# COMMONFIXED POINT THEOREMS INVOLVING CONTRACTIVE CONDITIONS OF RATIONAL TYPE IN DISLOCATED QUASI METRIC SPACES

**A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILMENT FOR REQUIREMETS OF THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS**


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## DECLARATION

I undersigned declare that, the research entitled ‘‘Common fixed point theorems involving contractive condition of rational type in dislocated quasi metric space” is original and it has not been submitted to any institution elsewhere for the award of any degree or like, where other sources of information that have been used, they have been acknowledged.

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## ABSTRACT

The aim of this research work was to establishsome common fixed point results in complete dislocated quasi metric spaces. In this study we established common fixed point theorems for a pair of self- maps involving contractive condition of rational type in complete dislocated quasi metric spaces by extending the work of Rahman and Sarwar. Our resultextend and improves the result of Rahman and Sarwar. In this research undertaking, we followed analytical study design. Secondary sources of data such as journal, internet and books were used for the study. The analysis techniques which we adopted for the successful completion of this study were that of Rahman and Sarwar. We provided examples in support of our main findings.

## CHAPETER ONE

## INTRODUCTION

### 1.1 Background of the study

Fixed point theory is one of the most important topics in the development of nonlinear functional analysis. It is the mixture of analysis, topology and geometry.

**Definition 1.1:** Let $X$ be a non-empty set. A map $T: \mathrm{X} \rightarrow \mathrm{X}$ is said to be a self-map ofX if $Tx = x$. An element $x$ in $X$ is called a fixed point ofT.

**Definition 1.2:** Let$(\mathrm{X}, \mathrm{d})$be metric space. A self-map $\mathrm{T}: \mathrm{X} \rightarrow \mathrm{X}$ is said to be a contraction map if there exists a constant $k \in [0,1)$ such that

$d(Tx, Ty) \leq kd(x, y)$forall $x, y \in X$, where k is contraction constant.

The fixed point theorem in metric spaces plays an essential role to construct method to solve problems in mathematics and other sciences involving certain mapping and space structures required in various areas such as economics, chemistry, biology, computer science, engineering and others (Dugudji, 1982).

In this area the first and the significant result was proved by Banach, (1922) for contraction mapping in metric spaces which is called Banach contraction principle.

**Theorem 1.1:** (Banach Contraction Principle)Let$(X, d)$ be a complete metric space, then each contraction map $T: X \rightarrow X$ has a unique fixed point.

The Banach contraction theorem is importantasa source of existence and uniqueness theorems in different branch of science and the sequence of successive approximation converges to a solution of the problem. The theorem provides an illustration of the unifying power of functional analytic methods and usefulness of fixed point theory in analysis.

Since a number of fixed point theorems for different types of nonlinear contraction mappings have been investigated and proved by various authors and various generalizations of this theorem have been established. Many researchers have obtained fixed point theorems, common fixed point theorems and other fixed point results in metric spaces, cone metric spaces, quasi metric spaces, dislocated metric spaces, dislocated quasi metric spaces and other spaces (Chandoks, 2015). Also Dass and Gupta,(1975) generalized the Banach contraction in metric space for some rational type contractions. Most of the works of Banach contraction theorems involves continuity of self-mappings for different type of contracions. Then a natural question arises whether the continuity mapping is essential for the existence of fixed points. Thisquestion has been affirmatively answered by Kannan, (1968). In 1968,Kannan proved a fixed point theorem for operators that need not be continuous.

**Theorem 1.2:** Kannan, (1968) Let $(X, d\ )$ be a nonempty complete metric space and a self-map $T: X \rightarrow X$ satisfying:

$d\ (Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)]$forall $x, y \in X$,where $\alpha \in [0, \frac{1}{2})$.

Then T has a unique fixed point. Mappings satisfying this inequality are called Kannan type mappings.

Further,Chatterjea,(1972) also proved a fixed point theorem for maps which are actually dual of Kannan type mappings.

**Theorem 1.3:** Chatterjea, (1972) Let $(X, d\ )$be a nonempty complete metric space and a self-map $T$ satisfying:

$d(Tx, Ty) \leq \alpha[d(x, Ty) + d(y, Tx)]$forall $x, y \in X$, where $\alpha \in [0, \frac{1}{2})$.

Then T has a unique fixed point. And the mapping satisfying the above inequality is called Chatterjea type mappings.

Zamfirescu, (1972)proved the following fixed point theorem by combiningtheorems1.1, 1.2 and 1.3 as follows.

**Theorem 1.4:** Zamfirescu (1972)Let $(X, d)$ be a complete metric space and $T: X \rightarrow X$ be a map for which there exist the real numbers $a$ , $b$ and $c$ satisfyinga $\in [0,1)$, $b \in \left[0, \frac{1}{2}\right)$ and $c \in [0, \frac{1}{2})$ such that for each pair $x, y \in X$ at least one of the following hold.

$(Z_1)$ $d(Tx, Ty) \leq a\ d(x, y)$.

$(Z_2)$ $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$.

$(Z_3)$ $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)$.Then $T$ has a unique fixed point.

Zamfirescu's theorem,(1972) is a generalization of Banach,(1922),Kannan,(1968) and Chatterjea,(1972).

In 1977, Rhoades compared various definitions of contractive mapping on a complete metric space which were used to generalize the contraction mapping principle. Due to the wide applications of fixed point theorems in different fields, the study of existence and uniqueness of fixed points and common fixed points have become a subject of great interest.There are a good number of common fixed point theorems available for commuting as well as non-commuting mappings in metric spaces satisfying different contractive conditions.

Jungck, (1976), gave a common fixed point theorem for commuting mappings in metric spaces, which generalized Banach's contractiontheorem.Dass and Gupta, (1975) generalized the Banach contraction principle in metric space for some rational type contractive conditions.

In general, various results pertaining to common fixed point have been investigated for maps satisfying different contractive conditions in different settings, equations such as differential and integral equations.

Kirishna and Deheri, (2015) obtained common fixed points of two continuous mappings in a complete metric spaces. Jha andpanths, (2012) obtained common fixed points of self-maps in dislocated metric spaces. Also Mannoj and Daheriya, (2015) obtained a common fixed point theorem for three maps in a d-metric and dq-metric spaces. Recently, many researchers have obtained fixed point and common fixed point theorems in dislocated quasi metric spaces.

Also Rahaman and Sarwar, (2016) established the following fixed point result in complete dislocated quasi metric space.

**Theorem 1.5:** Let $(X, d)$ be a complete dislocated quasi metric space. And $T: X \to X$ be self-mapping satisfying

$$d(Tx,Ty) \leq a\varphi\big(d(x,y)\big) + b\varphi(\max\{d(x,Tx), d(x,y)\}) + c\varphi\left(\frac{d(x,y)\left[1+\sqrt{d(x,y)d(x,Tx)}\right]^2}{\big(1+d(x,y)\big)^2}\right).$$

for all $x, y \in X$ , $a, b, c \geq 0$ with $a + b + c < 1$ and $\varphi$ is a comparison function. Then T has a unique fixed point.

So this researcher is motivated and inspired by the work of Rahman and Sarwar, (2016) to establish common fixed point theoremsinvolving contractive conditions of rational type for a pair of self-maps in dislocated quasi- metric spaces by extending the result of Rahman and Sarwar, (2016) and to verify the result of the study obtained by particular examples.

## 1.2. Objective of study

### 1.2.1. General Objective

The general objective of this study is to establish common fixed point theorems for a pair of self-maps satisfying certain contractive condition of rational type in dislocated quasi-metric space.

### 1.2.2.Specific objectives

- To prove the existence and uniqueness of common fixed points of self-maps satisfying certain contractive condition of rational type in dislocated quasi-metric spaces.
- To identify additional conditions required to assure uniqueness of common fixed point.
- To verify the applicability of the result using particular examples.

## 1.3. Significance of the Study

- The result obtained may contribute to research activities on the study area.
- It may develop basic research skills of the researcher and also may serve as reference for individuals who have interest to be engaged research activities in this line of research.
- It is applicable in solving some integral and differential equations.

## 1.4. Delimitation of the Study

The study was delimited to find common fixed point results in dq-metric spaces for a pair of self-maps and focused on developing a scheme to prove existence and uniqueness of common fixed point in complete dq-metric spaces. The study was conducted under the stream of functional analysis.

## CHAPTER TWO

## LITERATURE REVIEW

Fixed point theory is very important in diverse disciplines of mathematics since it can be applied for solving various problems and it is one of the most dynamic research subjects in nonlinear analysis. A very interesting and useful result in fixed point theory is the Banach contraction principles. This theorem has witnessed numerous generalizations and extensions in the literature because of its simplicity and contractive approaches. For this reason generalization of Banach's contraction principle have been investigated heavily by many researchers (Sintunavarat et al., 2013). Consequently, a number of generalizations of Banach's contraction principles have appeared. In the fixed point theory, contraction is one of the main tools to prove the existence and uniqueness of a fixed point.Banach's contraction principle, which gives an answer on the existence and uniqueness of the solution of an operator, $Tx = x$ is used in all analysis. The advantage of topology in logic programming is recognized (Hitzler and Seda, 2000). Particularly topological methods are applied to obtain fixed point semantics for logic programs. Such considerations motivated the concept of dislocated metric space.

**Definition2.1:**Zeyada et al., (2005) LetXbe anon-empty set and $d : X \times X \rightarrow [0, \infty)$ be a function satisfying:

$d_1$) $d(x, x) = 0$.

$d_2$) $d(x, y) = d(y, x) = 0$,imply that$x = y$.

$d_3$) $d(x, y) = d(y, x)$.

$d_4$) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

The pair $(X, d)$ is metric space if d satisfies$d_1$-$d_4$.

The pair $(X, d)$ is quasi metric space if d satisfies$d_1, d_2$ $and$ $d_4$.

The pair $(X, d)$ is dislocated metric space if $d$ satisfies $d_2, d_3$ $and$ $d_4$.

The pair $(X, d)$ is dq-metric if $d$ satisfies $d_2$ $and$ $d_4$ only.

The concept of Quasi- metric space was introduced by Wilson, (1931) as a generalization of metric spaces. Hitzler and Seda, (2000) introduced dislocated metric spaces as a generalization of Metric spaces.

Hitzler and seda (2000) investigated the useful application of dislocated topology in the context of logic programming semantics. Inorder to obtain a unique supported model for these programs, they introduced the notion of dislocated metric space and generalized the Banach contraction principle in such spaces. Furthermore, Zeyada,etal.(2005) generalized the result of Hitzler and Seda (2000) and introduced the concept of complete dislocated quasi metric spaces.

Aage and Salunke (2008) derived the following fixed point theorem with Kannan type contraction in the setting of dislocated quasi metric spaces.

**Theorem 2.1**: Aage and Salunke(2008)Let (X; d) be a complete dq- metric space and suppose there exist non negativeConstants $\alpha_1, \alpha_2, \alpha_3$ with $\alpha_1 + \alpha_2 + \alpha_3 < 1$. Let $f: X \rightarrow X$ be a continuous mapping Satisfying

$d(fx, fy) \leq \alpha_1 d(x, y) + \alpha_2 d(x, fx) + \alpha_3 d(y, fy)$.

For all $x, y \in X$, then $f$ has a unique fixed point.

Similarly Isufati, (2010) proved the following fixed point result for continuous contractive condition with rational type expression in the context of dislocated quasi metric spaces.

**Theorem 2.2:**Isufati, (2010) Let $(X, d)$ be a complete dq-metric space and $T: X \rightarrow X$ be a continuous self–mappings satisfying the following conditions:

$$d(Tx, Ty) \leq ad(x, y) + bd(y, Tx) + cd(x, Ty)$$

For all x, y$\in X$ where a, b, c > 0, with sup {a+2b+2c} < 1.Then T has a unique fixed point.

Kohil, et al. (2010) investigated a fixed point theorem which generalizes the result of Isufati.

In 2012, Zoto gave some new result in dislocated and dislocated quasi metric spaces.

For continuous self-mappings the fixed point theorem in dislocated quasi metric space was investigated by Shrivastava,et al.(2010).

In 2013 Patel constructed the following new fixed point results in a dislocated quasi metric space.

**Theorem 2.3:** Let $(X, d)$be a complete dislocated quasi metric space and $T: X \rightarrow X$ be continuous self-mapping satisfying the following condition:

$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)]$for all $x$, y$\in X$, where $a \in [0, \frac{1}{2})$, Then T has a unique fixed point.

Tiwari and Sarwar, (2015), also established fixed point and common fixed point theorems on dq-metric spaces.

In addition Tiwari and Vishuw, (2017), established the following common fixed point theorem on a complete dislocated quasi b-metric space.

**Theorem 2.4:** Let$(X, d)$be a complete dislocated quasi b-metric space. Let$T_1 : X \to X$and $T_2 : X \to X$ are continuous functions on $X$ .for $k \geq 1$ satisfying

$$d(T_1 x, T_2\, y) \leq \varphi\, d(x, y) \text{for all } x\,, y, \in\ X.$$

Where, $\varphi$is a comparison function,then$T_1$ and $T_2$ have a unique common fixed point in $X$.

**CHAPTER THREE**

# METHODOLOGY

This chapter contains study design, description of the research methodology, data collection procedures and data analysis process.

## 3.1.Study period and site

The study was conducted from October, 2016 to October, 2017GCinJimma University under Mathematics department.

## 3.2.Study Design

In order to achieve the objective of this research analytical design method was used.

## 3.3.Source of information

To conduct this research secondary data was used. Hence, the available sources of information are different mathematics reference books, published journals and published research worksand articles from internet.

## 3.4.Mathematical procedures

In this study we followed the analysis and the techniques used by Rahman and Sarwar, (2016), and also using additional techniques from the other related references like Manoj and Deheriya, (2015), Jha and Panths, (2012).

# CHAPTER FOUR

# DISCUSSION AND RESULT

## 4.1 Preliminaries

**Definition 4.1:** Zeyada et al., (2005)Let$X$ be a non- empty set. A function $\mathrm{d}: \mathrm{X} \times \mathrm{X} \rightarrow [0, \infty)$ is called a dislocated quasi metric provided that for all $x$, y, z $\in X$

$d_1$) $d(x, y) = d(y, x) = 0$,implies$x = y$.

$$d_2)\ d(x, y) \leq d(x, z) + d(z, y).$$

The pair $(X, d)$ is called dq-metric space. A dq-metric space is a function $d: X \times X \rightarrow [0, \infty)$ satisfying all the conditions of metric space with the exception of self-distance and symmetry. It should be noted that the class of dq- metric spaces is effectively general than that of metric spaces. Hence every metric space is dq-metric space but the converse is not true.

**Example4.1:** Let $X = \mathrm{R}$ define $d(x, y) = |x| + y^2$.It is adq-metric but not metric for all $x, y, z \in X$.

**Proof**:We need to show that $d$ satisfies definition 4.1

$d_1$) $d(x, y) = |x| + y^2$, $d(y, x) = |y| + x^2$

$|x| + y^2 = |y| + x^2 = 0$ implies $x = y = 0$,

$d(x, y) = d(y, x) = 0$implies $x = y$

$d_2$) $d(x, y) = |x| + y^2 \leq |x| + z^2 + |z| + y^2$, since $z^2, |z| \geq 0$

$$\leq d(x, z) + d(z, y).$$

Then $d(x, y) \leq d(x, z) + d(z, y)$for all $x, y \in X$.

Thus $d$satisfies both properties of definition (4.1) . Thenthe pair $(X, d)$ isdq-metricspace.

The pair $(X, d)$ does not satisfy conditions($d_1$) and ($d_3$) of definition 2.1.

That means self-distance is zero only when $x$ is zero and also $d$ is not symmetric. Hence $d$ is not metric. Therefore a dq-metric space is more general than metric space.

**Definition 4.2:**Zeyada et al., (2005) A sequence $\{x_n\}$ in a dq-metric space$(X,d)$ is called Cauchy sequence if for all $\varepsilon > 0,$ there exists $n_0 \in N$ such that for $m,n \geq n_0$, we have $d(x_n, x_m) < \varepsilon$ and $d(x_m, x_n) < \varepsilon$.

**Proposition 4.3:**Zeyada et al., (2005) Let $(X,d)$ be a dq-metric space$\{x_n\}$ be a sequence in$X$, and$x \in X$. Thena sequence $\{x_n\}$ converges to$x$ if and only if $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0$. In this case $x$ is called the limit of $\{x_n\}$ and we write$x_n \to x$.

**Proof:** Suppose that $\{x_n\}$ is a dq-convergent sequence. Then there exists$x \in X$ such that $\lim_{n\to\infty} x_n = x$, that is $\lim_{n\to\infty} d(x_n, x) = \lim_{m\to\infty} d(x, x_m) = 0$.

Hence, $\{x_n\}$ converges to $x$ (or dq-convergent).

**Proposition 4.4:** Zeyada et al., (2005) Limit of a convergent sequence in a dq-metric space is unique.

**Proof:** Let x and y be limits of the sequence $\{x_n\}$, then using triangular inequality it follows that $d(x,y) \leq d(x, x_n) + \mathrm{d}(x_n, y)$.

ByProposition 4.4$[d(x,x_n) + d(x_n, y)] \to 0$ hence $d(x,y) = 0$ and by property $(d_1)$ of definition (4.1) it follows that $x = y$.Therefore, limit of a convergent sequence in a dq-metric space is unique.

**Proposition 4.5**: Zeyada et al., (2005)Every convergent sequence in a dq-metric space is a dq-Cauchy.

**Proof:** Let $\{x_n\}$ be a sequence which converges to some $x$, and $\varepsilon > 0$, then there exist

$n_0 \in N$,With $d\,(x_n, x) < \frac{\varepsilon}{2}$ for all $n \geq n_0$.

For m, n$\geq n_0$, we obtain $d(x_n,\ x_m) \leq d(x_n,\ x) + d(x, x_m) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$d\left(x_{n,}x_m\right) < \varepsilon$. Hence $\{x_n\}$ is a dq- Cauchy.

**Definition 4.6:**Zeyada et al., (2005) A dislocated quasi-metric space $(X, d)$ is said to be complete if every Cauchy sequence in $X$ converges to a point in $X$.

**Dentition 4.7:**Berind, (2003)A map $\varphi: [0, \infty) \rightarrow [0, \infty)$ is called comparison function if it satisfies:

*i)* $\varphi$ is monotonically increasing.

*ii)* The sequence $\{\varphi^n \ (\mathrm{t})_{n=0}^{\infty}\}$converges to zero for all $t \in [0, \infty)$.

If $\varphi$ satisfies

*iii)* $\sum_{k=0}^{\infty} \varphi^k (t)$Converges for all $t \in [0, \infty)$.

Then $\varphi$ is called (c)comparison function. Every comparison function is (c) comparison.Prototype example for comparison function is $\varphi(t) = \alpha t$ , $t \in [0, \infty)$ and $\alpha \in [0,1)$.

**Lemma 4.8:** For every comparison function $\varphi; [0, \infty) \rightarrow [0, \infty)$
$\varphi(t) < t$for all$t > 0$, and $\varphi(t) = 0$ if and only if $t = 0$.

**Definition 4.9:** Let $X$ be a nonempty subset of a metric space $(X, d)$. A point $x \in X$ is a common fixed point of self-maps $T, f: X \rightarrow X$ if $fx = Tx = x$.

**Definition 4.10:**Jungck (1976) Two self-maps $T$ and $f$ on a nonempty set $X$ are said to be commuting if $Tfx = fTx$ for all$x \in X$.

If $Tx = fx$ for some $x \in X$, then $x$ is called coincidence point of $T$and $f$.

We call$z$ a point of coincidence of $f$ and $T$ if $z = Tx = fx$.If $z = x$, $x$ is called the common fixed point of $f$ and $T$.The set of coincidence points of $f$ and $T$ is denoted by $c(T, f)$.

**Definition 4.11:**Jungck (1986)Two self-mappings $f$ and $T$ of a metric space $(X, d)$are called compatibleif $\lim_{n\rightarrow\infty} \mathrm{d}\,(fTx_n, Tfx_n) = 0$ whenever $\{x_n\}_{n=0}^{\infty}$ is a sequence in X such that $\lim_{n\rightarrow\infty} fx_n = \lim_{n\rightarrow\infty} Tx_n = \mathrm{t}$ for some $t \in X$.

**Definition 4.12:** Jungck and Rhoades (1998)Two self- maps $f$ and $T$ of a metric space $(X, d)$ are called weakly compatible if they commute at their coincidence points. That is, $fu = Tu \ for\, u \in X$, then $Tfu = fTu$, for $u \in X$.

## 4. 2.Main result

**Theorem 4.1:** Let $(X, d)$ be a complete dislocated quasi metric spaces and $T, f: X \to X$ be self-maps satisfying the following condition.

$$i) \ TX \subseteq fX.$$

$ii)$ $f \ and \ T$are weakly compatible and $fX$ is closed subset of $X$.

$$iii) d(Tx, Ty) \leq a\varphi\big(d(fx, fy)\big) + b\varphi(\max\{d(fx, fy), d(fx, Tx)\}) +$$

$$c\varphi\left(\frac{d(fx,fy)\left[1+\sqrt{d(fx,fy)d(fx,Tx)}\right]^2}{(1+d(fx,fy))^2}\right).$$

For all $x, y \in X \ and \ a, b, c \geq 0 \ with \ a + b + c < 1 \ and\ \varphi$ is a comparison function, then $T$ and $f$ have a coincidence points. If $f$ and $T$ commute at their coincidence points then $f$ and $T$ have a unique common fixed point.

**Proof:** Let $x_0 \in X$, so that $y_0 = Tx_0 = fx_1$. Again since $Tx_1 \in fX$ there exist $x_2 \in X$.

Such that $y_1 = Tx_1 = fx_2$. Continuing this process we construct sequences $\{x_n\}$ and

$\{y_n\}$ such that $y_n = Tx_n = fx_{n+1}$ for $n = 0, 1, 2, \ldots$

To proof we consider two cases.

**<u>Case (i)</u>:** Suppose$y_n = y_{n+1}$for some $n = 0,1,2, \ldots$, then

$Tx_n = Tx_{n+1} = fx_{n+1}$. So that $x_{n+1}$ is coincidence point of $T$ and $f$.

Let$Tx_n = Tx_{n+1} = fx_{n+1} = u$, for some $u \in fX$.

Then by the weakly compatibility of $T \ and \ f$ we have

$Tu = Tfx_{n+1} = fTx_{n+1} = fu.$ (4.1)

Which shows $T$ *and* $f$ commute and $u$ is the coincidence point of $T$ *and* $f$.

Now we show $d\,(Tu, Tu) = 0$.

$$d(Tu,Tu) \le a\varphi\big(d(fu,fu)\big) + b\varphi(\max\{d(fu,fu), d(fu,Tu)\}) +$$

$$c\varphi\left(d(fu,fu)\frac{\left(1+\sqrt{d(fu,fu)d(fu,Tu)}\right)^2}{\big(1+(fu,fu)\big)^2}\right).$$

$$= a\varphi\big(d(Tu,Tu)\big) + b\varphi(\max\{d(Tu,Tu), d(Tu,Tu)\}) +$$

$$c\varphi\left(d(Tu,Tu)\frac{\left(1+\sqrt{d(Tu,Tu)d(Tu,Tu)}\right)^2}{\big(1+(Tu,Tu)\big)^2}\right).$$

Since $\varphi(t) \le t$, for all $t \ge 0$, we have

$$d(Tu,Tu) \le ad(Tu,Tu) + b\,d(Tu,Tu) + cd(Tu,Tu).$$

$$d(Tu,Tu) \le (a+b+c)d(Tu,Tu).$$

$$[1-(a+b+c)]d(Tu,Tu) \le 0.$$

Since $a+b+c < 1$ then $[1-(a+b+c)] > 0$.

This implies that $d(Tu,Tu) \le 0$, but $d(Tu,Tu) \ge 0$.

Hence $d(Tu,Tu) = 0$. (4.2)

Now we claim $Tu = u$.

$$d(Tu,u) = d(Tu,Tx_{n+1}) \le a\varphi\big(d(fu,fx_{n+1})\big) + b\varphi(max\{d(fu,Tu), d(fu,fx_{n+1})\}) +$$

$$c\varphi\left(d(fu,fx_{n+1})\frac{\left(1+\sqrt{d(fu,fx_{n+1})d(fu,Tu)}\right)^2}{\big(1+(fu,fx_{n+1})\big)^2}\right).$$

$$= a\varphi\big(d(Tu,u)\big) + b\varphi(max\{ d(Tu,Tu), d(Tu,u)\}) +$$

$$c\varphi\left( d(Tu,u)\frac{\left(1+\sqrt{d(Tu,u)d(Tu,Tu)}\right)^2}{\big(1+(Tu,u)\big)^2}\right).$$

Since$d(Tu,Tu) = 0, max\{ d(Tu,Tu), d(Tu,u)\}\ is\ \ d(Tu,u)$.

Then$\left( d(Tu,u)\frac{\left(1+\sqrt{d(Tu,u)d(Tu,Tu)}\right)^2}{\big(1+(Tu,u)\big)^2}\right) \le d(Tu,u)$.(4.3)

Since $\varphi(t) \le t$ for all $t \ge 0$ and using (4.3) we have

$$d(Tu,u) \le ad(Tu,u) + bd(Tu,u) + cd(Tu,u) = (a+b+c)d(Tu,u).$$

$$d(Tu,u) \le (a+b+c)\, d(Tu,u).$$

Since $a+b+c > 0$, then $d(Tu,u) \le 0$, but $d(Tu,u) \ge 0$.

Hence $d(Tu,u) = 0$. (4.4)

Similarly

$$d(u,Tu) = d(Tx_{n+1},Tu) \le a\varphi\big(d(fx_{n+1},fu)\big) + b\varphi(max\{d(fx_{n+1},fu), d(fx_{n+1},Tx_{n+1})\})$$

$$+c\varphi\left( d(fx_{n+1},fu)\frac{\left(1+\sqrt{dfx_{n+1},fu)d(fx_{n+1},Tx_{n+1})}\right)^2}{\big(1+d(fx_{n+1},fu)\big)^2}\right).$$

$$= a\varphi(d(u,Tu)) + b\varphi(\max\{ d(u,Tu)d(u,u)\} +$$

$$c\varphi\left( d(u,Tu)\frac{\left(1+\sqrt{d(u,Tu)d(u,u)}\right)^2}{\big(1+(u,Tu)\big)^2}\right).$$

Since $d(u,u) = 0$, then max$\{ d(u,Tu), d(u,u)\}$ is $d(u,Tu)$.

This impliesthat$d(u,Tu) \le a\varphi(d(u,Tu)) + b\varphi(d(u,Tu)) + c\varphi(d(u,Tu))$.

Since$\varphi(t) \le t, for\ all\ t \ge 0$,$d(u,Tu) \le (a+b+c)d(u,Tu)$.

Since $a+b+c>0$, $d(u,Tu)\le 0$, but $d(u,Tu)\ge 0$.

Hence $d(u,Tu)=0$. (4.5)

So,from (4.4) and (4.5) we get $u=Tu$.

By (4.1)$Tu=fu=u$.

Therefore $u$ is a common fixed point of $T\ and\ f$.

Now we prove the uniqueness of the common fixed point.

Suppose $u\ and\ v$ are two distinct fixed points of $T\ and\ f$.

That means $fu=Tu=u$ and $fv=Tv=v$.

$$d(u,v)=d(Tu,Tv)\le a\varphi(d(fu,fv))+b\varphi(\max\{d(fu,fv),d(fu,Tu)\})+$$

$$c\varphi\left(d(fu,fv)\frac{\left(1+\sqrt{d(fu,fv)d(fu,Tu)}\right)^2}{\left(1+(fu,fv)\right)^2}\right).$$

$$=a\varphi(d(u,v))+b\varphi(\max\{d(u,v),d(u,u)\})+$$

$$c\varphi\left(d(u,v)\frac{\left(1+\sqrt{d(u,v)d(u,u)}\right)^2}{\left(1+(u,v)\right)^2}\right).$$

$$d(u,v)\le a\varphi(d(u,v))$$
$$+b\varphi(\max\{d(u,v),d(u,u)\})+c\varphi\left(d(u,v)\frac{\left(1+\sqrt{d(u,v)d(u,u)}\right)^2}{\left(1+(u,v)\right)^2}\right).$$

Since $\max\{d(u,v),d(u,u)\}=d(u,v)$, then

$$d(u,v)\frac{\left(1+\sqrt{d(u,v)d(u,u)}\right)^2}{\left(1+(u,v)\right)^2}\le d(u,v).$$

This implies $d(u,v)\le a\varphi(d(u,v))+b\varphi(d(u,v))+c\varphi(d(u,v))$.

Since $\varphi(t)\le t,\ for\ all\ t\ge 0$, then $d(u,v)\le(a+b+c)d(u,v)$.

$a+b+c>0$, this implies that $d(u,v)\le 0$but $d(u,v)\ge 0$.

Hence $d(u,v)=0.$ (4.6)

Similarly

$$d(v,u) = d(Tv,Tu) \leq a\varphi(d(fv,fu)) + b\varphi(\max\{d(fv,fu),d(fv,Tv)\})$$

$$+c\varphi\left(d(fv,fu)\frac{\left(1+\sqrt{d(fv,fu)d(fv,Tv)}\right)^2}{\left(1+(fv,fu)\right)^2}\right)$$

$$= a\varphi(d(v,u)) + b\varphi(\max\{d(v,u),d(v,v)\}) + c\varphi\left(d(v,u)\frac{\left(1+\sqrt{d(v,u)d(v,v)}\right)^2}{\left(1+(v,u)\right)^2}\right).$$

$$d(v,u) \leq a\varphi(d(v,u)) + b\varphi(\max\{d(v,u),d(v,v)\}) + c\varphi\left(d(v,u)\frac{\left(1+\sqrt{d(v,u)d(v,v)}\right)^2}{\left(1+d(v,u)\right)^2}\right).$$

Since $\max\{d(v,u),d(v,v)\} = d(v,u)$.

This implies that $d(v,u)\frac{\left(1+\sqrt{d(v,u)d(v,v)}\right)^2}{\left(1+(v,u)\right)^2} \leq d(v,u)$.

Then,$d(v,u) \leq a\varphi d(v,u) + b\varphi\big(d(v,u) + c\varphi d(v,u)\big)$.

Since $\varphi(t) \leq t, for\ all\ t \geq 0 d(v,u) \leq ad(v,u) + bd(v,u) + cd(v,u)$

=$(a+b+c)d(v,u)$.

Then,$d(v,u) \leq (a+b+c)d(v,u)$.

Since$a+b+c > 0$,then $d(v,u) \leq 0$, but$d(v,u) \geq 0$.

Hence,$d(v,u) = 0$. (4.7)

From (4.6)and(4.7)we get$u = v$.

Hence $u$ is a unique common fixed point of $f\ and\ T$.

**<u>Case (ii)</u>:** Suppose$y_n \neq y_{n+1}$ for each $n = 0,1,2,...$

$$d(y_n,y_{n+1}) = d(Tx_n,Tx_{n+1}) \leq a\varphi\big(d(fx_n,fx_{n+1})\big) + b\varphi(\max\{d(fx_n,fx_{n+1}),d(fx_n,Tx_n)\})$$

$$+c\varphi\left(d(fx_n,fx_{n+1})\frac{\left(1+\sqrt{d(fx_n,fx_{n+1})d(fx_n,Tx_n)}\right)^2}{\left(1+d(fx_n,fx_{n+1})\right)^2}\right).$$

$$= a\varphi(d(y_{n-1},y_n)) + b\varphi(\max\{d(y_{n-1},y_n),d(y_{n-1},y_n)\}) +$$

$$c\varphi\left(d(y_{n-1},y_n)\frac{\left(1+\sqrt{d(y_{n-1},y_n)d(y_{n-1},y_n)}\right)^2}{\left(1+d(y_{n-1},y_n)\right)^2}\right). \qquad (4.8)$$

Since $\varphi(t) \le t, for\ all\ \ t \ge 0,$ (4.8)becomes

$d(y_n,y_{n+1}) \le ad(y_{n-1},y_n) + b\, d(y_{n-1},y_n) + c\, d(y_{n-1},y_n)$

$= (a+b+c)d(y_{n-1},y_n).$

Let$a+b+c = h < 1$

Then $d(y_n, y_{n+1}) \le hd(y_{n-1}, y_n)$. (4.9)

Similarly$d(y_{n-1},y_n) \le h\, d(y_{n-2}, y_{n-1})$.

Then we have $d(y_n, y_{n+1}) \le h^2 d(y_{n-2},y_{n-1})$.

Now, continuing this process we get$d(y_n,y_{n+1}) \le h^n d\left(y_{0,}y_1\right)$.

Since $h < 1$we have $\lim_{n\to\infty} h^n\, d\left(y_{0,}y_1\right) = 0.$ (4.10)

Similarly

$$d(y_{n+1},y_n) = d(Tx_{n+1}, Tx_n)$$

$$\le a\varphi(d(fx_{n+1},fx_n)) + b\varphi(\max\{d(fx_{n+1},fx_n), d(fx_{n+1},Tx_{n+1})\}) +$$

$$c\varphi\left(d(fx_{n+1},fx_n)\frac{\left(1+\sqrt{d(fx_{n+1},fx_n)d(fx_{n+1},Tx_{n+1})}\right)^2}{\left(1+d(fx_{n+1},fx_n)\right)^2}\right).$$

$$= a\varphi(d(y_n y_{n-1})) + b\varphi(\max\{d(y_n,y_{n-1}), d(y_n,y_{n+1})\}) +$$

$$c\varphi\left(d(y_n y_{n-1})\frac{\left(\mathbf{1}+\sqrt{\boldsymbol{d}(\boldsymbol{y_n},\boldsymbol{y_{n-1}})\boldsymbol{d}(y_n,\boldsymbol{y_{n+1}})}\right)^{\mathbf{2}}}{\left(\mathbf{1}+\boldsymbol{d}(y_n y_{n-1})\right)^{\mathbf{2}}}\right).$$

Since $\varphi(t) \le t, for\ all\ t \ge 0$

$d(y_{n+1},y_n) \le ad(y_n y_{n-1}) + b\max\{d(y_n,y_{n-1}), d(y_n,y_{n+1})\} +$

$c\left(d(y_n y_{n-1})\frac{\left(1+\sqrt{d(y_n,y_{n-1})d(y_n,y_{n+1})}\right)^2}{\left(1+d(y_n y_{n-1})\right)^2}\right)$.

(4.11)From(4.9)max{$d(y_n,y_{n-1}), d(y_n, y_{n+1})$}is $d(y_n, y_{n-1})$.

Then $\left(1+\sqrt{d(y_n, y_{n-1})d(y_n, y_{n+1})}\right)^2 \leq \left(1+d(y_{n-1}, y_n)\right)^2$.

This implies $d(y_n, y_{n-1})\frac{\left(1+\sqrt{d(y_{n-1},y_n)d(y_n,y_{n+1})}\right)^2}{\left(1+d(y_{n-1},y_n)\right)^2} \leq d(y_n, y_{n-1})$. (4.12)

From (4.11) and (4.12) we have

$d(y_{n+1},y_n) \leq ad(y_n,y_{n-1}) + b\, d(y_n,y_{n-1}) + c\, d(y_n,y_{n-1})$

$= (a+b+c)d(y_n,y_{n-1})$.

Let $a+b+c = h < 1$.

Thus $d(y_{n+1}, y_n) \leq hd(y_n, y_{n-1})$.

Similarly $d(y_n, y_{n-1}) \leq hd(y_{n-1}, y_{n-2})$.

Then $d(y_{n+1}, y_n) \leq h^2\, d(y_{n-1}, y_{n-2})$.

Continuing this process we get $d(y_{n+1},y_n) \leq h^n d(y_1, y_0)$.

Since $h < 1$, we have $\lim_{n\to\infty} h^n d(y_1, y_0) = 0$ . (4.13).

So, from (4.10) and (4.13) $\lim_{n\to\infty} h^n\, d\left(y_{0,}y_1\right) = \lim_{n\to\infty} h^n d(y_1, y_0) = 0$. (4.14)

Now we show $\{y_n\}$is a Cauchy sequence in $X$.

Let $m, n \in N$ with $m > n$.

Applying triangular inequality$d(y_n, y_m) \leq d(y_n,y_{n+1}) + d(y_{n+1}, y_m)$

$\leq d(y_n,y_{n+1}) + d(y_{n+1},y_{n+2}) + \ldots + d(y_{m-1}, y_m)$

$\leq h^n d(y_0, y_1) + h^{n+1} d(y_0, y_1) + \ldots + h^{m-1} d(y_0, y_1)$

$\leq h^n(1 + h + \cdots + h^{m-n-1})d(y_0, y_1)$

$$= h^n \left( \sum_{\mathrm{i=0}}^{\mathrm{m-n-1}} \mathrm{h^i} \right) d(y_0, y_1)$$

$$\leq h^n \left( \sum_{\mathrm{i=0}}^{\infty} \mathrm{h^i} \right) d(y_0, y_1)$$

$$= \frac{h^n}{1-h} d(y_0, y_1).$$

Thus $d(y_n, y_m) \leq \frac{h^n}{1-h}\, d\big(y_{0,}y_1\big)$.

Since h< 1 $\lim_{h\to 0} \frac{h^n}{1-h} d(y_0, y_1) = 0$ .

Then $\lim_{n,m\to\infty} d(y_n, y_m) = 0$ . (4.15)

By similar proof, let m, n $\in N, m > n$.

$d(y_m, y_n) \leq d(y_m, y_{m-1}) + d(y_{m-1}, y_n)$.

$\leq d(y_m, y_{m-1}) + d(y_{m-1}, y_{m-2}) + \cdots + d(y_{n+1}, y_n)$

$$\leq h^{m-1} d(y_1, y_0) + h^{m-2} d(y_1, y_0) + \cdots + h^n d(y_1, y_0)$$

$$\leq h^n (h^{m-1-n} + h^{m-2-n} + \cdots + h + 1)\, d(y_1, y_0)$$

$$= h^n \left( \sum_{i=0}^{m-n-1} h^i \right) d(y_1, y_0) .$$

Thus $d(y_m, y_n) \leq h^n \left(\sum_{i=0}^{\infty} h^i\right) d(y_1, y_0) = \frac{h^n}{1-h} d(y_1, y_0)$ .

This implies that $d(y_m, y_n) \leq \frac{h^n}{1-h} d(y_1, y_0)$.

Since h< 1, $\lim_{h\to 0} \frac{h^n}{1-h} d(y_1, y_0) = 0$ .

Then $\lim_{n\to\infty} d(y_m, y_n) = 0$. (4.16)

Hence from (4.15) and (4.16) we have

$\lim_{m,n\to\infty} d(y_n y_m) = \lim_{m,n\to\infty} d(y_m, y_n) = 0.$

Thus the sequence $\{y_n\}$ is a Cauchy sequence in $X$ for $n = 0, 1, 2, \ldots$ (4.17)

Since $X$ is complete, there exists $z \in X$, such that $\lim_{n\to\infty} y_n = z$.

Thus$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} fx_{n+1} = z$.

Since $fX$ is a closed subset of $X$, there exist $u \in fX$ such that$z = fu$.

Now we show that $Tu = z$.

$$d(Tu, z) = d(Tu, Tx_n) \le a\varphi(\, d(fu, fx_n)) + b\varphi(maxd(fu, fx_n), (fu, Tu)\})$$

$$+ c\varphi\left(d(fu, fx_n)\left(\frac{\left(1+\sqrt{d(fu,fx_n)d(fu,Tu)}\right)^2}{\left(1+d(fu,fx_n)\right)^2}\right)\right).$$

From (4.1)wehave$Tu = fu$, then

$$d(Tu, z) \le a\varphi(\, d(Tu, fx_n)) + b\varphi(\max\{d(Tu, fx_n), d(Tu, Tu)\}) +$$

$$c\varphi\left(d(Tu, fx_n)\left(\frac{\left(1+\sqrt{d(Tu,fx_n)d(Tu,Tu)}\right)^2}{\left(1+d(Tu,fx_n)\right)^2}\right)\right).$$

Letting $n \to \infty, d(Tu, z) \le a\varphi(\, d(Tu, z)) + b\varphi(\max\{d(Tu, z), d(Tu, Tu)\}) +$

$$c\varphi\left(d(Tu, z)\left(\frac{\left(1+\sqrt{d(Tu,z)d(Tu,Tu)}\right)^2}{\left(1+d(Tu,z)\right)^2}\right)\right).$$

Since $d(Tu, Tu) = 0$, $\max\{\, d(Tu, z), d(Tu, Tu)\} = d(Tu, z)$.

This implies that $\left(d(Tu, z)\frac{\left(1+\sqrt{d(Tu,z)d(Tu,Tu)}\right)^2}{\left(1+d(Tu,z)\right)^2}\right) \le d(Tu, z)$.

Then, $d(Tu, z) \le a\varphi(\, d(Tu, z)) + b\varphi(\, d(Tu, z)) + c\varphi(\, d(Tu, z))$.

Since$\varphi(t) \le t$forall$t \ge 0, d(Tu, z) \le (a + b + c)d(Tu, z)$.

Since $a + b + c > 0, d(Tu, z) \le 0$, but $d(Tu, z) \ge 0$.

Hence $d(Tu, z) = 0.$ (4.18)

Similarly $d(z, Tu) = d(Tx_n, Tu) \leq a\varphi(\ d(fx_{n,}fu)) + b\varphi(\max\{d(fx_{n,}fu),\ d(fx_{n,}Tx_n)\})$

$$+c\varphi\left(d(fx_n, fu)\frac{\left(1+\sqrt{d(fx_{n,}fu)d(fx_{n,}Tx_n))}\right)^2}{\left(1+d(fx_n, fu)\right)^2}\right).$$

Since $Tu = fu$ andtaking $n \to \infty Tx_n = fx_n = z$

$$d(z, Tu) \leq a\varphi(\ d(z, Tu)) + b\varphi(\max\{d(z, Tu), d(z, z)\}) +$$

$c\varphi\left(d(Tu, z)\left(\frac{\left(1+\sqrt{d(z,Tu)d(z,z)}\right)^2}{(1+d(z,Tu))^2}\right)\right).$ (4.19)

Since $\max\{d(z, Tu), d(z, z)\}$ is $d(z, Tu)$ and $\varphi(t) \leq t$, for all $t \geq\ 0$,

then(4.19)becomes$d(z, Tu) \leq (a + b + c)d(z, Tu)$.

Since $a + b + c > 0, d(z, Tu) \leq 0$ but $d(z, Tu) \geq 0$.

Hence$d(z, Tu) = 0.$ (4.20)

By (4.18) and (4.20) we have $z = Tu$.

Then $z = Tu = fu$.

By the weakly compatibility of$f\ and\ T, we\ have\ Tfu = fTu$.

Then $Tz = Tfu = fTu = fz$.

Which implies $f\ and\ T$ commute and z is the coincidence point of $f\ and\ T$.

Now we claim that $z$ is a common fixed point of $f\ and\ T$.

Consider $d(Tz, z) = d(Tz, Tu)$

$$\leq a\varphi(\ d(fz, fu)) + b\varphi(\max\{d(fz, fu), d(fz, Tz)\}) +$$

$c\varphi\left(d(fz,fu)\left(\frac{\left(1+\sqrt{d(fz,fu)d(fz,Tz))}\right)^2}{\left(1+d(fz,fu)\right)^2}\right)\right).$

$$= a\varphi(\ d(Tz,z)) + b\varphi(\max\{d(Tz,z), d(Tz,Tz)\}) +$$

$c\varphi\left(d(Tz,z)\left(\frac{\left(1+\sqrt{d(Tz,z)d(Tz,Tz))}\right)^2}{\left(1+d(Tz,z)\right)^2}\right)\right).$

$$d(Tz,z) \leq a\varphi(\ d(Tz,z)) + b\varphi(\max\{d(Tz,z), d(Tz,Tz)\}) +$$

$c\varphi\left(d(Tz,z)\left(\frac{\left(1+\sqrt{d(Tz,z)d(Tz,Tz))}\right)^2}{\left(1+d(Tz,z)\right)^2}\right)\right).$ (4.21)

Since ma$x\{d(Tz,z), d(Tz,Tz)\}$is$d(Tz,z)$,

$d(Tz,z)\left(\frac{\left(1+\sqrt{d(Tz,z)d(Tz,Tz))}\right)^2}{\left(1+d(Tz,z)\right)^2}\right) \leq d(Tz,z).$ (4.22)Using$\varphi(t) \leq t$,and (4.22)in (4.21) we get

$$d(Tz,z) \leq (a+b+c)d(Tz,z).$$

But$a+b+c > 0$, then $d(Tz,z) \leq 0$, but $d(Tz,z) \geq 0$.

Hence $d(Tz,z) = 0$(4.23)

Similarly

$$d(z,Tz) = d(Tu,Tz) \leq a\varphi(\ d(fu,fz)) + b\varphi(\max\{d(fu,fz), \quad d(fu,Tu)\})$$

$$+ c\varphi\left(d(fu,fz)\left(\frac{\left(1+\sqrt{d(fu,fz)d(fu,Tu))}\right)^2}{\left(1+d(fu,fz)\right)^2}\right)\right).$$

$$= a\varphi(\ d(z,Tz)) + b\varphi(\max\{d(z,Tz), d(z,z)\})$$

$$+ c\varphi\left(d(z,Tz)\left(\frac{\left(1+\sqrt{d(z,Tz)d(z,z))}\right)^2}{\left(1+d(z,Tz)\right)^2}\right)\right).$$

Sincemax$\{d(z,Tz)\ d(z,z)\}$ is $d(z,Tz)$and $\varphi(t) \leq t$, we have

$$d(z,Tz) \leq ad(z,Tz) + bd(z,Tz) + c\,d(z,Tz) = (a+b+c)d(z,Tz).$$

Then $(1-(a+b+c))\, d(z,Tz) \leq 0$.

But $a+b+c<1$, this implies that $d(z,Tz) \leq 0$, but $d(z,Tz) \geq 0$.

Hence $d(z,Tz) = 0$. (4.24)

So, from (4.23) and (4.24) we have $Tz = z$.

Then $Tz = fz = z$.

Therefore $z$ is common fixed point of $f$ and $T$.

Now we show the uniqueness of common fixed point of $f$ and $T$ in $X$.

Let $w$ be another common fixed point of $f$ and $T$ in $X$, that means $Tw = fw = w$.

Then we have

$$d(z,w) = d(Tz,Tw) \leq a\varphi\,(d(fz,fw)) + b\varphi(\max\{d(fz,fw), d(fz,Tz)\}) +$$

$$c\varphi\left(d(fz,fw)\left(d(fz,fw)\frac{\left(1+\sqrt{d(fz,fw)d(fz,Tz))}\right)^2}{\left(1+d(fz,fw)\right)^2}\right)\right).$$

$$= a\varphi\,(d(z,w)) + b\varphi(\max\{d(z,w), d(z,z)\}) +$$

$$c\varphi\left(d(z,w)\left(d(z,w)\frac{\left(1+\sqrt{d(z,w)d(z,z))}\right)^2}{\left(1+d(z,w)\right)^2}\right)\right).$$

$$d(z,w) \leq a\varphi\,(d(z,w)) + b\varphi(\max\{d(z,w), d(z,z)\}) +$$

$$c\varphi\left(d(z,w)\frac{\left(1+\sqrt{d(z,w)d(z,z))}\right)^2}{\left(1+d(z,w)\right)^2}\right) \quad . (4.25)$$

Since $max\{d(z,w), d(z,z)\}$ is $d(z,w)$ and since $\varphi(t) \leq t$

(4.25) become $d(z,w) \leq (a+b+c)d(z,w)$.

Since $a+b+c>0$, $d(z,w) \leq 0$, but $d(z,w) \geq 0$.

Hence $d(z,w) = 0$.(4.26) using similar line of proof$d(w,z) = d(Tw,Tz) \leq a\varphi( d(fw,fz)) +$ $b\varphi(\max\{d(fw,fz)\ , d(fw,Tw)\} +$

$$c\varphi\left(d(fw,fz)\left(\frac{\left(1+\sqrt{d(fw,fz)d(fw,Tw))}\right)^2}{\left(1+d(fw,fz)\right)^2}\right)\right).$$

$$d(w,z) \leq a\varphi( d(w,z)) + b\varphi(\max\{d(w,z), d(w,w)\}) +$$

$$c\varphi\left(d(w,z)\left(\frac{\left(1+\sqrt{d(w,z)d(w,w))}\right)^2}{\left(1+d(w,z)\right)^2}\right)\right). \qquad (4.27)$$

Since$max\{d(w,z), d(w,w)\}$ is $d(w,z)$,

then$$\left(d(w,z)\frac{\left(1+\sqrt{d(w,z)d(w,w))}\right)^2}{\left(1+d(w,z)\right)^2}\right) \leq d(w,z). \qquad (4.28)$$

Using $\varphi(t) \leq t$, and (4.28) in (4.27) we get

$$d(w,z) \leq (a+b+c)d(w,z).$$

$$[1-(a+b+c)]d(w,z) \leq 0.$$

Since $a+b+c < 1$,then $d(w,z) \leq 0$, but $d(w,z) \geq 0$.

Therefore $d(w,z) = 0.$ (4.29)

From (4.26) and (4.29)we conclude that$w = z$.

So,$z$is a unique common fixed point of$T\ and\ f\ in\ X$.

The following is an example in support of Theorem 4.1.

**Example 4.2:** Let $X = [0,4]$, define $d: X \times X \to [0,\infty)$ by

$d(x,y) = x^2 + y$and$T, f: X \to X$ by

$$T(x) = \begin{cases} 0, if\ 0 \leq x \leq 3 \\ \frac{1}{4}, if\ 3 < x \leq 4 \end{cases} \text{And} f(x) = \begin{cases} \frac{x}{3},\ if\ 0 \leq x \leq 3 \\ 4,\ if\ 3 < x \leq 4 \end{cases}$$

If $a = \frac{1}{2}, b = \frac{1}{5}, c = \frac{1}{5}$ ,*and* $\varphi(t) = \frac{3}{4}t$, for all $t \geq 0$.

In fact both properties of definition 4.1 holds true.

$i)$ $d(x,y) = x^2 + y = y^2 + x = d(y,x) = 0$ implies$x = y = 0$.

$ii)$ $d(x,y) = x^2 + y \leq x^2 + z + z^2 + y = d(x,z) + d(z,y)$.

We observe that $(X,d)$ is a dq-metric space and also $T0 = f0$. This implies 0 is coincidence point of $T$ *and* $f$. Moreover $fX = \{4\} \cup [0,1]$and $TX = \left\{0, \frac{1}{4}\right\}$. Hence $TX \subseteq fX$, also

$T0 = Tf0 = fT0 = f0 = 0$. Then$f$ and $T$are weakly compatible.

To see contractive condition of theorem 4.1 we consider the following cases.

**<u>Case($\boldsymbol{i}$):</u>**Suppose$x, y \in [0,3]$. Then,

$$d(Tx,Ty) = d(0,0) = 0.$$

$d(fx,fy) = d\left(\frac{x}{3}, \frac{y}{3}\right) = \left(\frac{x}{3}\right)^2 + \frac{y}{3} = \frac{x^2}{9} + \frac{y}{3} = \frac{x^2+3y}{9}$.

$d(fx,Tx) = d\left(\frac{x}{3}, 0\right) = \frac{x^2}{9}$.

From the contractive condition (2) we have

$$d(Tx,Ty) \leq a\varphi(\, d(fx,fy)) + b\varphi(\max\{d(fx,fy), d(fx,Tx)\}) +$$

$$c\varphi\left(d(fx,fy)\frac{\left(1+\sqrt{d(fx,fy), d(fx,Tx)}^2\right)}{\left(1+d(fx,fy)\right)^2}\right).$$

This implies $\quad 0 \leq \frac{3}{8}\left(\frac{x^2+3y}{9}\right) + \frac{3}{20}\left(\max\left\{\frac{x^2+3y}{9}, \frac{x^2}{9}\right\}\right) + \frac{3}{20}\left(\frac{x^2+3y}{9}\right)$.

Always true *for all* $x, y \in [0,3)$

**<u>Case ($\boldsymbol{ii}$)</u>**: Suppose $x, y \in (3,4]$. In this case

$$d(Tx,Ty) = d\left(\frac{1}{4}, \frac{1}{4}\right) = \left(\frac{1}{4}\right)^2 + \frac{1}{4} = \frac{5}{16}$$

$$d(fx,fy) = d(4,4) = (4)^2 + 4 = 20$$

$$d(fx,Tx) = d\left(4,\frac{1}{4}\right) = \frac{65}{4}$$

$$d(Tx,Ty) \leq a\varphi(d(fx,fy)) + b\varphi\,(\max\{\, d(fx,fy), d(fx,Tx)\}) +$$

$$c\varphi\left(d(fx,fy)\frac{\left(1+\sqrt{d(fx,fy),d(fx,Tx)}^{2}\right)}{\left(1+d(fx,fy)\right)^{2}}\right).$$

$\frac{5}{16} \leq \frac{3}{8}(20) + \frac{3}{20}\left(\max\left\{20,\frac{65}{4}\right\}\right) + \frac{3}{20}\left(\left(\frac{65}{4}\right)\frac{\left(1+\sqrt{20\left(\frac{65}{4}\right)}\right)^{2}}{\left(1+\frac{65}{4}\right)^{2}}\right)$.True for all $x, y \in (3, 4]$

**Case(iii):** If $x \in [0,3]$ and $y \in (3, 4]$, then

$$d(Tx,Ty) = d\left(0,\frac{1}{4}\right) = \frac{1}{4}.$$

$d(fx,fy) = d\left(\frac{x}{3},4\right) = \left(\frac{x}{3}\right)^2 + 4 = \frac{x^2}{9} + 4 = \frac{x^2+36}{9}$.

$d(fx,Tx) = d\left(\frac{x}{3},0\right) = \left(\frac{x}{3}\right)^2 = \frac{x^2}{9}$.

$$d(Tx,Ty) \leq a\varphi\big(\, d(fx,fy)\big) + b\varphi(\max\{\, d(fx,fy), d(fx,Tx)\}) +$$

$$c\varphi\left(d(fx,fy)\frac{\left(1+\sqrt{d(fx,fy),d(fx,Tx)}^{2}\right)}{\left(1+d(fx,fy)\right)^{2}}\right).$$

$\frac{1}{4} \leq \frac{3}{8}(\frac{x^2+36}{9}) + \frac{3}{20}\left(\frac{x^2+36}{9}\right) + \frac{3}{20}\left(\left(\frac{x^2+36}{9}\right)\frac{\left(1+\sqrt{\left(\frac{x^2}{9}\right)\frac{x^2+36}{9}}\right)^{2}}{\left(1+\frac{x^2+36}{9}\right)^{2}}\right).$

True for all $x \in [0,3]$ *and* $y \in (3,4]$.

**Case (iv):** Suppose $x \in (3,4]$ and $y \in [0, 3]$, then we have

$d(Tx,Ty) = d\left(\frac{1}{4},0\right) = \frac{1}{16}$ .

$d(fx,fy) = d\left(4,\frac{y}{3}\right) = (4)^2 += 16 + \frac{y}{3}$.

$$d(fx,Tx) = d\left(4,\frac{1}{4}\right) = 4^2 + \frac{1}{4} = \frac{65}{4}\ .$$

$$d(Tx,Ty) \le a(\varphi\, d(fx,fy)) + b\varphi(\max\{\, d(fx,fy), d(fx,Tx)\}) +$$

$$c\varphi\left(d(fx,fy)\frac{\left(1+\sqrt{d(fx,fy),d(fx,Tx)}\right)^2}{\left(1+d(fx,fy)\right)^2}\right).$$

$$\frac{1}{16} \le \frac{3}{8}(16+\frac{1}{3}\mathrm{y}) + \frac{3}{20}\left(16+\frac{1}{3}\mathrm{y}\right) + \frac{3}{20}\left(\left(16+\frac{1}{3}\mathrm{y}\right)\frac{\left(1+\sqrt{\left(16+\frac{1}{3}y\right)\frac{65}{4}}\right)^2}{\left(1+16+\frac{1}{3}y\right)^2}\right).$$

True for all $x \in (3,4]$ $and$ $y \in [0,3]$.

From cases (i) $-$ (iv) all the conditions of theorem 4.1 are satisfied and 0 is the unique common fixed point of $f$ $and$ $T$.

**Theorem 4.2:** Let$(X,d)$ be a complete dq-metric space. $T,f\colon X \to X$be continuous self-mappings satisfying the contractive condition of theorem 4.1.Then $T$ $and$ $f$ have a unique common fixed point.

Proof: Followingas in the proof of Theorem 4.1 we construct a sequence $\{y_n\}$. Let $\{x_{2n}\}$ $and$ $\{x_{2n+1}\}$ be subsequences of the sequence $\{y_n\}$.As in the theorem 4.1 we define $x_{2n+1} = Tx_{2n}$and$x_{2n} = fx_{2n-1.}$.

Similarly as shown in the proof of theorem 4.1 we can show that the sequence $\{y_n\}$ is a Cauchy sequence.

By the completeness of $X$ one can find that $\lim_{n\to\infty} y_n = u$ for some $u \in X$.

Since $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are subsequences of $\{y_n\}$, $\lim_{n\to\infty} x_{2n+1} = u,$ also $\lim_{n\to\infty} x_{2n} = u$.

Next since $T and$ $f$ are continuous we arrive at

$$Tu = T\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} T\,x_{2n}$$

$$= \lim_{n\to\infty} x_{2n+1}$$

$$= u\,.$$

Then $Tu = u.$ (4.30)

Similarly $fu = f\ \lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} fx_{2n-1}$

$$= \lim_{n\to\infty} x_{2n}$$

$= u.$

Then $fu = u$.(4.31)

So, from (4.38) and (4.39) we get $Tu = u = fu$.

Therefore $u$ is common fixed point of $T and f$.

Now we show uniqueness of the common fixed point.

Let q be another common fixed point of $f$ and $T$. that is $fq = Tq = q$.

Then by the contractive condition (2) we have

$$d(u,q) = d(Tu,Tq) \leq a\varphi(\, d(fu,fq)) + b\varphi(\max\{\, d(fu,fq), d(fu,Tu)\}) +$$

$$c\varphi\left( d(fu,fq)\frac{\left(1+\sqrt{d(fu,fq),d(fu,Tu)}^{2}\right)}{\left(1+d(fu,fq)\right)^{2}}\right).$$

$$= a\varphi(\, d(u,q)) + b\varphi(\max\{\, d(u,q), d(u,u)\}) + c\varphi\left( d(u,q)\frac{\left(1+\sqrt{d(u,q),d(u,u)}^{2}\right)}{\left(1+d(u,q)\right)^{2}}\right).$$

$$d(u,q) \leq a\varphi(\, d(u,q)) + b\varphi(\max\{\, d(u,q), d(u,u)\}) +$$

$$c\varphi\left( d(u,q)\frac{\left(1+\sqrt{d(u,q),d(u,u)}^{2}\right)}{\left(1+d(u,q)\right)^{2}}\right). \qquad (4.32)$$

Since $\max\{\, d(u,q), d(u,u)\}$is $d(u,q)$, and Since $\varphi(t) \leq t$

then(4.32) becomes $d(u,q) \leq a\, d(u,q) + b\, d(u,q) + cd(u,q)$.

$d(u,q) \leq (a+b+c)d(u,q)$, Since$a+b+c > 0, d(u,q) \leq 0$

but$d(u,q) \geq 0$. Hence $d(u,q) = 0$.(4.33)

Similarly

$$d(q,u) = d(Tq,Tu) \leq a\varphi(\, d(fq,fu)) + b\varphi(\max\{\, d(fq,fu), d(fq,Tq)\})$$

$$+c\varphi\left( d(fq,fu)\frac{\left(1+\sqrt{d(fq,fu),d(fq,Tq)}^{2}\right)}{\left(1+d(fq,fu)\right)^{2}}\right).$$

$d(q,u) \leq$

$a\varphi(\ d(q,u)) + b\varphi(\max\{\ d(q,u), d(q,q)\}) +$

$c\varphi\left(d(q,u)\frac{\left(1+\sqrt{d(q,u),d(q,q)}^2\right)}{\left(1+d(q,u)\right)^2}\right)$. Since $max\{\ d(q,u), d(q,q)\}\ \ is\ \ d(q,u)$ and $\varphi(t) \leq t$, for all

$t \geq 0$,

Thus $(q,u) \leq ad(q,u) + \mathrm{b}\ d(q,u) + \mathrm{c}d(q,u)$.

Since $a + b + c > 0$, then $d(q,u) \leq 0.$ but $d(q,u) \geq 0$.

Hence $d(q,u) = 0.$ (4.34)

So, from (4.33) and (4.34) we have $u = q$.

Thus $u$ is a unique common fixed point of $f$ and $T$.

Examplesupporting the result of theorem 4.2

**Example 4.3**: Let $X = [0,1] d: X \times X \to [0,\infty)$ by

$d(x,y) = x$and define $f,T: X \to X$ by

$Tx = \frac{x}{9}$, $fx = \frac{8x}{9}$, and $\varphi(t) = \frac{4}{5}$t, where $a = \frac{3}{4}$, b= $\frac{1}{6}$ and $c = \frac{1}{16}$.

$$d(Tx,Ty) = d(\frac{x}{9}, \frac{y}{9}) = \frac{x}{9}.$$

$d(fx,fy) = d\left(\frac{8x}{9}, \frac{8y}{9}\right) = \frac{8x}{9}$.

$d(fx,Tx) = d(\frac{8x}{9}, \frac{8x}{9}) = \frac{8x}{9}$.

From the contractive condition of the theorem we have

$$d(Tx,Ty) \leq a\varphi\ (d(fx,fy)) + b\varphi(\max\{\ d(fx,fy), d(fx,Tx)\}) +$$

$$c\varphi\left(d(fx,fy)\frac{\left(1+\sqrt{d(fx,fy),d(fx,Tx)}^2\right)}{\left(1+d(fx,fy)\right)^2}\right).$$

$$\frac{x}{9} \leq \frac{12}{20}\left(\frac{8x}{9}\right) + \frac{4}{30}\left(\max\{\frac{8x}{9}, \frac{8x}{9}\}\right) + \frac{4}{80}\left(\left(\frac{8x}{9}\right)\frac{\left(1+\sqrt{(\frac{8x}{9})(\frac{8x}{9})}\right)^2}{\left(1+\frac{8x}{9}\right)^2}\right).$$

$\frac{x}{9} \leq \frac{24x}{45} + \frac{16x}{135} + \frac{2x}{45}$which is true for all $x \in [0,1]$.

$T$and$f$ satisfies all the conditions of the Theorem 4.2 and $T$ and $f$ have a unique fixed point.

$T0 = f0 = 0$.Also$Tf0 = 0 = Tf0$.

Therefore 0 is the unique common fixed point of $T$and$f$ .

**Remark 1:** For $f = I(I =$identity map on $X$) from contractive condition of theorem 4.1 we get

$$d(Tx,Ty) \leq a\varphi\big(d(x,y)\big) + b\varphi(\max\{d(x,y), d(x,Tx)\}) + c\varphi\left(d(x,y)\frac{\left(1+\sqrt{d(x,y)d(x,Tx)}\right)^2}{\left(1+d(x,y)\right)^2}\right).$$

Whenever $f = I$contractive condition of theorem 4.1 is simplified to theorem 1.5.
Hence theorem 1.5 follows asa corollary to theorem 4.1.

# CHAPTER FIVE

# CONCLUSIONS AND FUTURE SCOPE

## 5.1. Conclusion

In this Thesis, we have explored the properties of dislocated quasi-metric spaces and also discuss the difference between metric space and generalizations of metric space. We established two common fixed point theorems for a pair of self-maps in complete dislocated quasi metric spaces under contractive conditions of rational type. We also obtained sufficient conditions for existence of points of coincidence and common fixed points of two self –mappings in dislocated quasi metric spaces. We have supported the result of this work by particular examples. Our works extend and generalize some of the results of Rahman and Sarwar(2016).

## 5.2 Future scope

Fixed point theory is one of the active and vigorous areas of research in mathematics and other sciences. There are several published results related to existence of fixed points of self-maps defined on dislocated quasi metric space. There are also few results related to the existence of common fixed points for a pair or more self-maps in this space. The researcher believes the search for the existence of coincidence and common fixed points of self-maps satisfying different conditions in dislocated quasi metric space is an active area of study. So, forthcoming postgraduate students of department of Mathematics or any researcher persons who are interested can exploit this opportunity and conduct their research work in this area.

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