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Common fixed point theorems involving contractive conditions of rational type in dislocated quasi metric spaces

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Abstract: The purpose of this paper is to establish new common fixed point results for a pair of self- maps involving contractive condition of rational type in complete dislocated quasi metric space by extending the result of Rahman and Sarwar.

Keywords: dislocated quasi metric space; compatible maps; weakly compatible mappings; coincidence points and common fixed point.

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1. Introduction

Fixed point theory is one of the most dynamic research subjects in nonlinear analysis and can be used to many discipline branches such as; control theory, convex optimization, differential equation, integral equation, economics etc. In this area, the first important and remarkable result was presented by Banach in 1922 for a contraction mapping in a complete metric space. Dass and Gupta [17] generalized the Banach contraction principle in a metric space for some rational type

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contractive conditions. Hitzler and Seda [19] investigated the useful applications of dislocated topology in the context of logic programming semantics. Dislocated metric space has a key role in logic programming and electronics engineering. Furthermore, Zeyada et al. [3] generalized the results of Hitzler and Seda [19] and introduced the concept of complete dislocated quasi metric space. Aage and Salunke [18] derived some fixed point theorems in dislocated quasi metric spaces. In this paper, we prove fixed point result in the setting of dislocated quasi-metric spaces for a pair of self-mappings which generalize the result of Rahman and Sarwar [16].

2. Preliminaries

Definition 2.1[3]. Let X be a non- empty set and $d : \mathrm{X} \times \mathrm{X} \to [0, \infty)$ be a function satisfying the following conditions:

$$d_1)\ d(x, y) = d(y, x) = 0, \text{ implies } x = y\ ;$$
$$d_2)\ d(x, y) \le d(x, z) + d(z, y), \text{ for all } x, y, z \in X.$$

Then d is called a dislocated quasi-metric (dq-metric) on X A pair (X, d) is called dq-metric space (dq-metric space).

Definition 2.2 [3]. A sequence $\{x_n\}$ in a dq-metric space (X, d) is called Cauchy sequence if for all $\varepsilon > 0,$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ or $d(x_m, x_n) < \varepsilon$ for all $m, n \ge n_0.$

Definition 2.3 [3]. A sequence $\{x_n\}$ in a dq-metric space (X, d) is said to be dislocated quasi-convergent (dq-convergent) to x if $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0.$ In this case, x is called a dq-limit of sequence $\{x_n\}$ and we write $x_n \to x$ as $n \to \infty.$

Dentition 1.4 [3]. A dq-metric space (X, d) is called complete if every Cauchy sequence in it is dq-convergent.

Definition 2.5[3]. Let $\left(X,d\right)$ be a dq-metric space. A self-map $T:X\to X$ said to be a contraction map if there exists a constant $k\in[0,1)$ such that

$d\left(Tx,Ty\right)\le kd\left(x,y\right)$ for all x,y in $\left(X,d\right)$.

Lemma 2.6[3]. Limit of a convergent sequence in a dq-metric space is unique.

In the following theorem, Zeyada et al. [3] generalized the Banach contraction principle in dislocated quasi metric space.

Theorem 2.7[3]. Let $\left(X,d\right)$ be a complete dq-metric space, $T:X\to X$ be a contraction, then T has a unique fixed point in X.

Dentition 2.8[7]. A map $\varphi:\left[0,\infty\right)\to\left[0,\infty\right)$ is called comparison function if it satisfies:

i) φ is monotonically increasing;

ii) The sequence $\left\{\varphi^n\left(\mathrm{t}\right)_{n=0}^{\infty}\right\}$ converges to zero for all $t\in\left[0,\infty\right)$.

If φ satisfies

iii) $\sum_{k=0}^{\infty}\varphi^k\left(t\right)$ Converges for all $t\in\left[0,\infty\right)$, then φ is called (c) comparison function.

Every comparison function is (c) comparison.

Prototype example for comparison function is $\varphi\left(t\right)=\alpha t, t\in\left[0,\infty\right)$ and $\alpha\in[0,1)$.

Lemma1.9[7]. For every comparison function $\varphi:\left[0,\infty\right)\to\left[0,\infty\right)$

$\varphi\left(t\right)<t$ for all $t>0,$ and $\varphi\left(t\right)=0$ if and only if $t=0.$

Theorem 1.10[8]. Let (X,d) be a complete dislocated quasi b-metric space. Let $T_1:X\to X$ and $T_2:X\to X$ are continuous functions on $X,$ for $k\ge1$ satisfying

$$d\left(T_1x,T_2y\right)\le\varphi d\left(x,y\right) \text{ for all } x,y\in X.$$

Where φ is a comparison function, then T_1 and T_2 have a unique common fixed point in X.

Theorem 2.11[16]. Let (X,d) be a complete dislocated quasi metric space. And $T:X\to X$ be self-mapping satisfying

$$d(Tx,Ty)\le a\varphi(d(x,y))+b\varphi(\max\{d(x,Tx),d(x,y)\})+ c\varphi\left(\frac{d(x,y)\left[1+\sqrt{d(x,y)d(x,Tx)}\right]^2}{(1+d(x,y))^2}\right);$$

for all $x,y\in X$ and $a,b,c\ge 0$ with $a+b+c<1$, and φ is a comparison function. Then T has a unique fixed point.

We denote $C(T,f)=\{x\in X: Tx=fx\}$ and $F(T,f)=\{x\in X: Tx=fx=x\}$.

Where $C(T,f)$ is the set of coincidence point and $F(T,f)$ is the set of common fixed point of T and f.

In the sequel we need the following dentitions.

Definition 2.12 [4]. Two self-maps T and f on a nonempty set X are said to be commuting if $Tfx=fTx$ for all $x\in X$.

Definition 2.13[5]. Two self-mappings T and f of a metric space (X,d) are called compatible if $\lim_{n\to\infty} d(fTx_n,Tfx_n)=0$ whenever $\{x_n\}_n^\infty$ is a sequence in X such that $\lim_{n\to\infty} fx_n=\lim_{n\to\infty} Tx_n=t$ for some $t\in X$.

Definition 2.14[6]. Two self- maps T and f of a metric space (X,d) are called weakly compatible if they commute at their coincidence points. That is, $fu=Tu$ *for* $u\in X$, then $Tfu=fTu$, for $u\in X$.

3. Main results

Theorem 3.1. Let (X,d) be a complete dislocated quasi metric spaces and $T, f : X \to X$ be self-maps satisfying the following conditions

$i)$ $TX \subseteq fX$;

$ii)$ T and f are weakly compatible and fX is closed subset of X;

$iii)$ $$d(Tx,Ty) \le a\varphi\big(d(fx, fy)\big) + b\varphi\big(\max\{d(fx, fy), d(fx,Tx)\}\big) + c\varphi\left(\frac{d(fx, fy)\left[1+\sqrt{d(fx, fy)d(fx,Tx)}\right]^2}{\big(1+d(fx, fy)\big)^2}\right);$$

for all $x, y \in X$ and $a,b,c \ge 0$ with $a+b+c<1$, and φ is a comparison function. Then T and f have a unique common fixed point if T and f commute at their coincidence points.

Proof

Let $x_0 \in X$, so that $y_0 = Tx_0 = fx_1$ and using condition (i) $Tx_1 \in fX$, then there exist $x_2 \in X$ such that $y_1 = Tx_1 = fx_2$. Continuing this process we construct a sequence $\{x_n\}$ and $\{y_n\}$ such that $y_n = \mathrm{T}x_n = fx_{n+1}$ for $n \in \{0,1,2,\ldots\}$.

Now we consider two cases.

Case (i):

Suppose $y_n = y_{n+1}$ for some $n \in \{0,1,2,\ldots\}$, then $y_n = Tx_n = Tx_{n+1} = y_{n+1} = fx_{n+1}$.

So that x_{n+1} is coincidence point of T and f. Let $Tx_n = Tx_{n+1} = fx_{n+1} = u$, for some $u \in fX$.

Then by the weakly compatibility of T and f we obtain

(3.1) $$Tu = Tfx_{n+1} = fTx_{n+1} = fu.$$

Therefore, $Tfu = fTu$ for $u \in X$. which shows that T and f commute at their coincidence point u.

Now we claim that $d(Tu,Tu) = 0.$

By using condition (iii) of Theorem 3.1

$$
\begin{aligned}
d(Tu,Tu) \le\ & a\varphi\big(d(fu, fu)\big) + b\varphi\big(\max\{d(fu, fu), d(fu,Tu)\}\big) + \\
& c\varphi\left(d(fu, fu)\frac{\left(1+\sqrt{d(fu, fu)d(fu,Tu)}\right)^2}{\big(1+(fu, fu)\big)^2} \right) \\
=\ & a\varphi\big(d(Tu,Tu)\big) + b\varphi\big(\max\{d(Tu,Tu), d(Tu,Tu)\}\big) + \\
& c\varphi\left(d(Tu,Tu)\frac{\left(1+\sqrt{d(Tu,Tu)d(Tu,Tu)}\right)^2}{\big(1+(Tu,Tu)\big)^2} \right);
\end{aligned}
$$

Since $\varphi(t) \le t$ for all $t \ge 0,$ then we obtain

$$
\begin{aligned}
d(Tu,Tu) &\le ad(Tu,Tu) + bd(Tu,Tu) + cd(Tu,Tu) \\
&\le (a+b+c)d(Tu,Tu).
\end{aligned}
$$

Since $0 \le a+b+c < 1$, so the above inequality is possible if

(3.2) $$d(Tu,Tu) = 0.$$

We claim that $Tu = u.$

$$
\begin{aligned}
d(Tu,u) &= d(Tu,Tx_{n+1}) \\
&\le a\varphi\big(d(fu, fx_{n+1})\big) + b\varphi\big(max\{d(fu,Tu), d(fu, fx_{n+1})\}\big) + \\
&\quad c\varphi\left(d(fu, fx_{n+1})\frac{\left(1+\sqrt{d(fu, fx_{n+1})d(fu,Tu)}\right)^2}{\big(1+(fu, fx_{n+1})\big)^2} \right) \\
&= a\varphi\big(d(Tu,u)\big) + b\varphi\big(\max\{d(Tu,Tu), d(Tu,u)\}\big) + \\
&\quad c\varphi\left(d(Tu,u)\frac{\left(1+\sqrt{d(Tu,u)d(Tu,Tu)}\right)^2}{\big(1+(Tu,u)\big)^2} \right);
\end{aligned}
$$

Since $d(Tu,Tu)=0,$ then $\max\{d(Tu,Tu),d(Tu,u)\}$ is $d(Tu,u)$ and

$$\left(d(Tu,u)\frac{\left(1+\sqrt{d\left(Tu,u\right)d\left(Tu,Tu\right)}\right)^2}{\left(1+\left(Tu,u\right)\right)^2} \right) \le d\left(Tu,u\right).$$

Thus,

$$d\left(Tu,u\right) \le a\varphi\left(d\left(Tu,u\right)\right)+b\varphi\left(d(Tu,u)\right)+c\varphi\left(d(Tu,u)\right).$$

Since $\varphi\left(t\right)\le t$ for all $t\ge 0,$ then we obtain

$$\begin{aligned} d\left(Tu,u\right) &\le ad\left(Tu,u\right)+bd\left(Tu,u\right)+cd\left(Tu,u\right) \\ &\le \left(a+b+c\right)d\left(Tu,u\right). \end{aligned}$$

Since $0\le a+b+c<1,$ so the above inequality is possible if

(3.3) $$d(Tu,u)=0.$$

Similarly,

$$\begin{aligned} d\left(u,Tu\right) &= d\left(Tx_{n+1},Tu\right) \\ &\le a\varphi\left(d\left(fx_{n+1}, fu\right)\right)+b\varphi\left(\max\left\{d\left(fx_{n+1}, fu\right),d\left(fx_{n+1},Tx_{n+1}\right)\right\}\right)+ \\ &\quad c\varphi\left(d\left(fx_{n+1}, fu\right)\frac{\left(1+\sqrt{dfx_{n+1}, fu)d\left(fx_{n+1},Tx_{n+1}\right)}\right)^2}{\left(1+d\left(fx_{n+1}, fu\right)\right)^2} \right) \\ &= a\varphi(d\left(u,Tu\right))+b\varphi(\max\{d\left(u,Tu\right)d(u,u)\}+ \\ &\quad c\varphi\left(d\left(u,Tu\right)\frac{\left(1+\sqrt{d\left(u,Tu\right)d\left(u,u\right)}\right)^2}{\left(1+\left(u,Tu\right)\right)^2} \right); \end{aligned}$$

Since $d\left(u,u\right)=0,$ then $\max\{d\left(u,Tu\right),d(u,u)\}$ is $d\left(u,Tu\right)$ and

$$d\left(u,Tu\right)\frac{\left(1+\sqrt{d\left(u,Tu\right)d\left(u,u\right)}\right)^2}{\left(1+\left(u,Tu\right)\right)^2} \le d\left(u,Tu\right).$$

Thus,

$$d(u,Tu) \le a\varphi\big(d(u,Tu)\big) + b\varphi\big(d(u,Tu)\big) + c\varphi\big(d(u,Tu)\big).$$

Since $\varphi(t) \le t$ for all $t \ge 0,$ then we obtain

$$\begin{aligned} d(u,Tu) &\le ad(u,Tu) + bd(u,Tu) + cd(u,Tu) \\ &\le (a+b+c)d(u,Tu). \end{aligned}$$

Since $0 \le a+b+c < 1,$ so the above inequality is possible if

(3.4) $$d(u,Tu) = 0.$$

From (3.3) and (3.4), we obtain $Tu = u.$

By (3.1) $Tu = fu = u.$

Therefore, u is a common fixed point of T and f.

Now we prove the uniqueness of the common fixed point.

Suppose u and v are two distinct fixed points of T and f.

That means $Tu = fu = u$ and $Tv = fv = v.$

By using condition (iii) of Theorem 3.1

$$\begin{aligned} d(u,v) &= d(Tu,Tv) \\ &\le a\varphi\big(d(fu, fv)\big) + b\varphi\big(max\{d(fu, fv), d(fu,Tu)\}\big) + \\ &\quad c\varphi\left(d(fu, fv)\frac{\left(1+\sqrt{d(fu, fv)d(fu,Tu)}\right)^2}{\big(1+(fu, fv)\big)^2} \right) \\ &= a\varphi\big(d(u,v)\big) + b\varphi\big(\max\{d(u,v), d(u,u)\}\big) + \\ &\quad c\varphi\left(d(u,v)\frac{\left(1+\sqrt{d(u,v)d(u,u)}\right)^2}{\big(1+d(u,v)\big)^2} \right); \end{aligned}$$

Since $d(u,u)=0,$ then $\max\{d(u,v),d(u,u)\}=d(u,v)$ and

$$d(u,v)\frac{\left(1+\sqrt{d(u,v)d(u,u)}\right)^2}{\left(1+d(u,v)\right)^2}\le d(u,v).$$

Thus,

$$d(u,v)\le a\varphi\left(d(u,v)\right)+b\varphi\left(d(u,v)\right)+c\varphi\left(d(u,v)\right).$$

Since $\varphi(t)\le t$ for all $t\ge 0,$ then we obtain

$$\begin{aligned} d(u,v) &\le ad(u,v)+bd(u,v)+cd(u,v) \\ &\le (a+b+c)d(u,v). \end{aligned}$$

Since $0\le a+b+c<1,$ so the above inequality is possible if

(3.5) $$d(u,v)=0.$$

Similarly,

$$\begin{aligned} d(v,u) &= d(Tv,Tu) \\ &\le a\varphi(d(fv,fu))+b\varphi(\max\{d(fv,fu),d(fv,Tv)\})+ \\ &\quad c\varphi\left(d(fv,fu)\frac{\left(1+\sqrt{d(fv,fu)d(fv,Tv)}\right)^2}{\left(1+d(fv,fu)\right)^2}\right) \\ &\le a\varphi(d(v,u))+b\varphi(\max\{d(v,u),d(v,v)\})+ \\ &\quad c\varphi\left(d(v,u)\frac{\left(1+\sqrt{d(v,u)d(v,v)}\right)^2}{\left(1+d(v,u)\right)^2}\right); \end{aligned}$$

Since $d(v,v)=0,$ then $\max\{d(v,u),d(v,v)\}=d(v,u)$ and

$$d(v,u)\frac{\left(1+\sqrt{d(v,u)d(v,v)}\right)^2}{\left(1+d(v,u)\right)^2}\le d(v,u).$$

Thus,

$$d(v,u) \le a\varphi(d(v,u)) + b\varphi(d(v,u)) + c\varphi(d(v,u)).$$

Since $\varphi(t) \le t$ for all $t \ge 0$, then we obtain

$$\begin{aligned} d(v,u) &\le ad(v,u) + bd(v,u) + cd(v,u) \\ &\le (a+b+c)d(v,u). \end{aligned}$$

Since $0 \le a+b+c < 1$, so the above inequality is possible if

(3.6) $$d(v,u) = 0.$$

From (3.5) and (3.6), we get $u = v$.

Hence, u is a unique common fixed point of T and f.

Case (ii): Suppose $y_n \ne y_{n+1}$ for each $n \in \{0,1,2,\ldots\}$.

$$\begin{aligned} d(y_n, y_{n+1}) &= d(Tx_n, Tx_{n+1}) \\ &\le a\varphi(d(fx_n, fx_{n+1})) + b\varphi(\max\{d(fx_n, fx_{n+1}), d(fx_n, Tx_n)\}) + \\ &\quad c\varphi\left(d(fx_n, fx_{n+1}) \frac{\left(1+\sqrt{d(fx_n, fx_{n+1})d(fx_n, Tx_n)}\right)^2}{\left(1+d(fx_n, fx_{n+1})\right)^2} \right) \\ &= a\varphi(d(y_{n-1}, y_n)) + b\varphi(\max\{d(y_{n-1}, y_n), d(y_{n-1}, y_n)\}) + \\ &\quad c\varphi\left(d(y_{n-1}, y_n) \frac{\left(1+\sqrt{d(y_{n-1}, y_n)d(y_{n-1}, y_n)}\right)^2}{\left(1+d(y_{n-1}, y_n)\right)^2} \right); \end{aligned}$$

Since $\varphi(t) \le t$ for all $t \ge 0$, then we obtain

$$\begin{aligned} d(y_n, y_{n+1}) &\le ad(y_{n-1}, y_n) + bd(y_{n-1}, y_n) + cd(y_{n-1}, y_n) \\ &\le (a+b+c)d(y_{n-1}, y_n); \end{aligned}$$

Let $h = a+b+c$.

(3.7) $$d(y_n, y_{n+1}) \le hd(y_{n-1}, y_n).$$

Since, $0 \le h < 1.$ we obtain

$d\left(y_{n-1}, y_n\right) \le hd\left(y_{n-2}, y_{n-1}\right).$

Similarly,

$d(y_n, y_{n+1}) \le h^2 d(y_{n-2}, y_{n-1}).$

Now, continuing this process we get

$d(y_n, y_{n+1}) \le h^n d\left(y_{0,} y_1\right).$

Since, $0 \le h < 1$ we have

$\lim_{n\to\infty} h^n d\left(y_{0,} y_1\right) = 0.$

Thus,

(3.8) $$\lim_{n\to\infty} d\left(y_n, y_{n+1}\right) = 0.$$

Similarly, we can show

(3.9) $$\lim_{n\to\infty} d\left(y_{n+1}, y_n\right) = 0.$$

Now we can show $\{y_n\}$ is a Cauchy sequence in X.

Let $m, n \in \mathbb{N}$ with $m > n.$

Applying triangular inequality

$$\begin{aligned} d(y_n, y_m) &\le d(y_n, y_{n+1}) + d\left(y_{n+1}, y_m\right) \\ &\le d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \dots + d\left(y_{m-1}, y_m\right) \\ &\le h^n d(y_0, y_1) + h^{n+1} d\left(y_0, y_1\right) + \dots + h^{m-1} d\left(y_0, y_1\right) \\ &\le h^n \left(1 + h + \ldots + h^{m-n-1}\right) d\left(y_0, y_1\right) \\ &= h^n \left(\sum_{\mathrm{i}=0}^{\mathrm{m-n-1}} h^{\mathrm{i}}\right) d\left(y_0, y_1\right) \\ &\le h^n \left(\sum_{\mathrm{i}=0}^{\infty} h^{\mathrm{i}}\right) d\left(y_0, y_1\right) \\ &= \frac{h^n}{1-h} d\left(y_0, y_1\right); \end{aligned}$$

Thus,

$$d(y_n, y_m) \le \frac{h^n}{1-h} d\left(y_0, y_1\right).$$

Since $0 \le h < 1$, then $\lim_{h \to 0} \frac{h^n}{1-h} d\left(y_0, y_1\right) = 0.$

This implies,

(3.10) $$\lim_{n,m \to \infty} d\left(y_n, y_m\right) = 0.$$

Let $m, n \in \mathbb{N}$ with $n > m.$

By similar line of proof we obtain.

(3.11) $$\lim_{n,m \to \infty} d\left(y_m, y_n\right) = 0.$$

Hence from (3.10) and (3.11) we have

$$\lim_{m,n \to \infty} d\left(y_n, y_m\right) = \lim_{m,n \to \infty} d\left(y_m, y_n\right) = 0.$$

Thus, the sequence $\{y_n\}$ is a Cauchy sequence in X for $n \in \{0, 1, 2, \ldots\}.$

Since, X is complete there exists $z \in X$, such that $\lim_{n \to \infty} y_n = z.$

Thus, $\lim_{n \to \infty} Tx_n = \lim_{n \to \infty} fx_{n+1} = z.$

Since fX is a closed subset of X, there exist $u \in fX$ such that $z = fu$

Now we show that $Tu = z.$

$$\begin{aligned} d\left(Tu, z\right) &= d(Tu, Tx_n) \\ &\le a\varphi(d\left(fu, fx_n\right)) + b\varphi(max d\left(fu, fx_n\right), \left(fu, Tu\right)\}) \\ &\quad + c\varphi\left(d\left(fu, fx_n\right)\left(\frac{\left(1+\sqrt{d\left(fu, fx_n\right) d\left(fu, Tu\right)}\right)^2}{\left(1+d\left(fu, fx_n\right)\right)^2}\right)\right); \end{aligned}$$

Since $Tu = fu$, then

$$d(Tu,z) \leq a\varphi(d(Tu, fx_n)) + b\varphi(\max\{d(Tu, fx_n), d(Tu,Tu)\}) + c\varphi\left(d(Tu, fx_n)\left(\frac{\left(1+\sqrt{d(Tu, fx_n)\,d(Tu,Tu)}\right)^2}{\left(1+d(Tu, fx_n)\right)^2}\right)\right);$$

Letting $n \to \infty$, then $\lim_{n\to\infty} fx_n = z$.

$$d(Tu,z) \leq a\varphi(d(Tu,z)) + b\varphi(\max\{d(Tu,z), d(Tu,Tu)\}) + c\varphi\left(d(Tu,z)\left(\frac{\left(1+\sqrt{d(Tu,z)\,d(Tu,Tu)}\right)^2}{\left(1+d(Tu,z)\right)^2}\right)\right).$$

Since by (3.2) $d(Tu,Tu) = 0$, then $\max\{d(Tu,z), d(Tu,Tu)\} = d(Tu,z)$ and

$$\left(d(Tu,z)\frac{\left(1+\sqrt{d(Tu,z)\,d(Tu,Tu)}\right)^2}{\left(1+d(Tu,z)\right)^2}\right) \leq d(Tu,z).$$

Thus,

$$d(Tu,z) \leq a\varphi(d(Tu,z)) + b\varphi(d(Tu,z)) + c\varphi(d(Tu,z)).$$

Since $\varphi(t) \leq t$ for all $t \geq 0$, then we obtain

$$\begin{aligned} d(Tu,z) &\leq ad(Tu,z) + bd(Tu,z) + cd(Tu,z) \\ &\leq (a+b+c)\,d(Tu,z). \end{aligned}$$

Since $0 \leq a+b+c < 1$, so the above inequality is possible if

(3.12) $$d(Tu,z) = 0.$$

Similarly,

$$\begin{aligned} d(z,Tu) &= d(Tx_n,Tu) \\ &\le a\varphi(d(fx_{n,}fu)) + b\varphi(\max\{d(fx_{n,}fu), d(fx_{n,}Tx_n)\}) + \\ &+ c\varphi\left(d(fx_n, fu)\frac{\left(1+\sqrt{d(fx_{n,}fu)d(fx_{n,}Tx_n))}\right)^2}{\left(1+d(fx_n, fu)\right)^2} \right); \end{aligned}$$

Since $Tu = fu,$ then

$$\begin{aligned} d(z,Tu) \le a\varphi(d(fx_n,Tu)) + b\varphi(\max\{d(fx_n,Tu), d(fx_n,Tx_n)\}) + \\ + c\varphi\left(d(fx_n,Tu)\frac{\left(1+\sqrt{d(fx_n,Tu)d(fx_n,Tx_n)}\right)^2}{\left(1+d(fx_n,Tu)\right)^2} \right); \end{aligned}$$

Letting $n \to \infty,$ then $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} fx_n = z.$

$$\begin{aligned} d(z,Tu) \le a\varphi(d(z,Tu)) + b\varphi(\max\{d(z,Tu), d(z,z)\}) + \\ + c\varphi\left(d(z,Tu)\frac{\left(1+\sqrt{d(z,Tu)d(z,z)}\right)^2}{\left(1+d(z,Tu)\right)^2} \right); \end{aligned}$$

Since $d(z,z) = 0,$ then $\max\{d(z,Tu), d(z,z)\}$ is $d(z,Tu)$ and

$$\left(d(z,Tu)\frac{\left(1+\sqrt{d(z,Tu)d(z,z)}\right)^2}{\left(1+d(z,Tu)\right)^2} \right) \le d(z,Tu).$$

Thus,

$$d(z,Tu) \le a\varphi(d(z,Tu)) + b\varphi(d(z,Tu)) + c\varphi(d(z,Tu)).$$

Since $\varphi(t) \le t$ for all $t \ge 0,$ then we obtain

$$\begin{aligned} d(z,Tu) &\le ad(z,Tu) + bd(z,Tu) + cd(z,Tu) \\ &\le (a+b+c)d(z,Tu). \end{aligned}$$

Since $0 \le a+b+c<1,$ so the above inequality is possible if

(3.13) $$d(z,Tu)=0.$$

Using (3.12) and (3.13) we have $z=Tu.$

Then, $z=Tu=fu.$

By the weakly compatibility of T and f, we have $Tfu=fTu.$

Then $Tz=Tfu=fTu=fz.$

Which implies that z is a coincidence point of T and f.

Consider

$$\begin{aligned} d(Tz,z) &= d(Tz,Tu) \\ &\le a\varphi(d(fz,fu))+b\varphi(\max\{d(fz,fu),d(fz,Tz)\})+ \\ &\quad c\varphi\left(d(fz,fu)\left(\frac{\left(1+\sqrt{d(fz,fu)d(fz,Tz))}\right)^2}{\left(1+d(fz,fu)\right)^2}\right)\right) \\ &= a\varphi(d(Tz,z))+b\varphi(\max\{d(Tz,z),d(Tz,Tz)\})+ \\ &\quad c\varphi\left(d(Tz,z)\left(\frac{\left(1+\sqrt{d(Tz,z)d(Tz,Tz))}\right)^2}{\left(1+d(Tz,z)\right)^2}\right)\right). \end{aligned}$$

Since $\varphi(t)\le t,$ $\max\{d(Tz,z),d(Tz,Tz)\}$ is $d(Tz,z)$ and

$$d(Tz,z)\left(\frac{\left(1+\sqrt{d(Tz,z)d(Tz,Tz))}\right)^2}{\left(1+d(Tz,z)\right)^2}\right)\le d(Tz,z).$$

We get

$$d(Tz,z)\le(a+b+c)\,d(Tz,z).$$

Since $0 \le a+b+c<1,$ so the above inequality is possible if

$$(3.14) \qquad d\left(Tz,z\right)=0.$$

Similarly,

$$\begin{aligned} d\left(z,Tz\right) &= d(Tu,Tz) \\ &\le a\varphi(d\left(fu, fz\right))+b\varphi(\max\left\{d\left(fu, fz\right), d\left(fu,Tu\right)\right\})+ \\ &\quad c\varphi\left(d\left(fu, fz\right)\left(\frac{\left(1+\sqrt{d\left(fu, fz\right)d\left(fu,Tu\right)}\right)^2}{\left(1+d\left(fu, fz\right)\right)^2}\right)\right) \\ &= a\varphi(d\left(z,Tz\right))+b\varphi(\max\left\{d\left(z,Tz\right), d\left(z,z\right)\right\})+ \\ &\quad c\varphi\left(d\left(z,Tz\right)\left(\frac{\left(1+\sqrt{d\left(z,Tz\right)d\left(z,z\right)}\right)^2}{\left(1+d\left(z,Tz\right)\right)^2}\right)\right); \end{aligned}$$

Since, $\max\left\{d\left(z,Tz\right)d\left(z,z\right)\right\}$ is $d\left(z,Tz\right)$ and $\varphi\left(t\right)\le t.$

Then we have

$$\begin{aligned} d\left(z,Tz\right) &\le ad\left(z,Tz\right)+bd\left(z,Tz\right)+cd\left(z,Tz\right) \\ &\le \left(a+b+c\right)d\left(z,Tz\right). \end{aligned}$$

Since $0 \le a+b+c<1,$ so the above inequality is possible if

$$(3.15) \qquad d\left(z,Tz\right)=0.$$

So, from (3.14) and (3.15) we have $Tz = z.$

Thus, $Tz = fz = z.$

Therefore, z is a common fixed point of T and f.

Now we show the uniqueness of common fixed point of T and f in X.

Let w be another common fixed point of T and f in X. That is $Tw = fw = w.$

Consider

$$\begin{aligned} d(z,w) &= d(Tz,Tw) \\ &\le a\varphi(d(fz, fw)) + b\varphi(\max\{d(fz, fw), d(fz,Tz)\}) + \\ &\quad c\varphi\left(d(fz, fw)\left(d(fz, fw)\frac{\left(1+\sqrt{d(fz, fw)d(fz,Tz))}\right)^2}{\left(1+d(fz, fw)\right)^2}\right)\right) \\ &= a\varphi(d(z,w)) + b\varphi(\max\{d(z,w), d(z,z)\}) + \\ &\quad c\varphi\left(d(z,w)\left(d(z,w)\frac{\left(1+\sqrt{d(z,w)d(z,z))}\right)^2}{\left(1+d(z,w)\right)^2}\right)\right); \end{aligned}$$

Since $max\{d(z,w), d(z,z)\}$ is $d(z,w)$ and $\varphi(t) \le t$, then the above inequality becomes

$$d(z,w) \le (a+b+c)\,d(z,w).$$

Since $0 \le a+b+c < 1$, so the above inequality is possible if

(3.16) $$d(z,w) = 0.$$

Using similar line of proof

$$\begin{aligned} d(w,z) &= d(Tw,Tz) \\ &\le a\varphi(d(fw, fz)) + b\varphi(\max\{d(fw, fz), d(fw,Tw)\} + \\ &\quad c\varphi\left(d(fw, fz)\left(\frac{\left(1+\sqrt{d(fw, fz)d(fw,Tw))}\right)^2}{\left(1+d(fw, fz)\right)^2}\right)\right) \\ &= a\varphi(d(w,z)) + b\varphi(\max\{d(w,z), d(w,w)\}) + \\ &\quad c\varphi\left(d(w,z)\left(\frac{\left(1+\sqrt{d(w,z)d(w,w))}\right)^2}{\left(1+d(w,z)\right)^2}\right)\right) \end{aligned}$$

Since $\varphi(t) \le t$ and $max\{d(w,z), d(w,w)\}$ is $d(w,z)$, then the above inequality becomes

$$d(w,z) \le ad(w,z) + bd(w,z) + cd(w,z)$$
$$\le (a+b+c)d(w,z).$$

Since $0 \le a+b+c < 1$, so the above inequality is possible if

(3.17) $$d(w,z) = 0.$$

From (3.16) and (3.17), we conclude that $w = z$.

So, z is a unique common fixed point of T and f in X.

The following is an example in support of Theorem 3.1.

Example 4.2: Let $X = [0,4]$, define $d : X \times X \to [0,\infty)$ by

$d(x,y) = x^2 + y$ and $T, f : X \to X$ by

$$T(x) = \begin{cases} 0\ , & if\ 0 \le x \le 3 \\ \frac{1}{4}\ , & if\ 3 < x \le 4 \end{cases} \text{ and } f(x) = \begin{cases} \frac{x}{3}, & if\ 0 \le x \le 3 \\ 4\ , & if\ 3 < x \le 4 \end{cases}$$

If $a = \frac{1}{2}, b = \frac{1}{5}, c = \frac{1}{5}$ and $\varphi(t) = \frac{3}{4}t$, for all $t \ge 0.$

In fact both properties of Definition 2.2 holds true.

$i) d(x,y) = x^2 + y = y^2 + x = d(y,x) = 0$ implies $x = y = 0.$

$ii) d(x,y) = x^2 + y \le x^2 + z + z^2 + y = d(x,z) + d(z,y)$.

We observe that (X,d) is a dq-metric space and also $T0 = f0 = 0.$

This implies 0 is a coincidence point of T and f. Moreover $fX = \{4\} \cup [0,1]$ and $TX = \left\{0, \frac{1}{4}\right\}.$

Hence, $TX \subseteq fX$ and also $T0 = Tf0 = fT0 = f0 = 0,$ then f and T are weakly compatible.

To see contractive condition (iii) of Theorem 3.1 we consider the following cases.

Case (i): Suppose $x, y \in [0,3]$. Then,

$$d(Tx,Ty) = d(0,0) = 0.$$

$$d(fx, fy) = d\left(\frac{x}{3}, \frac{y}{3}\right) = \left(\frac{x}{3}\right)^2 + \frac{y}{3} = \frac{x^2}{9} + \frac{y}{3} = \frac{x^2+3y}{9}.$$

$$d(fx, Tx) = d\left(\frac{x}{3}, 0\right) = \frac{x^2}{9}.$$

Then by contractive condition (iii) of Theorem 3.1 we have

$$d(Tx,Ty) \le a\varphi(d(fx, fy)) + b\varphi(\max\{d(fx, fy), d(fx, Tx)\}) + c\varphi\left(d(fx, fy)\frac{\left(1+\sqrt{d(fx, fy)d(fx,Tx)}\right)^2}{\left(1+d(fx, fy)\right)^2}\right).$$

This implies $0 \le \frac{3}{8}\left(\frac{x^2+3y}{9}\right) + \frac{3}{20}\left(\max\left\{\frac{x^2+3y}{9}, \frac{x^2}{9}\right\}\right) + \frac{3}{20}\left(\frac{x^2+3y}{9}\right).$

Always true for all $x, y \in [0,3)$.

Case (ii): Suppose $x, y \in (3,4]$. In this case,

$$d(Tx,Ty) = d\left(\frac{1}{4}, \frac{1}{4}\right) = \left(\frac{1}{4}\right)^2 + \frac{1}{4} = \frac{5}{16}$$

$$d(fx, fy) = d(4,4) = (4)^2 + 4 = 20$$

$$d(fx, Tx) = d\left(4, \frac{1}{4}\right) = \frac{65}{4}$$

$$d(Tx,Ty) \le a\varphi(d(fx,fy)) + b\varphi(\max\{d(fx,fy), d(fx,Tx)\}) + c\varphi\left(d(fx,fy)\frac{\left(1+\sqrt{d(fx,fy)d(fx,Tx)}\right)^2}{\left(1+d(fx,fy)\right)^2}\right).$$

$$\frac{5}{16} \le \frac{3}{8}(20) + \frac{3}{20}\left(\max\left\{20, \frac{65}{4}\right\}\right) + \frac{3}{20}\left(\left(\frac{65}{4}\right)\frac{\left(1+\sqrt{20\left(\frac{65}{4}\right)}\right)^2}{\left(1+\frac{65}{4}\right)^2}\right).$$ True for all $x, y \in (3,4]$.

Case (iii): If $x \in [0,3]$ and $y \in (3,4]$, then

$$d(Tx,Ty) = d\left(0, \frac{1}{4}\right) = \frac{1}{4}.$$

$$d(fx,fy) = d\left(\frac{x}{3}, 4\right) = \left(\frac{x}{3}\right)^2 + 4 = \frac{x^2}{9} + 4 = \frac{x^2+36}{9}.$$

$$d(fx,Tx) = d\left(\frac{x}{3}, 0\right) = \left(\frac{x}{3}\right)^2 = \frac{x^2}{9}.$$

$$d(Tx,Ty) \le a\varphi\left(d(fx,fy)\right) + b\varphi(\max\{d(fx,fy), d(fx,Tx)\}) + c\varphi\left(d(fx,fy)\frac{\left(1+\sqrt{d(fx,fy)d(fx,Tx)}\right)^2}{\left(1+d(fx,fy)\right)^2}\right).$$

$$\frac{1}{4} \le \frac{3}{8}\left(\frac{x^2+36}{9}\right) + \frac{3}{20}\left(\frac{x^2+36}{9}\right) + \frac{3}{20}\left(\left(\frac{x^2+36}{9}\right)\frac{\left(1+\sqrt{\left(\frac{x^2}{9}\right)\frac{x^2+36}{9}}\right)^2}{\left(1+\frac{x^2+36}{9}\right)^2}\right).$$

True for all $x \in [0,3]$ and $y \in (3,4]$.

Case (iv): Suppose $x \in (3,4]$ and $y \in [0,3]$, then we have

$$d(Tx,Ty) = d\left(\frac{1}{4},0\right) = \frac{1}{16}.$$

$$d(fx,fy) = d\left(4,\frac{y}{3}\right) = 16 + \frac{y}{3}.$$

$$d(fx,Tx) = d\left(4,\frac{1}{4}\right) = 4^2 + \frac{1}{4} = \frac{65}{4}.$$

$$d(Tx,Ty) \le a(\varphi d(fx,fy)) + b\varphi(\max\{d(fx,fy),d(fx,Tx)\}) + c\varphi\left(d(fx,fy)\frac{\left(1+\sqrt{d(fx,fy)d(fx,Tx)}\right)^2}{\left(1+d(fx,fy)\right)^2}\right).$$

$$\frac{1}{16} \le \frac{3}{8}(16+\frac{1}{3}y) + \frac{3}{20}\left(16+\frac{1}{3}\mathrm{y}\right) + \frac{3}{20}\left(\left(16+\frac{1}{3}\mathrm{y}\right)\frac{\left(1+\sqrt{\left(16+\frac{1}{3}y\right)\frac{65}{4}}\right)^2}{\left(1+16+\frac{1}{3}y\right)^2}\right).$$

True for all $x \in (3,4]$ and $y \in [0,3]$.

From cases (i)–(iv) all the conditions of Theorem 3.1 are satisfied and 0 is the unique common fixed point of f and T.

Theorem 3.2. Let (X,d) be a complete dq-metric space. $T, f : X \to X$ be continuous self-mappings satisfying the contractive condition of Theorem 3.1. Then f and T have a unique common fixed point.

Proof

Following as in the proof of Theorem 3.1 we construct a sequence $\{y_n\}$. Let $\{x_{2n}\}$ and $\{x_{2n+1}\}$ be subsequences of the sequence $\{y_n\}$. As in the Theorem 3.1we define $x_{2n+1} = Tx_{2n}$ and $x_{2n} = fx_{2n-1}$.

Similarly, as shown in the proof of Theorem 3.1we can show that the sequence $\{y_n\}$ is a Cauchy sequence.

By the completeness of X one can show that $\lim_{n\to\infty} y_n = u$ for some $u \in X$.

Since $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are subsequences of $\{y_n\}$, then $\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} x_{2n+1} = u$.

Next since, f and T are continuous we arrive at

$$Tu = T \lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} x_{2n+1} = u.$$

Then

$$Tu = u. \tag{3.18}$$

Similarly,

$$fu = f \lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} fx_{2n-1} = \lim_{n\to\infty} x_{2n} = u.$$

Then

$$fu = u. \tag{3.19}$$

So, from (3.18) and (3.19) we get $Tu = u = fu$.

Therefore u is common fixed point of f and T.

Now we want to show uniqueness of the common fixed point.

Let q be another common fixed point of f and T. That is $fq = Tq = q$.

Then by the contractive condition (iii) of Theorem 3.1 we have

$$\begin{aligned} d(u,q) &= d(Tu,Tq) \\ &\le a\varphi(d(fu,fq)) + b\varphi(\max\{d(fu,fq), d(fu,Tu)\}) + \\ &\quad c\varphi\left(d(fu,fq)\frac{\left(1+\sqrt{d(fu,fq)d(fu,Tu)}\right)^2}{\left(1+d(fu,fq)\right)^2} \right) \\ &\le a\varphi(d(u,q)) + b\varphi(\max\{d(u,q), d(u,u)\}) + \\ &\quad c\varphi\left(d(u,q)\frac{\left(1+\sqrt{d(u,q)d(u,u)}\right)^2}{\left(1+d(u,q)\right)^2} \right). \end{aligned}$$

Since, $\max\{d(u,q), d(u,u)\}$ is $d(u,q)$ and $\varphi(t) \le t$ for all $t \ge 0$.

Then the above inequality becomes

$$\begin{aligned} d(u,q) &\le ad(u,q) + bd(u,q) + cd(u,q) \\ &\le (a+b+c)d(u,q). \end{aligned}$$

Since $0 \le a+b+c < 1$, so the above inequality is possible if

(3.20) $$d(u,q) = 0.$$

Similarly,

$$\begin{aligned} d(q,u) &= d(Tq,Tu) \\ &\le a\varphi(d(fq,fu)) + b\varphi(\max\{d(fq,fu), d(fq,Tq)\} + \\ &\quad c\varphi\left(d(fq,fu)\frac{\left(1+\sqrt{d(fq,fu)d(fq,Tq)}\right)^2}{\left(1+d(fq,fu)\right)^2} \right) \\ &\le a\varphi(d(q,u)) + b\varphi(\max\{d(q,u), d(q,q)\}) + \\ &\quad c\varphi\left(d(q,u)\frac{\left(1+\sqrt{d(q,u)d(q,q)}\right)^2}{\left(1+d(q,u)\right)^2} \right). \end{aligned}$$

Since, $max\{d(q,u), d(q,q)\}$ is $d(q,u)$ and $\varphi(t) \le t$ for all $t \ge 0$.

Then the above inequality becomes

$$(q,u) \le ad(q,u) + \mathrm{b}d(q,u) + \mathrm{c}d(q,u)$$
$$\le (a+\mathrm{b}+\mathrm{c})d(q,u).$$

Since $0 \le a+b+c<1$, so the above inequality is possible if

(3.21) $$(q,u)=0.$$

So, from (3.20) and (3.21) we obtain $u=q$.

Thus u is a unique common fixed point of f and T.

The following is an example in support of Theorem 3.2.

Example 2.2: Let $X=[0,1]$ define $d: X\times X \to [0,\infty)$ by

$d(x,y)=x$ for all $x,y\in X$.

And also define $f,T: X\to X$ by

$Tx=\frac{x}{9},\ fx=\frac{8x}{9},$ and $\varphi(t)=\frac{4}{5}t.$

Let $a=\frac{3}{4}, b=\frac{1}{6}$ and $c=\frac{1}{16}.$

$$d(Tx,Ty)=d(\frac{x}{9},\frac{y}{9})=\frac{x}{9}.$$

$$d(fx,fy)=d\left(\frac{8x}{9},\frac{8y}{9}\right)=\frac{8x}{9}.$$

$$d(fx,Tx)=d(\frac{8x}{9},\frac{8x}{9})=\frac{8x}{9}.$$

From the contractive condition (iii) of the Theorem 3.1 we have

$$d\left(Tx,Ty\right)\le a\varphi(d\left(fx,fy\right))+b\varphi(\max\left\{d\left(fx,fy\right),d\left(fx,Tx\right)\right\})+$$
$$c\varphi\left(d\left(fx,fy\right)\frac{\left(1+\sqrt{d\left(fx,fy\right)d\left(fx,Tx\right)}\right)^2}{\left(1+d\left(fx,fy\right)\right)^2}\right);$$

$$\frac{x}{9}\le\frac{12}{20}\left(\frac{8x}{9}\right)+\frac{4}{30}\left(\max\{\frac{8x}{9},\frac{8x}{9}\}\right)+\frac{4}{80}\left(\left(\frac{8x}{9}\right)\frac{\left(1+\sqrt{(\frac{8x}{9})(\frac{8x}{9})}\right)^2}{\left(1+\frac{8x}{9}\right)^2}\right).$$

$$\frac{x}{9}\le\frac{24x}{45}+\frac{16x}{135}+\frac{2x}{45}.$$

Which, is true for all $x\in\left[0,1\right]$.

f and T satisfies all the conditions of the Theorem 3.2 and f and T have a unique fixed point.

$T0=f0=0.$

Therefore 0 is the unique common fixed point of f and T.

Remark 1: For $f=I$ (I identity map on X) from contractive condition of Theorem 3.1 we get

$$d\left(Tx,Ty\right)\le a\varphi\left(d\left(x,y\right)\right)+b\varphi\left(\max\left\{d\left(x,y\right),d\left(x,Tx\right)\right\}\right)+$$
$$c\varphi\left(d\left(x,y\right)\frac{\left(1+\sqrt{d\left(x,y\right)d\left(x,Tx\right)}\right)^2}{\left(1+d\left(x,y\right)\right)^2}\right).$$

Whenever, $f=I$ contractive condition of Theorem 3.1 is simplified to Theorem 2.11.

Hence, Theorem 2.11 follows as a corollary to Theorem 3.1.

Conflict of Interests

The authors declare that there is no conflict of interests.

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