contraction condition (1) we have above,

$$\psi\big(d(Tx, Ty)\big) = \alpha\big(d(Tx, Ty)\big)^2 \leq \beta d(fx, fy) = \phi\big(d(fx, fy)\big)$$

which implies that,

$$\alpha\big(d(Tx, Ty)\big)^2 \leq \beta d(fx, fy)$$

From the conditions of theorem 4.2.1 since the set of all coincidence point of f and T is non-empty, $C(f, T) \neq \emptyset$ it implies that f and T are occasionally weakly compatible maps, then f and T have common fixed point.

5. CONCLUSION AND RECOMMENDATION

5.1. CONCLUSION

In this research work, we proved three fixed point theorems namely Theorem 4.2.1, Theorem 4.2.2 and Theorem 4.2.4 on the existence of common fixed points for f-contraction mapping on complete metric spaces endowed with a partial order by using generalized altering distance function in partially ordered metric spaces. By imposing additional condition to theorems 4.2.1 and 4.2.2 we also proved the uniqueness of common fixed point for f-contraction mapping on complete metric spaces by using generalized altering distance function in partially ordered metric spaces.

1. By remark 1, we conclude that theorem 4.1.1 is a corollary to Theorem 4.2.1.
2. By remark 2, we conclude that theorem 4.2.2 is more general than theorem 4.1.2

Our result extends and improved the extension of the work of Yangfu Su [26] and also our work in this paper is the new and original work on the common fixed points of f-contraction mappings on complete metric spaces endowed with a partial order by using generalized altering distance function in partially ordered metric spaces.

5.2. RECOMMENDATION

Now a day the fixed point theory is the most desirable area of study. The existence of common fixed points of contraction mappings using altering distance function and generalized altering distance function in partially ordered metric spaces were the research papers recently published. There are a number of published research papers related to this area of study. So we recommend the upcoming Post Graduate students and any other researcher of the department who will interested to do their research work in this area of study.

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