A Common Fixed Point Theorem for Generalized Weakly Contractive Mappings in Multiplicative Metric Spaces

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS

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Declaration

I, the undersigned declare that, this research paper entitled ”A Common Fixed Point Theorem for Generalized Weakly Contractive Mappings in Multiplicative Metric Spaces ” is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged.

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Abstract

In this thesis, we introduced generalized weakly contractive mappings, established a common fixed point result and proved the existence and uniqueness of a common fixed point for a pair of self-mappings in setting of multiplicative metric spaces. We employed analytical design and used secondary sources of data such as published articles and related books. Finally, we provide an example in support of our main finding.

Contents

Chapter 1

Introduction

1.1 Background of the study

Fixed point theory provides the most important and traditional tools for proving the existence of solutions of many problems in both pure and applied mathematics. The study of fixed point theory plays an important role in application areas not only in different branches of mathematics but also other fields such as Biology, Chemistry, Physics, and almost all engineering fields.

One of the basic and most widely applied fixed point theorems in mathematical analysis is Banach Contraction Mapping Principle or "Banachs Fixed PointTheorem" by (Banach, 1922) .

Let (X,d) be a metric space, a self-mapping $T : X \to X$ is said to be a contraction map if there exists $\lambda \in [0,1)$ such that

$$d(Tx,Ty) \leq \lambda d(x,y),$$

for all $x,y \in X$.

The Banach contraction mapping principle is stated as follows. Let (X,d) be a complete metric space and $T : X \to X$ be a contraction map. Then T has a unique fixed point.

Following this concept in 1968, Kannan introduced the following mapping for metric space X.

Let (X,d) be a metric space and $T : X \to X$ be a mapping satisfying

$$d(Tx,Ty) \leq \lambda[d(x,Tx)+d(y,Ty)]$$

for all $x,y \in X$ and $\lambda \in [0,\frac{1}{2})$. Then T is called Kanna type mapping and if (X,d) is complete, then T has a fixed point.

In 1972, a new concept which is different from that of Banach and Kannan for contraction type mapping was introduced by Chatterjea which gives a new direction to the study of fixed-point theory as follows:

Let (X,d) be a metric space and $T : X \to X$ be a mapping satisfying

$$d(Tx,Ty) \leq \lambda[d(x,Ty)+d(y,Tx)]$$

for all $x,y \in X$ and $\lambda \in [0,\frac{1}{2})$. Then T is called Chatterjea type mapping and if (X,d) is complete, then T has a fixed point.

In 1972, Zamfirescu generalizes Banachs, Kannans and Chatterjeas fixed point theorems as follows. Let (X,d) be a complete metric space and $f : X \longrightarrow X$ be a mapping for which there exists $a,k,c \in R$ with $a \in [0,1)$ and $k,c \in [0,\frac{1}{2})$, such that for all $x,y \in X$, at least one of the following is true:

i. $d(f(x),f(y)) \leq ad(x,y)$;

ii. $d(f(x),f(y)) \leq k[d(x,f(x))+d(y,f(y))]$;

iii. $d(f(x),f(y)) \leq c[d(x,f(y))+d(y,f(x))]$. Then f is a Picard operator.

Banach contraction principle has been extended and generalized by (Caristiet *et al.*, 1976), by modifying contractive conditions .

In 1972, Michael Grossman and Robert Katz gave definitions of a new kind of derivative and integral, moving the roles of subtraction and addition to division and multiplication, and thus established a new calculus, called multiplicative calculus. Following this concept in 2008, Bashirov et al. studied the concept of multiplicative calculus and proved the fundamental theorem of multiplicative . Furthermore, they gave application of multiplicative calculus, defined multiplicative absolute value, multiplicative distance between two positive real numbers and finally they introduced the notion of multiplicative metric spaces. In 2012, Florack *et al.* explored the advantage of multiplicative calculus in biomedical image analysis. In 2011, Bashirov *et al.* discussed the simplicity of solving multiplicative differential equations than ordinary differential equations in different fields. In 2012, Ozavsar and Cevikel gave the definition of multiplicative contraction and proved Banach contraction principle in the setting of multiplicative metric spaces and also they studied multiplicative metric toplology.

Let f be a mapping of a multiplicative metric space (X,d) into itself. Then f is said to be a multiplicative contraction if there exists a real constant $\lambda \in [0,1)$ such that

$$d(fx,fy) \leq d^{\lambda}(x,y)$$

for all $x,y \in X$.

The concept of a weakly contractive mapping was introduced in 1997 by Alber and Guerre-Delabriere as follows:
Let (X,d) be a metric space. A map $f : X \longrightarrow X$ is called a φ-weakly contractive if there exists a continuous and non-decreasing function $\varphi(t)$ defined on R^+ such that φ is positive on $R^+ \backslash \{0\}$, $\varphi(0) = 0$ and $\lim_{t\to+\infty} \varphi(t) = +\infty$,

$$d(fx;fy) \leq d(x;y) - \varphi(d(x;y)) \; for\; each\; x,y \in X.$$

Many authors obtained generalizations and extensions of the weak contraction principle. For example, in 2011 Choudhury *et al.* introduced generalized weakly contractive mappings using altering distance function as follows:
Let (X,d) be a metric space, T a self-mapping of X. We shall call T a generalized weakly contractive mapping if for all $x,y \in X$,

$$\psi(d(Tx,Ty)) \leq \psi(m(x,y)) - \phi(max\, d(x,y), d(y,Ty)),$$

where $m(x,y) = max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty) + d(y,Tx)]\}$,
$\psi : [0,\infty) \longrightarrow [0,\infty)$ is an altering distance function which is monotone increasing with $\psi(t) = 0$ if and only if $t = 0$ and ϕ:$[0,\infty) \longrightarrow [0,\infty)$is a continuous function with $\phi(t) = 0$ if and only if t = 0.
In 2015, Abbas *et al.* obtained several fixed and common fixed point results of self-maps satisfying certain generalized contractive conditions in the framework of multiplicative metric space.
Recently in 2018, Cho introduced generalized weakly contractive mappings and proved the existence and uniqueness of fixed point result in the setting of metric spaces.
In this research, motivated and inspired by the work of Cho, (2018), the notion of generalized weakly contractive mappings has been introduced and the existence and uniqueness of a common fixed point result for a pair of self-mappings in the setting of multiplicative metric spaces have been proved.

1.2 Statements of the problem

This study focused on establishing and proving a common fixed point theorem for a pair of generalized weakly contractive mappings.

1.3 Objectives of the study

1.3.1 General objective

The main objective of this study was to establish a common fixed point theorem for a pair of generalized weakly contractive mappings in the setting of multiplicative metric spaces.

1.3.2 Specific objectives

This study has the following specific objectives:

- To prove the existence of a common fixed point for a pair of self-mappings.
- To show the uniqueness of common fixed point for a pair of self-mappings.
- To verify the applicability of the main result obtained using an example.

1.4 Significance of the study

The study may have the following importance:

- The outcome of this study may contribute to research activities in the study area.
- It may provide basic research skills to the researcher.
- It may help to show existence and uniqueness of solutions of problems involving multiplicative differential and integral equations.

1.5 Delimitation of the Study

The study focused only to prove the existence and uniqueness of common fixed point for a pair of generalized weakly contractive mapping in the setting of multiplicative metric spaces.

Chapter 2

Review of Related Literatures

In 2008, A. E. Bashirov, E. M. Kurplnara, A. Ozyapici introduced the notion of Multiplicative metric space and many authors proved fixed point and a common fixed point results using different contractive method for different types of maps.
Definition 2.1 Let X be a non empty set. A mapping d :XxX $\longrightarrow R^+$ is said to be a multiplicative metric on X if for any $x, y, z \in X$,the following conditions hold:

i. $d(x,y) \geq 1$ and $d(x,y) = 1$ if and only if $x = y$.

ii. $d(x,y) = d(y,x)$.

iii. $d(x,y) \leq d(x,z).d(z,y)$.

Then the mapping d together with X, that is, (X,d) is a multiplicative metric space.
In 2012, Özavsar and Cevikel gave the concept of multiplicative contraction mappings and they proved the Banach Contraction Principle in the setting of multiplicative metric spaces as follows:
Theorem 2.1 Let f be a multiplicative contraction mapping of a complete multiplicative metric space (X,d) into itself. Then f has a unique fixed point.
Definition 2.2 Let $X \neq \varnothing$. Then $T : X \longrightarrow X$ is said to be self-mapping with domain of $T = D(T) = X$ and range of $T = R(T) = T(X) \subseteq X$.
Definition 2.3 Let $f, g : X \longrightarrow X$ be self-mappings. A point $x \in X$ is called
(1) fixed point of f if $fx = x$;
(2) coincidence point of the pair $\{f, g\}$ if $fx = gx$;
(3) common fixed point of the pair $\{f, g\}$ if $x = fx = gx$.
In 2015, Abbas *et al.* obtained several fixed and common fixed point results of self-mappings satisfying certain generalized contractive conditions in the framework of multiplicative metric space by establishing the following theorem.
Theorem 2.2: Let (X,d) be a complete multiplicative metric space and $f : X \longrightarrow X$. Suppose that there exist control functions ψ and φ such that

$$\psi(d(fx, fy)) \leq \frac{\psi(M_f((x,y))}{\varphi(M_f(x,y))},$$

for all $x, y \in X$, where $M_f((x,y) = max\{d(x,y), d(fx,x), d(fy,y), \{d(fx,y).d(fy,x)\}^{\frac{1}{2}}\}$, $\psi : [1,\infty) \longrightarrow [1,\infty)$ is a continuous non-decreasing function with $\psi(t) = 1$ if and only if $t = 1\}$ and $\phi : [1,\infty) \longrightarrow [1,\infty)$ is a lower semi-continuous function with $\phi(t) = 1$ if and only if $t = 1$. Then f has a unique fixed point.

In 2015, Gu *et al.* proved Common fixed point results for four maps satisfying ϕ-contractive condition in multiplicative metric spaces for the following theorem.

Theorem 2.3 Let (X,d) be a complete multiplicative metric space,S, T, A, and B be four mappings of X into itself. Suppose that there exists $\lambda \in (0, \frac{1}{2})$
such that S(X) $\subset$ B(X), T(X) $\subset$ A(X), *and*
d(Sx, Ty)$\leq \phi(d^{\lambda}(Ax,By), d^{\lambda}(Ax,Sx), d^{\lambda}(By,Ty), d^{\lambda}(Sx,By), d^{\lambda}(Ax,Ty))$
for all $x, y \in X$.

In 2015, Kang *et al.* gave the notion of compatible mappings in the setting of multiplicative metric space and proved common fixed point for compatible mappings and its variants.

Using these notions, Nagpal *et al.*, (2016) proved several fixed point theorems for expansive mappings with a pair of maps.

Definition 2.4 (Gu *et al.*, 2003) Suppose that S, T are two self-mappings of a multiplicative metric space (X,d); S, T are called weak commutative mappings if it holds that for all $x \in X$, $d(STx, TSx) \leq d(Sx, Tx)$.

In 2014, He *et al.* proved a common fixed point theorems for four self-mappings in multiplicative metric space for weak commutative mappings for the following theorem.

Theorem 2.4 Let S, T, A and B be self-mappings of a complete multiplicative metric space X; they satisfy the following conditions:

i. $SX \subset BX$, $TX \subset AX$;

ii. A and S are weak commutative, B and T also are weak commutative;

iii. One of S, T, A and B is continuous;

iv. $d(Sx,Ty) \leq \{max\{d(Ax,By), d(Ax,Sx), d(By,Ty), d(Sx,By), d(Ax,Ty)\}\}^{\lambda}$,
$\lambda \in (0, \frac{1}{2})$, for all $x, y \in X$.Then S, T, A and B have a unique common fixed point.

In 2015, Abbas *et al.* proved several common fixed point results of self-mappings on multiplicative closed balls in the framework of multiplicative metric spaces.

Recently, Yong *et al.*, (2017) introduced various types of compatible mappings in multiplicative metric space and they proved some common fixed point results.

Chapter 3

Methodology

The chapter contains study period, study design, source of information, description of the research methodology and mathematical procedures.

3.1 Study period and site

The study was conducted at Jimma University under the department of mathematics from Sept, 2018 G.C. to June, 2019 G.C.

3.2 Study Design

In this research work, we employed analytical design.

3.3 Source of Information

The relevant sources of information for this study were books and published articles related to the study area.

3.4 Mathematical Procedure of the Study

In this study we followed the standard procedures used in published work of (Abbas *et al.*, 2015, Choudhury *et al.*, 2011, Kang *et al.*, 2015) in the setting of multiplicative metric spaces. That is,

- Introducing generalized weakly contractive mappings.
- Establishing a theorem for the mappings introduced.
- Constructing a sequence.
- Showing the sequence is multiplicative Cauchy.

- Proving the existence of coincidence point of maps considered.
- Proving the existence of a common fixed point.
- Showing uniqueness of the common fixed point.
- Giving an example in support of the main result.
- Giving conclusion and recommendation .

Chapter 4

Preliminaries and Main Results

4.1 Preliminaries

Definition 4.1.1 *(Bashirov et al., 2008) Let X be a non-empty set. A mapping $d : XxX \longrightarrow R^+$ is said to be a multiplicative metric on X if for any $x,y,z \in X$,the following conditions hold:*

i. $d(x,y) \geq 1$ *and* $d(x,y) = 1$ *if and only if* $x = y$.

ii. $d(x,y) = d(y,x)$.

iii. $d(x,y) \leq d(x,z).d(z,y)$.

Then the mapping d together with X, that is, (X,d) *is a multiplicative metric space.*

Example 4.1 (Özavsar *et al., 2012*) Let R^n_+ be the collections of $n-$ tuples of positive real numbers. Let $d^* : R^n_+ \times R^n_+ \longrightarrow R$ be defined as follows:

$$d^*(x,y) = \left|\frac{x_1}{y_1}\right|^* . \left|\frac{x_2}{y_2}\right|^* ...... \left|\frac{x_n}{y_n}\right|^*,$$

where $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n) \in R^n_+$ and $|.|^* : R_+ \longrightarrow R_+$ is defined by

$$|a|^* = \begin{cases} a & \text{if } a \geq 1 \\ \frac{1}{a} & \text{if } a < 1. \end{cases}$$

Then it is clear that all conditions of definition of 4.1.1 are satisfied. Therefore (R^n_+, d^*) is a multiplicative metric space.

Example 4.2 (Sarwar *et al.*, 2014) Let $d : R \times R \longrightarrow [1, \infty)$ be defined as $d(x,y) = a^{|x-y|}$, where $x, y \in R$ and $a > 1$. Then d is a multiplicative metric and (R,d) is a multiplicative metric space. It is taken as usual multiplicative metric spaces for all real numbers.

Example 4.3 (Sarwar *et al.*, 2014) Let (X,d) be a metric space. Define a mapping

d_a on X by

$$d_a(x,y) = a^{d(x,y)} = \begin{cases} 1 & \text{if } x = y \\ a & \text{if } x \neq y, \end{cases}$$

where $x, y \in X$ and $a > 1$. Then d_a is a multiplicative metric and (X, d_a) is known as the discrete multiplicative metric space.

Remark 4.1 (Sarwar *et al.*, 2014) We note that multiplicative metric spaces and metric spaces are independent. Indeed, the mapping d^* defined in Example 4.1 is multiplicative metric but not metric as it does not satisfy the triangular inequality. Consider

$$d^*\left(\frac{1}{3}, \frac{1}{2}\right) + d^*\left(\frac{1}{2}, 3\right) = \frac{3}{2} + 6 = 7.5 < 9 = d^*\left(\frac{1}{3}, 3\right).$$

On the other hand the usual metric on R is not multiplicative metric as it doesnt satisfy multiplicative triangular inequality, since

$$d(2,3).d(3,6) = 3 < 4 = d(2,6).$$

Definition 4.1.2 *(Özavsar et al., 2012) Let (X, d) be a multiplicative metric space, $x \in X$ and $\varepsilon > 1$. We now define a set $B_\varepsilon(x) = \{y \in X \mid d(x,y) < \varepsilon\}$, which is called multiplicative open ball of radius ε with center x. Similarly, one can describe multiplicative closed ball as $\bar{B}_\varepsilon(x) = \{y \in X \mid d(x,y) \leq \varepsilon\}$.*

Definition 4.1.3 *(Özavsar et al., 2012) Let (X, d) be a multiplicative metric space. Then a sequence $\{x_n\}$ in X said to be*
(1) a multiplicative convergent to $x \in X$ if for every multiplicative open ball $B_\varepsilon(x) = \{y \mid d(x,y) < \varepsilon\}, \varepsilon > 1$, there exists a natural number N such that $n \geq N$, then $x_n \in B_\varepsilon(x)$, that is, $d(x_n, x) \longrightarrow 1$ as $n \longrightarrow \infty$.
(2) a multiplicative Cauchy sequence if for all $\varepsilon > 1$, there exists $N \in N_0$ such that $d(x_n, x_m) < \varepsilon$ for all $m, n \geq N$, that is, $d(x_n, x_m) \longrightarrow 1$ as $n, m \longrightarrow \infty$.

Definition 4.1.4 *(Özavsar et al., 2012) We call a multiplicative metric space is complete if every multiplicative Cauchy sequence in it is multiplicative convergent to $x \in X$.*

Theorem 4.1.1 *(Özavsar et al., 2012) Let (x_n) be a multiplicative Cauchy sequence* *in a multiplicative metric space* (X,d). *If the sequence* (x_n) *has a subsequence* (x_{n_k}) *such that* $x_{n_k} \longrightarrow *x \in X$ *as* $n_k \longrightarrow \infty$, *then* $x_n \longrightarrow *x \in X$ *as* $n \longrightarrow \infty$.

Remark 4.2 (Kang *et al.*, 2015) The set of positive real numbers $R_+ = (0,\infty)$ is not complete according to the usual metric. Let $X = R_+$ and the sequence $\{x_n\} = \{\frac{1}{n}\}$. It is obvious $\{x_n\}$ is a Cauchy sequence in X with respect to usual metric and X is not a complete metric space, since $0 \notin R_+$. In the case of a multiplicative metric space, we take a sequence $\{x_n\} = \{a^{\frac{1}{n}}\}$, where $a > 1$. Then $\{x_n\}$ is a multiplicative Cauchy sequence since for $n \geq m$,

$$\begin{aligned} d(x_n,y_m) &= \left|\frac{x_n}{y_m}\right| = \left|\frac{a^{\frac{1}{n}}}{a^{\frac{1}{m}}}\right| = \left|a^{\frac{1}{n}-\frac{1}{m}}\right| \\ &\leq a^{\frac{1}{n}-\frac{1}{m}} < a^{\frac{1}{m}} < \varepsilon \; if \; m > \frac{loga}{log\varepsilon}, \end{aligned}$$

$$|a| = \begin{cases} a & \text{if } a \geq 1 \\ \frac{1}{a} & \text{if } a < 1 \end{cases}.$$

Also, $\{x_n\} \longrightarrow 1$ as $n \longrightarrow \infty$ and $1 \in R_+$ Hence (X,d) is a complete multiplicative metric space.

Definition 4.1.5 *Let* $f,g : X \longrightarrow X$ *be self-mappings. A point* $x \in X$ *is called*
(1) fixed point of f *if* $fx = x$;
(2) coincidence point of the pair $\{f,g\}$ *if* $fx = gx$;
(3) common fixed point of the pair $\{f,g\}$ *if* $x = fx = gx$.

Definition 4.1.6 *(Jungck, 1996) Two self-maps S and T on a nonempty set X are called weakly compatible if they commute at their coincidence point.*

Definition 4.1.7 *A function* $f : X \longrightarrow [0,\infty)$, *where X is a metric space, is called lower semi-continuous if, for all* $x \in X$ *and* $x_n \in X$ *with* $\lim_{n\to\infty} x_n = x$,
we have

$$f(x) \leq \lim_{n\to\infty} inf f(x_n).$$

Definition 4.1.8 *(Abbas et al., 2015) The control functions* ψ *and* ϕ *are defined as follows:*

i. $\Psi = \{\psi : [1,\infty) \longrightarrow [1,\infty) \mid \psi$ *is a continuous non-decreasing function with*

$\psi(t) = 1$ *if and only if* $t = 1\}$.

ii. $\Phi = \{\phi : [1,\infty) \longrightarrow [1,\infty) \mid \phi$ *is is a lower semi-continuous function with* $\phi(t) = 1$ *if and only if* $t = 1\}$.

In 2018, Cho proved the following fixed point theorem for generalized weakly contractive mappings in metric spaces as follows:

Theorem 4.1.2 *Let* X *be complete metric spaces and* T *satisfies the following conditions:*
$\psi(d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)) \leq \psi(m(x,y,d,T,\varphi)) - \phi(l(x,y,d,T,\varphi))$, *for all* $x,y \in X$*, where* $\Psi = \{\psi : [0,\infty) \longrightarrow [0,\infty) \mid \psi$ *is a continuous and* $\psi(t) = 0$ *if and only if* $t = 0$*,* $\Phi = \{\phi : [0,\infty) \longrightarrow [0,\infty) \mid \phi$ *is is a lower semi-continuous function and* $\phi(t) = 0$ *if and only if* $t = 0$*,*

$$
\begin{aligned}
m(x,y,d,T,\varphi) \quad = \quad & max\{(d(x,y)+\varphi(x)+\varphi(y), d(x,Tx)+\varphi(x)+\varphi(Tx), \\
& d(y,Ty)+\varphi(y)+\varphi(Ty), \\
& \{d(x,Ty)+\varphi(x)+\varphi(Ty)+d(y,Tx)+\varphi(y)+\varphi(Tx)\}^{\frac{1}{2}}\}.
\end{aligned}
$$

and

$$
l(x,y,d,T,\varphi) = max((d(x,y)+\varphi(x)+\varphi(y), d(y,Ty)+\varphi(y)+\varphi(Ty)).
$$

Then there exists $z \in X$ *such that* $z = Tz$ *and* $\varphi(z) = 0$*.*

4.2 Main Results

In this section, we introduce generalized weakly contractive mappings, establish a common fixed point theorem for the mappings introduced and prove the existence and uniqueness a common fixed point result in the setting of multiplicative metric spaces.

Definition 4.2.1 *Let* (X,d) *be a multiplicative metric space with metric d, let* $S,T : X \longrightarrow X$*, and let* $\varphi : X \to [1,\infty)$ *be a lower semi-continuous function. Then S and T are called a generalized weakly contractive mapping if they satisfies the following condition:*

$$\psi(d(Tx,Ty).\varphi(Tx).\varphi(Ty)) \leq \frac{\psi(m(Sx,Sy,d,T,\varphi))}{\phi(l(Sx,Sy,d,T,\varphi))},\ for\ all x,y \in X, \quad (4.1)$$

where $\psi \in \Psi, \phi \in \Phi$ and

$$\begin{aligned} m(Sx,Sy,d,T,\varphi) &= max\{d(Sx,Sy).\varphi(Sx).\varphi(Sy), d(Sx,Tx).\varphi(Sx).\varphi(Tx), \\ &\quad d(Sy,Ty).\varphi(Sy).\varphi(Ty), \\ &\quad \{d(Sx,Ty).\varphi(Sx).\varphi(Ty).d(Sy,Tx).\varphi(Sy).\varphi(Tx)\}^{\frac{1}{2}}\} \end{aligned}$$

and

$$l(Sx,Sy,d,T,\varphi) = max(d(Sx,Sy).\varphi(Sx).\varphi(Sy), d(Sy,Ty).\varphi(Sy).\varphi(Ty)). \quad (4.2)$$

Theorem 4.2.1 *Let (X,d) be a multiplicative metric space and S and T are generalized weakly contractive mappings. Assume that*
i. $T(X) \subseteq S(X)$;
ii. S and T are weakly compatible mappings. If one of the subspaces $T(X)$ or $S(X)$ is complete, then S and T have a unique common fixed point $z \in X$ such that $z = Tz = Sz$ and $\varphi(z) = 1$.

Proof: Let $x_0 \in X$ be fixed. Since $T(X) \subseteq S(X)$, choose $x_1 \in X$ such that $Sx_1 = Tx_0$. In general, choose $x_{n+1} \in X$ and define a sequence $\{x_n\}$ by $y_{n+1} = Sx_{n+1} = Tx_n$ for all $n = 0,1,2,...$
If $y_n = y_{n+1}$ for some n, we have $Sx_n = Sx_{n+1} = Tx_n$ and x_n is a coincidence point of S and T. Since S and T are weakly compatible, we have

$$TSx_n = STx_n = SSx_n \qquad (*)$$

Here, Sx_n is a coincidence point of S and T. Now by setting $x = x_{n+1}$ and $y = Sx_n$ in (4.2), we have

$$\begin{aligned} m(Sx,Sy,d,T,\varphi) &= max\{d(Sx_{n+1},SSx_n).\varphi(Sx_{n+1}).\varphi(SSx_n), d(Sx_{n+1},Tx_{n+1}).\varphi(Sx_{n+1}).\varphi(Tx_{n+1}), \\ &\quad d(SSx_n,TSx_n).\varphi(SSx_n).\varphi(TSx_n), \\ &\quad \{d(Sx_{n+1},TSx_n).\varphi(Sx_{n+1}).\varphi(TSx_n).d(SSx_n,Tx_{n+1}).\varphi(SSx_n).\varphi(Tx_{n+1})\}^{\frac{1}{2}}\} \end{aligned}$$

and

$$l(Sx,Sy,d,T,\varphi)=max(d(Sx_{n+1},SSx_n).\varphi(Sx_{n+1}).\varphi(SSx_n),d(SSx_n,TSx_n).\varphi(SSx_n).\varphi(TSx_n)).$$

Which implies that

$$m(Sx,Sy,d,T,\varphi) = d(Sx_{n+1},SSx_n).\varphi(Sx_{n+1}).\varphi(SSx_n)$$

and

$$l(Sx,Sy,d,T,\varphi)=d(Sx_{n+1},SSx_n).\varphi(Sx_{n+1}).\varphi(SSx_n).$$

Hence, (4.1) becomes $\psi(d(Tx_{n+1},TSx_n).\varphi(Tx_{n+1}).\varphi(TSx_n))\leq\frac{\psi(d(Sx_{n+1},SSx_n).\varphi(Sx_{n+1}).\varphi(SSx_n))}{\phi(d(Sx_{n+1},SSx_n).\varphi(Sx_{n+1}).\varphi(SSx_n))}$.
Using $(*)$, we have
$\phi(d(Sx_{n+1},SSx_n).\varphi(Sx_{n+1}).\varphi(SSx_n))=1$. From this, $d(Sx_{n+1},SSx_n)=1$ and Sx_n is a fixed point of S. Again using $(*)$, $d(Sx_{n+1},TSx_n)=1$ and Sx_n is a fixed point of T. Therefore, Sx_n is a common fixed point of S and T.
Suppose $y_n\neq y_{n+1}$. Plunging $x=x_n$ and $y=x_{n+1}$ in (4.2) we have,

$$\begin{aligned}
m(Sx_n,Sx_{n+1},d,T,\varphi) &= max\{d(Sx_n,Sx_{n+1}).\varphi(Sx_n).\varphi(Sx_{n+1}),d(Sx_n,Tx_n).\varphi(Sx_n).\varphi(Tx_n),\\
&\quad d(Sx_{n+1},Tx_{n+1}).\varphi(Sx_{n+1}).\varphi(Tx_{n+1}),\\
&\quad \{d(Sx_n,Tx_{n+1}).\varphi(Sx_n).\varphi(Tx_{n+1}).d(Sx_{n+1},Tx_n).\varphi(Sx_{n+1}).\varphi(Tx_n)\}^{\frac{1}{2}}\}\\
&= max\{(d(y_n,y_{n+1}).\varphi(y_n).\varphi(y_{n+1}),d(y_n,y_{n+1}).\varphi(y_n).\varphi(y_{n+1}),\\
&\quad d(y_{n+1},y_{n+2}).\varphi(y_{n+1}).\varphi(y_{n+2}),\\
&\quad \{d(y_n,y_{n+2}).\varphi(y_n).\varphi(y_{n+2}).d(y_{n+1},y_{n+1}).\varphi(y_{n+1}).\varphi(y_{n+1})\}^{\frac{1}{2}}\}.
\end{aligned}$$

Since

$$\begin{aligned}
\{d(y_n,y_{n+2}).\varphi(y_n).\varphi(y_{n+2}).\varphi(y_{n+1}).\varphi(y_{n+1})\}^{\frac{1}{2}}\} &\leq \{d(y_n,y_{n+1}).d(y_{n+1},y_{n+2}).\\
&\quad \varphi(y_n).\varphi(y_{n+2}).\varphi(y_{n+1}).\varphi(y_{n+1})\}^{\frac{1}{2}}\}\\
&\leq max\{d(y_n,y_{n+1}).\varphi(y_n).\varphi(y_{n+1}),d(y_{n+1},y_{n+2}).\\
&\quad \varphi(y_{n+1}).\varphi(y_{n+2})\},
\end{aligned}$$

$$m(Sx,Sy,d,T,\varphi) = max\{d(y_n,y_{n+1}).\varphi(y_n).\varphi(y_{n+1}), d(y_{n+1},y_{n+2}).\varphi(y_{n+1}).\varphi(y_{n+2})\} \tag{4.3}$$

and

$$l(Sx,Sy,d,T,\varphi) = max\{d(y_n,y_{n+1}).\varphi(y_n).\varphi(y_{n+1}), d(y_{n+1},y_{n+2}).\varphi(y_{n+1}).\varphi(y_{n+2})\}. \tag{4.4}$$

Then by using (4.3) and (4.4), (4,1) becomes

$$\psi(d(Tx,Ty).\varphi(Tx).\varphi(Ty)) \leq \frac{\psi(max\{d(y_n,y_{n+1}).\varphi(y_n).\varphi(y_{n+1}), d(y_{n+1},y_{n+2}).\varphi(y_{n+1}).\varphi(y_{n+2})\}}{\phi(max\{d(y_n,y_{n+1}).\varphi(y_n).\varphi(y_{n+1}), d(y_{n+1},y_{n+2}).\varphi(y_{n+1}).\varphi(y_{n+2})\}}. \tag{4.5}$$

Now suppose $d(y_n,y_{n+1}).\varphi(y_n).\varphi(y_{n+1}) < d(y_{n+1},y_{n+2}).\varphi(y_{n+1}).\varphi(y_{n+2})$, for some positive integer n.
Then (4.5) becomes

$$\begin{aligned} \psi(d(y_{n+1},y_{n+2}).\varphi(y_{n+1}).\varphi(y_{n+2})) &\leq \frac{\psi(d(y_{n+1},y_{n+2}).\varphi(y_{n+1}).\varphi(y_{n+2}))}{\phi(d(y_{n+1},y_{n+2}).\varphi(y_{n+1}).\varphi(y_{n+2}))} \\ &< \psi(d(y_{n+1},y_{n+2}).\varphi(y_{n+1}).\varphi(y_{n+2})), \end{aligned}$$

which is a contradiction. Thus

$$d(y_{n+1},y_{n+2}).\varphi(y_{n+1}).\varphi(y_{n+2}) \leq d(y_n,y_{n+1}).\varphi(y_n).\varphi(y_{n+1}), \tag{4.6}$$

and (4.5) becomes

$$\psi(d(y_{n+1},y_{n+2}).\varphi(y_{n+1}).\varphi(y_{n+2})) \leq \frac{\psi(d(y_n,y_{n+1}).\varphi(y_n).\varphi(y_{n+1}))}{\phi(d(y_n,y_{n+1}).\varphi(y_n).\varphi(y_{n+1}))}. \tag{4.7}$$

Hence, the sequence $\{d(y_{n+1},y_{n+2}).\varphi(y_{n+1}).\varphi(y_{n+2})\}$ is monotone decreasing. Thus, there exists $r \geq 1$ such that

$$\lim_{n\to\infty}(d(y_{n+1},y_{n+2}).\varphi(y_{n+1}).\varphi(y_{n+2})) \to r. \tag{4.8}$$

Now we show $r = 1$. Assume $r > 1$. Letting $n \to \infty$ in (4.7), by the continuity of ψ and the lower semi-continuity of ϕ it follows that
$\psi(r) \leq \dfrac{\psi(r)}{\lim_{n\to\infty} inf\phi(d(y_n,y_{n+1}).\varphi(y_n).\varphi(y_{n+1}))} \leq \frac{\psi(r)}{\phi(r)}$.

This implies that $\phi(r) \leq \frac{\psi(r)}{\psi(r)} = 1$, which is a contradiction since $r > 1$, from property of ϕ. Hence, $r = 1$ and (4.8) becomes

$$\lim_{n\to\infty}(d(y_{n+1}, y_{n+2})) \longrightarrow 1, \lim_{n\to\infty} \varphi(y_{n+1}) \to 1, and \lim_{n\to\infty} \varphi(y_{n+2})) \to 1. \tag{4.9}$$

Now we prove that the sequence $\{y_n\}$ is a multiplicative Cauchy sequence. By using (4.9), it is sufficient to prove that $\{y_{2n}\}$ is a multiplicative Cauchy sequence. To prove this, suppose $\{y_{2n}\}$ is not a multiplicative Cauchy sequence, that is there exist $\varepsilon > 1$ for which we can find two sequences of positive integers $2m(k)$ and $2n(k)$ such that for all positive integer k,
$2n(k) > 2m(k) > k$,

$$d(y_{2m(k)}, y_{2n(k)}) \geq \varepsilon \; and \; d(y_{2m(k)}, y_{2n(k-2)}) < \varepsilon. \tag{4.10}$$

Now using the triangle inequality,
$\varepsilon \leq d(y_{2m(k)}, y_{2n(k)}) \leq d(y_{2m(k)}, y_{2n(k)-2}).d(y_{2n(k)-2}, y_{2n(k)-1}).d(y_{2n(k)-1}, y_{2n(k)})$.
This implies that $\varepsilon \leq d(y_{2m(k)}, y_{2n(k)}) < \varepsilon.d(y_{2n(k)-2}, y_{2n(k)-1}).d(y_{2n(k)-1}, y_{2n(k)})$.
Letting $k \to \infty$ in the above inequalities and using (4.9), we have

$$\lim_{k\to\infty}(d(y_{2m(k)}, y_{2n(k)})) \leq \varepsilon. \tag{4.11}$$

Now letting $k \to \infty$ in (4.10) and using (4.11), we have

$$\lim_{k\to\infty}(d(y_{2m(k)}, y_{2n(k)})) = \varepsilon. \tag{4.12}$$

Again,
$d(y_{2m(k)}, y_{2n(k)}) \leq d(y_{2m(k)}, y_{2m(k)+1}).d(y_{2m(k)+1}, y_{2n(k)+1}).d(y_{2n(k)+1}, y_{2n(k)})$
and
$d(y_{2m(k)+1}, y_{2n(k)+1}) \leq d(y_{2m(k)+1}, y_{2m(k)}).d(y_{2m(k)}, y_{2n(k)}).d(y_{2n(k)}, y_{2n(k)+1})$.
Letting $k \to \infty$, using (4.9) and (4.12), we have

$$\lim_{k\to\infty}(d(y_{2m(k)+1}, y_{2n(k)+1})) = \varepsilon. \tag{4.13}$$

Again,
$d(y_{2n(k)+2}, y_{2m(k)+1}) \leq d(y_{2n(k)+2}, y_{2n(k)+1}).d(y_{2n(k)+1}, y_{2m(k)+1})$

and
$d(y_{2n(k)+1}, y_{2m(k)+1}) \leq d(y_{2n(k)+1}, y_{2n(k)+2}).d(y_{2n(k)+2}, y_{2m(k)+1})$.
Similarly,
$d(y_{2m(k)}, y_{2n(k)+1}) \leq d(y_{2m(k)}, y_{2n(k)}).d(y_{2n(k)}, y_{2n(k)+1})$
and
$d(y_{2m(k)}, y_{2n(k)}) \leq d(y_{2m(k)}, y_{2n(k)+1}).d(y_{2n(k)+1}, y_{2n(k)})$.
Letting $k \to \infty$ in the above inequalities, using (4.9), (4.12) and (4.13), we have

$$\lim_{k\to\infty}(d(y_{2n(k)+2}, y_{2m(k)+1})) = \varepsilon, \tag{4.14}$$

$$\lim_{k\to\infty}(d(y_{2m(k)}, y_{2n(k)+2})) = \varepsilon, \tag{4.15}$$

$$\lim_{k\to\infty}(d(y_{2m(k)}, y_{2n(k)+1})) = \varepsilon. \tag{4.16}$$

By setting $x = x_{2m(k)}$ and $y = x_{2n(k)+1}$ in (4.2), we have

$$\begin{aligned} m(Sx_{2m(k)}, Sx_{2n(k)+1}, d, T, \varphi) &= max\{d(Sx_{2m(k)}, Sx_{2n(k)+1}).\varphi(Sx_{2m(k)}).\varphi(Sx_{2n(k)+1}),\\ &\quad d(Sx_{2m(k)}, Tx_{2m(k)}).\varphi(Sx_{2m(k)}).\varphi(Tx_{2m(k)}),\\ &\quad d(Sx_{2n(k)+1}, Tx_{2n(k)+1}).\varphi(Sx_{2n(k)+1}).\varphi(Tx_{2n(k)+1}),\\ &\quad \{d(Sx_{2m(k)}, Tx_{2n(k)+1}).\varphi(y_{2m(k)}).\varphi(Ty_{2n(k)+1}).\\ &\quad d(Sx_{2n(k)+1}, Tx_{2m(k)}).\varphi(Sx_{2n(k)+1}).\varphi(Tx_{2m(k)})\}^{\frac{1}{2}}\}\\ &= max\{(d(y_{2m(k)}, y_{2n(k)+1}).\varphi(y_{2m(k)}).\varphi(y_{2n(k)+1}),\\ &\quad d(y_{2m(k)}, y_{2m(k)+1}).\varphi(y_{2m(k)}).\varphi(y_{2m(k)+1}),\\ &\quad d(y_{2n(k)+1}, y_{2m(k)+2}).\varphi(y_{2n(k)+1}).\varphi(y_{2m(k)+2}),\\ &\quad \{d(y_{2m(k)}, y_{2m(k)+2}).\varphi(y_{2m(k)}).\varphi(y_{2m(k)+2}).\\ &\quad d(y_{2n(k)+1}, y_{2m(k)+1}).\varphi(y_{2n(k)+1}).\varphi(y_{2m(k)+1})\}^{\frac{1}{2}}\} \end{aligned}$$

and

$$\begin{aligned} l(Sx_{2m(k)}, Sx_{2n(k)+1}, d, T, \varphi) &= max(d(Sx_{2m(k)}, Sx_{2n(k)+1}).\varphi(Sx_{2m(k)}.\varphi(Sx_{2n(k)+1}),\\ &\quad d(Sx_{2n(k)+1}, Tx_{2n(k)+1}).\varphi(Sx_{2n(k)+1}).\varphi(Tx_{2n(k)+1})\\ &= max(d(y_{2m(k)}, y_{2n(k)+1}).\varphi(y_{2m(k)}.\varphi(y_{2n(k)+1}),\\ &\quad d(y_{2n(k)+1}, y_{2n(k)+2}).\varphi(y_{2n(k)+1}).\varphi(y_{2n(k)+2}). \end{aligned}$$

Letting $k \to \infty$ in the above inequalities, using (4.9), and (4.12) - (4.16), we have

$$\lim_{k\to\infty}(m(Sx_{2m(k)}, Sx_{2n(k)+1}, d, T, \varphi)) = \varepsilon \textit{ and } \lim_{k\to\infty}(l(Sx_{2m(k)}, Sx_{2n(k)+1}, d, T, \varphi)) = \varepsilon. \tag{4.17}$$

Thus from (4.1), we have
$\psi(d(Tx_{2m(k)}, Tx_{2n(k)+1}).\varphi(Tx_{2m(k)}).\varphi(Tx_{2n(k)+1})) \leq \frac{\psi(m(Sx_{2m(k)}, Sx_{2n(k)+1}, d, T, \varphi))}{\phi(l(Sx_{2m(k)}, Sx_{2n(k)+1}, d, T, \varphi))}$.
Or
$\psi(d(y_{2m(k)+1}, y_{2n(k)+2}).\varphi(y_{2m(k)+1}).\varphi(y_{2n(k)+2})) \leq \frac{\psi(m(Sx_{2m(k)}, Sx_{2n(k)+1}, d, T, \varphi))}{\phi(l(Sx_{2m(k)}, Sx_{2n(k)+1}, d, T, \varphi))}$.
Letting $k \to \infty$, using (4.9), (4.14) and (4,17), applying continuity of ψ and lower semi-continuity of ϕ, we have
$\psi(\varepsilon) \leq \frac{\psi(\varepsilon)}{\lim_{k\to\infty} inf \phi(l(Sx_{2m(k)}, Sx_{2n(k)+1}, d, T, \varphi))} \leq \frac{\psi(\varepsilon)}{\phi(\varepsilon)}$. This implies that,
$\psi(\varepsilon) \leq \frac{\psi(\varepsilon)}{\phi(\varepsilon)} < \psi(\varepsilon)$, which is a contradiction from property of ϕ.
Therefore $\{y_{2n}\}$ is a multiplicative Cauchy sequence. Hence by (4.9), $\{y_n\}$ is a multiplicative Cauchy sequence. Now since $S(X)$ is a complete subspace of X, it has multiplicative convergent subsequence of $\{y_n\}$. That is, there exists $p \in X$ such that

$$Sp = z. \tag{4.18}$$

As $\{y_n\}$ is a multiplicative Cauchy sequence containing a convergent multiplicative subsequence, therefore the sequence $\{y_n\}$ also converges to $z \in X$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z. \tag{4.19}$$

Since φ is lower semi- continuous, $\varphi(z) \leq \lim_{n\to\infty} inf \varphi(y_n) = 1$. But $\varphi(z) \geq 1$, which implies that $\varphi(z) = 1$.
Now we show $Tp = z$.
By setting $x = x_n$ and $y = p$ in (4.2), we have

$$\begin{aligned} m(Sx_n, Sp, d, T, \varphi) &= max\{d(Sx_n, Sp).\varphi(Sx_n).\varphi(Sp), d(Sx_n, Tx_n).\varphi(Sx_n).\varphi(Tx_n), \\ &\quad d(Sp, Tp).\varphi(Sp).\varphi(Tp), \\ &\quad \{d(Sx_n, Tp).\varphi(Sx_n).\varphi(Tp).d(Sp, Tx_n).\varphi(Sp).\varphi(Tx_n)\}^{\frac{1}{2}}\} \end{aligned}$$

and

$$l(Sx_n, Sp, d, T, \varphi) = max(d(Sx_n, Sp).\varphi(Sx_n).\varphi(Sp), d(Sp, Tp).\varphi(Sp).\varphi(Tp).$$

Letting $n \to \infty$, using (4.19) and applying lower semi-continuty of φ, we have

$$\begin{aligned}\lim_{n\to\infty} m(Sx_n, Sp, d, T, \varphi) &= max\{d(z,Sp).\varphi(z).\varphi(Sp), d(z,z).\varphi(z).\varphi(z),\\ &\quad d(Sp,Tp).\varphi(Sp).\varphi(Tp),\\ &\quad \{d(z,Tp).\varphi(z).\varphi(Tp).d(Sp,z).\varphi(Sp).\varphi(z)\}^{\frac{1}{2}}\}\\ &= d(z,Tp).\varphi(Tp)\end{aligned}$$

and

$$l(Sx_n, Sp, d, T, \varphi) = d(z,Tp).\varphi(Tp). \tag{4.20}$$

Then using (4.1), we have
$\psi(d(Tx_n,Tp).\varphi(Tx_n).\varphi(Tp)) \leq \frac{\psi(m(Sx_n,Sp,d,T,\varphi))}{\phi(l(Sx_n,Sp,d,T,\varphi))}$.
Letting $n \to \infty$, using (4.20) and by applying the continuity of ψ, the lower semi-continuity of ϕ, we have
$\psi(d(z,Tp).\varphi(Tp)) \leq \frac{\psi(d(z,Tp).\varphi(Tp)}{\phi(d(z,Tp).\varphi(Tp)}$.
Which implies that $\phi(d(z,Tp).\varphi(Tp)) = 1$. Then from property of ϕ, we have

$$d(z,Tp).\varphi(Tp) = 1.$$

$$Hence, d(z,Tp) = 1 \ implies\ that\ z = Tp\ and\ \varphi(Tp) = 1. \tag{4.21}$$

Therefore, from (4.18) and (4.21), we have $Sp = Tp = z$.
Since S and T are weakly compatible, we have

$$STp = TSp = Sz = Tz. \tag{4.22}$$

Now we show $Tz = z$.
Again by setting $x = z$ and $y = x_n$ in (4.2), we have

$$\begin{aligned}m(Sz, Sx_n, d, T, \varphi) &= max\{d(Sz,Sx_n).\varphi(Sz).\varphi(Sx_n), d(Sz,Tz).\varphi(Sz).\varphi(Tz),\\ &\quad d(Sx_n,Tx_n).\varphi(Sx_n).\varphi(Tx_n),\\ &\quad \{d(Sz,Tx_n).\varphi(Sz).\varphi(Tx_n).d(Sx_n,Tz).\varphi(Sx_n).\varphi(Tz)\}^{\frac{1}{2}}\}.\end{aligned}$$

and

$l(Sz, Sx_n, d, T, \varphi) = max(d(Sz, Sx_n).\varphi(Sz).\varphi(Sx_n), d(Sx_n, Tx_n).\varphi(Sx_n).\varphi(Tx_n).$

Letting $n \to \infty$, using (4.22) and applying lower semi-continuity of φ, we have

$$\begin{aligned} m(Tz, z, d, T, \varphi) &= max\{d(Tz, z).\varphi(Tz).\varphi(z), d(Tz, Tz).\varphi(Tz).\varphi(Tz), \\ &\quad d(z, z).\varphi(z).\varphi(z), \\ &\quad \{d(Tz, z).\varphi(Tz).\varphi(z).d(z, Tz).\varphi(z).\varphi(Tz)\}^{\frac{1}{2}}\} \\ &= d(Tz, z).\varphi(Tz). \end{aligned}$$

and

$$l(Tz, z, d, T, \varphi) = d(Tz, z).\varphi(Tz). \tag{4.23}$$

Then using (1.1), we have
$\psi(d(Tz, Tx_n).\varphi(Tz).\varphi(Tx_n)) \leq \frac{\psi(m(Sz, Sx_n, d, T, \varphi))}{\phi(l(Sz, Sx_n, d, T, \varphi))}$. Letting $n \to \infty$, using (4.22), (4.23) and applying lower semi-continuity of φ, we have

$$\psi(d(Tz, z).\varphi(Tz)) \leq \frac{\psi(d(Tz, z).\varphi(Tz))}{\phi(d(Tz, z).\varphi(Tz))}.$$

Which implies that $\phi(d(Tz, z).\varphi(Tz)) = 1$. Then from property of ϕ, we have $d(Tz, z) = 1$ and hence, $Tz = z$. Therefore z is a fixed point of T. Using (4.22), $Tz = z = Sz$.
Hence z is a common fixed point of T and S.

Uniqueness.
Suppose there is another common fixed point of T and S say u with $Tu = u$ and

$Su = u$. Setting $x = z$ and $y = u$ in (4.2) and applying semi-continuity of φ, we have

$$\begin{aligned} m(Sz,Su,d,T,\varphi) &= max\{d(Sz,Su).\varphi(Sz).\varphi(Su), d(Sz,Tz).\varphi(Sz).\varphi(Tz), \\ &\quad d(Su,Tu).\varphi(Su).\varphi(Tu), \\ &\quad \{d(Sz,Tu).\varphi(Sz).\varphi(Tu).d(Su,Tz).\varphi(Su).\varphi(Tz)\}^{\frac{1}{2}}\} \\ &= max\{(d(z,u).\varphi(z).\varphi(u), d(z,z).\varphi(z).\varphi(z), \\ &\quad d(u,u).\varphi(u).\varphi(u), \\ &\quad \{d(z,u).\varphi(z).\varphi(u).d(u,z).\varphi(u).\varphi(z)\}^{\frac{1}{2}}\} \\ &= d(z,u). \end{aligned}$$

and

$$\begin{aligned} l(Sz,Su,d,T,\varphi) &= max(d(Sz,Su).\varphi(Sz).\varphi(Su), d(Su,Tu).\varphi(Su).\varphi(Tu) \\ &= max(d(z,u).\varphi(z).\varphi(u), d(u,u).\varphi(u).\varphi(u) \\ &= d(z,u). \end{aligned}$$

Using (4.1), we have
$\psi(d(Tz,Tu).\varphi(Tz).\varphi(Tu)) \leq \psi(d(z,u).\varphi(z).\varphi(u)) \leq \frac{\psi(m(Sz,Su,d,T,\varphi))}{\phi(l(Sz,Su,d,T,\varphi))} \leq \frac{\psi(d(z,u))}{\phi(d(z,u))}$.
By applying lower semi-continuity of φ to the left side, we have

$$\psi(d(z,u)) \leq \frac{\psi(d(z,u))}{\phi(d(z,u))}.$$

Which implies that $\phi(d(z,u)) = 1$. Then from property of ϕ, we have
$d(z,u) = 1$ and hence, $z = u$.
Therefore, T and S have a unique common fixed point z.
The following is an example in support of our main result.
Example Let $X = [1,\infty)$ with the usual multiplicative metric d. Define S and T : $X \longrightarrow X$ by

$$S(x) = \begin{cases} x & \text{if } 1 \leq x \leq 5; \\ 12 & \text{if } x > 5; \end{cases}$$

and

$$T(x) = \begin{cases} \sqrt{x} & \text{if } 1 \leq x \leq 5; \\ 5 & \text{if } x > 5; \end{cases}$$

for all $x \in X$. Let $\phi, \psi : [1,\infty) \longrightarrow [1,\infty)$ defined by $\psi(t) = t^2$ for $t \in [1,\infty)$,

$$\varphi(t) = \begin{cases} 2t & \text{if } t > 5; \\ t & \text{if } t \leq 5; \end{cases}$$

and

$$\phi(t) = \begin{cases} t^{\frac{1}{2}} & \text{if } t > 5; \\ 1 & \text{if } t \leq 5. \end{cases}$$

Now we show condition (4.1) as follows.

Case 1: Let $x, y \in (5,\infty)$. Then

$$\begin{aligned}
i.\psi(d(Tx,Ty).\varphi(Tx).\varphi(Ty)) &= \psi(d(5,5).\varphi(5).\varphi(5)) \\
&= \psi\left(\left(\left|\frac{5}{5}\right|^*\right).10.10\right) \\
&= \psi(100) \\
&= 10000. \\
ii.\ \ d(Sx,Sy).\varphi(Sx).\varphi(Sy) &= d(12,12).\varphi(12).\varphi(12) \\
&= \left|\frac{12}{12}\right|^* .24.24 \\
&= 576. \\
iii.\ \ d(Sx,Tx).\varphi(Sx).\varphi(Tx) &= d(12,5).\varphi(12).\varphi(5) \\
&= \left|\frac{12}{5}\right|^* .24.10 \\
&= 576.
\end{aligned}$$

Similarly,

$$iv.\ \ d(Sy,Ty).\varphi(Sy).\varphi(Ty) \quad = \quad 576.$$

$$v.\ \ (d(Sx,Ty).\varphi(Sx).\varphi(Ty).d(Sy,Tx).\varphi(Sy).\varphi(Tx))^{\frac{1}{2}} \quad = 576.$$

Thus, using $(ii),(iii),(iv)$ and (v), (4.2) becomes

$$\begin{aligned} m(Sx,Sy,d,T,\varphi) &= max\{576,576,576,576\} \\ &= 576 \end{aligned}$$

and

$$l(Sx,Sy,d,T,\varphi) = 576 \tag{4.24}$$

Hence, using (i) and (4.24), (4.1) becomes
$10,000 \leq \frac{\psi(576)}{\phi(576)} \leq 13,824$.
Case 2. Let $x,y \in [1,5]$ with $x \geq y$. Then

$$\begin{aligned} i.\psi(d(Tx,Ty).\varphi(Tx).\varphi(Ty)) &= \psi\left(d\left(x^{\frac{1}{2}},y^{\frac{1}{2}}\right).\varphi\left(x^{\frac{1}{2}}\right).\varphi\left(y^{\frac{1}{2}}\right)\right) \\ &= \psi\left(\left|\frac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}}\right|^{*}.x^{\frac{1}{2}}.y^{\frac{1}{2}}\right) \\ &= x^2. \\ ii.\ d(Sx,Sy).\varphi(Sx).\varphi(Sy) &= d(x,y).\varphi(x).\varphi(y) \\ &= \left|\frac{x}{y}\right|^{*}.x.y \\ &= x^2. \\ iii.\ d(Sx,Tx).\varphi(Sx).\varphi(Tx) &= d(x,\sqrt{x}).\varphi(x).\varphi(\sqrt{x}) \\ &= \left|\frac{x}{\sqrt{x}}\right|^{*}.x.\sqrt{x} \\ &= x^2. \end{aligned}$$

Similarly,

$$\begin{aligned} iv.\ d(Sy,Ty).\varphi(Sy).\varphi(Ty) &= y^2 \\ v.\ (d(Sx,Ty).\varphi(Sx).\varphi(Ty).d(Sy,Tx).\varphi(Sy).\varphi(Tx))^{\frac{1}{2}} &= xy. \end{aligned}$$

Hence, (4.1) becomes

$$1 \leq x^2, for\ x \geq y,\ where\ equality\ holds\ for\ x = 1$$

and

$$1 \leq y^2, for\ x < y.$$

Case 3. Let $x \in (5,\infty)$ and $y \in [1,5]$.

$$\begin{aligned}
i.\psi(d(Tx,Ty).\varphi(Tx).\varphi(Ty)) &= \psi(d(5,y^{\frac{1}{2}}).\varphi(5).\varphi(y^{\frac{1}{2}})) \\
&= \psi\left(\left|\frac{5}{y^{\frac{1}{2}}}\right|^{*}.5.y^{\frac{1}{2}}\right) \\
&= 625. \\
ii.\ d(Sx,Sy).\varphi(Sx).\varphi(Sy) &= d(12,y).\varphi(12).\varphi(y) \\
&= \left|\frac{12}{y}\right|^{*}.24.y \\
&= 288. \\
iii.\ d(Sx,Tx).\varphi(Sx).\varphi(Tx) &= d(12,5).\varphi(12).\varphi(5) \\
&= \left|\frac{12}{5}\right|^{*}.24.5 \\
&= 288. \\
iv.\ d(Sy,Ty).\varphi(Sy).\varphi(Ty) &= d(y,\sqrt{y}).\varphi(y).\varphi(\sqrt{y}) \\
&= \left|\frac{y}{\sqrt{y}}\right|^{*}.y.\sqrt{y} \\
&= y^2.
\end{aligned}$$

Similarly,

$$v.\ (d(Sx,Ty).\varphi(Sx).\varphi(Ty).d(Sy,Tx).\varphi(Sy).\varphi(Tx)))^{\frac{1}{2}} = 60\sqrt{2}$$

Here,

$$\begin{aligned}
m(Sx,Sy,d,T,\varphi) &= max\{288,y^2,288,60\sqrt{2}\} \\
&= 288
\end{aligned}$$

and

$$l(Sx,Sy,d,T,\varphi) = max\{288,288\} = 288. \tag{4.25}$$

Hence, using (i) and (4.25), (4.1) becomes

$$625 \leq 4887.68.$$

Case 4. Let $y \in (5, \infty)$ and $x \in [1,5]$.

$$\begin{aligned}
i. \psi(d(Tx,Ty).\varphi(Tx).\varphi(Ty)) &= \psi(d(\sqrt{x},5).\varphi(\sqrt{x}).\varphi(5)) \\
&= \psi(\left|\frac{\sqrt{x}}{5}\right|^{*}.\sqrt{x}.5) \\
&= 625. \\
ii.\ d(Sx,Sy).\varphi(Sx).\varphi(Sy) &= d(x,12).\varphi(x).\varphi(12) \\
&= \left|\frac{x}{12}\right|^{*}.x.24 \\
&= 288. \\
iii.\ d(Sx,Tx).\varphi(Sx).\varphi(Tx) &= d(x,\sqrt{x}).\varphi(x).\varphi(\sqrt{x}) \\
&= \left|\frac{x}{\sqrt{x}}\right|^{*}.x.\sqrt{x} \\
&= x^2. \\
iv.\ d(Sy,Ty).\varphi(Sy).\varphi(Ty) &= d(12,5).\varphi(12).\varphi(5) \\
&= \left|\frac{12}{5}\right|^{*}.24.5 \\
&= 288.
\end{aligned}$$

Similarly,

$$v.\ (d(Sx,Ty).\varphi(Sx).\varphi(Ty).d(Sy,Tx).\varphi(Sy).\varphi(Tx))^{\frac{1}{2}} = 60\sqrt{2}$$

Here,

$$\begin{aligned}
m(Sx,Sy,d,T,\varphi) &= max\{288, x^2, 288, 60\sqrt{2}\} \\
&= 288
\end{aligned}$$

and

$$l(Sx,Sy,d,T,\varphi) = max\{288,288\} = 288. \tag{4.26}$$

Hence, using (i) and (4.26), (4.1) becomes

$$625 \leq 4887.68.$$

Therefore, condition (4.1) is satisfied.
Next $Sx = Tx$ at $x = 1$ and $STx = TSx = 1$. This shows that S and T are weakly compatible. Again $T(X) \subseteq S(X)$.
Thus all conditions of the Theorem 4.2.1 are satisfied and $x = 1$ is a unique common fixed point of S and T.

Chapter 5

Conclusion and Future Scope

5.1 Conclusion

In this thesis, we have discussed the historical back ground of multiplicative calculus with its applications in different fields and simplicity of its operation. Next, we have explored the properties of multiplicative metric spaces with its some of topological spaces, development of contraction and weak contraction in multiplicative metric spaces and also the independence of metric spaces and multiplicative metric spaces has been discussed. We introduced generalized weakly contractive mappings, establish a common fixed point theorem for the mappings introduced and prove the existence and uniqueness a common fixed point result in the setting of multiplicative metric spaces.
We have supported the result of this work by an example.

5.2 Future Scope

One of the basic and most widely applied fixed point theorems in mathematical analysis is Banach Contraction Mapping Principle or ”Banachs Fixed Point Theorem” by (Banach, 1922). This principle has been disclosed 97 years ago and become a century in 2022, but it is an active area of research work in mathematics and other sciences. There are several published results related to existence of fixed points of self-mappings defined in multiplicative metric spaces. There are also few results related to the existence of common fixed points for a pair or more self-mappings in different types of contractive conditions with application in this spaces. The researcher believes the search for the existence of coincidence and common fixed points of self-mappings satisfying different contractive type conditions in multiplicative metric spaces is an active area of study. So, any interested researchers can use this opportunity and conduct their research work in this area.

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