A COMMON FIXED POINT THEOREM FOR A PAIR OF MAPPINGS IN CONE b-HEPTAGONAL METRIC SPACES

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS

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Declaration

I, the undersigned declare that, this research paper entitled ”A common fixed point theorem for a pair of mappings in cone *b*-heptagonal metric spaces” is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged.

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Abstract

In this research work we establish a common fixed point result in cone b-heptagonal metric spaces and proved the existence and uniqueness of a common fixed point for a pair of self-mappings involving certain contractive type conditions in cone b-heptagonal metric spaces setting without assuming the normality condition of cone. Our result extends and generalizes the recent results announced by (Auwalu and Denker, 2017). In this research undertaking, we followed analytical study design and used secondary sources of data, such as published articles and related books. The analysis techniques which we adopted for the successful completion of this study were that of (Auwalu and Denker, 2017). Finally, we provide an example in support of our main findings.

Contents

Chapter 1

Introduction

1.1 Background of the study

Fixed point theory is one of the most powerful and popular tools of modern mathematics. It is the mixture of analysis, topology and geometry. The study of fixed point theory plays an important role in application areas not only in different branches of mathematics but also other fields such as Biology, Chemistry, Physics, and almost all engineering fields. The contraction mapping principle introduced by Banach (1922) is a base of applications in a fixed point theory. Let (X,d) be a metric space, a self-mapping $T : X \to X$ is said to be a contraction map if there exists $\lambda \in [0,1)$ such that

$$d(Tx,Ty) \leq \lambda d(x,y),$$

for all x,y in X.

The contraction mapping principle is stated as follows. Let (X,d) be a complete metric space and $T : X \to X$ be a contraction map. Then T has a unique fixed point. There are a number of extensions and generalizations of Banach Contraction theorem by many researchers, who have obtained fixed point and common fixed results in metric spaces, cone metric spaces, cone rectangular metric spaces, cone pentagonal metric spaces, cone hexagonal metric spaces, cone b-hexagonal metric spaces, cone heptagonal metric spaces, ordered metric spaces, quasi metric spaces, dislocated quasi metric spaces and others.

In 2017 Auwalu and Denker defined the following definition:

Let P be a cone and let Φ be the set of non-decreasing continuous functions, where $\varphi \in \Phi$ and $\varphi \colon P \to P$ satisfying:

(1) $0 < \varphi(t) < t$ for $t \in P \backslash \{0\}$.

(2) The series

$$\sum_{n \geq 0} \varphi^n(t)$$

converges for all $t \in P \backslash \{0\}$.
From (1), we have $\varphi(0) = 0$ and from (2), we have $\lim_{n \to \infty} \varphi^n(t) = 0$.
Recently, Ampadu (2017) introduced the notion of cone heptagonal metric space and proved Chatterjea contraction mapping principle in a normal cone heptagonal metric space setting.
Also Auwalu and Denker (2017) proved fixed point theorem for a mapping satisfying certain contractive conditions without assuming the normality of cone b-hexagonal metric spaces as follows:

Theorem 1.1.1 *(Auwalu and Denker, 2017) Let (X,d) be a complete cone b-hexagonal metric space with $s \geq 1$. Suppose the mapping $S : X \to X$ satisfy the contractive condition:*

$$d(Sx,Sy) \leq \varphi(d(x,y))$$

for all $x,y \in X$, where $\varphi \in \Phi$.
Then S has a unique fixed point in X.

Motivated and inspired by the research work of (Auwalu and Denker, 2017), it is our purpose in this thesis to continue the study of a common fixed point result for a pair of self-mappings involving contractive type condition in the setting of complete cone *b*-heptagonal metric spaces. Our results extends and generalizes the result of (Auwalu and Denker, 2017).

1.2 Statements of the problem

This study was focused on establishing and proving a common fixed point theorem for a pair of self-maps satisfying certain contractive condition in cone *b*-heptagonal metric spaces.

1.3 Objectives of the study

1.3.1 General objective

The main objective of this study was to establish a common fixed point theorem for a pair of self-mappings satisfying certain contractive type condition in cone *b*-

heptagonal metric spaces.

1.3.2 Specific objectives

This study has the following specific objectives:

- To prove the existence of a common fixed point for a pair of mappings satisfying certain contractive type condition in cone b-heptagonal metric spaces.
- To show the uniqueness of a common fixed point for a pair of mappings satisfying certain contractive type condition in cone b-heptagonal metric spaces.
- To verify the applicability of the main results obtained using specific example.

1.4 Significance of the study

The study may have the following importance:

- The outcome of this study may contribute to research activities in the study area.
- It may provide basic research skills to the researcher.
- It may help to show existence and uniqueness of problems involving integral and differential equations.

1.5 Delimitation of the Study

The study was delimited to find common fixed point results focuses only to prove the existence and uniqueness of common fixed point for a pair of self-mappings satisfying certain contractive condition in cone b-heptagonal metric spaces.

Chapter 2

Review of Related Literatures

In 2007 Huang and Zhang introduced the concept of a cone metric space; they replaced the set of real numbers by an ordered Banach space and proved some fixed point theorems for contractive type conditions in cone metric spaces. Later on many authors have proved some fixed point theorems for different contractive type conditions in cone metric spaces. Now a days, the concept of standard metric spaces plays a role of fundamental tool in fixed point theory and also attract many researchers because of development of fixed point theory in standard metric space. Several years later the theory of cone metric space was introduced by (Guang and Zhang, 2007) and established some fixed point theorems for contractive type mappings in a normal cone metric space. Hussain and Shah (2011) introduced cone b-metric spaces as a generalization of b-metric spaces and cone metric spaces.

Azam *et al.* (2008) A point $y \in X$ is called point of coincidence of two mappings $T, f : X \to X$ if there exists a point $x \in X$ such that $y = fx = Tx$. Let (X,d) be a complete cone metric space, P be a normal cone with normal constant k.

Suppose that the mappings $T, f : X \to X$ satisfy:

$d(Tx,Ty) \leq Ad(fx,fy) + Bd(fx,Tx) + Cd(fyTy) + Dd(fx,Ty) + Ed(fyTx)$, for all $x, y \in X$, where A,B.C,D and E are non-negative real numbers.

Subsequently, several other authors (Abbas and Jungck, 2008) Common fixed point results for non-commuting mappings without continuity in cone metric spaces.

Azam *et al.* (2009) introduced the notion of cone rectangular metric space and proved Banach contraction mapping principle in a normal cone rectangular metric space setting. In 2011 Hussain and Shah introduced the concept of cone b-metric space.

(Hussain and Shah, 2011) Let X be a nonempty set and E a real Banach space with cone P. A vector-valued function, $D : X \times X \to P$ is said to be a cone b-metric function on X with $k \geq 1$, if the following conditions are satisfied:

$M1$) $\theta \preceq D(x,y)$, for all $x, y \in X$, and $D(x,y) = \theta$ if and only if $x = y$,

$M2$) $D(x,y) = D(y,x)$ for all $x, y \in X$,

$M3$) $D(x,z) \preceq k[D(x,y) + D(y,z)]$ for all $x, y, z \in X$.

The pair (X, D) is called the cone b-metric space. Observe that if $k = 1$, then the ordinary triangle inequality in a cone metric space is satisfied, however it does not hold true when $k > 1$. Thus the class of cone b-metric spaces is effectively larger than that of the ordinary cone metric spaces. That is, every cone metric space is a cone b-metric space, but the converse need not be true. (Garg and Agarwal, 2012) introduced the notion of cone pentagonal metric space and proved Banach contraction mapping principle in a normal cone pentagonal metric space setting.

In 2014 Garg and Agarwal introduced the notion of cone hexagonal metric space and proved Banach contraction mapping principle in a normal cone hexagonal metric space setting.

In 2017 Auwalu and Denker introduced the notion of cone b-hexagonal metric space and proved Banach contraction mapping principle without normality in cone b-hexagonal metric space.

Chapter 3

Methodology

The chapter contains study period, study design, source of information, description of the research methodology and study procedures.

3.1 Study period and site

The study was conducted at Jimma University under the department of mathematics from October, 2017 G.C to September, 2018 G.C.

3.2 Study Design

In order to accomplish the objective of this research analytical design method was used.

3.3 Source of Information

The relevant sources of information for this study were books and published articles.

3.4 Mathematical Procedure of the Study

In this study we followed the standard procedures used in published work of (Auwalu and Denker, 2017) in the setting of cone b-hexagonal metric spaces. That is,

- Establishing a theorem.
- Constructing a sequence.
- Showing whether the sequence is Cauchy or not.
- Showing the convergence of the sequence.

- Proving the existence of a common fixed point.
- Showing uniqueness of the common fixed point.
- Giving an example in support of the main result.

Chapter 4

Preliminaries and Main Results

4.1 Preliminaries

Definition 4.1.1 *(Huang and Zhang, 2007) Let E be a real Banach space with the zero vector θ. A subset P of E is called a cone if the following conditions are satisfied:*

(i) P is closed, non-empty and $P \neq \{0\}$;

(ii) $a,b \in \mathbb{R}, a,b \geq 0, x,y \in P \Rightarrow ax+by \in P$;

(iii) $x \in P$ and $-x \in P \Rightarrow x = \theta$.

Given a cone $P \subset E$ we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y-x \in P$ and we write $x < y$ if $x \leq y$ and $x \neq y$. Likewise,we shall write $x \ll y$ if $y-x \in int(P)$, where int(P) denotes the interior of P.

Definition 4.1.2 *(Huang and Zhang, 2007) Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies:*

(i) $0 \leq d(x,y),\ for\ all\ x,y \in X\ and\ d(x,y) = 0\ if\ and\ only\ if\ x = y,$

(ii) $d(x,y) = d(y,x)\ for\ all\ x,y \in X,$

(iii) $d(x,y) \leq d(x,z)+d(z,y)\ for\ all\ x,y,z \in X$. Then d is called a cone metric on X and (X,d) is called a cone metric space.

Definition 4.1.3 *(Azam, et al., 2009)Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies:*

(i) $0 \leq d(x,y)$, for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$,

(ii) $d(x,y) = d(y,x)$ for all $x,y \in X$,

(iii) $d(x,y) \leq d(x,w)+d(w,z)+d(z,y)$ for all $x,y,z,w \in X$, for all distinct $w,z \in X-\{x,y\}$. (rectangular property)

Then d is called a cone rectangular metric on X and (X,d) is called a cone rectangular metric space.

Remark:1 It is clear that any cone metric space is a cone rectangular metric space but the converse is not true in general.

Definition 4.1.4 *(Auwalu, 2016) Let X be a nonempty set. Suppose the mapping* $d : X \times X \to E$ *satisfies:*

(i) $0 \leq d(x,y)$, *for all* $x,y \in X$ *and* $d(x,y) = 0$ *if and only if* $x = y$,

(ii) $d(x,y) = d(y,x)$ *for all* $x,y \in X$,

(iii) $d(x,y) \leq d(x,z) + d(z,w) + d(w,u) + d(u,y)$ *for all* $x,y,z,u,w \in X$, *for all distinct* $u,w,z \in X - \{x,y\}$. *[Pentagonal property]*

Then d is called a cone pentagonal metric on X and the pair (X,d) *is called a cone pentagonal metric space.*

Remark:2 Every cone rectangular metric space and so cone metric space is a cone pentagonal metric space. The converse is not necessarily true.

Definition 4.1.5 *(Garg, 2014) Let X be a nonempty set. Suppose the mapping* $d : X \times X \to E$ *satisfies:*

(i) $0 \leq d(x,y)$, *for all* $x,y \in X$ *and* $d(x,y) = 0$ *if and only if* $x = y$,

(ii) $d(x,y) = d(y,x)$ *for all* $x,y \in X$,

(iii) $d(x,y) \leq d(x,z) + d(z,w) + d(w,u) + d(u,v) + d(v,y)$ *for all* $x,y,z,u,v,w \in X$, *for all distinct* $u,v,w,z \in X - \{x,y\}$. *[Hexagonal property]*

Then d is called a cone hexagonal metric on X and (X,d) *is called a cone hexagonal metric space.*

Definition 4.1.6 *(Auwalu and Denker, 2017) Let X be a nonempty set and* $s \geq 1$. *Suppose the mapping* $d : X \times X \to E$ *satisfies:*

(i) $0 \leq d(x,y)$, *for all* $x,y \in X$ *and* $d(x,y) = 0$ *if and only if* $x = y$,

(ii) $d(x,y) = d(y,x)$ *for all* $x,y \in X$,

(iii) $d(x,y) \leq s[d(x,z) + d(z,w) + d(w,u) + d(u,v) + d(v,y)]$ *for all* $x,y,z,u,v,w \in X$, *for all distinct* $u,v,w,z \in X - \{x,y\}$. *[b-hexagonal property]*

Then d is called a cone b-hexagonal metric on X and (X,d) *is called a cone b-hexagonal metric space.*

Definition 4.1.7 *(Auwalu, 2017) Let X be a nonempty set. Suppose the mapping* $d : X \times X \to E$ *satisfies:*

(i) $0 \leq d(x,y)$, *for all* $x,y \in X$ *and* $d(x,y) = 0$ *if and only if* $x = y$,

(ii) $d(x,y) = d(y,x)$ *for all* $x,y \in X$,

(iii) $d(x,y) \leq d(x,z)+d(z,w)+d(w,u)+d(u,v)+d(v,h)+d(h,y)$ *for all* $x,y,z,u,v,w,h \in X$, *for all distinct* $u,v,w,z,h \in X - \{x,y\}$. *[Heptagonal property]*

Then d is called a cone heptagonal metric on X and (X,d) *is called a cone heptagonal metric space.*

Remark:3 Every cone hexagonal metric space, cone pentagonal metric space and so cone rectangular metric space is cone heptagonal metric space. The converse is not true.

Definition 4.1.8 *Let* (X,d) *be a cone b-heptagonal metric space. Let* $\{x_n\}$ *be a sequence in* X *and* $x \in X$. *If for every* $c \in E$ *with* $0 \ll c$ *there exist* $n_0 \in N$ *and that for all* $n > n_0$, $d(x_n,x) \ll c$, *then* $\{x_n\}$ *is said to be convergent and* $\{x_n\}$ *converges to x, and x is the limit of* $\{x_n\}$. *We denote this by*

$\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

Definition 4.1.9 *Let* (X,d) *be a cone b-heptagonal metric space. Let* $\{x_n\}$ *be a sequence in* X *and* $x \in X$. *If for every* $c \in E$ *with* $0 \ll c$ *there exist* $n_0 \in N$ *such that for all* $n,m > n_0$, $d(x_n,x_m) \ll c$, *then* $\{x_n\}$ *is called Cauchy sequence in* X.

Definition 4.1.10 *Let* (X,d) *be a cone b-heptagonal metric space. If every Cauchy sequence is convergent in* (X,d) *then X is called a complete cone b-heptagonal metric space.*

Definition 4.1.11 *(Jankovic, et al., 2010) A pair of self-mappings* (f,g) *on a cone metric space* (X,d) *is said to be compatible, if for arbitrary sequence* $\{x_n\} \subset X$, *such that,*

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \in X,$$

and for arbitrary $c \in intP$, there exists $n_0 \in N$ such that
$d(fgx_n, gfx_n) \ll c$, whenever $n > n_0$. It is said to be weakly compatible if

$$fx = gx \Rightarrow fgx = gfx.$$

Lemma 4.1.1 *(Auwalu and Denker, 2017) Let T and S be weakly compatible self-mappings of nonempty set X. If T and S have a unique point of coincidence $w = Tx = Sx$, then w is the unique common fixed point of T and S.*

4.2 Main Results

In this section, we drive the main result of our work, which is a common fixed point theorem for a pair of maps in cone b-heptagonal metric spaces.

Definition 4.2.1 *Let X be a nonempty set and $s \geq 1$. Suppose the mapping $d : X \times X \to E$ satisfies:*

(i) $0 \leq d(x,y)$, for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$,

(ii) $d(x,y) = d(y,x)$ for all $x,y \in X$,

(iii) $d(x,y) \leq s[d(x,z) + d(z,w) + d(w,u) + d(u,v) + d(v,h) + d(h,y)]$ for all $x,y,z,u,v,w,h \in X$, for all distinct $u,v,w,z,h \in X - \{x,y\}$. [b-heptagonal property] Then d is called a cone b-heptagonal metric on X and (X,d) is called a cone b-heptagonal metric space.

Theorem 4.2.1 *Let (X,d) be a cone b-heptagonal metric space with $s \geq 1$. Suppose the mappings $S, g : X \to X$ satisfy the contractive condition:*

$$d(Sx,Sy) \leq \varphi(d(gx,gy)) \tag{4.1}$$

for all $x,y \in X$, where $\varphi \in \Phi$. Suppose that $S(X) \subseteq g(X)$, and $g(X)$ is a complete subspace of X, then the mappings S and g have a unique point of coincidence in X. Moreover, if S and g are weakly compatible, then S and g have a unique common fixed point in X.

Proof: Let x_0 be arbitrary point in X. Since $S(X) \subseteq g(X)$, we can choose $x_1 \in X$ such that

$$gx_1 = Sx_0.$$

Also we can choose $x_2 \in X$ such that $gx_2 = Sx_1$.
Continuing this process, having chosen x_n in X, we obtain x_{n+1}
such that, $gx_{n+1} = Sx_n$ and $gx_{n+2} = Sx_{n+1}$, for all $n = 0,1,2,\cdots$.

If for some n, $x_n = x_{n+1}$ then $gx_n = gx_{n+1}$ this implies $gx_n = Sx_n$ and x_n is a coincidence point of S and g.

Hence, we assume that $x_n \neq x_{n+1}$ for all $n \in N$. It follows from (4.1) that

$$\begin{aligned} d(gx_n, gx_{n+1}) &= (d(Sx_{n-1}, Sx_n)) \\ &\leq \varphi(d(gx_{n-1}, gx_n)) \\ &\leq \varphi^2(d(gx_{n-2}, gx_{n-1})) \\ &\quad . \\ &\quad . \\ &\quad . \\ &\leq \varphi^n(d(gx_0, gx_1)). \end{aligned} \tag{4.2}$$

In similar way, it again follows that

$$d(gx_n, gx_{n+2}) \leq \varphi^n(d(gx_0, gx_2)) \tag{4.3}$$
$$d(gx_n, gx_{n+3}) \leq \varphi^n(d(gx_0, gx_3)) \tag{4.4}$$
$$d(gx_n, gx_{n+4}) \leq \varphi^n(d(gx_0, gx_4)) \tag{4.5}$$
$$d(gx_n, gx_{n+5}) \leq \varphi^n(d(gx_0, gx_5)). \tag{4.6}$$

In similarly way, for $k = 1, 2, 3, \ldots$ we get

$$d(gx_n, gx_{n+5k+1}) \leq \varphi^n(d(gx_0, gx_{5k+1})) \tag{4.7}$$
$$d(gx_n, gx_{n+5k+2}) \leq \varphi^n(d(gx_0, gx_{5k+2})) \tag{4.8}$$
$$d(gx_n, gx_{n+5k+3}) \leq \varphi^n(d(gx_0, gx_{5k+3})) \tag{4.9}$$
$$d(gx_n, gx_{n+5k+4}) \leq \varphi^n(d(gx_0, gx_{5k+4})) \tag{4.10}$$
$$d(gx_n, gx_{n+5k+5}) \leq \varphi^n(d(gx_0, gx_{5k+5})). \tag{4.11}$$

By using (4.2) and b-heptagonal property, we have

$$
\begin{aligned}
d(gx_0,gx_6) &\leq s[d(gx_0,gx_1)+d(gx_1,gx_2)+d(gx_2,gx_3)\\
&\quad +d(gx_3,gx_4)+d(gx_4,gx_5)+d(gx_5,gx_6)]\\
&\leq s[d(gx_0,gx_1)+\varphi(d(gx_0,gx_1))+\varphi^2(d(gx_0,gx_1))\\
&\quad +\varphi^3(d(gx_0,gx_1))+\varphi^4(d(gx_0,gx_1))\\
&\quad +\varphi^5(d(gx_0,gx_1))]\\
&\leq s[\sum_{i=0}^{5}\varphi^i(d(gx_0,gx_1))].
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
d(gx_0,gx_{11}) &\leq s[d(gx_0,gx_1)+d(gx_1,gx_2)+d(gx_2,gx_3)+d(gx_3,gx_4)\\
&\quad +d(gx_4,gx_5)+d(gx_5,gx_6)+d(gx_6,gx_7)+d(gx_7,gx_8)\\
&\quad +d(gx_8,gx_9)+d(gx_9,gx_{10})+d(gx_{10},gx_{11})]\\
&\leq s[d(gx_0,gx_1)+\varphi(d(gx_0,gx_1))+\varphi^2(d(gx_0,gx_1))\\
&\quad +\varphi^3(d(gx_0,gx_1))+\varphi^4(d(gx_0,gx_1))+\varphi^5(d(gx_0,gx_1))\\
&\quad +\varphi^6(d(gx_0,gx_1))+\varphi^7(d(gx_0,gx_1))+\varphi^8(d(gx_0,gx_1))\\
&\quad +\varphi^9(d(gx_0,gx_1))+\varphi^{10}(d(gx_0,gx_1))]\\
&\leq s[\sum_{i=0}^{10}\varphi^i(d(gx_0,gx_1))].
\end{aligned}
$$

Now by induction, we obtain for each $k=1,2,3,...$

$$
d(gx_0,gx_{5k+1})\leq s[\sum_{i=0}^{5k}\varphi^i(d(gx_0,gx_1))]. \tag{4.12}
$$

Also, by (4. 2) and (4. 3) and b-heptagonal property, we have

$$\begin{aligned} d(gx_0, gx_7) &\le s[d(gx_0, gx_1) + d(gx_1, gx_2) + d(gx_2, gx_3) + d(gx_3, gx_4) \\ &\quad + d(gx_4, gx_5) + d(gx_5, gx_7)] \\ &\le s[d(gx_0, gx_1) + \varphi(d(gx_0, gx_1)) + \varphi^2(d(gx_0, gx_1)) \\ &\quad + \varphi^3(d(gx_0, gx_1)) + \varphi^4(d(gx_0, gx_1)) + \varphi^5(d(gx_0, gx_2))] \\ &\le s[\sum_{i=0}^{4} \varphi^i(d(gx_0, gx_1)) + \varphi^5 d(gx_0, gx_2)] \end{aligned}$$

Similarly,

$$\begin{aligned} d(gx_0, gx_{12}) &\le s[d(gx_0, gx_1) + d(gx_1, gx_2) + d(gx_2, gx_3) + d(gx_3, gx_4) \\ &\quad + d(gx_4, gx_5) + d(gx_5, gx_6) + d(gx_6, gx_7) + d(gx_7, gx_8) \\ &\quad + d(gx_8, gx_9) + d(gx_9, gx_{10}) + d(gx_{10}, gx_{12})] \\ &\le s[d(gx_0, gx_1) + \varphi(d(gx_0, gx_1)) + \varphi^2(d(gx_0, gx_1)) \\ &\quad + \varphi^3(d(gx_0, gx_1)) + \varphi^4(d(gx_0, gx_1)) + \varphi^5(d(gx_0, gx_1)) \\ &\quad + \varphi^6(d(gx_0, gx_1)) + \varphi^7(d(gx_0, gx_1)) + \varphi^8(d(gx_0, gx_1)) \\ &\quad + \varphi^9(d(gx_0, gx_1)) + \varphi^{10}(d(gx_0, gx_2))] \\ &\le s[\sum_{i=0}^{9} \varphi^i(d(gx_0, gx_1)) + \varphi^{10}(d(gx_0, gx_2))]. \end{aligned}$$

Now by induction, we obtain for each $k = 1, 2, 3, ...$

$$d(gx_0, gx_{5k+2}) \le s[\sum_{i=0}^{5k-1} \varphi^i(d(gx_0, gx_1)) + \varphi^{5k}(d(gx_0, gx_2))]. \tag{4.13}$$

Again by (4.2), (4.4) b-heptagonal property, we have

$$\begin{aligned} d(gx_0, gx_8) &\le s[d(gx_0, gx_1) + d(gx_1, gx_2) + d(gx_2, gx_3) + d(gx_3, gx_4) \\ &\quad + d(gx_4, gx_5) + d(gx_5, gx_8)] \\ &\le s[d(gx_0, gx_1) + \varphi(d(gx_0, gx_1)) + \varphi^2(d(gx_0, gx_1)) \\ &\quad + \varphi^3(d(gx_0, gx_1)) + \varphi^4(d(gx_0, gx_1)) + \varphi^5(d(gx_0, gx_3))] \\ &\le s[\sum_{i=0}^{4} \varphi^i(d(gx_0, gx_1)) + \varphi^5(d(gx_0, gx_3))]. \end{aligned}$$

Similarly,

$$\begin{aligned} d(gx_0,gx_{13}) &\le s[d(gx_0,gx_1)+d(gx_1,gx_2)+d(gx_2,gx_3)+d(gx_3,gx_4)\\ &\quad +d(gx_4,gx_5)+d(gx_5,gx_6)+d(gx_6,gx_7)+d(gx_7,gx_8)\\ &\quad +d(gx_8,gx_9)+d(gx_9,gx_{10})+d(gx_{10},gx_{13})]\\ &\le s[d(gx_0,gx_1)+\varphi(d(gx_0,gx_1))+\varphi^2(d(gx_0,gx_1))\\ &\quad +\varphi^3(d(gx_0,gx_1))+\varphi^4(d(gx_0,gx_1))+\varphi^5(d(gx_0,gx_1))\\ &\quad +\varphi^6(d(gx_0,gx_1))+\varphi^7(d(gx_0,gx_1))+\varphi^8(d(gx_0,gx_1))\\ &\quad +\varphi^9(d(gx_0,gx_1))+\varphi^{10}(d(gx_0,gx_3))]\\ &\le s[\sum_{i=0}^{9}\varphi^i(d(gx_0,gx_1))+\varphi^{10}(d(gx_0,gx_3))]. \end{aligned}$$

Now by induction, we obtain for each $k=1,2,3,\ldots$

$$d(gx_0,gx_{5k+3}) \le s[\sum_{i=0}^{5k-1}\varphi^i(d(gx_0,gx_1))+\varphi^{5k}(d(gx_0,gx_3))]. \tag{4.14}$$

In fact, by (4.2), (4.5) and b-heptagonal property, we have

$$\begin{aligned} d(gx_0,gx_9) &\le s[d(gx_0,gx_1)+d(gx_1,gx_2)+d(gx_2,gx_3)+d(gx_3,gx_4)\\ &\quad +d(gx_4,gx_5)+d(gx_5,gx_9)]\\ &\le s[d(gx_0,gx_1)+\varphi(d(gx_0,gx_1))+\varphi^2(d(gx_0,gx_1))\\ &\quad +\varphi^3(d(gx_0,gx_1))+\varphi^4(d(gx_0,gx_1))+\varphi^5(d(gx_0,gx_4))]\\ &\le s[\sum_{i=0}^{4}\varphi^i(d(gx_0,gx_1))+\varphi^5(d(gx_0,gx_4))]. \end{aligned}$$

Similarly,

$$\begin{aligned}
d(gx_0,gx_{14}) &\leq s[d(gx_0,gx_1)+d(gx_1,gx_2)+d(gx_2,gx_3)+d(gx_3,gx_4)\\
&\quad +d(gx_4,gx_5)+d(gx_5,gx_6)+d(gx_6,gx_7)+d(gx_7,gx_8)\\
&\quad +d(gx_8,gx_9)+d(gx_9,gx_{10})+d(gx_{10},gx_{14})]\\
&\leq s[d(gx_0,gx_1)+\varphi(d(gx_0,gx_1))+\varphi^2(d(gx_0,gx_1))\\
&\quad +\varphi^3(d(gx_0,gx_1))+\varphi^4(d(gx_0,gx_1))+\varphi^5(d(gx_0,gx_1))\\
&\quad +\varphi^6(d(gx_0,gx_1))+\varphi^7(d(gx_0,gx_1))+\varphi^8(d(gx_0,gx_1))\\
&\quad +\varphi^9(d(gx_0,gx_1))+\varphi^{10}(d(gx_0,gx_4))].\\
&\leq s[\sum_{i=0}^{9}\varphi^i(d(gx_0,gx_1))+\varphi^{10}(d(gx_0,gx_4))].
\end{aligned}$$

Now by induction, we obtain for each $k=1,2,3,...$

$$d(gx_0,gx_{5k+4})\leq s[\sum_{i=0}^{5k-1}\varphi^i(d(gx_0,gx_1))+\varphi^{5k}(d(gx_0,gx_4))]. \tag{4.15}$$

In fact, by (4.2), (4.6) and b-heptagonal property, we have

$$\begin{aligned}
d(gx_0,gx_{10}) &\leq s[d(gx_0,gx_1)+d(gx_1,gx_2)+d(gx_2,gx_3)+d(gx_3,gx_4)\\
&\quad +d(gx_4,gx_5)+d(gx_5,gx_{10})]\\
&\leq s[d(gx_0,gx_1)+\varphi(d(gx_0,gx_1))+\varphi^2(d(gx_0,gx_1))\\
&\quad +\varphi^3(d(gx_0,gx_1))+\varphi^4(d(gx_0,gx_1))+\varphi^5(d(gx_0,gx_5))]\\
&\leq s[\sum_{i=0}^{4}\varphi^i(d(gx_0,gx_1))+\varphi^5(d(gx_0,gx_5))].
\end{aligned}$$

Similarly,

$$\begin{aligned}
d(gx_0, gx_{15}) &\leq s[d(gx_0, gx_1) + d(gx_1, gx_2) + d(gx_2, gx_3) + d(gx_3, gx_4) \\
&\quad + d(gx_4, gx_5) + d(gx_5, gx_6) + d(gx_6, gx_7) + d(gx_7, gx_8) \\
&\quad + d(gx_8, gx_9) + d(gx_9, gx_{10}) + d(gx_{10}, gx_{15})] \\
&\leq s[d(gx_0, gx_1) + \varphi(d(gx_0, gx_1)) + \varphi^2(d(gx_0, gx_1)) \\
&\quad + \varphi^3(d(gx_0, gx_1)) + \varphi^4(d(gx_0, gx_1)) + \varphi^5(d(gx_0, gx_1)) \\
&\quad + \varphi^6(d(gx_0, gx_1)) + \varphi^7(d(gx_0, gx_1)) + \varphi^8(d(gx_0, gx_1)) \\
&\quad + \varphi^9(d(gx_0, gx_1)) + \varphi^{10}(d(gx_0, gx_5))] \\
&\leq s[\sum_{i=0}^{9} \varphi^i(d(gx_0, gx_1)) + \varphi^{10}(d(gx_0, gx_5))].
\end{aligned}$$

Now by induction, we obtain for each $k = 1, 2, 3, ...$

$$d(gx_0, gx_{5k+5}) \leq s[\sum_{i=0}^{5k-1} \varphi^i(d(gx_0, gx_1)) + \varphi^{5k}(d(gx_0, gx_5))]. \tag{4.16}$$

Using inequality (4.7) and (4.12) for each $k = 1, 2, 3, ...$ we have

$$\begin{aligned}
d(gx_n, gx_{n+5k+1}) &\leq \varphi^n(d(gx_0, gx_{5k+1})) \\
&\leq \varphi^n(s[\sum_{i=0}^{5k} \varphi^i(d(gx_0, gx_1))]) \\
&\leq \varphi^n(s[\sum_{i=0}^{5k} \varphi^i(d(gx_0, gx_1) + d(gx_0, gx_2) + d(gx_0, gx_3) \\
&\quad + d(gx_0, gx_4) + d(gx_0, gx_5))]) \\
&\leq \varphi^n(s[\sum_{i=0}^{\infty} \varphi^i(d(gx_0, gx_1) + d(gx_0, gx_2) + d(gx_0, gx_3) \\
&\quad + d(gx_0, gx_4) + d(gx_0, gx_5))]).
\end{aligned} \tag{4.17}$$

Using inequality (4.8) and (4.13) for each $k = 1, 2, 3, \ldots$ we have

$$\begin{aligned}
d(gx_n, gx_{n+5k+2}) &\leq \varphi^n(d(gx_0, gx_{5k+2})) \\
&\leq \varphi^n(s[\sum_{i=0}^{5k-1} \varphi^i(d(gx_0, gx_1)) + \varphi^{5k}(d(gx_0, gx_2))]) \\
&\leq \varphi^n(s[\sum_{i=0}^{5k-1} \varphi^i(d(gx_0, gx_1) + d(gx_0, gx_2) + d(gx_0, gx_3) + d(gx_0, gx_4) \\
&\quad + d(gx_0, gx_5) + \varphi^{5k}(d(gx_0, gx_1) + d(gx_0, gx_2) + d(gx_0, gx_3) + d(gx_0, gx_4) \\
&\quad + d(gx_0, gx_5))]) \\
&\leq \varphi^n(s[\sum_{i=0}^{5k} \varphi^i(d(gx_0, gx_1) + d(gx_0, gx_2) + d(gx_0, gx_3) + d(gx_0, gx_4) + d(gx_0, gx_5))]) \\
&\leq \varphi^n(s[\sum_{i=0}^{\infty} \varphi^i(d(gx_0, gx_1) + d(gx_0, gx_2) + d(gx_0, gx_3) + d(gx_0, gx_4) \\
&\quad + d(gx_0, gx_5))]).
\end{aligned} \tag{4.18}$$

Again for $k = 1, 2, 3, \ldots$ inequalities (4.9) and (4.14) imply that

$$\begin{aligned}
d(gx_n, gx_{n+5k+3}) &\leq \varphi^n(d(gx_0, gx_{5k+3})) \\
&\leq \varphi^n(s[\sum_{i=0}^{5k-1} \varphi^i(d(gx_0, gx_1)) + \varphi^{5k}(d(gx_0, gx_3))]) \\
&\leq \varphi^n(s[\sum_{i=0}^{5k-1} \varphi^i(d(gx_0, gx_1) + d(gx_0, gx_2) + d(gx_0, gx_3) + d(gx_0, gx_4) \\
&\quad + d(gx_0, gx_5) + \varphi^{5k}(d(gx_0, gx_1) + d(gx_0, gx_2) + d(gx_0, gx_3) + d(gx_0, gx_4) \\
&\quad + d(gx_0, gx_5))]) \\
&\leq \varphi^n(s[\sum_{i=0}^{5k} \varphi^i(d(gx_0, gx_1) + d(gx_0, gx_2) + d(gx_0, gx_3) + d(gx_0, gx_4) + d(gx_0, gx_5))]) \\
&\leq \varphi^n(s[\sum_{i=0}^{\infty} \varphi^i(d(gx_0, gx_1) + d(gx_0, gx_2) + d(gx_0, gx_3) + d(gx_0, gx_4) \\
&\quad + d(gx_0, gx_5))]).
\end{aligned} \tag{4.19}$$

Again for $k = 1,2,3,...$ inequalities (4. 10) and (4. 15) implies that

$$\begin{aligned} d(gx_n, gx_{n+5k+4}) &\leq \varphi^n(d(gx_0, gx_{5k+4})) \\ &\leq \varphi^n(s[\sum_{i=0}^{5k-1} \varphi^i(d(gx_0, gx_1)) + \varphi^{5k}(d(gx_0, gx_4))]) \\ &\leq \varphi^n(s[\sum_{i=0}^{5k-1} \varphi^i(d(gx_0, gx_1) + d(gx_0, gx_2) + d(gx_0, gx_3) + d(gx_0, gx_4) \\ &\quad + d(gx_0, gx_5) + \varphi^{5k}(d(gx_0, gx_1) + d(gx_0, gx_2) + d(gx_0, gx_3) + d(gx_0, gx_4) \\ &\quad + d(gx_0, gx_5))]) \\ &\leq \varphi^n(s[\sum_{i=0}^{5k} \varphi^i(d(gx_0, gx_1) + d(gx_0, gx_2) + d(gx_0, gx_3) + d(gx_0, gx_4) + d(gx_0, gx_5))]) \\ &\leq \varphi^n(s[\sum_{i=0}^{\infty} \varphi^i(d(gx_0, gx_1) + d(gx_0, gx_2) + d(gx_0, gx_3) + d(gx_0, gx_4) \\ &\quad + d(gx_0, gx_5))]). \end{aligned} \tag{4.20}$$

Similarly for $k = 1,2,3,...$ inequalities (4.11) and (4.16) implies that

$$\begin{aligned} d(gx_n, gx_{n+5k+5}) &\leq \varphi^n(d(gx_0, gx_{5k+5})) \\ &\leq \varphi^n(s[\sum_{i=0}^{5k-1} \varphi^i(d(gx_0, gx_1)) + \varphi^{5k}(d(gx_0, gx_5))]) \\ &\leq \varphi^n(s[\sum_{i=0}^{5k-1} \varphi^i(d(gx_0, gx_1) + d(gx_0, gx_2) + d(gx_0, gx_3) + d(gx_0, gx_4) \\ &\quad + d(gx_0, gx_5) + \varphi^{5k}(d(gx_0, gx_1) + d(gx_0, gx_2) + d(gx_0, gx_3) + d(gx_0, gx_4) \\ &\quad + d(gx_0, gx_5))]) \\ &\leq \varphi^n(s[\sum_{i=0}^{5k} \varphi^i(d(gx_0, gx_1) + d(gx_0, gx_2) + d(gx_0, gx_3) + d(gx_0, gx_4) + d(gx_0, gx_5))]) \\ &\leq \varphi^n(s[\sum_{i=0}^{\infty} \varphi^i(d(gx_0, gx_1) + d(gx_0, gx_2) + d(gx_0, gx_3) + d(gx_0, gx_4) \\ &\quad + d(gx_0, gx_5))]). \end{aligned} \tag{4.21}$$

Thus, by inequalities (4. 17), (4. 18), (4. 19), (4. 20), and (4. 21) we have, for each m,n and, $m \neq n$

$$d(gx_n, gx_{n+m}) \leq \varphi^n(s[\sum_{i=0}^{\infty} \varphi^i(d(gx_0,gx_1)+d(gx_0,gx_2)+d(gx_0,gx_3) + d(gx_0,gx_4)+d(gx_0,gx_5))]). \tag{4.22}$$

Since

$$(s[\sum_{i=0}^{\infty} \varphi^i(d(gx_0,gx_1)+d(gx_0,gx_2)+d(gx_0,gx_3)+d(gx_0,gx_4)+d(gx_0,gx_5))])$$

converges, by definition Φ, where
$d(gx_0,gx_1)+d(gx_0,gx_2)+d(gx_0,gx_3)+d(gx_0,gx_4)+d(gx_0,gx_5) \in P\backslash\{0\}$, P is closed.
Then,

$$(s[\sum_{i=0}^{\infty} \varphi^i(d(gx_0,gx_1)+d(gx_0,gx_2)+d(gx_0,gx_3)+d(gx_0,gx_4)+d(gx_0,gx_5))]) \in P\backslash\{0\}.$$

Hence,

$$\lim_{n\to\infty} \varphi^n(s[\sum_{i=0}^{\infty} \varphi^i(d(gx_0,gx_1)+d(gx_0,gx_2)+d(gx_0,gx_3)+d(gx_0,gx_4)+d(gx_0,gx_5))]) = 0.$$

Then, for a given $c \gg 0$, there is a natural N_1 such that,

$$\varphi^n(s[\sum_{i=0}^{\infty} \varphi^i(d(gx_0,gx_1)+d(gx_0,gx_2)+d(gx_0,gx_3)+d(gx_0,gx_4)+d(gx_0,gx_5))]) \ll c \text{for all } n \geq N_1. \tag{4.23}$$

Thus from (4.22) and (4.23) we have,
$d(gx_n, gx_{n+m}) \ll c$, for all $n \geq N_1$.
Therefore $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Since $g(X)$ is a complete subspace of X, there exist $u, v \in g(X)$ such that

$$\lim_{n\to\infty} gx_n = v = gu.$$

Now we show that $gu = Su$. Given $c \gg 0$, we choose a numbers N_2, N_3 such that $d(v, gx_n) \ll \frac{c}{6s}$, for all $n \geq N_2$ and $d(v, gx_n) \ll \frac{c}{6s}$, for all $n \geq N_3$. Since $x_n \neq x_m$ for

$n \neq m$, by b-heptagonal property we have that:

$$\begin{aligned} d(gu,Su) &\leq s[d(gu,gx_n)+d(gx_n,gx_{n+1})+d(gx_{n+1},gx_{n+2})+d(gx_{n+2},gx_{n+3}) \\ &\quad +d(gx_{n+3},gx_{n+4})+d(gx_{n+4},Su)] \\ &= s[d(v,gx_n)+d(gx_n,gx_{n+1})+d(gx_{n+1},gx_{n+2})+d(gx_{n+2},gx_{n+3}) \\ &\quad +d(gx_{n+3},gx_{n+4})+d(Sx_{n+3},Su)] \\ &\leq s[d(v,gx_n)+d(gx_n,gx_{n+1})+d(gx_{n+1},gx_{n+2})+d(gx_{n+2},gx_{n+3}) \\ &\quad +d(gx_{n+3},gx_{n+4})+\varphi(d(gu,gx_{n+3}))] \\ &< s[d(v,gx_n)+d(gx_n,gx_{n+1})+d(gx_{n+1},gx_{n+2})+d(gx_{n+2},gx_{n+3}) \\ &\quad +d(gx_{n+3},gx_{n+4})+d(v,gx_{n+3})] \end{aligned}$$

$\ll s[\frac{c}{6s}+\frac{c}{6s}+\frac{c}{6s}+\frac{c}{6s}+\frac{c}{6s}+\frac{c}{6s}] = c$ for all $n \geq N$, where $N := max\{N_2, N_3\}$.
Since c is arbitrary, we have $d(gu,Su) \ll \frac{c}{m}$, for all $m \in N$.
Since $\frac{c}{m} \to 0$ as $m \to \infty$, we conclude that $\frac{c}{m} - d(gu,Su) \to d(gu,Su)$ as $m \to \infty$.
Since P is closed, $-d(gu,Su) \in P$, hence $d(gu,Su) \in P \cap -P = \{\theta\}$.
By the definition of a cone we get that $d(gu,Su) = 0$, and so

$$gu = Su = v.$$

Hence, v is a point of coincidence of S and g. i.e.,

$$gu = Su = v.$$

Next, we show that v is unique. For suppose v' be another point of coincidence of g and S that is, $Su' = gu' = v'$ for some $u' \in X$.
We show $u' = v'$, suppose $u' \neq v'$, then
$d(v,v') = d(Su,Su') \leq \varphi(d(gu,gu')) = \varphi(d(v,v')) < d(v,v')$, a contradiction.
Hence $v' = v$. Since the pair (S,g) is weakly compatible by Lemma 4.1 v is the unique common fixed point of S and g.
The following is an example in support of Theorem 4.2.1
Example: Let $X = \{1,2,3,4,5,6,7\}, E = R^2$ and $P = \{(x,y) \in E : x,y \geq 0\}$, then

P is a cone in E.
We define $d : X \times X \to E$ as follows:

$$\begin{aligned}
d(x,x) &= 0, \text{for all} x \in X, \\
d(1,2) &= d(2,1) = (22,44), \\
d(1,3) &= d(3,1) = d(1,4) = d(4,1) = d(1,5) = d(5,1) = d(1,6) = d(6,1), \\
&= d(2,3) = d(3,2) = d(2,4) = d(4,2) = d(2,5) = d(5,2) = d(2,6), \\
&= d(6,2) = d(3,4) = d(4,3) = d(3,5) = d(5,3) = d(3,6) = d(6,3), \\
&= d(4,5) = d(5,4) = d(4,6) = d(6,4) = d(5,6) = d(6,5) = (2,4), \\
d(1,7) &= d(7,1) = d(2,7) = d(7,2) = d(3,7) = d(7,3) = d(4,7) = d(7,4), \\
&= d(5,7) = d(7,5) = d(6,7) = d(7,6) = (6,12).
\end{aligned}$$

Then (X,d) is a complete cone b-heptagonal metric space with $s = 2$.
Let $s = 2$, then

$$\begin{aligned}
(22,44) = d(1,2) &< 2[d(1,3) + d(3,4) + d(4,5) + d(5,6) + d(6,7) \\
&\quad + d(7,2)] \\
&= 2[(2,4) + (2,4) + (2,4) + (2,4) + (6,12) + (6,12)] \\
&= 2(20,40) = (40,80).
\end{aligned}$$

Therefore $(22,44) < (40,80)$ is satisfied.
That is,$(40,80) - (22,44) = (18,36) \in intP$.
Also,

$$\begin{aligned}
d(1,3) &\leq 2[d(1,2) + d(2,4) + d(4,5) + d(5,6) + d(6,7) \\
&\quad + d(7,3)] \\
&= 2[(22,44) + (2,4) + (2,4) + (2,4) + (6,12) + (6,12)], thatis, \\
(2,4) &< 2(40,80) = (80,160).
\end{aligned}$$

That is, $(80,160)-(2,4)=(78,156)\in intP$.
Again,

$$\begin{aligned}
d(1,7) &\leq 2[d(1,2)+d(2,3)+d(3,4)+d(4,5)+d(5,6) \\
&\quad +d(6,7)] \\
&= 2[(22,44)+(2,4)+(2,4)+(2,4)+(2,4)+(6,12)] \\
(6,12) &< 2(36,72)=(72,144)
\end{aligned}$$

That is, $(72,144)-(6,12)=(66,132)\in intP$.
(X,d) is not a cone b-hexagonal metric space. Because it lacks b-hexagonal property.
For justification we consider the following:-

$$\begin{aligned}
(22,44)=d(1,2) &\nleq 2[d(1,3)+d(3,4)+d(4,5)+d(5,6)+d(6,2)] \\
&= 2[(2,4)+(2,4)+(2,4)+(2,4)+(2,4)] \\
&= 2(10,20)=(20,40).
\end{aligned}$$

Hence (X,d) does not satisfy b-hexagonal property.
We define mappings $S,g:X\to X$ as follows:

$$S(x)=\begin{cases} 4 & \text{if } x\neq 5, \\ 2 & \text{if } x=5 \end{cases}$$

$$g(x)=\begin{cases} 3 & \text{if } x=1, \\ 1 & \text{if } x=2, \\ 2 & \text{if } x=3, \\ 4 & \text{if } x=4, \\ 7 & \text{if } x=5, \\ 6 & \text{if } x=6,7. \end{cases}$$

We define $\varphi:P\to P$
by $\varphi(t)=\frac{1}{3}t$. Clearly $S(X)\subseteq g(X), g(X)$ is a complete subspace of X and the pairs

(S,g) is weakly compatible. The inequality 4.1 holds for all $x,y \in X$, with $s=2$ and 4 is the unique common fixed point of the mappings S and g.

Corollary 4.1 Let (X,d) be a complete cone b - hexagonal metric space with $s \geq 1$. Suppose the mapping $S: X \to X$ satisfy the contractive condition:

$$d(Sx,Sy) \leq \varphi(d(x,y))$$

for all $x,y \in X$ where $\varphi \in \Phi$. Then S has a unique fixed point in X.

Proof: The result follows by taking $g=I$ (Identity map on X) in Theorem 4.2.1.

Corollary 4.2 Let (X,d) be a cone hexagonal metric space, P be a normal cone, and the mapping $S: X \to X$ satisfy the following:

$$d(Sx,Sy) \leq \lambda(d(x,y))$$

for all $x,y \in X$, where $\lambda \in [0,1)$. Then S has a unique fixed point in X.

Proof: Define $\varphi : P \to P$ by $\varphi(t) = \lambda t, s=1$. Then it is clear that φ satisfies the conditions in Definition 4.1.12. Hence the results follow from Theorem 4.2.1

Chapter 5

Conclusion and Future Scope

5.1 Conclusion

In this thesis, we have explored the properties of cone b-hexagonal metric spaces and also discuss the difference between cone metric spaces, cone rectangular metric spaces, cone pentagonal metric spaces, cone hexagonal metric spaces, cone b-metric spaces, rectangular cone b-metric spaces, cone heptagonal metric spaces and cone b-heptagonal metric spaces. We established a common fixed point theorem for a pair of self-mappings in cone b-heptagonal metric spaces satisfying certain contractive type condition. We also obtained sufficient conditions for existence of points of coincidence and common fixed point for a pair of self-mappings in cone b-heptagonal metric spaces. We have supported the result of this work by particular example. Our results extend and generalize the recent results announced by (Auwalu and Denker, 2017).

5.2 Future Scope

Fixed point theory is one of the most active areas of research work in mathematics and other sciences. There are several published results related to existence of fixed points of self-mappings defined in cone b-heptagonal metric spaces. There are also few results related to the existence of common fixed points for a pair or more self-mappings in this space. The researcher believes the search for the existence of coincidence and common fixed points of self-mappings satisfying different contractive type conditions in cone b-heptagonal metric spaces is an active area of study. So, any other interested researchers can use this opportunity and conduct their research work in this area.

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