JIMMA UNIVERSITY

COLLEGE OF NATURAL SCIENCES

DEPARTMENT OF MATHEMATICS

COMMON FIXED POINT THEOREMS FOR A PAIR OF MAPS SATISFYING CONTRACTIVE CONDITION OF INTEGRAL TYPE

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DECLARATION

I, the undersigned declare that, the research entitled **"Common fixed point theorems for a pair maps satisfying contractive condition of integral type"** is original and it has not been submitted to any institution elsewhere for the award of any degree or like, where other sources of information that have been used, they have been acknowledge.

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ABSTRACTS

In this research, we introduced a pair of self-maps satisfying contractive condition of integral type in complete metric spaces and establish the existence and uniqueness of common fixed points for those maps. We used analytical design in our work. The analysis technique we adopted for the successful completion of this study was by extending a single map of Zeqing Liu , Heng Wu ,Jeong Sheok Ume and Shin Min Kang [20] to a pair of maps. Secondary source of data such as journal articles, books and internet was used to carry out the study. We also provided examples in support of our results. This study was conducted from September 2014 to September 2015.

Key words: Fixed point, Common fixed point, complete metric space, contractive condition of integral type and Lebesgue integrable maps.

CHAPTER ONE

1. INTRODUCTION

1.1 BACKGROUND OF THE STUDY

Fixed point theory is one of the famous and traditional theories in Mathematics and has a broad set of applications. In this theory contraction is one of the main tools to prove existence and uniqueness of a fixed point. Banach contraction principle, which gives an answer on the existence and uniqueness of a solution of an operator equation, $Tx = x$ is most widely used tool in the study of nonlinear equations. There are many extension of Banach contraction principle [1, 9. 15].

Let X be a nonempty set and $T: X \rightarrow X$ a map, we call T a self-map of X. An element $x\ in\ X$ is called a fixed point of T if $Tx = x$. Let (X, d) be a metric space. A self-map T is said to be a contraction if there is a real number $k\ in\ [0, 1)$ such that for all $x, y \in X$

$d(Tx, Ty) \leq kd(x, y)$.

In this case k is called a contraction constant.

The Banach contraction principle [3] which states that a contraction map on a complete metric space has a unique fixed point is one of the pivotal results of analysis. It is widely considered as the source of metric fixed point theory. Also, its significance lies in its vast applicability in a number of branches of Mathematics. In Banach contraction principle the map T is continuous.

In 1968, Kannan [11] introduced a different contraction, where the map $T: X \rightarrow X$ may not be continuous which is stated as follows:

Let (X, d) be a complete metric space and T be a self-map on X and if there exists a constant a in $[0, \frac{1}{2})$ such that

$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)]$ for all $x, y \in X$

then T has a unique fixed point.

In 1972, Chatterjea [7] gave the dual of Kannan fixed point theorem as follows.

Let (X, d) be a complete metric space and T be a self-map on X and if there exists a constant ∂ in $[0, \frac{1}{2})$ such that

$d(Tx, Ty) \leq \partial[d(x, Ty) + d(y, Tx)]$ for all $x, y \in X$

then T has a unique fixed point.

In 1977, Rhoades [16] showed that Banach contraction principle, Kannan mapping

And Chatterjea are independent.

Definition1.2: Let (X, d) be a metric space. A mapping $T: X \to X$ is said to be weakly

contraction $d(Tx,Ty) \leq d(x,y) - \varphi(d(x,y))$ for all $x,\ y \in X$
where $\varphi: R_+ \to R_+$ is continuous and non-decreasing function and $\varphi(t) = 0$ if and only if $t = 0$.
In 1997, Alber and Guerre Delabrere [1] introduced the concept of weakly contraction in Hilbert spaces and proved the corresponding fixed point result.

Rhoades [16] extended this concept to Banach spaces and proved the existence of fixed point of weakly contractive maps in complete metric space.

In recent years there has been an increasing interest in the study of fixed points and common fixed points using different contractive conditions in [4, 5,10,19] and others continued the study of Rhoades[16] and Branciari [6] which proved fixed point for weakly contraction mappings and contractive maps satisfying contractive condition of integral type, respectively which are generalization of the Banach fixed point theorem and they extended the idea to prove some fixed point and common fixed point theorems for various generalization of weakly contraction mappings and contractive mappings of integral type in a complete metric space. Branciari [6] studied fixed point theorem of integral type. For more results in this directions of study see [2,8,13,15,21] and the reference therein. Rhoades [17] and Liu et al. [12] extended the work of Branciari [6] and obtained fixed point theorems for the contractive mappings of integral type. Inspired and motivated by the results of [20] on fixed point theorems for maps satisfying contractive condition of integral type. The researcher planned to study about the existence and uniqueness of common fixed point results for a pair of maps satisfying contractive condition of integral type.

1.2 **STATEMENT OF THE PROBLEM**

The purpose of the research is to establish common fixed point for a pair of self-maps satisfying contractive condition of integral type. The research answered the following basic questions.

1. How can we prove the existence of common fixed points of pair of self-maps satisfying contractive condition of integral type?
2. How can we assure the uniqueness of the common fixed point?
3. How can we support the results obtained by providing applicable examples?

1.3 OBJECTIVE

1.3.1 General Objective

The general objective of this study was to establish the existence of common fixed points for a pair of self-maps satisfying contractive condition of integral type, by extending the works of [20] of 2014.

1.3.2 Specific Objectives

1. To prove the existence of common fixed points of a pair of self-maps satisfying contractive condition of integral type.
2. To discuss conditions required to assure uniqueness of common fixed point.
3. To provide examples in support of the results obtained.

1.4 SIGNIFICANCE OF THE STUDY

Recently wide interest in the study of fixed points and common fixed points of mappings satisfying contractive conditions of integral type with wide range of applications in several fields are observed (see [2, 8, 12,13, 20]). Hence, the results of this study may contribute in giving some background information and guiding activities for researchers who need to conduct further research in this line of research. The researcher has gained basic research skills in pure mathematics which will help him to be engaged in research activities in his future career.

1.5 DELIMITATION OF THE STUDY

This study was conducted under the Streams of Functional Analysis and is delimited to the existence of common fixed point of maps satisfying contractive condition of integral type.

CHAPTER TWO

2. LITERATURE REVIEW

The theory of fixed point in metric space is originated from the well-known Banach contraction principle [3] which is a very popular tool for solving existence problems in many branches of Mathematical analysis that has many applications in solving non-linear equations which can be stated as, "on a complete metric space (X, d) if a self-mapping T satisfies

$d(Tx,Ty) \le kd(x,y)$ for all $x, y \in X$,

where k is a constant in $[0,1)$. Then T has a unique fixed point $x \in X$.

Let X be a set and $d : X^2 \to R_+$ be a function with the following properties:

For all $x, y, z \in X$

(i) $d(x,y) \ge 0$
(ii) $d(x,y) = 0 \Leftrightarrow x = y$
(iii) $d(x,y) = d(y,x)$
(iv) $d(x,z) \le d(x,y) + d(y,z)$.

Then we say that d is a metric on X and (X,d) is a metric space.

Related to this many researchers studied and gave generalization of the Banach contraction principle (see [1, 4, 5, 9, 10, 16, 19]).

In 1997 Alber and Guerre-Delabrere [1] introduced the concept of on weaks contraction map in Hilbert spaces and established a fixed point theorem. Rhoads [16] showed that the result of [1] is also valid in complete metric spaces. That is, he introduced the notion of weakly contractive maps in the setting of metric spaces and proved that any weakly contractive map defined on a complete metric spaces has a unique fixed point, and he stated the theorem as follows:

A mapping $T : X \to X$, where (X, d) is a metric space is said to be weakly contractive if

$d(Tx,Ty) \le d(x,y) - \varphi(d(x,y))$ for all $x, y \in X$

where $\varphi : R_+ \to R_+$ is

a continuous and non-decreasing function such that $\varphi(t)=0$ if and only $t=0$. Weak inequalities of the above type have been used to establish fixed point results in number of subsequent works. For example Zang and Song [19] used generalized φ-weak contraction which is defined for two mappings and gave conditions for the existence of a common fixed point which is stated as " let (X,d) be a complete metric space, and $f,g:X\to X$ be two maps such that for all $x,\ y\in X$

$$d(fx,gy)\le M(x,y)-\varphi(M(x,y))$$

where $\varphi:R_+\to R_+$ is a lower semi-continuous function with $\varphi(t)>0$ for any t in

$(0,\infty)$ and $\varphi(0)=0$,

$$M(x,y)=\max\{d(x,y),d(x,fx),\ d(y,gy),\ \frac{1}{2}[d(x,gy)+d(y,fx)]\}$$

then there exists a unique point $u\in X$ such that $u=fu=gu$.

Branciari [6] introduced the first contractive mappings of integral types as

$$\int_0^{d(Tx,Ty)}\varphi(t)dt\le c\int_0^{d(x,y)}\varphi(t)dt \quad \text{for all } x,\ y\in X$$

where c in $[0,1)$ is a constant, φ is Lebesgue integrable, and $T:X\to X$ is a self-map and proved the existence of fixed point on complete metric spaces. Recently, from these conditions an increasing interest in the study of fixed point and common fixed points of mappings satisfying contractive conditions of integral type, see for example [2, 8, 12, 13, 14, 17, 20, 21] have been branched out (developed).

Researchers in [17] and [12] extended Branciari's result and proved some fixed point and common fixed point theorems for various generalized weakly contraction mappings and contractive mappings of integral type of complete metric spaces. From this [20] and [21] introduced new classes of contractive mappings of integral type in complete metric spaces and study the existence and uniqueness and iterative approximations of fixed points for maps satisfying as follows:

Let (X,d) be a complete metric space and let (φ,ψ) in $\emptyset_1\times\emptyset_2$ and $T:X\to X$ satisfying

$$\int_0^{d(Tx,Ty)}\varphi(t)dt\le\int_0^{d(x,y)}\varphi(t)dt-\int_0^{\psi(d(x,y))}\varphi(t)dt \quad \text{for all } x,\ y\in X,$$

where $\varphi : R_+ \rightarrow R_+$ is Lebesgue integrable, summable and $\int_0^{\varepsilon} \varphi(t)dt > 0$ for each $\varepsilon > 0$ and $\psi : R_+ \rightarrow R_+$ is a lower semi- continuous function with $\psi(t) = 0$ and $\psi(t) > 0$ for each $t > 0$

then T has unique fixed point a in X .

Inspired and motivated by the result in [21] and [20] in this paper we introduced a new classes of contractive mappings of integral type in complete metric spaces extending [20] to pair of self-maps, i.e. common fixed point theorems for a pair of self-mappings satisfying contractive condition of integral type.

CHAPTER THREE

3. METHODOLOGY

The chapter gives the direction (address) study design, description of the research methodology, data collection procedures and data analysis process

3.1. STUDY SITE

The study was conducted from September 2014 to September 2015 in Jimma university under mathematics department.

3.2. STUDY DESIGN

In order to achieve the objectives of the study, analytical design method was used.

3.3. SOURCE OF INFORMATION

In this study document materials, so available source of information for the study such as books, journals, deferent study related to the topic and internet services were used.

3.4. PROCEDURE OF THE STUDY

This study intended to establish common fixed point theorems for maps satisfying contractions of integral type by using the standard techniques similar to that of Zeqing L., Heng, W., Ume, J.S., and Kang, S.M.,[20] of one self-map to a pair of self-maps and also technique of Sastry, K.P.R., Babu, G.V.R., and Kidane, K.T.,[18]

3.5. ETHICAL CONSIDERATION

The researcher has taken a cooperation letter from mathematics department of Jimma University to get consent from the institute (s) where books, journals, internet and other related materials where available for this study to collect related information. Moreover, kept rules and regulations of the institute(s) from where the researcher got this materials.

CHAPTER FOUR

4. preliminaries and results

4.1. Preliminaries

Notation In this thesis we denote:

i. $R_+ = [0,\infty)$, $N_0 = \{0\} \cup N$
where N denotes the set of positive integers.

ii. $\phi_1 = \{\varphi : \varphi : R_+ \to R_+$ is Lebesgue integrable, summable on each compact subset of R_+ and $\int_0^\varepsilon \varphi(t)dt > 0$ for each $\varepsilon > 0\}$.

iii. $\phi_2 = \{\psi : \psi : R_+ \to R_+$ is a lower semi-continuous function with $\psi(0) = 0$ and $\psi(t) > 0$ for each $t > 0\}$.

Theorem 4.1.1. [16] Let (X, d) be a complete metric space and T is a self-map on X satisfying

$$d(Tx,Ty) \le d(x,y) - \psi(d(x,y)), \text{ for all } x,\ y \in X, \tag{4.1.1.1}$$

where $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and non-decreasing such that ψ is positive on $R_+ \setminus \{0\}$, $\psi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$.

Then T has a unique fixed point $x \in X$.

Theorem 4.1.2 [6] let T be a mapping from a complete metric space (X, d) in to itself satisfying

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \le k \int_0^{d(x,y)} \varphi(t)dt \text{ for all } x, y \text{ in } X, \tag{4.1. 2.1}$$

where $k \in (0,1)$ is a constant and $\varphi \in \phi_1$.Then T has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} T^n x = a$ for each $x \in X$.

Definition 4.1.3. Let (X, d) be a metric space, $x \in X$ and $\{x_n\}$ be a sequence in X, then we say that

i. $\{x_n\}$ converges to $x \in X$ or equivalently $\lim_{n\to\infty} x_n = x$, if for every $\varepsilon > 0$ there exist a positive integer n_0 such that $n > n_0$ implies $d(x_n, x) < \varepsilon$.

ii. $\{x_n\}$ is a Cauchy sequence in X, if for every $\varepsilon > 0$ there is appositive integer n_o such that $n, m > n_0$ implies $d(x_n, x_m) < \varepsilon$.

Lemma 4.1.4 [12] Let $\varphi \in \phi_1$ and $\{r_n\}_{n\varepsilon N}$ be non-negative sequence with $\lim_{n\to\infty} r_n = a$.

Then $\lim_{n\to\infty} \int_0^{r_n} \varphi(t)dt = \int_0^a \varphi(t)dt$.

Lemma 4.1.5 [12] Let $\varphi \in \phi_1$ and $\{r_n\}_{n\varepsilon N}$ be nonnegative sequence. Then $\lim_{n\to\infty} \int_0^{r_n} \varphi(t)dt = 0$ if and only if $\lim_{n\to\infty} r_n = 0.$

Theorem 4.1.6 [20] Let (φ, ψ) be in $\phi_1 \times \phi_2$ and T be a mapping from a complete metric space (X, d) in to itself satisfying,

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \le \int_0^{d(x,y)} \varphi(t)dt - \int_0^{\psi(d(x,y))} \varphi(t)dt, \text{ for all } x, y \in X,$$

then T has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} T^n x = a$ for each $a \in X$.

Theorem 4.1.7 [20] Let (φ, ψ) be in $\phi_1 \times \phi_2$ and T be a mapping from a complete metric space (X, d) in to itself satisfying,

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \le \int_0^{M(x,y)} \varphi(t)dt - \int_0^{\psi(M(x,y))} \varphi(t)dt \quad \text{for all } x, y \in X$$

where, $M(x, y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty) + d(y,Tx)]\}$.

Then T has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} T^n x = a$ for each $x \in X$.

4.2 Main Results

Theorem 4.2.1. Let (φ,ψ) be in $\phi_1 \times \phi_2$ and (X,d) be a complete metric space and let $T,S: X \to X$ be mapping satisfying

$$\int_0^{d(Tx,Sy)} \varphi(t)dt \leq \int_0^{d(x,y)} \varphi(t)dt - \int_0^{\psi(d(x,y))} \varphi(t)dt \text{ for all } x,\ y \in X. \qquad (4.2.1.1)$$

Then T and S have a unique common fixed point $a \in X$ such that

$\lim_{n\to\infty} Tx_{2n-2} = \lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} x_{2n-1} = a$ and $\lim_{n\to\infty} Sx_{2n-1} = \lim_{n\to\infty} x_{2n} = a$ for each $x \in X$,

where $Tx_{2n-2} = x_{2n-1}$, and $Sx_{2n-1} = x_{2n}$ for n=1,2,3,…

Proof: For self-maps T and S in metric space (X,d) and $(x,y,n) \in X^2 \times N$, let $x_0 \in X$. Since $T,S: X \to X$, we can choose $x_1 \in X$ such that $x_1 = Tx_0$. Corresponding to x_1, we can choose $x_2 \in X$ such that $x_2 = Sx_1$. Continuing this process, we can construct a sequence $\{x_n\}_{n\geq 1}$ given by:

$$x_{2n-1} = Tx_{2n-2} \text{ and } x_{2n} = Sx_{2n-1} \text{ for } n \geq 1 . \qquad (4.2.1.2)$$

Denote $d_n = d(x_n, x_{n+1})$ for n=0,1,2,… Now, if there exists some $n_0 \in N$ with

$x_{n_o+1} = x_{n_o}$, then we shall consider the following two cases:

Case 1:- if n_o is odd, then

$x_{n_o+1} = Sx_{n_o} = Sx_{n_o+1}$.

Claim: $Tx_{n_o+1} = x_{n_o+1}$

$$0 \leq \int_0^{d(Tx_{n_o+1},\, x_{n_o+1})} \varphi(t)dt$$

$$= \int_0^{d(Tx_{n_o+1},\, Sx_{n_o})} \varphi(t)dt$$

$$\leq \int_0^{d(x_{n_o+1},\, x_{n_o})} \varphi(t)dt - \int_0^{\psi(d(x_{n_o+1},\, x_{n_o}))} \varphi(t)dt = 0$$

That is, $d(Tx_{n_o+1},\ x_{n_o+1}) = 0$.

$$\Rightarrow Tx_{n_o+1} = x_{n_o+1}.$$

Case 2:- if n is even then

$$x_{n_o+1} = Tx_{n_o} = Tx_{n_o+1}.$$

<u>Claim:</u> $Sx_{n_o+1} = x_{n_o+1}$

$$0 \leq \int_0^{d(Sx_{n_o+1},\, x_{n_o+1})} \varphi(t)dt$$

$$= \int_0^{d(Sx_{n_o+1},\, Tx_{n_o})} \varphi(t)dt$$

$$= \int_0^{d(\, Tx_{n_o},\, Sx_{n_o+1})} \varphi(t)dt$$

$$\leq \int_0^{d(x_{n_o},\, x_{n_o+1})} \varphi(t)dt - \int_0^{\psi(d(x_{n_o},\, x_{n_o+1}))} \varphi(t)dt = 0$$

$$\Rightarrow d(sx_{n_o+1},\, x_{n_o+1}) = 0.$$

This shows that $sx_{n_o+1} = x_{n_o+1}$.

Hence , x_{n_o} is a common fixed point T and S .

Suppose $x_n \neq x_{n+1}$ for all $n \geq 0$. Now to show that $\{d_n\}_{n\in \mathrm{N}}$ is non-increasing sequence, we shall again consider two cases using (4.2.1.1) and $(\varphi,\psi) \in \phi_1 \mathrm{X} \phi_2$.

Case 1:- if n is even

$$\int_0^{d_n} \varphi(t)dt = \int_0^{d(x_{n+1},x_n)} \varphi(t)dt = \int_0^{d(Tx_n,Sx_{n-1})} \varphi(t)dt$$

$$\leq \int_0^{d(x_n,x_{n-1})} \varphi(t)dt - \int_0^{\psi(d(x_n,x_{n-1}))} \varphi(t)dt$$

$$\leq \int_0^{d_{n-1}} \varphi(t)dt - \int_0^{\psi(d_{n-1})} \varphi(t)dt$$

$$< \int_0^{d_{n-1}} \varphi(t)dt \text{ for all n} \geq 1,$$

which yields $d_n < d_{n-1}$, for all $n \geq 0$.

Case 2:- if n is odd

$$\int_0^{d_n} \varphi(t)dt = \int_0^{d(x_{n+1},x_n)} \varphi(t)dt = \int_0^{d(Sx_n,Tx_{n-1})} \varphi(t)dt = \int_0^{d(Tx_{n-1},Sx_n)} \varphi(t)dt$$

$$\leq \int_0^{d(x_{n-1},x_n)} \varphi(t)dt - \int_0^{\psi(d(x_{n-1},x_n))} \varphi(t)dt$$

$$\leq \int_0^{d_{n-1}} \varphi(t)dt - \int_0^{\psi(d_{n-1})} \varphi(t)dt$$

$$< \int_0^{d_{n-1}} \varphi(t)dt, \text{ for all } n \geq 0,$$

which yields $d_n < d_{n-1}$, for all $n \geq 1$.

Thus, the sequence $\{d_n\}_{n\in N}$ is a non-increasing sequence of real numbers which is bounded below.

This implies that there exists a constant $c \geq 0$ with $\lim\limits_{n\to\infty} d_n = c$, which also implies $\lim\limits_{n\to\infty} d_{2n-1} = \lim\limits_{n\to\infty} d_{2n-2} = c$. Suppose that $c > 0$. *Put* $\lim\limits_{n\to\infty}\Psi(d_{2n-2}) = \alpha$. *and we can observe that there exists a sub-sequence* $(d_{2n(k)-2})_{n\in N}$ of $(d_{2n-2})_{n\in N}$ *satisfying* $\lim\limits_{k\to\infty}\Psi(d_{2n(k)-2}) = \alpha$.

Since ψ a lower semi-continuous and $\psi \in \phi_2$, it follows that $\alpha \geq \psi(c)$.

Using (4.2.1.1), Lemma 4.1.4 and $(\varphi,\psi) \in \phi_1 \times \phi_2$, we get

$$0 < \int_0^c \varphi(t)dt = \limsup_{k\to\infty} \int_0^{d_{2n(k)-1}} \varphi(t)dt$$

$$= \limsup_{k\to\infty} \int_0^{d(x_{2n(k)-1},\, x_{2n(k)})} \varphi(t)dt$$

$$= \limsup_{k\to\infty} \int_0^{d(Tx_{2n(k)-2},\, Sx_{2n(k)-1})} \varphi(t)dt$$

$$\leq \limsup_{k\to\infty} \left(\int_0^{d(x_{2n(k)-2}\, x_{2n(k)-1})} \varphi(t)dt - \int_0^{\psi(d(x_{2n(k)-2},x_{2n(k)-1})} \varphi(t)dt \right)$$

$$= \limsup_{k\to\infty} \int_0^{(d(x_{2n(k)-2},\, x_{2n(k)-1})} \varphi(t)dt - \liminf_{k\to\infty} \int_0^{\psi(d(x_{2n(k)-2},\, x_{2n(k)-1},))} \varphi(t)dt$$

$$= \int_0^c \varphi(t)dt - \int_0^{\alpha} \varphi(t)dt$$

$$\leq \int_0^c \varphi(t)dt - \int_0^{\psi(c)} \varphi(t)dt$$

$$< \int_0^c \varphi(t)dt,$$

which is impossible. Hence $c = 0$, that means, $\lim_{n\to\infty} d_{2n(k)-2} = 0.$ (4.2.1.3)

Now, to prove that $\{x_n\}_{n\in N}$ is a Cauchy sequence it is sufficient to show

that $\{x_{2n}\}_{n\in N}$ is Cauchy. Suppose it is not a Cauchy sequence. Then there exist a constant $\varepsilon > 0$ and two subsequences $\{x_{2m(k)}\}_{k\in N}$ and $\{x_{2n(k)}\}_{k\in N}$ of $\{x_{2n}\}_{n\in N}$ such that $n(k)$ is minimal in the sense that $n(k) > m(k) > k$ and $d(x_{2m(k)}, x_{2n(k)}) > \varepsilon$. It follows that

$$d(x_{2m(k)}, x_{2n(k)-2}) \le \varepsilon. \qquad (4.2.1.4)$$

$$\varepsilon < d\big(x_{2m(k)}, x_{2n(k)}\big)$$

$$\le d\big(x_{2n(k)}, x_{2n(k)-1}\big) + d\big(x_{2n(k)-1}, x_{2n(k)-2}\big) + d\big(x_{2n(k)-2}, x_{2m(k)}\big), \quad (4.2.1.5)$$

and

$$| d(x_{2m(k)-1}, x_{2n(k)}) - d(x_{2m(k)}, x_{2n(k)}| \le d_{2m(k)-1.} \qquad (4.2.1.6)$$

taking $k \to \infty$ in (4.2.1.5) and (4.2.1.6) and using (4.2.1.3) and (4.2.1.4) we have

$$\lim_{k\to\infty} d(x_{2m(k)}, x_{2n(k)}) = \lim_{k\to\infty} d(x_{2m(k)-1}, x_{2n(k)}) = \varepsilon. \qquad (4.2.1.7)$$

And again,

$$\varepsilon < d\big(x_{2m(k)}, x_{2n(k)}\big) \le d\big(x_{2m(k)}, x_{2m(k)-1}\big) + d\big(x_{2m(k)-1}, x_{2m(k)-2}\big)$$
$$+ d\big(x_{2m(k)-2}, x_{2n(k)-1}\big) + d\big(x_{2n(k)-1}, x_{2n(k)}\big),$$

and

$$d\big(x_{2m(k)-2}, x_{2n(k)-1}\big) \le d\big(x_{2m(k)-2}, x_{2m(k)-1}\big) + d\big(x_{2m(k)-1}, x_{2n(k)}\big)$$
$$+ d\big(x_{2n(k)}, x_{2n(k)-1}\big).$$

Taking $k \to \infty$ in the above inequalities and using (4.2.1.3) and (4.2.1.7), we have

$$\lim_{k\to\infty} d\big(x_{2m(k)-2}, x_{2n(k)-1}\big) = \varepsilon. \qquad (4.2.1.8)$$

Putting

$\lim_{k\to\infty} inf\ \psi\,(d\big(x_{2m(k)-2}, x_{2n(k)-1}\big)) = \theta.$ From this there exists a subsequence

$\left\{d(x_{2m(k_j)-2}, x_{2n(k_j)-1})\right\}_{j\in N}$ of $\left\{d\left(x_{2m(k)-2}, x_{2n(k)-1}\right)\right\}_{k\in N}$ such that

$$\lim_{j\to\infty} \psi\left(d(x_{2m(k_j)-2}, x_{2n(k_j)-1}\right) = \theta. \qquad (4.2.1.9)$$

Since ψ is lower semi–continuous, it follows from (4.2.1.8) and $\psi \in \emptyset_2$ that

$\theta \geq \psi(\epsilon) > 0$. By equations (4.2.1.1), (4.2.1.8), Lemma 4.1.4 and $\varphi \in \emptyset_1$, we deduce that

$$0 < \int_0^{\varepsilon} \varphi(t)dt$$

$$= \lim_{j\to\infty} \sup \int_0^{d(x_{2m(k_j)-1}, x_{2n(k_j)})} \varphi(t)dt$$

$$= \lim_{j\to\infty} \sup \int_0^{d(Tx_{2m(k_j)-2}, Sx_{2n(k_j)})} \varphi(t)dt$$

$$\leq \lim_{j\to\infty} \sup\left(\int_0^{d(x_{2m(k_j)-2}, x_{2n(k_j)})} \varphi(t)dt - \int_0^{\psi(d(x_{2m(k_j)-2}, x_{2n(k_j)}))} \varphi(t)dt\right)$$

$$= \lim_{j\to\infty} \sup \int_0^{d(x_{2m(k_j)-2}, x_{2n(k_j)})} \varphi(t)dt - \lim_{j\to\infty} \inf \int_0^{\Psi(d(x_{2m(k_j)-2}, x_{2n(k_j)}))} \varphi(t)dt$$

$$= \lim_{j\to\infty} \sup \int_0^{d(x_{2m(k_j)-2}, x_{2n(k_j)-1})} \varphi(t)dt - \lim_{j\to\infty} \inf \int_0^{\Psi(d(x_{2m(k_j)-2}, x_{2n(k_j)-1}))} \varphi(t)dt$$

$$= \int_0^{\varepsilon} \varphi(t)dt - \int_0^{\theta} \varphi(t)dt$$

$$\leq \int_0^{\varepsilon} \varphi(t)dt - \int_0^{\psi(\varepsilon)} \varphi(t)dt$$

$$< \int_0^{\varepsilon} \varphi(t)dt,$$

which is a contradiction. Thus $\{x_{2n}\}_{n\in N}$ is a Cauchy sequence and hence $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists a in X such that $\lim_{n\to\infty} x_n = a$. It follows that

$$\lim_{n\to\infty} Sx_{2n-1} = a \quad \text{and} \quad \lim_{n\to\infty} Tx_{2n-2} = a\,. \qquad (4.2.1.10)$$

Next we prove that a is common fixed point of T and S. First we show that Ta = a In view of (4.2.1.1), (4.2.1.10) and lemma 4.1.5, we obtain

$$0 < \int_0^{d(a,Ta)} \varphi(t)dt = \lim_{n\to\infty} \int_0^{d(Sx_{2n-1},T(Tx_{2n-2}))} \varphi(t)dt$$

$$\leq \lim_{n\to\infty} \left(\int_0^{d(x_{2n-1},Tx_{2n-2})} \varphi(t)dt - \int_0^{\psi(d(x_{2n-1},Tx_{2n-2})} \varphi(t)dt\right)$$

$$= \lim_{n\to\infty} \int_0^{d(x_{2n-1},x_{2n-1})} \varphi(t)dt - \lim_{n\to\infty} \int_0^{\psi(d(x_{2n-1},x_{2n-1}))} \varphi(t)dt = 0.$$

It follows that

$\int_0^{d(a,Ta)} \varphi(t)dt = 0,$ then $d(a,Ta) = 0$ that yields $Ta = a$ this shows a is a fixed point of T.

Now, to show $Ta = Sa$.

Suppose $a = Ta \neq Sa$, then

$$0 < \int_0^{d(Ta,Sa)} \varphi(t)dt = \lim_{n\to\infty} \int_0^{d(T(Tx_{2n-2}),S(Sx_{2n-1}))} \varphi(t)dt$$

$$\leq \lim_{n\to\infty} \left(\int_0^{d(Tx_{2n-2},Sx_{2n-1})} \varphi(t)dt - \int_0^{\psi(d(Tx_{2n-2},Sx_{2n-1}))} \varphi(t)dt\right)$$

$$= \lim_{n\to\infty} \int_0^{d(x_{2n-1},x_{2n})} \varphi(t)dt - \lim_{n\to\infty} \int_0^{\psi(d(x_{2n-1},x_{2n}))} \varphi(t)dt$$

$$= \int_0^{(d(a,a)} \varphi(t) - \int_0^{\psi(d(a,a))} \varphi(t)dt = 0,$$

which means that $0 < \int_0^{d(Ta,Sa)} \varphi(t)dt \leq 0,$

which is a contradiction.

Thus, $d(Ta,Sa) = 0,$ i.e., $Ta = Sa = a.$

Hence, a is a common fixed point of T and S.

Now, we show that a is unique.

Suppose b in X is also another common fixed point of S and T that is

$b = Sb = Tb \neq a$ then

$$0 < \int_0^{d(a,b),} \varphi(t)dt = \int_0^{d(Ta,Sa)} \varphi(t)dt \leq \int_0^{d(a,b)} \varphi(t)dt - \int_0^{\psi(d(a,b))} \varphi(t)dt$$
$$< \int_a^{d(a,b)} \varphi(t)dt.$$

This is a contradiction. Therefore, $d(a,b) = 0.$

Thus, a = b.

Hence, the common fixed point a of T and S is unique.

Now, we give an example in support of Theorem 4.2.1

Let X = [0, 1] and d: $X \times X \rightarrow R_+$ be given by $d(x,y) = |x-y|$, then (X,d) is a complete metric space. Let T, S: $X \rightarrow X$, φ, ψ: $R_+ \rightarrow R_+$ be defined by

$Tx = \frac{x+1}{2}$, $Sx = \frac{x+3}{4}$, φ(t) = 2t, t ∈ R_+ and

$$\psi(t) = \begin{cases} \sqrt{\frac{15}{16}t^2 - \frac{3}{16}}, & t \geq \sqrt{\frac{1}{5}} \\ 0, & 0 \leq t < \sqrt{\frac{1}{5}} \end{cases}.$$

Now, we show that the inequality (4.2.1.1) is satisfied.

$$\int_0^{d(Tx,Sy)} \varphi(t)dt = \int_0^{d(\frac{xt1}{2},\ \frac{y+3}{4})} 2\ tdt = (\frac{2(x-y)}{8} - \frac{2(1-x)}{8})^2$$

$$= \frac{(x-y)^2}{16} - \frac{2(1-x)(x-y)}{16} + \frac{(1-x)^2}{16}.$$

Cases 1: If x , y ∈ [0, 1] and x – y < 0, then

$$\int_0^{d(Tx,Sy)} \varphi(t)dt = \frac{(x-y)^2}{16} - \frac{2(1-x)(x-y)}{16} + \frac{(1-x)^2}{16}$$

$$= \frac{(y-x)^2}{16} + \frac{2(1-x)\ \ (y-x)}{16} + \frac{(1-x)^2}{16} \qquad (y-x>0)$$

$$\leq (y-x)^2 - \frac{15(y-x)^2}{16} + \frac{1}{8} + \frac{1}{16} \qquad (\text{max} \quad when \quad x=0\,, y=1)$$

$$= (y-x)^2 - (\frac{15}{16}(y-x)^2 - \frac{3}{16})$$

$$= \int_0^{d(x,y)} \varphi(t)dt - \int_0^{\Psi(d(x,y))} \varphi(t)dt.$$

Case 2 : If x, y ∈ [0, 1] and x – y > 0. Then

$$\int_0^{d(Tx,Sy)} \varphi(t)dt = \frac{(x-y)^2}{16} - \frac{2(1-x)(x-y)}{16} + \frac{(1-x)^2}{16}$$

$$\leq \frac{(x-y)^2}{16} + \frac{1}{16}$$

$$\leq \frac{(x-y)^2}{16} + \frac{3}{16}$$

$$= (x-y)^2 - \frac{15}{16}(x-y)^2 + \frac{3}{16} = (x-y)^2 - \frac{15}{16}((x-y)^2 - \frac{3}{16})$$

$$= \int_0^{d(x,y)} \varphi(t)dt - \int_0^{\Psi(d(x,y))} \varphi(t)dt.$$

Thus the inequality (4.2.1.1) is satisfied and

$1 \in X$ is a unique common fixed point of T and S such that

$$T(1) = \frac{1+1}{2} = 1 = S(1) = \frac{1+3}{4}.$$

And $\lim_{n\to\infty} T_n\, x = 1$ *and* $\lim_{n\to\infty} S_n\, x = 1.$

Since,

$$T_1 x_0 = \frac{x_0+1}{2} = x_1 \ for\ x_0 \in X$$

$$Sx_1 = \frac{\frac{x_0+1}{2}+3}{4} = \frac{x_0+7}{8} = x_2 = S_1 x$$

$$Tx_2 = \frac{\frac{x_0+7}{8}+1}{2} = \frac{x_0+15}{16} = x_3 = T_2 x$$

$$Sx_3 = \frac{\frac{x_0+15}{16}+3}{4} = \frac{x_0+127}{128} = x_4 = s_2 x$$

$Tx_4 = \frac{\frac{x_0+127}{128}+1}{2} = \frac{x_0+255}{256} = x_5 = T_3 x.$ Continuing in this way we have

$$T_n x = \frac{x+2^{3n-2}-1}{2^{3n-2}} \qquad for\ all\ n \geq 1\ , n \in N.$$

And $S_n x = \frac{x+2^{4n-1}-1}{2^{4n-1}} \qquad for\ all\ n \geq 1.$

Hence $\lim_{n\to\infty} Tx_{2n-2} = \lim_{n\to\infty} T_n x = \lim_{n\to\infty} \frac{x+2^{3n-2}-1}{2^{3n-2}} = 1,\ for\ \ each\ x\ in\ X.$

And $\lim_{n\to\infty} Sx_{2n-1} = \lim_{n\to\infty} S_n x = \lim_{n\to\infty} \frac{x+2^{4n-1}-1}{2^{4n-1}} = 1,\ for\ \ each\ \ x\ in\ X$

Therefore it satisfies Theorem 4.2.1.

And also $\lim_{n \to \infty} T^n x = \lim_{n \to \infty} T(T(T(\cdots(T(x))) = \lim_{n \to \infty} \frac{2^{2^{(n-1)}}-1+x}{2^{2^{(n-1)}}} = 1.$

Since $T^1 x = \frac{x+1}{2},\ T^2 x = \frac{x+3}{4},\ T^3 x = \frac{x+15}{16} \cdots\cdots\cdots\cdots T^n(x) = \frac{2^{2^{(n-1)}}-1+x}{2^{2^{(n-1)}}}$

And $\lim_{n \to \infty} S^n x = \lim_{n \to \infty}(s(s(s(\cdots\cdots(s(x))\cdots) = \lim_{n \to \infty} \frac{2^{2^n}-1+x}{2^{2^n}} = 1.$

(Since $s^1 x = \frac{x+3}{4},\ s^2 x = s(s(x)) = \frac{\frac{x+3}{4}+3}{4} = \frac{x+15}{16},\ \ s^3 x = s(s^2 x) = \frac{\frac{x+15}{16}+3}{4} = \frac{x+63}{64} \ldots..$
$s^n x = \frac{2^{2^n}-1+x}{2^{2^n}}$)

It satisfies also theorem (4.1.6) [20].

Remark : In Theorem 4. 2.1 if we take T = S we get Theorem (4. 1 .6) [20].

Theorem 4.2.2. Let (φ,ψ) be in $\phi_1 \times \phi_2$ and (X, d) be a complete metric space and let $T, S : X \to X$ be mapping satisfying:

$$\int_0^{d(Tx,Sy)} \varphi(t)dt \leq \int_0^{M(x,y)} \varphi(t)dt - \int_0^{\Psi(M(x.y)} \varphi(t)dt \tag{4.2.2.1}$$

for all $x, y\ in\ X$ where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Sy), \frac{1}{2}[(d(x,Sy) + d(y,Tx)]\right\}.$$

Then T and S have a unique common fixed point $a \in X$ such that

$$\lim_{n \to \infty} Tx_{2n-2} = a \text{ and } \lim_{n \to \infty} Sx_{2n-1=a.}$$

Where Tx_{2n-2}=x_{2n-1} and $Sx_{2n-1} = x_{2n}$.

Proof: Let $x_o \in X$. since $T, S : X \to X$, we can choose $x_1 \epsilon X$ such that $x_1 = Tx_o$. Corresponding to x_1 we can choose $x_2 \in X$ such that $x_2 = Sx_1$. continuing in this process we can construct a sequence$\{x_n\}_{n \in N}$,

$$x_{2n-1=} Tx_{2n-2}\ ,\ \ x_{2n} = Sx_{2n-1}.$$

Now Let x be an arbitrary point in X.

If $x_{2n_0-1} = x_{2n_0-2}$ for some $n_o \in$ N, then one can show that $Tx_{2n_{0-1}} = Sx_{2n_{o-1}} = x_{2n_o-1}$.

Suppose that $x_{2n-2} \neq x_{2n-1}$ for all $n \in N$. We observe that

$M\left(x_{2n-2,}\, x_{2n-1}\right)$

$$= \max\{d\left(x_{2n-2,} x_{2n-1}\right), d\left(x_{2n-2,} Tx_{2n-2}\right), d\left(x_{2n-1,} Sx_{2n-1}\right),$$

$$\frac{1}{2}[d\left(x_{2n-2,} Sx_{2n-1}\right) + d(x_{2n-1,} Tx_{2n-2})]\}$$

$$= \max\{d\left(x_{2n-2,} x_{2n-1}\right), d\left(x_{2n-2,} x_{2n-1}\right), \; d\left(x_{2n-1,} x_{2n}\right),$$

$$\frac{1}{2}[d\left(x_{2n-2,} x_{2n}\right) + d(x_{2n-1}, x_{2n-1})]\}$$

$$= \max\{d\left(x_{2n-2,} x_{2n-1}\right), d\left(x_{2n-1,} x_{2n}\right), \frac{1}{2}[d(x_{2n-2}, x_{2n}]\}$$

$$= \max\left\{d\left(x_{2n-2,} x_{2n-1}\right), d\left(x_{2n-1,} x_{2n}\right)\right\}$$

since $\left(\frac{1}{2} d(x_{2n-2}, x_{2n}) \quad \leq \frac{1}{2}[(dx_{2n-2,} x_{2n-1}) + d(x_{2n-1,} x_{2n)]}\right)$

$$= \max\{d_{2n-2,}\, d_{2n-1}\}, \tag{4.2.2.2}$$

Now, if $M(x_{2n-2},\; x_{2n-1})\} = d_{2n-1,}$ then

$$\int_0^{d_{2n-1}} \varphi(t)dt = \int_0^{d(x_{2n-1,} x_{2n})} \varphi(t)dt$$

$$= \int_0^{d(Tx_{2n-2,}\, Sx_{2n-1})} \varphi(t)dt$$

$$\leq \int_0^{M(x_{2n-2,} x_{2n-1})} \varphi(t)dt - \int_0^{\Psi(M(x_{2n-2},\; x_{2n-1}))} \varphi(t)dt$$

$$= \int_0^{d(x_{2n-1,},\; x_{2n})} \varphi(t)dt - \int_0^{\Psi(d(x_{2n-1},\; x_{2n})} \varphi(t)dt$$

$$= \int_0^{d_{2n-1}} \varphi(t)dt - \int_0^{\Psi(d_{2n-1})} \varphi(t)dt$$

$$< \int_0^{d_{2n-1}} \varphi(t)dt.$$

It is a contradiction, hence

$M\left(x_{2n-2,} x_{2n-1}\right) = d(x_{2n-2}, x_{2n-1})$. Consecutively

$$\int_0^{d_{2n-1}} \varphi(t)dt = \int_0^{d\left(x_{2n-1},\; x_{2n}\right)} \varphi(t)dt$$

$$= \int_0^{d(Tx_{2n-2},\; Sx_{2n-1})} \varphi(t)dt$$

$$\leq \int_0^{M(x_{2n-2}\,, x_{2n-1})} \varphi(t)dt - \int_0^{\Psi(M(x_{2n-2},\; x_{2n-1}\,)} \varphi(t)dt$$

$$= \int_0^{d_{2n-2}} \varphi(t)dt - \int_0^{\Psi(d_{2n-2})} \varphi(t)dt$$

$$< \int_0^{d_{2n-2}} \varphi(t)dt.$$

This shows that$d_{2n-1} < d_{2n-2}$ $\forall n \in N$. (4.2.2.3)

Thus there exists a constant $c \geq 0$ such that

$$\lim_{n \to \infty} d_{2n-2} = c. \tag{4.2.2.4}$$

Suppose that $c > 0$. Set $\lim_{n \to \infty} inf \Psi(d_{2n-2}) = \delta$. Obviously, there exists a subsequence $\{d_{2n(k)-2}\}_{n \in N}$ of $\{d_{2n-2}\}_{n \in N}$ such that $\lim_{k \to \infty} \Psi\big(d_{2n(k)-2}\big) = \delta$. Since Ψ is lower semi-continuous, it follows from $\Psi \in \emptyset_2$ that $\delta \geq \Psi(c) > 0$. On account of (4.2.2.1), (4.2.2.4), Lemma 4.1.4 and $\varphi \in \emptyset_1$, we arrive at

$$0 < \int_0^c \varphi(t)dt\, dt$$

$$= \lim_{k \to \infty} sup \int_0^{d_{2n(k)-1}} \varphi(t)$$

$$= \lim_{k \to \infty} \sup \int_0^{d(x_{2n(k)-1}, x_{2n(k)})} \varphi(t)dt$$

$$= \lim_{k \to \infty} \sup \int_0^{d(Tx_{2n(k)-2}, Sx_{2nk-1})} \varphi(t)dt$$

$$\leq \lim_{k \to \infty} \sup(\int_0^{M(x_{2n(k)-2}, x_{2n(k)-1})} \varphi(t)dt - \int_0^{\Psi(M(x_{2n(k)-2}, x_{2n(k)-1})} \varphi(t)dt)$$

$$= \lim_{k \to \infty} \sup \int_0^{M(x_{2n(k)-2}, x_{2n(k)-1})} \varphi(t)dt - \lim_{k \to \infty} \inf \int_0^{\Psi(M(x_{2n(k)-2}, x_{2n(k)-1})} \varphi(t)dt$$

$$= \lim_{k \to \infty} \sup \int_0^{d_{2n(k)-2}} \varphi(t)dt - \lim_{k \to \infty} \inf \int_0^{\Psi(d2n(k)-2)} \varphi(t)dt$$

$$= \int_0^c \varphi(t)dt - \int_0^\delta \varphi(t)dt$$

$$\leq \int_0^c \varphi(t)dt - \int_0^{\Psi(c)} \varphi(t)dt < \int_0^c \varphi(t)dt.$$

It is contradiction. Hence c = 0.

Now, to show that $\{x_n\}_{n\epsilon N}$ is a Cauchy sequence it suffice to show $\{x_{2n}\}_{n\epsilon N}$ is Cauchy sequence. Suppose that $\{x_{2n}\}_{n\epsilon N}$ is not Cauchy sequence. It follows that there exist $\varepsilon > 0$ and two subsequence

$\{x_{2m(k)}\}_{k\epsilon N}$ and $\{x_{2n(k)}\}_{k\epsilon N}$ of $\{x_{2n}\}_{n\epsilon N}$ such that $n(k)$ is minimal in the sense n(k)> m(k)> k and $d(x_{2m(k)}, x_{2n(k)}) > \varepsilon$ which follows that $d\left(x_{2m(k)}, x_{2n(k)-2}\right) \leq \varepsilon$.

Form this,

$$\varepsilon < d(x_{2m(k)}, x_{2n(k)})$$

$$\leq d(x_{2m(k)}, x_{2m(k)-1}) + d(x_{2m(k)-1,x_{2m(k)-2}}) + d(x_{2m(k)-2}, x_{2n(k)-1})$$

$$+d(x_{2n(k)-1}, x_{2n(k)})$$

$$\leq d_{2m(k)-1} + 2d_{2m(k)-2} + d_{2n(k)-1} + d\,(x_{2m(k)-1}, x_{2m(k)}) + d\left(x_{2m(k)}, x_{2n(k)-2}\right)$$

$$+d\left(x_{2n(k)-2}, x_{2n(k)-1}\right)$$

$$= 2d_{2m(k)-1} + 2d_{2m(k)-2}) + d_{2n(k)-1} + d\left(x_{2m(k)}, x_{2n(k)-2}\right) + d_{2n(k)-2} \qquad (4.2.2.5)$$

and $|d(x_{2m(k)-1}, x_{2n(k)}) - d(x_{2m(k)}, x_{2n(k)}| \leq d_{2m(k)-1 \rightarrow 0}$ as $k \rightarrow \infty$ which shows that

$$\lim_{k \rightarrow \infty}\left(x_{2m(k)-1}, x_{2n(k)}\right) = \varepsilon \qquad (4.2.2.6)$$

Also

$$\varepsilon = \lim_{k \rightarrow \infty} d\left(x_{2m(k)-1}, x_{2n(k)}\right)$$

$$\varepsilon \leq d(x_{2m(k)-1}, x_{2m(k)-2}) + d(x_{2m(k)-2}, x_{2n(k)-1})$$

$$+ d(x_{2n(k)-1,}x_{2n(k)})$$

$$\leq d_{2m(k)-2} + d(x_{2m(k)-2,}\ x_{2n(k)}) + 2d_{2n(k)-1}$$

$$\leq 2d_{2m(k)-2} + d(x_{2m(k)-1,}x_{2n(k)-1}) + 3d_{2n(k)-1}$$

$$\leq 2d_{2m(k)-2} + d_{2n(k)-1,} + d(x_{2m(k)-1}, x_{2n(k)}) + 3d_{nk-1} \ldots \qquad (4.2.2.7)$$

Taking $k \rightarrow \infty$ for (4.2.2.5) , (4.2.2.6) and (4.2.2.7) we have

$$\lim_{k \rightarrow \infty} d(x_{2m(k)-1}, x_{2n(k)-1})$$

$$= \lim_{k \rightarrow \infty} d(x_{2m(k)-1}, x_{2n(k)})$$

$$= \lim_{k \rightarrow \infty} d(x_{2m(k)-2}, x_{2n(k)}) = \lim_{k \to \infty} d(x_{2m(k)-2,}\ x_{2nk-1})$$

$$= \lim_{k\to\infty} d\big(x_{2m(k)}, x_{2n(k)}\big) = \lim_{k\to\infty}\big(x_{2m(k)}, x_{2n(k)-2}\big) = \varepsilon.$$

Thus,

$$M(x_{2m(k)-2}, x_{2n(k)-1})$$

$$= max\{d(x_{2m(k)-2},\ \ x_{2n(k)-1}), d(x_{2m(k)-2}, Tx_{2m(k)-2}),$$

$$d(x_{2m(k)-1}, Sx_{2n(k)-1}), \frac{1}{2}\big[d(x_{2m(k)-2}, Sx_{2n(k)-1}\big)$$

$$+d(x_{2n(k)-1}, Tx_{2m(k)-2})]\}$$

$$= \max\{d(x_{2m(k)-2}, x_{2n(k)-1}), d(x_{2m(k)-2}, x_{2n(k)-1}), d(x_{2n(k)-1}, x_{2n(k)})$$

$$+ \frac{1}{2}[d(x_{2m(k)-2},\ \ x_{2n(k)}) +\ \ d(x_{2n(k)-1}, x_{2m(k)-1})]\}$$

$$= \max\left\{\varepsilon, 0, 0, \frac{1}{2}(\varepsilon + \epsilon)\right\} = \varepsilon \quad as\ k \to \infty \qquad (4.2.2.8)$$

Put $\liminf_{j\to\infty} \Psi\left(M(x_{2m(k_j)-2,}, x_{2n(k_j)-1}\right) = \alpha.$ (4.2.2.9)

Then there exists a subsequence

$M(x_{2m(k_j)-2,}x_{2n(k_j)-1})_{j\in N}$ of $\{M(x_{2m(k)-2,}\ x_{2n(k)-1})\}_{k\in N}$ such that
$\lim_{j\to\infty} \Psi(M(x_{2m(k_j)-2,}\ \ x_{2n(k_j)-1}) = \alpha \geq \Psi(\varepsilon)$

Combining (4.2.2.3), (4.2.2.1), (4.2.2.5), (4.2.2.6), (4.2.2.7), Lemma 4.1.4 and $(\varphi, \psi) \in \emptyset_1 \times \emptyset_2$ we get

$$0 < \int_0^{\varepsilon} \varphi(t)dt$$

$$= \limsup_{j\to\infty} \int_0^{d\left(x_{2m(k_j)-1,}\ x_{2n(k_j)}\right)} \varphi(t)dt$$

$$= \limsup_{j\to\infty} \int_0^{d(Tx_{2m(k_j)-2},\ Sx_{2n(k_j)-1})} \varphi(t)dt$$

$$\leq \limsup_{j\to\infty}\left(\int_0^{M(x_{2m(k_j)-2},\, x_{2n(k_j)-1})} \varphi(t)dt - \int_0^{\Psi(M(x_{2m(k_j)-2},\, x_{2n(k_j)-1}))} \varphi(t)dt\right)$$

$$= \limsup_{j\to\infty} \int_0^{M\left(x_{2m(k_j)-2},\, x_{2n(k_j)-1}\right)} \varphi(t)dt \quad -\liminf_{j\to\infty}\int_0^{\Psi(M(x_{2m(k_j)-2},\, x_{2n(k_j)-1})} \varphi(t)dt$$

$$= \int_0^{\varepsilon} \varphi(t)dt - \int_0^{\alpha} \varphi(t)dt$$

$$\leq \int_0^{\varepsilon} \varphi(t)dt - \int_0^{\Psi(\varepsilon)} \varphi(t)dt$$

$$< \int_0^{\varepsilon} \varphi(t)dt.$$

It is a contradiction. Hence $\{x_{2n}\}_{n\epsilon N}$ is a Cauchy sequence and hence $\{x_n\}_{n\epsilon N}$ is a Cauchy sequence and the completeness of (X, d) ensures that there exists $a\ in\ X$ such that $\lim_{n\to\infty} x_n$= a

Now, suppose $d(a, Ta) > 0$.

Now,

$$\lim_{n\to\infty} \mathrm{M}(\mathrm{a}, \mathrm{x}_{2\mathrm{n}-1})$$

$$= \lim_{n\to\infty} \max \{\mathrm{d}(\mathrm{a}, \mathrm{x}_{2\mathrm{n}-1}), \mathrm{d}(\mathrm{a}, \mathrm{Ta}), \mathrm{d}(\mathrm{x}_{2\mathrm{n}-1,} \mathrm{Sx}_{2\mathrm{n}-1}),$$

$$\frac{1}{2}[\mathrm{d}(\mathrm{a}, \mathrm{Sx}_{2\mathrm{n}-1}) + \mathrm{d}(\mathrm{x}_{2\mathrm{n}-1,} Ta)\}$$

$$= \max\{0, (\mathrm{a}, \mathrm{Ta}), \mathrm{d}(\mathrm{a}, \mathrm{a}), \tfrac{1}{2}(\mathrm{d}(\mathrm{a}, \mathrm{a}) + \mathrm{d}\ (\mathrm{a}, \mathrm{Ta}))\}$$

$$= \mathrm{d}\ (\mathrm{a}, \mathrm{Ta}) \qquad (4.2.2.10)$$

Put $\lim_{n\to\infty} inf\ \psi(M(a, x_{2n-1}) = \alpha$, clearly , there exists a subsequence

$\{\mathrm{M}(x_{2n(k)-1}, a)\}_{k\in N}$ of $\{\mathrm{M}(x_{2n-1}, a)\}_{\mathrm{n}\in N}$ such that

$$\lim_{k\to\infty} \psi(M(x_{2n(k)-1}, a))=\alpha \geq \psi(\mathrm{d(a,Ta)}). \qquad (2.2.11)$$

In virtue of (4.2.2.1),(4.2.2.10),(4.2.2.11) and Lemma 4.1.4, we conclude that

$$0 < \int_0^{\mathrm{d(a,Ta\ \ )}} \varphi(t)dt = \lim_{k\to\infty} sup \int_0^{\mathrm{d}(\mathrm{x}_{2\mathrm{n(k)}},\ \mathrm{Ta})} \varphi(t)dt$$

$$= \lim_{k\to\infty} sup \int_0^{\mathrm{d}(\mathrm{Sx}_{2\mathrm{n(k)}-1,}\mathrm{Ta}\ )} \varphi(t)dt$$

$$= \lim_{k\to\infty} \sup \int_0^{d(Ta,\ Sx_{2n(k)-1})} \varphi(t)dt$$

$$\leq \lim_{k\to\infty} \sup(\int_0^{M(x_{2n(k)-1},a)} \varphi(t)dt - \lim_{n\to\infty} \int_0^{\Psi(M(x_{2n(k)-1}, a))} \varphi(t)dt)$$

$$\leq \lim_{k\to\infty} \sup \int_0^{M(x_{2n(k)-1},a)} \varphi(t)dt - \lim_{n\to\infty} inf \int_0^{\Psi(M(x_{2n(k)-1}, a))} \varphi(t)dt$$

$$= \int_0^{d(a,Ta)} \varphi(t)dt - \int_0^{\alpha} \varphi(t)dt$$

$$\leq \int_0^{d(a,Ta)} \varphi(t)dt - \int_0^{\Psi(d(a,Ta))} \varphi(t)dt$$

$$< \int_0^{d(a,Ta)} \varphi(t)\,\mathrm{dt}.$$

That is, $0 < \int_0^{d(a,Ta)} \varphi(t)dt < \int_0^{d(a,Ta)} \varphi(t)dt$ which is impossible. Consequently,

$a = Ta$. That is, a is the fixed point of T in X.

Now to show that a is also a fixed point of S, that is $a = Sa$,

Suppose $Sa \neq a = Ta(Ta \neq Sa)$. Thus,

$$0 < \int_0^{d(Ta,Sa)} \varphi(t)dt$$

$$\leq \int_0^{M(a,a)} \varphi(t)dt - \int_0^{\Psi(M(a,a))} \varphi(t)dt \qquad (4.2.2.12)$$

$$M(a,a) = max\left\{d(a,a), d(a,Ta), d(a,Sa), \frac{1}{2}[\, d(a,Sa) + d(a,Ta)]\right\}$$

$$= max\left\{0, 0, d(a,Sa), \frac{1}{2}\big(d(a,Sa)\big)\right\} = d(a,Sa)$$

$$= d(Ta,Sa). \qquad (4.2.2.13)$$

Thus using equations (4.2.2.12) and (4.2.2.13) we have,

$$\int_0^{d(Ta,Sa)} \varphi(t)dt \leq \int_0^{M(a,a)} \varphi(t)dt - \int_0^{\Psi(M(a,a))} \varphi(t)dt$$

$$= \int_0^{d(Ta,Sa)} \varphi(t)dt - \int_0^{\Psi(M(Ta,Sa))} \varphi(t)dt$$

$$< \int_0^{d(Ta,Sa)} \varphi(t)dt.$$

This is impossible, hence $Ta = Sa = a$ that confirms a is a common fixed point of T and S. Now to show the uniqueness of a common fixed point.

Suppose b is also a common fixed point of T and S such that

$b \neq a$ $(Tb = Sb = b)$. Then,

$$0 < \int_0^{d(a,b)} \varphi(t)dt = \int_0^{d(Ta,Sb)} \varphi(t)dt$$

$$\leq \int_0^{M(a,b)} \varphi(t)dt - \int_0^{\psi\,(d(a,b))} \varphi(t)dt. \qquad (4.2.2.14)$$

where

$$M(a,b) = max\left\{d(a,b), d(a,Ta), d(b,Sb), \tfrac{1}{2}[\,d(a,Sb) + d(b,Ta)]\right\}$$

$$= max\left\{d(a,b), 0, 0,\ \tfrac{1}{2}\big(d(a,b) + d(b,a)\big)\right\} = d(a,b).$$

Hence, $0 < \int_0^{d(a,b)} \varphi(t)dt = \int_0^{d(Ta,Sb)} \varphi(t)dt \leq \int_0^{M(a,b)} \varphi(t)dt - \int_0^{\psi\,(M(a,b))} \varphi(t)dt.$

$$= \int_0^{d(a,b)} \varphi(t)dt - \int_0^{\psi\,(d(a,b))} \varphi(t)dt < \int_0^{d(a,b)} \varphi(t)dt$$

It is a contradiction, thus $d(a,b) = 0$, that is $a = b$. Hence, a is a unique common fixed point of T and S.

We now give an example in support of Theorem 4.2.2.

Let $X = [0,1] \cup \{4\}$ be endowed with the Euclidean metric $T, S : X \to X$ and

$\varphi, \psi : R_+ \to R_+\ be$ defined by:

$$Tx = \begin{cases} \frac{x}{2}, x \in [0,1] \\ 1, x = 4 \end{cases} \qquad Sx = \frac{x}{4}\ \ for\ x \in [0,1] \cup \{4\},\ \varphi(t) = 8t \ \text{ and } \psi(t) = \begin{cases} \frac{\sqrt{3}}{4}t\ \ , t \in [0,1], \\ \sqrt{1 - \frac{1}{1+t}},\ \ t > 1 \end{cases}$$

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Sy), \frac{1}{2}[d(x,Sy) + d(y,Tx)]\}$$

$$= \max\{\,|x-y|, |x - \tfrac{x}{2}|,\ |y - \tfrac{y}{4}|, \tfrac{1}{2}(|x - \tfrac{y}{4}| + |y - \tfrac{x}{2}|)\} \ \text{ for x ,y} \neq 4$$

$$= \max\{|x-y|, \tfrac{x}{2}, \tfrac{3}{4}y, \tfrac{1}{2}(|x - \tfrac{y}{4}| + |y - \tfrac{x}{2}|)\}.$$

Clearly, $(\varphi, \psi) \in \phi_1 X \phi_2$, for $x, y, \in X$ we consider the following cases:

Case 1: Let $x, y, \in [0,1]$ and $0 \leq x \leq \frac{y}{4}$, then

$M(x,y) = y - x \leq 1$ and

$$\int_0^{d(Tx,Sy)} \varphi(t)dt = \int_0^{\left|\frac{x}{2}-\frac{y}{4}\right|} 8t\,dt = [4t^2]_0^{\frac{1}{2}\left|x-\frac{y}{2}\right|} = 4(\tfrac{1}{4}(x-\tfrac{y}{2})^2$$

$$= (x-\tfrac{y}{2})^2 = x^2 - xy + \tfrac{y^2}{4} = (x-y)^2 + x.y - \tfrac{3}{4}y^2$$

$$\leq (x-y)^2 + \tfrac{y}{4}.y - \tfrac{3}{4}y^2 \qquad (\text{since } x \leq \frac{y}{4})$$

$$= (x-y)^2 - \tfrac{2}{4}y^2$$

$$\leq 4(x-y)^2 - 2y^2$$

$$\leq 4(x-y)^2 - \sqrt{3}(y-x)^2 \qquad (y \geq y-x)$$

$$= 4(x-y)^2 - 4(\frac{\sqrt{3}}{4}(y-x)^2)$$

$$= 4(M(x,y))^2) - 4((\psi(M(x,y))^2)$$

$$= \int_0^{M(x,y)} \varphi(t)dt \text{ - } \int_0^{\psi(M((x,y)^2)} \varphi(t)dt.$$

Case 2: Let $x, y \in [0,1]$ and $\frac{y}{4} < x \leq \frac{3}{4}y$

Then $M(x,y) = \frac{3}{4}y$ and

$$\int_0^{d(Tx,Sy)} \varphi(t)dt = \int_0^{\left|\frac{x}{2}-\frac{y}{4}\right|} 8t\,dt$$

$$= 4(\frac{1}{2}(x-\frac{y}{2}))^2 = (x-\frac{y}{2})^2 = x^2 - x.y + \frac{y^2}{4}$$

$$\leq x^2 - \frac{y}{4}.y + \frac{y^2}{4} \text{ (Since } \frac{y}{4} < x)$$

$$= x^2 \leq (\frac{3}{4}y)^2 = \frac{9}{16}y^2 = \frac{9}{4}y^2 - \frac{27}{16}y^2 \leq \frac{9}{4}y^2 - \frac{27}{64}y^2$$

$$= 4(\frac{3}{4}y)^2 - 4(\frac{\sqrt{3}}{4}(\frac{3}{4}y))^2$$

$$= 4(M(x,y)^2) - 4(\psi(m(x,y))^2$$

$$= \int_0^{M(x,y)} \varphi(t)dt - \int_0^{\psi(M(x,y)} \varphi(t)dt.$$

Case 3 : Let $x, y \in [0,1]$ with $\frac{3}{4}y < x \leq y$

Then, $M(x,y) = \frac{3}{4}y$ and

$$\int_0^{d(Tx,Sy)} \varphi(t)dt = \int_0^{\left|\frac{x}{2}-\frac{y}{4}\right|} 8tdt$$

$$= 4\left| \frac{1}{2}(x - \frac{y}{2})\right|^2$$

$$= x^2 - x.y + \frac{y^2}{4}$$

$$\leq x^2 - \frac{3}{4}y.y + \frac{y^2}{4} \quad \text{(Since } x > \frac{3}{4}y)$$

$$\leq y^2 - \frac{1}{2}y^2 = \frac{y^2}{2} \qquad (y \geq x)$$

$$= \frac{9}{4}y^2 - \frac{7}{4}y^2$$

$$= \frac{9}{4}y^2 - \frac{112}{64}y^2$$

$$\leq \frac{9y^2}{4} - \frac{27}{64}y^2$$

$$= 4(\frac{3}{4}y)^2 - 4(\frac{\sqrt{3}}{4}(\frac{3}{4}y))^2 \qquad = \int_0^{\frac{3}{4}y} 8tdt - \int_0^{\frac{\sqrt{3}}{4}(\frac{3}{4}y)} 8tdt$$

$$= \int_0^{M(x,y)} \varphi(t) - \int_0^{\psi(M(x,y))} \varphi(t)dt.$$

Case 4: Let $x, y \in [0,1]$ with $0 \leq y \leq \frac{1}{2}x$, then

$M(x,y) = x - y \leq 1$ and

$$\int_0^{d(Tx,Sy)} \varphi(t) = \int_0^{\left|\frac{x}{2}-\frac{y}{4}\right|} 8tdt$$

$$= (x - \frac{y}{2})^2 \leq \frac{13}{4}(x-y)^2 = 4(x-y)^2 - \frac{3}{4}(x-y)^2$$

$$= \int_0^{M(x,y)} \varphi(t) - \int_0^{\psi(M(x,y))} \varphi(t)dt$$

Case 5: Let $x, y \in [0,1]$ with $\frac{3}{4}y \leq \frac{1}{2}x < y \leq x$ then

$M(x,y) = \frac{x}{2}$ and

$$\int_0^{d(Tx,Sy)} \varphi(t)dt = \int_0^{\left|\frac{x}{2}-\frac{y}{4}\right|} 8tdt$$

$$= (x - \frac{y}{2})^2$$

$$= x^2 - xy + \frac{y^2}{4}$$

$$\leq x^2 - x.\frac{x}{2} + \frac{y^2}{4} \quad \text{(Since } y > \frac{x}{2}\text{)}$$

$$= \frac{x^2}{2} + \frac{y^2}{4}$$

$$\leq \frac{x^2}{2} + \frac{x^2}{4} \qquad (y \leq x)$$

$$= \frac{3x^2}{4} = \frac{12x^2}{16} \leq x^2 - \frac{x^2}{16}$$

$$= 4(\frac{x}{2})^2 - 4(\frac{\sqrt{3}}{4}(\frac{x}{2}))^2$$

$$= \int_0^{\frac{x}{2}} 8tdt - \int_0^{\frac{\sqrt{3}}{4}(\frac{x}{2})} 8tdt$$

$$= \int_0^{M(x,y)} \varphi(t)dt - \int_0^{\psi(M(x,y))} \varphi(t)dt.$$

Case 6: Let $\frac{1}{2}x < \frac{3}{4}y < y \le x$ and $x, y \in [0,1]$,then

$M(x,y) = \frac{3}{4}y$ and

$$\int_0^{d(Tx,Sy)} \varphi(t)dt = \int_0^{\left|\frac{x}{2}-\frac{y}{4}\right|} 8t dt$$

$$= x^2 - x.\ y + \frac{1}{4}y^2$$

$$\le x^2 - x.\frac{1}{2}x^2 + \frac{1}{4}y^2 \quad \text{(Since } y > \tfrac{x}{2}\text{)}$$

$$= \frac{x^2}{2} + \frac{y^2}{4}$$

$$= 2(\frac{x}{2})^2 + \frac{y^2}{4}$$

$$\le 2(\frac{9}{16}y^2) + \frac{y^2}{4} \qquad \text{(Since } \frac{x}{2} < \frac{3}{4}y\text{)}$$

$$= \frac{11}{8}y^2 \le \frac{117}{64}y^2 = \frac{9}{4}y^2 - \frac{27}{64}y^2$$

$$= 4(\frac{3}{4}y)^2 - 4(\frac{\sqrt{3}}{4}(\frac{3}{4}y))^2$$

$$= 4(M(x,y)^2) - 4(\psi(x,y))^2)$$

$$= \int_0^{M(x,y)} \varphi(t)dt - \int_0^{\psi(M(x,y)} \varphi(t)dt.$$

Case 7: Let $y \in [0,1]$ and $x = 4$, then

$$M(x,y) = \max\{d(x,y), d(x,Tx), (y,Sy), \tfrac{1}{2}[d(x,Sy) + d(y,Tx)]\}$$
$$M(x,y) = \max\{4 - y,\ 3,\ \frac{3}{4}y, \frac{1}{2}(5 - \frac{5}{4}y)\}$$

$= 4 - y \ge 3$ and

$$\int_0^{d(Tx,Sy)} \varphi(t)dt \;=\; \int_0^{|Tx,Sy|} 8tdt$$

$$= [4t^2]_0^{\left|1-\frac{y}{4}\right|}$$

$$= \; 4(1-\frac{y}{4})^2 \le 4 \le (4-y)^2 - (1-\frac{1}{1+(4-y)})$$

$$\le 4(4-y)^2 - 4(1-\frac{1}{1+4-y})$$

$$= \; 4(M(x,y) - 4(\psi(M(x,y))^2)$$

$$= \int_0^{M(x,y)} \varphi(t)dt \text{ - } \int_0^{\psi(M(x,y))} \varphi(t)dt$$

Case 8: Let $x = y = 4$, then

$$M(x,y) = \max\left\{ \;|4-4|, |4-Tx|\,, \left|4-\frac{4}{4}\right|, \frac{1}{2}\left[\; \left|4-\frac{4}{2}\right| + \left|4-\frac{4}{2}\right| \;\right] \right\}$$

$$= \max \; \{0,3,3,2\}$$

$$= 3 \quad \text{and}$$

$$\int_0^{d(Tx,Sy)} \varphi(t)dt = \int_0^{\left|1-\frac{4}{4}\right|} 8tdt = 0 \;\le 36-3 = 4(3^2) - 4(1-\frac{1}{1+3})$$

$$= \; 4(M(x,y)^2 - 4(\psi(M(x,y))^2)$$

$$= \int_0^{M(x,y)} \varphi(t)dt \text{ - } \int_0^{\psi(M(x,y))} \varphi(t)dt.$$

This shows that the example satisfies all the conditions of the Theorem 4.22.2 and T and S have a unique a common fixed point $0 \in X$.

That is $(T(0) = \frac{0}{2} = 0 = S(0) = \frac{0}{4})$. Also we have

$$\lim_{n \to \infty} T^n x = \lim_{n \to \infty} \frac{x}{2^n} = 0 \text{ and } \lim_{n \to \infty} S^n x = \lim_{n \to \infty} \frac{x}{4^n} = 0 \text{ for every } x \in X$$

Thus the example holds true.

To check $\lim_{n \to \infty} T_n x = \lim_{n \to \infty} S_n x = a = 0$ the fixed point for each $x \in X$

Case 1: Let $x_O \in [0,1]$

$$Tx_0 = \frac{x_0}{2} = x_1 = T_1 x$$

$$Sx_1 = \frac{\left(\frac{x_0}{2}\right)}{4} = \frac{x_0}{8} = x_2 = S_1 x$$

$$Tx_2 = \frac{\left(\frac{x_0}{8}\right)}{2} = \frac{x_0}{16} = x_3 = T_2 x$$

$$Sx_3 = \frac{\left(\frac{x_0}{16}\right)}{4} = \frac{x_0}{64} = x_4 = S_2 x$$

$$Tx_4 = \frac{\left(\frac{x_0}{64}\right)}{2} = \frac{x_0}{128} = x_5 = T_3 x.$$

Continuing the steps we have,

$Tx_{2n-2} = T_n x = \frac{x}{2^{3n-2}}$ and $Sx_{2n-1} = S_n x = \frac{x}{2^{3n}}$ for n = 1, 2,… and for each $x \in [0,1]$

Hence, $\lim_{n \to \infty} T_n x = \lim_{n \to \infty} \frac{x}{2^{3n-2}} = \lim_{n \to \infty} S_n x = \lim_{n \to \infty} \frac{x}{4^{3n}} = 0$

Case 2: For $x_0 = 4$, $Tx_0 = 1 = x_1 = T_1 x$

$$Sx_1 = \frac{1}{4} = x_2 = S_1 x$$

$$Tx_2 = \frac{(\frac{1}{4})}{2} = \frac{1}{8} = x_3 = T_2 x$$

$$Sx_3 = \frac{(\frac{1}{8})}{4} = \frac{1}{32} = x_4 = S_2 x.$$

Continuing the steps we have,

$Tx_{2n-2} = T_n x = \frac{1}{2^{3(n-1)}}$ and S $x_{2n-1} = S_n x = \frac{1}{2^{3n-1}}$

Hence $\lim_{n\to\infty} T_n x = \lim_{n\to\infty} \frac{1}{2^{(3n-2)}} = \lim_{n \to \infty} S_n x = \lim_{n \to \infty} \frac{1}{2^{3n-1}} = 0$ Therefore, all the condition of the Theorem 4.2.2 are satisfied.

Remark : In Theorem 4. 2.2 if we take T = S we get Theorem (4. 1 .7) [20].

CHAPTER FIVE

5. CONCLUSION AND FUTURE SCOPE

5.1. CONCLUSION

In 2014, Zeqing Liu, Heng Wu, Jeong Sheok Ume and Shin Min Kang [20] have established the existence and uniqueness of fixed points for a single map satisfying contractive condition of integral type. In this research work we extended the works of [20] to a pair of self-maps and proved the existence and uniqueness of common fixed points for the maps under consideration. The results we established were supported by examples.

5.2. FUTURE SCOPES

The existence of common fixed point of pair of maps satisfying contractive conditions of integral type is one of the area of study in analysis. Recently, there are a number of published research papers related to this area of study. So the researcher recommends the upcoming post graduate students and other researchers to do their research work in this area of study.

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