Available online at http://www.journalcra.com

INTERNATIONAL JOURNAL OF CURRENT RESEARCH

International Journal of Current Research
Vol. 8, Issue, 03, pp.27356-27360, March, 2016

ISSN: 0975-833X

RESEARCH ARTICLE

THE COMPARISON OF RUNGE-KUTTA AND ADAMS-BASHFORH-MOULTON METHODS FOR THE FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

*Solomon Gebregiorgis and Genanew Gofe

Department of Mathematics, Jimma University, Ethiopia

ARTICLE INFO

Article History:
Received 15th December, 2015
Received in revised form
27th January, 2016
Accepted 25th February, 2016
Published online 16th March, 2016

Key words:

Computation time,
Relative error,
Efficiency, and accuracy.

ABSTRACT

The performances of Runge-Kutta (RK4) and Adams-Bashforth-Moulton(ABM) methods were compared by considering first order ordinary differential equations. Moreover the effectiveness of modifiers in the ABM method has been validated. The result of this research show that ABM method is the most efficient method for first order ODE but in terms of accuracy there is no one best method. So it is not possible to make generalizations. But it is possible to conclude that the performance of a given method depend on the characteristics of the ODEs we are considering such as stiffness and stability. Regarding the modifiers in the corrector and predictor formulas of the ABM method, they are effective in improving the accuracy of ABM method in most cases.

Citation: Solomon Gebregiorgis and Genanew Gofe, 2016. "The comparison of runge-kutta and Adams-Bashforh-Moulton methods for the first order ordinary differential equations", International Journal of Current Research, 8, (03), 27356-27360.

INTRODUCTION

Numerical methods are most widely being utilized to solve the equations arising in the fields of applied medical sciences, engineering and technology due to the advancements in the field of computational mathematics. There are several numerical methods to solve ordinary differential equations of the type initial value problem that is the single-step methods and the multiple-step methods (Chapra and Canale, 1989). Many contributions have been made in the area of numerical methods for ordinary differential equations especially in the area of the comparison of numerical methods. For comparison purpose we need to consider the problem to be solved, methods to be considered, and comparison criteria (Hull *et al.*, 1972). The major factors to be considered in comparing different numerical methods are the accuracy of the numerical solution and its computation time (Bedet *et al.*, 1975). They further indicated that it is important to note that the comparison of numerical methods is not so simple because their performances may depend on the characteristic of the problem at hand. It should also be noted that there are other factors to be considered, such as stability, versatility, proof against run-time error, and so on which are being considered in most of the MATLAB built-in routines (Yang *et al.*, 2005).

***Corresponding author: Solomon Gebregiorgis,**
Department of Mathematics, Jimma University, Ethiopia

Here the methods selected are the explicit Runge-Kutta method of fourth order which is a single step method and Adams-Bashforth-Moulton predictor corrector method of fourth order which is a multistep method. The methods selected are among the best methods available (Hull *et al.*, 1972). Each Runge-Kutta methods are derived from an appropriate Taylor method in such a way that the final global error is of order $O(h^N)$ (Mathews. *et al.*, 2004). In this method several function evaluations is performed at each step and eliminate the necessity to compute the higher derivatives. These methods can be considered for any order N. The Runge-Kutta method of order N = 4 is most popular. It is a good choice for common purposes because it is quite accurate, stable, and easy to program. Hence it is not necessary to go to a higher-order method because the increased accuracy is offset by additional computational effort (Mathews. *et al.*, 2004). If more accuracy is required, then either a smaller step size or an adaptive method should be used. A desirable feature of a multistep method is that the local truncation error can be determined and a correction term can be included, which improves the accuracy of the answer at each step. Also it is possible if the step size is small enough to obtain an accurate value for y_{k+1}, yet large enough so that unnecessary and time-consuming calculations are eliminated. Using the combination of a predictor and corrector requires only two function evaluations of f(x, y) per step. By obtaining the predictor/corrector errors it is possible to derive Adams–Bashforth–Moulton method with modification formulas (Yang *et al*., 2005). The performance of numerical methods depend on the characteristics of the ODE considered (Hull *et al*., 1972; Bedet *et al*., 1975; Butcher, 2000; Yang *et al*., 2005; Petzoid, 2006; Clement *et al*., 2009; Abdul, 2013; Polla, 2013; and Muhammmad and Arshad, 2013). While the central activity of numerical analysts is providing accurate and efficient general purpose numerical methods and algorithms, there has always been a realization that some problem types have distinctive features that they will need their own special theory and techniques (Butcher, 2000).

The first problem considered in this research is a first-order differential equation $y' = -y$ with $y(0) = 1$ (Hull *et al.*, 1972). It has the following form of analytical solution $y = e^{-x}$. The second problem is $y' = -y^3/2,\ y(0) = 1$ is a special case of the Riccati equation and whose solution is given by $y = \frac{1}{\sqrt{x+1}}$ (Davis, 1963 as cited in Hull *et al*., 1972). The third problem is $y' = ycosx,\ y(0) = 1$ which is an oscillatory problem and whose solution is given by $y = e^{sinx}$ (Hull *et al.*, 1972). The fourth problem is $y' = \frac{y}{4}\left(1 - \frac{y}{20}\right),\ y(0) = 1$ whose solution is given by $y = \frac{20}{1 + 19e^{-\frac{x}{4}}}$ which is the logistic curve, (Davis, 1962 as cited in Hull *et al*., 1972). The fifth problem is $y' = \frac{y-x}{y+x},\ y(0) = 4$ whose solution is given by $r = 4e^{-\theta+\frac{\pi}{2}}$ which is a spiral curve (Davis, 1962 as cited in Hull *et al*., 1972). As it is shown above the problems selected have exact solutions. This is helpful to compare the approximated values with the exact values and to calculate relative errors. The selection covers a realistically broad spectrum of problem types (Hull *et al*., 1972). The testing of ODE solvers will be done by using MATLAB after the problems are properly coded and inserted for analysis. Hence the main purpose of this research is to compare the accuracy and computation times of Runge-Kutta and Adams-Bashforth-Moulton methods for first order ordinary differential equations.

MATERIALS AND METHODS

The study involves entirely laboratory work with the help of a laptop and a MATLAB software. So it is an experimental research. The methods are coded and run using MATLAB software by properly inserting the problems and as a result numerical results are automatically generated. All algorithms have been made in the same condition, which use the same type of processor, having the same memory size, the same operating system, and using the same function. The processor used is Intel(R) 2.10 GHz, with 2 GB memory, with the 32-bit operating system (windows 7 home premium). The language program used is MATLAB version 7.14. Three major programs (codes) have been written to compare the computation times and accuracies of RK4, ABM, and Modified ABM (MABM) methods.

Computation time

MATLAB has two convenient commands that let us measure how long an operation takes to solve a given problem after certain iterations. To start (and reset) a timer, use the command tic;. To stop the timer and display the amount of time elapsed, use toc;. Computation times for RK4, ABM, and MABM using the five problems have been calculated by varying the number of steps.

Accuracy

To find which numerical method gives a more accurate approximation this study compared the relative errors obtained by using RK4, ABM, MABM for the five problems selected. Relative error (ε_r) is calculated using the formula

$$\varepsilon_r = \left|\frac{Exact\ Value - Approximated\ Value}{Exact\ Value}\right|.$$

According Mathews *et al*., (2004), the step size h for a fixed-point iteration using RK4 and ABM method must satisfy the following condition $h < \frac{0.75}{|f_y(x,y)|}$. Hence the value of h is in such a way that it satisfies this condition. Finally, data obtained by using MATLAB version 7.14 software about computation times and relative errors were analyzed after calculating the average and standard deviations. More over graphs of computation times and relative errors have been sketched for the purpose of analysis.

RESULTS

Comparison of computation times

Data about computation times (in seconds) obtained using RK4, ABM, and MABM methods by varying the number of steps for the five problems. ABM method has the greatest speed than RK4 and MABM and MABM is the second faster in approximating the solution of the ordinary differential equation for all the problems considered.

Comparison of accuracy

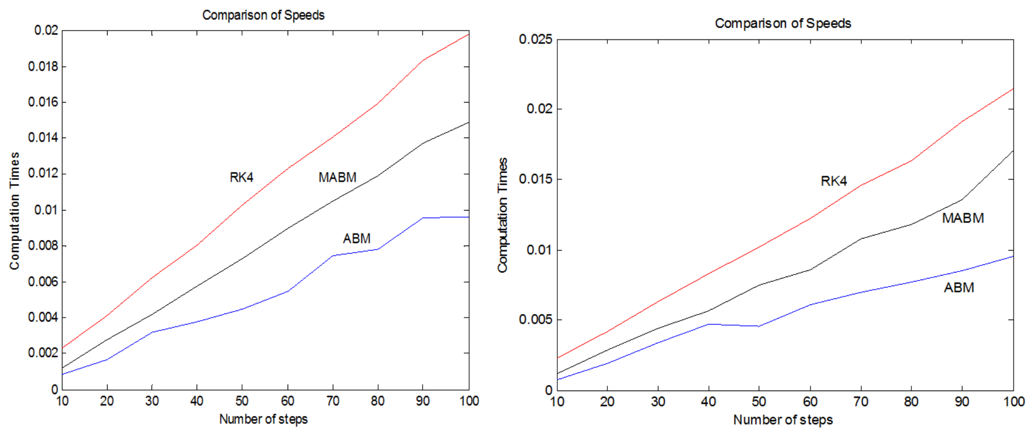
To compare the accuracies of the RK4, ABM, and MABM methods, relative errors are computed by taking the number of steps M = 40 for the five problems considered. Based on these data graphs are sketched for comparison purpose. Moreover, average and standard deviations are also calculated to further strengthen the comparison task. RK4 method has a better accuracy than ABM method for initial value problems $y' = -y,\ y(0) = 1$ and $y' = \frac{y-x}{y+x},\ y(0) = 4$ but with the help of modifiers the ABM method has a better accuracy than RK4 (table 1 and fig.2). This result also applies for the problem

$$y' = -\frac{y^3}{2}, \quad y(0) = 1.$$

RK4 method has the greatest accuracy than both ABM and MABM methods on the interval [0, 0.475] but ABM method is more accurate than RK4 on the interval (0.475, 1] for the problem $y' = ycosx,\ y(0) = 1$. Both RK4 and ABM methods have the same accuracy up to eight decimal places. More over the usage of modifiers decreases accuracy of ABM (table 2 and fig.3a). So there is no dominant method on the interval [0, 1]. The intensity of the problem becomes much greater when we increase the interval to [0, 10] (fig.3b).

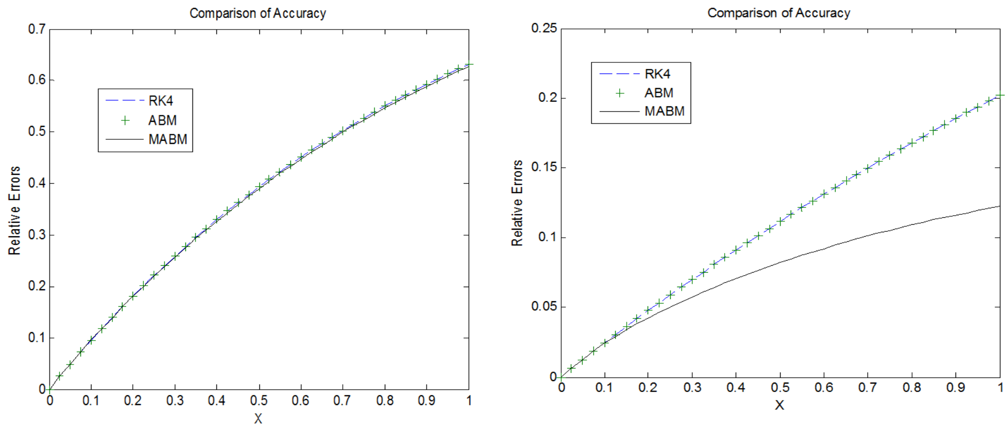
Table 1. Average and Standard Deviation of the relative errors obtained by RK4, ABM, and MABM methods by using the problems $y' = -y, y(0) = 1$ and $y' = \frac{y-x}{y+x},\ y(0) = 4$.

	Problems			
	$y' = -y, y(0) = 1$		$y' = \frac{y-x}{y+x},\ y(0) = 4$	
Method	Average	Standard Deviation	Average	Standard Deviation
RK4	0.36658343	0.188061591	0.10783153063	0.060281657417
ABM	0.36658344	0.188061593	0.10783153067	0.060281657400
MABM	0.36398505	0.186498033	0.07530830050	0.035248994175



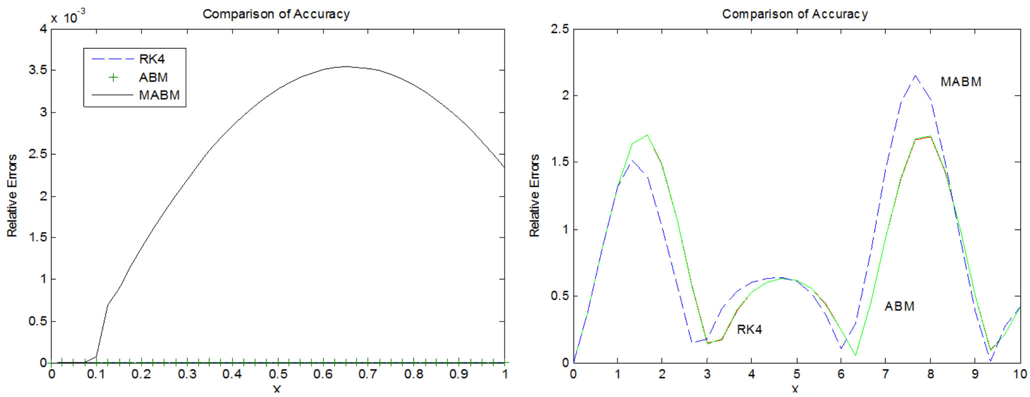
(a) (b)

Figure 1. The comparison of computation times of RK4, ABM, and MABM methods for the problems (a) $y' = -y, y(0) = 1$ and (b) $y' = -y^3/2,\ y(0) = 1$



(a) (b)

Figure 2. Comparison of accuracies of the RK4, ABM, MABM methods by using the problems (a) $y' = -y,\ y(0) = 1$ and (b) $y' = \frac{y-x}{y+x},\ y(0) = 4$



(a) (b)

Figure 3. Comparison of accuracies of RK4, ABM, MABM methods by using the problem $y' = y\cos x,\ y(0) = 1$ on the intervals (a) [0, 1] and (b) [0, 10]

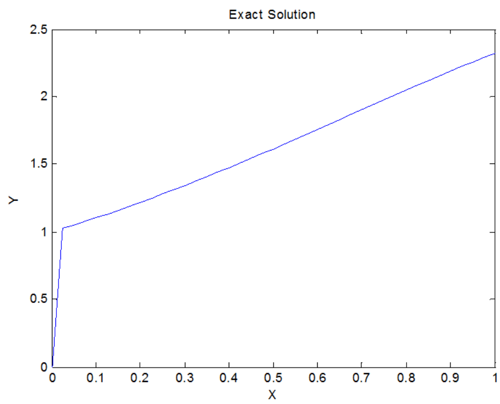
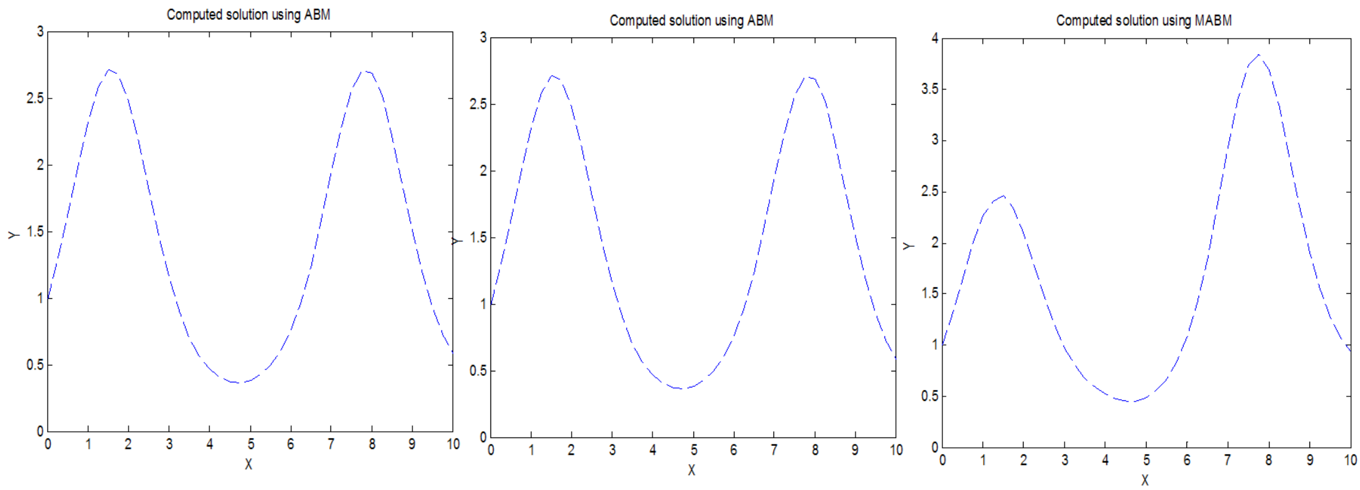


Figure 4. Plot of a stiff solution of $y' = y\cos x,\ y(0) = 1$



(a) (b) (c)

Figure 5. (a) Oscillations in the computed solution by using RK4; (b) Oscillations in the computed solution by using ABM; (c) Oscillations in the computed solution by using MABM

Table 2. Average and Standard Deviation of the relative errors obtained by RK4, ABM, and MABM methods by using the problem $y' = y\cos x,\ y(0) = 1$

Method	Average	Standard Deviation
RK4	1.0944e-09	5.04652e-10
ABM	9.93391e-10	6.96787e-10
MABM	0.00242	0.00118

RK4 method has the same accuracy as the ABM method up to twelve decimal places but ABM method has a better accuracy for the problem $y' = \frac{y}{4}\left(1 - \frac{y}{20}\right),\ y(0) = 1$. More over MABM method has the least accuracy.

DISCUSSION

Using the combination of a predictor and corrector requires only two function evaluations of f(x, y) per step and hence unnecessary and time-consuming calculations are eliminated (Mathews. *et al*., 2004). The ABM method registers the smallest computation time than RK4 and MABM methods. MABM method is slower than ABM method due to the addition of modifier formulas on the predictor and corrector parts but MABM is still faster than the RK4 method.

The number of iterations of the corrector is highly dependent on the accuracy of the initial prediction. Consequently, if the prediction is modified properly, we might reduce the number of iterations required to converge on the ultimate value of the corrector (Chapra and Canale, 1989). Stiffness is a special problem that can arise in the solution of ordinary differential equations. A stiff system is one involving rapidly changing components together with slowly changing ones (Chapra and Canale, 1989). In many cases, the rapidly varying components are ephemeral transients that die away quickly, after which the solution becomes dominated by the slowly varying components. Although the transient phenomena exist for only a short part of the integration interval, they can dictate the time step for the entire solution. The problem $y' = y\cos x$, $y(0) = 1$ is a stiff differential equation as can be illustrated by the graph of its solution which showed a fast transient from y = 0 to 1 that occurs in less than 0.001166 time unit. This transient is perceptible only when the response is viewed on the finer timescale in the inset (Fig.4). Stiffness causes instability in uniform interval methods like RK4 unless many very small intervals are used. Stiff equations are problems for which explicit methods don't work (Hairer and Wanner as cited in Higham and Trefethen, 1993). The high instability resulted due to the stiff nature of the given differential equation can also be clearly seen by increasing the interval from [0, 1] to [0, 10] and sketching the approximated solutions by RK4, ABM, MABM methods (Fig.5). So it is not possible to compare the accuracies of the three methods as their approximated solutions manifest oscillations. Even though the addition of the modifiers increases both the efficiency and accuracy of multistep methods, there are situations where the corrector modifier will affect the stability of the corrector iteration process (Chapra and Canale, 2010). The problem of determining when a method is stable is more complicated in the case of multistep methods, due to the interplay of previous approximations at each step (Faires and Burden, 2002). It is due to these reasons that the use of modifiers in the ABM method decreases its accuracy for the problem $y' = \frac{y}{4}\left(1 - \frac{y}{20}\right)$, $y(0) = 1$. But for the problems $y' = -y$ with $y(0) = 1$, $y' = -y^3/2$ with $y(O) = 1$ and $y' = \frac{y-x}{y+x}$ with $y(0) = 4$, the modifiers are effective as they improve the accuracy of ABM method to be better in accuracy than RK4 which was previously inferior to it. The comparisons of accuracies for the problems $y' = -y$, $y(0) = 1$ and $y' = \frac{y-x}{y+x}$, $y(0) = 4$, and $y' = -y^3/2$, $y(O) = 1$ reveal an interesting fact that, although the ABM method, even without modifiers, are theoretically expected to have better accuracy than the RK4 method, they turn out to work better than RK4 only with modifiers. Of course, it is not always the case.

Acknowledgements

The authors would like to thank the College of Natural Sciences, Jimma University for funding this research work.

REFERENCES

Abdul , S.S., Gurudeo, A.T., and Muhammad, M.S. 2013. A Comparison of Numerical Methods for Solving the Unforced Van Der Pol's Equation. *ISSN*. 3(2):66

Atkinson, K. E. 1978. *An Introduction to Numerical Analysis*, 2nd ed. John Wiley & Sons, Inc, pp. 325-355.

Bedet, R. A., Enright, W. H., and Hall, T. E. 1975. STIFF DETEST: A program for comparing numerical methods for stiff ordinary differential equations. *Computing Surveys* 17(1): 25-48

Butcher, J.C. 1999. Numerical methods for ordinary differential equations in the 20th century. *Journal of Computational and Applied Mathematics* 125:1-29

Chapra, S. C. and Canale, R. P. (2010. *Numerical Methods for Engineers*, 6th ed. McGraw-Hill Co., Inc., New York, pp.751-777

Clement, A., Amy, Clement, G., Greg, F., Jeff, F., and Jayson, L. 2008. Comparing Numerical Integration Methods. *SIAM* 2:1–9.

Faires, J.D. and Burden, R.L. 2002. *Numerical Methods*, 3rd ed. Brooks Cole, pp. 277-280.

Gerardus, P. 2013. Comparing Accuracy of Differential Equation Results between Runge-Kutta-Fehlberg and Adams-Moulton Methods. *Applied Mathematical Sciences* 7: 116-127

Hornberger, G.M. and Wiberg, P.L. 2005. *Numerical Methods in the Hydrological Sciences.* Johns Hopkins Press, Baltimore, pp.302.

Hull, T.E., Enright, W.H., Fellen, B.M. and Sedgwick, A.E. (1972. Comparing numerical methods for ordinary differential equations. *SIAM J. NumeR. AnaL.* 9: 603-637.

Jain, M.K., Iyengar, S.R.K., and Jain, R.K. 1984. *Numerical Methods for Scientific and Engineering Computation*. New Age International (P) Limited Publishers, New Delhi, pp.471-539

Jorba, A. and Zou, M. 2004. A software package for the numerical integration of ODEs by means of high-order Taylor methods. *ACM Trans. Math. Software* 8:114-144

Mathews, J.H., Kurtis D. 2004. *Numerical Methods Using MATLAB*, 4th ed. Pearson Education, Inc, pp.450-455.

Muhammad, A. And Arshad, H. (2013. Comparing Numerical Methods for Multi-component Gas Separation by Single Permeation Unit. *Chiang Mai J. Sci*. 41: 184-199.

Volino, P. and Magnenat-Thalmann, N . 2000. Comparing Efficiency of Integration Methods for Cloth Simulation. *VSMM* 5: 109-118.

Yang, W.Y., Cao, W., Chung, T. and Morris, J. 2005. *Applied Numerical Methods Using MATLAB*. John Wiley & Sons, Inc., Hoboken, New Jersey, pp.274-277

Yatim, S. A., Ibrahim, Z. B., Othman, K. I., and. Suleiman, M. B. 2011. A Numerical Algorithm for Solving Stiff Ordinary Differential Equations. *ISSN* 3(2):55

****