A COMMON FIXED POINT THEOREM FOR REICH TYPE CO-CYCLIC CONTRACTION IN DISLOCATED QUASI - METRIC SPACES

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ABSTRACT

The purpose of this research is to establish the existence of coincidence and common fixed points for Reich type co-cyclic contraction in dislocated quasi-metric space and to show the uniqueness of the common fixed point. In this research undertaking, we followed analytical design, secondary source of data such as journal, internet etc. was used. The study procedure we used is that of Karapinar and Erhan [14]; and Panthi et al. [19]. This study was conducted from September 2014 G.C to September 2015 G.C

Keywords: Dislocated quasi-metric space, coincidence points; common fixed point; Reich type co-cyclic contraction

CHAPTER ONE: INTRODUCTION

1.1 BACKGROUND OF THE STUDY

Fixed point theory provides the most important and traditional tools for proving the existence of solutions of many problems in both pure and applied mathematics. In metric fixed point theory, the interplay between contractive condition and the existence and uniqueness of a fixed point has been very strong and fruitful. The study of fixed points of mappings which satisfy certain contractive conditions has primary applications in the solution of differential and integral equations (see, e.g., [3, 4]).

The Banach Contraction Principle is a very popular tool which is used to solve existence problems in many branches of Mathematical Analysis and its applications. It is no surprise that there is a great number of generalizations of this fundamental theorem. In some generalizations, the contractive nature of the map is weakened; see ([9], [12], [13], [14], [15], [24]), and others. In other generalizations the ambient space is weakened; see ([1], [3], [6], [19], [26], [27]), and others. This celebrated theorem can be stated as follow.

Theorem 1.1 [5]: Let (X, d) be a complete metric space and T be a mapping of X into itself satisfying:

$$d(Tx, Ty) \leq kd(x, y), \text{ for all } x, y \in X, \tag{1.1}$$

where $k \in [0,1)$. Then, T has a unique fixed point $x^* \in X$.

Inequality (1.1) implies continuity of T. A natural question is whether we can find contractive conditions which will imply existence of a fixed point in a complete metric space but do not imply continuity. In the literature there is a great number of generalizations of the Banach contraction principle (see [2] and references cited therein).

In 2003, Kirk et al. [15] introduced cyclic contractions in metric spaces and investigated the existence of proximity points and fixed points for cyclic contraction mappings. Since then many results appeared in this field. (See [1], [2], [6], [7], [11], [14], [15], [16], [17], [19], [22], [27]).

Definition 1.2 [15]: A mapping $T : A \cup B \rightarrow A \cup B$ is called cyclic if $T(A) \subseteq B$ and $T(B) \subseteq A$, where A, B are nonempty subsets of a metric space (X, d).

Definition 1.3 [15] A mapping $T : A \cup B \rightarrow A \cup B$ is called a cyclic contraction if there exists $k \in [0,1)$ such that

$$d(Tx, Ty) \leq kd(x, y) \text{ for all } x \in A \text{ and } y \in B. \tag{1.2}$$

The concept of quasi-metric spaces was introduced by Wilson [25] in 1931 as a generalization of metric spaces, and in 2000 Hitzler and Seda [7] introduced dislocated metric spaces as a generalization of metric spaces. Furthermore, Zeyada et al. [26] generalized the results of Hitzler and Seda [7] and introduced the concept of complete dislocated quasi metric space. Aage and Salunke [1, 2] derived some fixed point theorems in dislocated quasi metric spaces. For more information about the fixed point results one can refer ([1], [2], [6], [7], [11], [17], [19], [20], [21], [22], [27]).

In 1976, Jungck [9] proved a common fixed point theorem for commuting maps by generalizing the Banach's fixed point theorem. The concept of the commutativity has been generalized in several ways. For this, Sessa [24] has introduced the concept of weakly commuting mappings and Jungck [10] initiated the concept of compatibility. When two mappings are commuting then they are compatible but not conversely. In 1998, Jungck and Rhoades [11] introduced the notion of weakly compatible mappings and showed that compatible maps are weakly compatible but not conversely. The study of common fixed point of mappings satisfying contractive type conditions has been a very active field of research activity.

In 2012, Chaipunya [6] introduced co-cyclic contractions as follows which is a guarantee for common fixed point theorem of a pair of self-mappings.

Definition 1.4 [6]: Let $T, f: A \cup B \rightarrow A \cup B$ be two self-mappings. $A \cup B$ is said to be co-cyclic representation between T and f if the following conditions are satisfied :

i. Both $A$and B are nonempty subsets of $A \cup B$,
ii. $T(A) \subset f(B)\ and\ T(B) \subset f(A)$.

In 2011, Karapinar et al. [14] introduced the following definition and established the theorem following it.

Definition 1.6 [14] Let A and B be non-empty subsets of a metric space (X, d). A cyclic map $T: A \cup B \rightarrow A \cup B$ is said to be Reich type cyclic contraction if:

$$d(Tx, Ty) \leq a\, d(x, y) + b\, d(Tx, x) + c\, d(Ty, y) \text{ for all } x \in A \text{ and } y \in B, \tag{1.3}$$

where a, b, c are non-negative real numbers satisfying$a + b + c < 1$.

The fixed point theorem of the Reich type cyclic contraction is given as follows.

Theorem 1.7 [14]: Let A and B be non-empty closed subsets of a complete metric space (X, d) and $T : A \cup B \rightarrow AUB$ be a Reich type cyclic contraction. Then T has a unique fixed point in $A \cap B$.

Inspired and motivated by the result of Karapinar and Erhan [14], in this research work, the researcher studied the existence of coincidence points and common fixed points of a pair of self-mappings satisfying the conditions of Reich type co-cyclic contraction in dislocated quasi-metric space. Also, the uniqueness of the common fixed points has been shown. An example has provided in support of our main result.

1.2. STATEMENT OF THE PROBLEM

The aim of this study is to prove the existence and uniqueness of common fixed point for Reich type co-cyclic contraction in dislocated quasi metric space.

This study will answer the following basic questions:

- How can we prove the existence of coincidence points for Reich type co-cyclic contraction maps in complete dislocated quasi metric space?
- How can we prove the existence of common fixed points for Reich type co-cyclic contraction maps in complete dislocated quasi metric space?
- How can we prove the uniqueness of common fixed point for Reich type co-cyclic contraction maps in complete dislocated quasi-metric space?
- How can we support the main result of this study with example?

1.3 OBJECTIVE OF THE STUDY

1.3.1 General objective

The main objective of this study is to establish the existence of coincidence point; and the existence and uniqueness of common fixed points for Reich type co-cyclic contraction in a complete dislocated quasi-metric space.

1.3.2 Specific objectives

i. To prove the existence of coincidence point for Reich type co-cyclic contraction maps in complete dislocated quasi metric space.
ii. To prove the existence of common fixed points for Reich type co-cyclic contraction maps in complete dislocated quasi metric space.
iii. To prove the uniqueness of common fixed point for Reich type co-cyclic contraction complete in dislocated quasi metric space.
iv. To give an example in support of the main result of this study.

1.4 SIGINIFICANCE OF THE STUDY

Fixed point theory is one of the most rapidly growing research areas in nonlinear functional analysis. We hope that the results obtained in this study may contribute to research activities in this area. Further, collaboration in this research will be useful for the graduate program of the department. The researcher may get benefit from this study since it uses to develop scientific research writing skill, scientific literature collecting skill and scientific communication in mathematics.

1.5 DELIMITATION OF THE STUDY

This study was conducted under the stream of functional Analysis and is delimited to the study of existence of coincidence point and common fixed point for Reich type co-cyclic contraction in complete dislocated quasi-metric space and the uniqueness of the common fixed points.

CHAPTER TWO: LITERATURE REVIEW

Let X be a nonempty set and $T: X \to X$ a self-map. We say that $x \in X$ is a fixed point of T if $T(x) = x$ and denote by F(T) or Fix(T) the set of all fixed points of T.

The study of common fixed points of mappings satisfying certain contractive conditions has been at the Centre of vigorous research activity, being the applications of fixed point is very important in several areas of Mathematics.

2.1 FIXED POINT THEOREM

Hitzler and Seda [7] investigated the useful applications of dislocated topology in the context of logic programming semantics. In order to obtain a unique supported model for these programs, they introduced the notation of dislocated metric space and generalized the Banach contraction principle in such spaces.

Furthermore, Zeyada *et al.* [26] generalized the results of Hitzler and Seda [7] and introduced the concept of complete dislocated quasi metric space. Aage and Salunke [1, 2] derived some fixed point theorems in dislocated quasi metric spaces. Zoto [27] gave some new results in dislocated and dislocated quasi metric spaces. Patel [20] constructed some new fixed point results in a dislocated quasi metric space.

Furthermore, Aage and Salunke [1, 2] derived the following fixed point theorems with a Kannan-type contraction and a generalized contraction in the setting of dislocated quasi metric spaces, respectively.

Theorem 2.1.1 [12]: Let (X, d)be a complete dq-metric space and $T: X \to X$ be a continuous self-mapping satisfying the following condition:

$$d(\mathrm{Tx}, \mathrm{Ty}) \leq a\,[\mathrm{d}(\mathrm{x}, \mathrm{Tx}) + \mathrm{d}(\mathrm{y}, \mathrm{Ty})] \text{ for all } x, y \in X,$$

where $a \geq 0$ with $a < \frac{1}{2}$. Then T has a unique fixed point.

Isufati [8] proved the following result in dislocated quasi metric spaces.

Theorem 2.1.2 [8]: Let (X, d) be a complete dq-metric space and $T : X \to X$ be a continuous Self- mapping satisfying the following condition

$$d(Tx, Ty) \leq ad(x, y) + bd(y, Tx) + cd(x, Ty),$$

where a, b, c > 0 which may depends on both x and y with $\sup\{a + 2b + 2c: x, y \in X\} < 1$. Then T has a unique fixed point.

In 1922, Banach [5] established a fixed point theorem for contraction mapping in metric space. Since then a number of fixed point theorems have been proved by many authors and various generalizations of this theorem have been established. In 1982, Sessa [24] introduced the concept of weakly commuting maps and Jungck [10] in 1986, initiated the concept of compatibility. In 1998, Jungck and Rhoades [11] initiated the notion of weakly compatible maps and pointed that compatible maps were weakly compatible but not conversely.

In 2012, Kumari et al. [18] have established the following result:

Theorem 2.2.1 [18]: Let (X, d) be a complete d-metric space and Let $S, T: X \to X$ be continuous mappings satisfying

$$d(Sx, Ty) \leq \alpha\, max\{d(Tx, Sy), d(Tx\ , Sx), d(SyTy)\},$$

for all $x, y \in X$, where $0 \leq a < \frac{1}{2}$, then S and T have a unique common fixed point in X.

The purpose of this research study is to establish the existence of coincidence points and common fixed points of a pair of self-mappings satisfying the conditions of Reich type co-cyclic contraction in dislocated quasi-metric space and the uniqueness of the common fixed points was investigated. Also, we provided an example in support of our main result. We believe that our main result extended the related existing result in the recent literatures.

CHAPTER THREE: METHODOLOGY OF THE STUDY

3.1 Study site and period

This study was conducted from September 2014 GC to September 2015 GC in Department of Mathematics, Jimma University.

3.2. Study design

In this study we followed analytical method of design.

3.3. Source of information

To conduct this research secondary data was used. Hence, the sources of these data are

- Different mathematics reference books.
- Unpublished Mathematics MSc. Theses in the department.
- Journals and Published research works from the internet.

3.4. Procedure of the study

In this study we followed the standard procedures used in the published work of Karapinar and Erhan [14]; and that of Panthi et al. [19].

3.5 Ethical consideration

Ethical consideration has to be considered in all stages of the research process. This study needs books, published journal articles and other related materials was collected from different sources. But there may be a problem in collecting all the above listed materials without any permission letters. Soto make the study legal, permission was taken from a research review and ethical committee of college of Natural science of Jimma University

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CHAPTER FOUR: RESULT AND DISCUSION

4.1 PRELIMINARIES

We recall the definition of complete metric space, quasi metric space, dislocated metric space, dislocated quasi metric space, the notion of convergence and other results that will be needed in the sequel.

Definition 4.1.1 [26]: Let X be a non-empty set. Suppose that the mapping d:$X \times X \to [0, \infty)$ satisfies the following conditions:

d_1: $d(x, x) = 0$, for all $x \in X$.

d_2: $d(x, y) = d(y, x) = 0$ implies $x = y$.

d_3: $d(x, y) = d(y, x)$, for all $x, y \in X$.

d_4: $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then, the pair (X, d) is called metric space. If d satisfies d_1, d_2 and d_4, then (X, d) is called quasi metric space [25]. If d satisfies d_2, d_3 and d_4, then (X, d) is called dislocated metric space [7].

If d satisfies d_2 and d_4, then (X, d) is called dislocated quasi metric space [26].

Here we note that every metric space are quasi metric space, dislocated metric space and dislocated quasi metric space but the converse is not necessarily true and every dislocated metric space are dislocated quasi metric space but the converse is not always true [26].

Definition 4.1.2 [26]: A sequence $\{x_n\}$ in a dq-metric space(X, d) is called Cauchy sequence if for all ε > 0, $\exists n_o \in N$ such that for $m, n \geq n_o$, we have $d(x_m, x_n) < \varepsilon$ *and* $d(x_n, x_m) < \varepsilon$.

Definition 4.1.3 [26]: A sequence $\{x_n\}$ in a dq-metric space (X, d) converges with respect to dq, if there exists x in X, such that $\lim_{n\to\infty} d(x_n, x)$= $\lim_{n\to\infty} d(x, x_n)$= 0. In this case x is called a dq limit of $\{x_n\}$ and we write as$x_n \to x$.

Definition 4.1.4 [26]: A dq-metric space (X, d) is called complete if every Cauchy sequence in it is convergent in X with respect to dq.

Lemma 4.1.5 [26]: Limits in a dq-metric space are unique.

Definition 4.1.6 [5]: Let (X, d) be a dq-metric space. A mapping $T: X \to X$ is called contraction if there exists $k \in [0,1)$ such that $d(Tx, Ty) \leq k\ d(x, y)$ for all $x, y\ in\ X$

Theorem 4.1.7 [26]: Let (X, d) be a complete dq-metric space and let $T: X \to X$ be a contraction mapping. Then, T has a unique fixed point.

Definition 4.1.8 [15]: Let A and B be non-empty subset of a dq-metric space (X, d) and $T: AUB \to AUB$ be a self -map. T is said to be cyclic map if and only if $T(A) \subseteq B$ and $T(B) \subseteq A$ and is said to be cyclic contraction if there exists $k \in [0,1)$ such that $d(Tx, Ty) \leq k\ d(x, y)$ for all x in A and y in B.

Definition 4.1.9 [11]: Let X be a non-empty set. Two self-maps $T, f : X \to X$ are said to be

i. Commuting if $Tfx = fTx$ for all x in X. If $Tx = fx$ for some x in X, then x is called coincidence point of T and f.
ii. Weakly compatible if they commute at their coincidence points .i.e. if u in X such that $Tu = fu$, then $Tfu = fTu$.

Definition 4.1.10 [11]: Let T and f be weakly compatible self-mappings of a set X. If $z = Tw = fw$, then z is called point of coincidence of f and T; and w is called coincidence point of f and T. If $z = w$, then z is called a unique common fixed point of T and f.

Example 4.1 [11]: Let $X = [0, 3]$ be equipped with a dq-metric $d(x, y) = |x - y|$.

Define $T, f : X \to X$ by $T(x) = \begin{cases} 3 - x, & 0 \leq x \leq 1 \\ 3, & 1 < x \leq 3 \end{cases}$ and $f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 3, & 1 < x \leq 3 \end{cases}$. Then for any x in , $fTx = Tfx$, showing that f, T are weakly compatible maps on X and 3 is a common fixed point of T and f.

Example 4.2 [11]: Let $X = R$ and define $T , f : R \to R$ by $Tx = x^2$ and $fx = \frac{x}{3}$ for x in R. Hence, 0 and 1/3 are two coincidence points of f and T . Note that f and T commute at 0, i.e., $fT(0) = Tf(0) = 0$ but $fT(\frac{1}{3}) = \frac{1}{27}$ and $Tf(\frac{1}{3}) = \frac{1}{81}$ and so f and T are not weakly compatible mappings on R.

4.2 MAIN RESULT

Definition 4.2.1: Let A and B be non -empty subsets of a dislocated quasi metric space (X, d). The selfmap $\mathrm{T} : \mathrm{X} \rightarrow \mathrm{X}$ is said to be a dq-Reich type co-cyclic contraction if there exists a selfmap $\mathrm{f}: \mathrm{X} \rightarrow \mathrm{X}$ such that

i. $\mathrm{X} = \mathrm{AUB}$ is a co-cyclic representation of X between T and f.

ii. $\mathrm{d}(\mathrm{Tx}, \mathrm{Ty}) \leq \mathrm{a}\, \mathrm{d}(\mathrm{fx}, \mathrm{fy}) + \mathrm{b}\, \mathrm{d}(\mathrm{Tx}, \mathrm{fx}) + \mathrm{c}\, \mathrm{d}(\mathrm{Ty}, \mathrm{fy})$ for all $\mathrm{x} \in \mathrm{A}$ and $\mathrm{y} \in \mathrm{B}$,

where a, b, c are nonnegative numbers such that $\mathrm{a} + \mathrm{b} + \mathrm{c} < 1$

Theorem 4.2.2: Let A and B be a non-empty subsets of a complete dislocated quasi dq- metric space(X, d). Let $T, f: AUB \rightarrow AUB$ be a dq-Reich type co-cyclic contraction. If $f : X \rightarrow X$ is injective and $f(A)$ and $f(B)$ are closed subsets of X, where T and f are weakly compatible mappings, then T and f have a unique common fixed point in $f(A) \cap f(B)$.

Proof: Let $x_0 \in A$ (fix), $T(A) \subseteq f(B)$, there exist $x_1 \in B$ such that $Tx_0 = fx_1 = y_0$ (say). Since $x_1 \in B$, there exist $x_2 \in A$ such that $Tx_1 = fx_2 = y_1$ (say).

On continuing this procedure inductively, we get a sequence $\{ y_n \}$in X such that $y_n = Tx_n = fx_{n+1}$for each $= 0, 1, 2, \cdots$, where $\{x_{2n}\} \subseteq A$ and $\{x_{2n-1}\} \subseteq B$ for each $n = 1, 2, \cdots$.

Now, we want to show that $\{ y_n\}$ for each $n = 0, 1, 2, \cdots$ is a Cauchy sequence in X.

Now consider

$$d(y_2, y_1) = d(Tx_2, Tx_1) \leq ad(fx_2, fx_1) + bd(Tx_2, fx_2) + cd(Tx_1, fx_1)$$
$$= ad(y_1 , y_0) + bd(y_2, y_1) + cd(y_1, y_0).$$

This implies,

$$(1 - b)\, d(y_2, y_1) \leq (a + c)d(y_1, y_0)$$
$$\Longrightarrow d(y_2, y_1) \leq [\tfrac{a+c}{1-b}]\, d(y_1, y_0), \tag{4.1}$$

where $\frac{a+c}{1-b} < 1$. Let k = $[\frac{a+c}{1-b}]$. Then 0 < k < 1.

So, (4.1) becomes

$$d(y_2, y_1) \leq k\, d(y_1, y_0)$$
$$\Longrightarrow \quad d(y_2, y_1) \leq k\, \alpha, \tag{4.2}$$

where $\alpha = \max \{d(y_1, y_0) , d\, (y_0, y_1)\}$.

Now again,

$$d(y_1, y_2) = d(Tx_1, Tx_2) \leq ad(fx_1, fx_2) + bd(Tx_1, fx_1) + cd(Tx_2, fx_2)$$

$$= ad\,(y_0, y_1) + \text{b}\, d(y_1, y_0) + \text{c}\, d(y_2, y_1). \qquad (4.3)$$

Using (4.1) in (4.3), we obtain,

$$d(y_1, y_2) \leq ad\,(y_0, y_1) + \text{b}\, d(y_1, y_0) + \text{c}\,\frac{(\text{a+c})}{1-b} d(y_1, y_0)$$

$$= ad(y_0, y_1) + [b + c\frac{(\text{a+c})}{1-b}]\, d(y_1, y_0)$$

$$\leq [a + b + c\frac{(\text{a+c})}{1-b}]\alpha \text{ (from (4. 2))}$$

$$= [\frac{a-ab+b-b^2+c\,(a+c)}{(1-b)}]\alpha$$

$$\leq [\frac{a-ab+b-b^2+c\,(1-b)}{(1-b)}]\alpha \qquad \text{(since a+ c < 1-b)}$$

$$= [\frac{a+c+b(1-b)-b\,(a+c)}{(1-b)}]\alpha$$

$$\leq [\frac{a+c+b(1-b)-b\,(1-b)}{(1-b)}]\alpha \qquad \text{(since a+ c < 1-b)}$$

$$= [\frac{\text{a+c}}{1-b}]\,\alpha.$$

This implies that

$$d(y_1, y_2) \leq [\frac{\text{a+c}}{1-b}]\,\alpha = \text{k}\,\alpha.$$

Thus,

$$d(y_1, y_2) \leq k\,\alpha. \qquad (4.4)$$

Similarly,

$$d(y_3, y_2) \leq k\, d(y_2, y_1) \leq k^2\,\alpha. \qquad (4.5)$$

and

$$d(y_2, y_3) \leq k^2\,\alpha. \qquad (4.6)$$

Inductively, for each $n \in \mathbb{N}$, we have

$$d(y_{n+1}, y_n) \leq k^n\,\alpha, \qquad (4.7)$$

and

$$d(y_n, y_{n+1}) \leq k^n\,\alpha. \qquad (4.8)$$

Letting $n \to \infty$ in (4.7) and (4.8), we obtain

$$d(y_{n+1}, y_n) \to 0, \qquad (4.9)$$

and

$$d(y_n, y_{n+1}) \to 0 \tag{4.10}$$

Now let $n, m \in \mathbb{N}$, with $m > n$, by using the triangular inequality , we have

$$\begin{aligned} d(y_m, y_n) &= d(Tx_m, Tx_n) \leq d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_n) \\ &\leq d(Tx_m, Tx_{n+2}) + d(Tx_{n+2}, Tx_{n+1}) + d(Tx_{n+1}, Tx_n) \\ &\quad \dots\dots\dots\dots \quad \dots\dots\dots\dots \quad \dots\dots\dots\dots \quad \dots\dots\dots\dots \\ &\leq d(Tx_m, Tx_{m-1}) + d(Tx_{m-1}, Tx_{m-2}) + \dots\dots + d(Tx_{n+1}, Tx_n) \\ &\leq k^{m-1}\alpha + k^{m-2}\alpha + \dots + k^n\alpha \\ &\leq (k^{m-1} + k^{m-2} + \dots + k^n)\alpha \\ &\leq k^n (k^{m-n-1} + k^{m-n-2} + \dots + k + 1)\alpha \\ &= k^n \left(\sum_{i=0}^{m-n-1} k^i\right)\alpha \\ &\leq \left(k^n \sum_{i=0}^{\infty} k^i\right)\alpha \\ &= k^n \frac{\alpha}{1-k}. \end{aligned}$$

This implies,

$$d(y_m, y_n) \leq k^n \frac{\alpha}{1-k}. \tag{4.11}$$

Taking $n \to \infty$ in (4.11), we obtain $d(y_m, y_n) \to 0$.

Similarly, let $n, m \in N$ with $m > n$ by a similar procedure, we obtain

$$d(y_n, y_m) = d(Tx_n, Tx_m) \leq k^n \frac{\alpha}{1-k}. \tag{4.12}$$

Taking $n \to \infty$ in (4.12), we obtain $d(y_n, y_m) \to 0$.

Thus, $\{y_n\}$ for each n = 0, 1, 2, $\cdots$ is a Cauchy sequence in X.

Since X is complete, there exists z in X such that $\lim_{n\to\infty} y_n = z$.

Thus, $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} fx_{n+1} = z$

Since the sequence $\{y_n\}$ as $n \to \infty$ converges to z , the subsequence $\{y_{2n}\}$, where $y_{2n} = Tx_{2n}$, for each n = 0, 1,2,$\cdots$, converges to z. But, $\{y_{2n}\} \subseteq T(A) \subseteq f(B)$. Since $f(B)$ is closed, $z \in f(B)$. So, there exists $u_1 \in B$ such that $fu_1 = z$.

Also, the subsequence $\{y_{2n-1}\}$ where $y_{2n-1} = Tx_{2n-1}$ for each n = 1, 2, $\cdots$ converges to z.

But, $\{y_{2n-1}\} \subseteq T(B) \subseteq f(A)$ for each n = 1, 2, $\cdots$ since $f(A)$ is closed, $z \in f(A)$. So, there exists $u_2 \in A$ such that $fu_2 = z$. Hence $f(A) \cap f(B) \neq \phi$ and $fu_1 = fu_2 = z$.

Since $f : X \to X$ is injective map, we get $u_1 = u_2 = u$ (say).

Hence $u \in A \cap B$ such that $z = fu$.

Now, we show that $d\ (z\ ,z)\ =\ 0$.

Consider,

$$d(Tx_{2n}, Tx_{2n-1}) \leq a\ d(fx_{2n}, fx_{2n-1}) + b\ d(Tx_{2n}, fx_{2n}) + c\ d(Tx_{2n-1}, fx_{2n-1}).$$

Taking $n \to \infty$, we obtain

$$d(z\ ,z) \leq\ ad(z,z) +\ b\ d(z\ ,z) +\ c\ d(z\ ,z).$$

This implies

$$d(z\ ,z)\ =\ 0. \tag{4.13}$$

Now, for each n = 0, 1, 2, ⋯, we have

$$\begin{aligned} d(Tu, y_{2n-1}) &= \mathrm{d}\ (Tu, Tx_{2n-1}) \\ &\leq a\ d\ (fu, fx_{2n-1}) + b\ d(Tu, fu) + c\ d(Tx_{2n-1}, fx_{2n-1}). \end{aligned}$$

Taking $n \to \infty$, we obtain

$$d(Tu, z) \leq\ ad(fu\ ,z\ ) +\ b\ d(Tu, fu) +\ c\ d(z, z).$$

This implies

$$(1-b)\ d(Tu\ ,z)\ \leq\ 0.$$

This implies

$$d(Tu\ ,z)\ =\ 0. \tag{4.14}$$

Similarly, for each n = 0, 1, 2, ⋯, we obtain

$$\begin{aligned} d(y_{2n-1}, Tu) &= d\ (Tx_{2n-1}, Tu\ ) \\ &\leq \mathrm{a\ d}\ (fx_{2n-1}, fu) + \mathrm{b\ d}\ (Tx_{2n-1}, fx_{2n-1}) + \mathrm{c\ d}\ (Tu, fu). \end{aligned}$$

Taking $n \to \infty$, we obtain

$$d(z\ ,Tu) \leq\ ad(z\ ,z\ ) +\ b\ d(z\ ,z) +\ c\ d(\ Tu\ ,z\ ).$$

This implies

$$d(z, Tu\ )\ =\ 0. \tag{4.15}$$

So, from (4.14) and (4.15), we get $Tu\ =\ z$.

Hence, $Tu\ =\ z\ =\ fu$.

Therefore, u is a coincidence point of T and f, hence $C(T\ , f\ )$ is a non-empty set.

Since T and f are weakly compatible mappings, $Tfu = fTu$,whenever $Tu\ =\ fu$.

So, $Tz\ =\ Tfu\ =\ fTu\ =\ fz$.

Now, we claim that $z\ =\ Tz\ =\ fz$.

Consider,

$$d(Tz\,,Tz\,)\ \le\ ad(fz\,,fz\,)\ +\ bd(Tz\,,fz)\ +\ cd(Tz\,,fz\,)$$
$$=\ ad(Tz\,,Tz) + \ bd(Tz\,,Tz) + \ cd(Tz\,,Tz).$$

This implies

$$d(Tz\,,Tz\,)\ =\ 0. \tag{4.16}$$

So,

$$d(Tz\,,z\,)\ = d(Tz,Tu) \le\ ad(fz\,,fu\,)\ +\ bd(Tz\,,fz)\ +\ cd(Tu,fu)$$
$$=\ d(Tz,Tu) \le\ ad(Tz\,,z) + \ bd(Tz\,,Tz) + \ cd(z,z).$$

From (4.13) and (4.16), we obtain

$$d(Tz\,,z\,)\ =\ 0. \tag{4.17}$$

Similarly,

$$d(z\,,Tz) = \ d(Tu\,,Tz) \le\ ad(fu\,,fz) + \ bd(Tu\,,fu) + \ cd(Tz,fz)$$
$$=\ ad(z\,,Tz\,)\ +\ bd(z\,,z)\ +\ cd(Tz,Tz)$$
$$\Rightarrow (1-a)\,d(z\,,Tz\,)\ \le 0.$$

This implies

$$d(z\,,Tz)\ =\ 0. \tag{4.18}$$

From (4.17) and (4.18), we obtain $Tz\ =\ z$.

Since $Tz\ = fz$, we obtain

$$Tz\ =\ fz\ = z.$$

Therefore, z is a common fixed point of T and f in X.

Now, we show the uniqueness of z.

Let w ∈ X be another common fixed point of T and f. Then we have

$$d\ (z,w)\ = d\ (Tz,Tw)\ \le\ ad\ (fz\,,fw\,)\ +\ b\,d(Tz\,,fz)\ +\ c\,d(Tw,fw)$$
$$=\ ad\ (z\,,w\,)\ +\ b\,d(z\,,z)\ +\ c\,d(w,w)$$
$$=\ a\,d(z\,,w).\ \text{[Since } d(z\,,z)\ = 0\text{, we have } d(w,w) = 0\text{]}.$$
$$\Rightarrow (1-a)\,d(z,w) \le\ 0.$$

This implies

$$d(z,w)\ =\ 0. \tag{4.19}$$

Similarly,

$$d(w,z) = d(Tw\,,Tz\,) \leq ad(fw\,,fz\,) + b\,d(Tw\,,fw) + c\,d(Tz,fz)$$
$$= a\,d\,(w,z) + b\,d(w\,,w) + c\,d(z,z)$$
$$= ad\,(w,z).$$
$$\Rightarrow (1-a)\,d\,(w\,,z) \leq 0.$$

This implies $d(w\,,z) = 0$ (4.20)

Thus, from (4.19) and (4.20) we obtain $z = w$.

Therefore, z is a unique common fixed point of T and f in X.

Corollary 4.2.2: Let A and B be non-empty subsets of a complete dislocated quasi (dq)-metric space (X,d). Let $T,f: AUB \to AUB$ be a dq co-cyclic contraction.If $f : X \to X$ is injective and $f(A)$ and $f(B)$are closed subsets of X, where T and f are weakly compatible mappings, then T and f have a unique common fixed point in $f(A) \cap f(B)$.

Proof: Taking $b = 0$ and $c = 0$, the proof follows from Theorem 4.2.2.

Corollary 4.2.3: Let A and B be a non-empty subsets of a complete dislocated quasi dq-metric space(X,d). Let $T,f: AUB \to AUB$ be a Kannan type dq co-cyclic contraction. If $f : X \to X$ is injective and $f(A)$ and $f(B)$ are closed subsets of X, where T and f are weakly compatible mappings, then T and f have a unique common fixed point in $f(A) \cap f(B)$.

Proof: Taking $a = 0$ and $b = c$, the proof follows from Theorem 4.2.2.

Remark 1: If $f: AUB \to AUB$ is the identity mapping, Theorem 4.2.2 become Theorem 1.7. This shows that Theorem 4.2.2 is an extension of Theorem 1.7.

The following is an example in support of Theorem 4.2.2.

Example 4.2.4: .Let $X = \mathbb{R}, A = (-\infty, 2]$ and $B = [\,0\,,\infty)$.

Let $d : X \times X \to [0\,,\infty)$ be a dislocated quasi metric space defined by

$$d(\,x\,,y) = |x-y| + |y| \text{ for all } x\,,y \text{ in } X.$$

Define $T,f: AUB \to AUB$ by $Tx = \begin{cases} 0, & \text{if } x \leq 2 \\ 1, & \text{if } x > 2 \end{cases}$ and $fx = \frac{3}{2}x$ for all x in AUB.

Then $T(A) = \{0\}\,, T(B) = \{0,1\}\,, f(A) = (-\infty, 3]$ and $f(B) = [\,0\,,\infty)$.

This implies T$(A) \subseteq f(B)$ and $T(B) \subseteq f(A)$. Hence, $X = AUB$ is a co-cyclic representation with respect to the pair (T, f) and collection $\{A , B\}$.
Now we check the inequality of the conditions of Reich type contraction by considering the following cases.

Case 1: Let $x \in A$ and $y \in B$.
We shall consider the following two subcases $x \in A, y \in [0, 2]$ and $x \in A, y > 2$.
If $x \in A$ and $y \in [0, 2]$, the inequality clearly holds.
If $x \in A$ and $y > 2$, we have

$$d(Tx, Ty) = |Tx - Ty| + |Ty| = 2;$$

$$d(fx, fy) = |fx - fy| + |fy| = 3y - \tfrac{3}{2}x;$$

$$d(Tx, fx) = |Tx - fx| + |fx| = |0 - \tfrac{3}{2}x| + |\tfrac{3}{2}x| = 3|x|; \text{ and}$$

$$d(Ty, fy) = |Ty - fy| + |fy| = |1 - \tfrac{3}{2}y| + |\tfrac{3}{2}y| = 3y - 1.$$

So, if we choose $a = \frac{1}{5}$, $b = \frac{1}{5}$ and $c = \frac{2}{5}$, we have

$$\frac{2}{5} d(Ty, fy) = \frac{2}{5}(3y - 1) \geq \frac{2}{5}(5) = 2 \ \forall y > 2.$$

This implies

$$d(Tx, Ty) \leq \tfrac{1}{5} d(fx, fy) + \tfrac{1}{5} d(Tx, fx) + \tfrac{2}{5} d(Ty, fy) \text{ for all } x \in A, \text{ and } y > 2.$$

Case 2: Let $x \in B$ and $y \in A$.
Here also we consider the following two sub-cases, i.e., $x \in [0,2]$ and $y \in A$; and $x > 2$ and $y \in A$.
If $x \in [0, 2]$ and $y \in A$, the contraction clearly holds.
If $x > 2$ and $y \in A$.Then

$$d(Tx, Ty) = |Tx - Ty| + |Ty| = 1;$$

$$d(fx, fy) = |fx - fy| + |fy| = |\tfrac{3}{2}x - \tfrac{3}{2}y| + |\tfrac{3}{2}y| = \tfrac{3}{2}[|x - y| + |y|];$$

$$d(Tx, fx) = |Tx - fx| + |fx| = |1 - \tfrac{3}{2}x| + |\tfrac{3}{2}x| = 3x - 1; \text{ and}$$

$$d(Ty, fy) = |Ty - fy| + |fy| = |0 - \tfrac{3}{2}y| + |\tfrac{3}{2}y| = 3|y|.$$

Now, if we choose $a = \frac{1}{5}, b = \frac{1}{5}$ and $c = \frac{2}{5}$, we have

$$\tfrac{1}{5} d(Tx, fx) = \tfrac{1}{5}(3x - 1) \geq \tfrac{1}{5}(5) = 1 \text{ for all x } > 2.$$

This implies

$$d(Tx,Ty) \leq \frac{1}{5}d(fx\,,fy) + \frac{1}{5}d(Tx\,,fx) + \frac{2}{5}d(Ty,fy) \text{ for all } x > 2 \text{ and } y \in A.$$

Hence T and f satisfy all conditions of our theorem. Moreover, 0 is a unique common fixed point of T and f in $f(A) \cap f(B)$.

CHAPTER FIVE: CONCLUSION AND FUTURE SCOPE

5.1 CONCLUSION

In 2011, Karapinar and Erhan [14] established the existence of fixed point for mapping satisfying the Reich type cyclic contraction in a complete metric space.

In this research work, we have established the existence of coincidence points and a unique common fixed point of Reich type co-cyclic contraction defined in dislocated quasi-metric space. We have supported the main result of this research work by example. Our work extends the main result in [14].

5.2 FUTURE SCOPE

There are several published results related to existence of fixed points self-maps defined on dislocated quasi metric space. There are also few results related to the existence of common fixed points for a pair or more self-maps in this space. The researcher believes the search for the existence of coincidence and common fixed points of maps satisfying different conditions in dislocated quasi metric space is an active area of study. So, the forthcoming postgraduate students of Department of Mathematics can exploit this opportunity and conduct their research work in this area.

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