Coupled Coincidence and Coupled Common Fixed Points of (ψ, ϕ) Contraction Type T-coupling in Metric Spaces

A Thesis Submitted to the Department of Mathematics in Partial Fulfillment for the Requirements of the Degree of Masters of Science in Mathematics

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Declaration

I, the undersigned declare that, this thesis entitled ”Coupled Coincidence and Coupled Common Fixed Points of (ψ,ϕ)-Contractive type T-couplings in Metric Spaces” is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged.

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Abstract

The purpose of this thesis was to establish coupled coincidence and coupled common fixed point theorem of (ψ, ϕ)-contraction type T-coupling in metric spaces and prove the existence and uniqueness of coupled coincidence and coupled common fixed point of (ψ, ϕ)-contraction type T-coupling. We employed analytical design and used secondary sources of data such as published articles and related books. The standard procedures used in the published work of Rao et al. (2013); Choudhury et al. (2017); Rashid & Khan (2018) have been used to prove our established theorem. Our result extend and generalize comparable results in the literature. Finally, we gave an example to illustrate our main results.

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Chapter 1

Introduction

1.1 Background of the study

Fixed point theory is an important tool in the study of nonlinear analysis as it is considered to be the key connection between pure and applied mathematics with wide applications in economics, physical sciences, such as Biology, Chemistry, Physics, differential, and almost all engineering fields.

The study of fixed points of mappings which satisfy certain contractive conditions has been at the center of rigorous research activity (Malhotra & Bansal, 2015). In 1922, Banach proved the famous fixed point theorem called Banach contraction principle which states that for a complete metric space (X,d) and a contraction mapping $\mathrm{T} : X \to X$ (where T satisfies the condition that $\mathrm{d}(\mathrm{T}x, \mathrm{T}y) \leq k\mathrm{d}(x, y)$ for all $x, y \in X$ and for some $k \in [0, 1)$), there exists a unique fixed-point $x_0 \in X$ of T. Due to the strong applications of fixed-point theory in nonlinear analysis, it is extended by several authors such as Caristi (1976); Sedghi et al. (2007).

The concept of coupled fixed point and the study of coupled fixed point problems appeared for the first time in Opoitsev & Khurodze (1984). Bhaskar & Lakshmikantham (2006) they established coupled point fixed results on partially ordered metric spaces.
Lakshmikantham & Ciric (2019) also introduced the concept of coupled coincidence point. The concept of coupling was introduced by Choudhury & Maity

(2014); Choudhury et al. (2017). The results on existence of coupled fixed point and coupled coincidence points appeared in many Papers Choudhury et al. (2017); Aydi et al. (2017); Rashid & Khan (2018). They proved the existence and uniqueness of strong coupled fixed point for couplings using Kannan type contractions for complete metric spaces. Aydi et al. (2017) proved the existence and uniqueness of strong coupled fixed point for (ψ, ϕ)-contraction type coupling in complete partial metric spaces.

Recently, Rashid & Khan (2018) generalized the result of Aydi et al. (2017) by introducing Strong Contractive Coupling (SCC)-Map for metric spaces. They also proved the existence and uniqueness of coupled fixed point for (ψ, ϕ)-contraction type coupling in complete metric spaces.

In this study, the results of Rashid & Khan (2018) have been generalized by introducing SCC-Map for metric spaces by proving the existence and uniqueness of coupled coincidence and coupled common fixed Points for (ψ, ϕ)-contraction type T-coupling in metric spaces.

1.2 Objectives of the study

1.2.1 General objective

The general objective of this study was to investigate Banach (ψ, ϕ)-contraction type T-coupling in complete metric spaces.

1.2.2 Specific objectives

This study has the following specific objectives

- To prove the existence of coupled coincidence and coupled common fixed point of Banach (ψ, ϕ)-contraction type T-coupling in metric spaces.
- To show the uniqueness of coupled common fixed point of Banach (ψ, ϕ)-contraction type T-coupling in metric spaces.
- To verify the applicability of the results obtained using a specific example.

1.3 Significance of the study

The result of this study may have the following importance

- The outcome of this study may contribute to research activities on the study area.
- It will provide basic research skills to the researcher.
- It will be applicable in studying the existence of unique solution to non-linear integral equations.

1.4 Delimitation of the Study

The study is focused on establishing and proving the existence and uniqueness of coupled coincidence point results of Banach (ψ, ϕ)-contraction type T-coupling in metric spaces.

Chapter 2

Review of Related Literature

The theoretical framework of metric fixed point theory has been an active research field and the contraction mapping principle is one the most important theorems in functional analysis. Fixed point theory has been studied extensively, which can be seen from the works of many authors Bhaskar & Lakshmikantham (2006); Caristi (1976); Malhotra & Bansal (2015); Sedghi et al. (2007). The concept of coupled fixed point and the study of coupled fixed point problems appeared for the first time in Opoitsev & Khurodze (1984). Several years later, the theory of coupled fixed points in the setting of an ordered metric space and under some contractive type conditions on the operator was re-considered by Bhaskar & Lakshmikantham (2006). Coupled fixed point theorems for nonlinear contractions in partially ordered G-metric spaces is generalized by Aydi Aydi et al. (2011).

The results on existence of coupled fixed point and coupled coincidence points appeared in many Papers (Choudhury et al., 2017; Aydi et al., 2017; Rashid & Khan, 2018). The concept of couplings is introduced recently by Choudhury et al. (2017). This nice concept follows when combining the notion of coupled and cyclic maps. They proved the existence and uniqueness of strong coupled fixed point for couplings using Kannan type contractions for complete metric spaces.

Cyclic representations and cyclic contractions were introduced by Kirk et al. (2003) and further used by several authors to obtain various fixed point results for not necessarily continuous mappings (Chen, 2012; Karapınar, 2011; Karapınar et al., 2012). Nashine et al. (2012) studied about cyclic generalized contractions and fixed point results with applications to an integral equation. Cyclic coupled fixed point result using Kannan type contractions was introduced by Choudhury & Maity (2014).

Aydi et al. (2017) proved the existence and uniqueness of strong coupled fixed point for (ψ, ϕ)-contraction type coupling in complete partial metric spaces. Recently Rashid & Khan (2018) generalized the result of Aydi et al. (2017) by introducing Strong Contractive Coupling (SCC)-Map for metric spaces not necessarily complete. They also proved the existence and uniqueness of coupled fixed point for (ψ, ϕ)-contraction type coupling in complete metric spaces.

Chapter 3

Methodology

This chapter contains study design, description of the research methodology, data collection procedures and data analysis process.

3.1 Study period and site

The study has been conducted from November 2017 to September 2018 in Jimma University under Mathematics department.

3.2 Study Design

In order to achieve the objectives stated, this study has employed analytical design.

3.3 Source of Information

In this study secondary data such as, different mathematics books related to the study area, published articles related to the topic and internet sources have been used.

3.4 Mathematical Procedure of the Study

In this study, the standard procedures used in the published works of Rao et al. (2013); Choudhury et al. (2017); Rashid & Khan (2018) have been followed.

The procedures are

1. Establishing a theorem.

2. Constructing sequences and showing that the constructed sequences are Cauchy and convergent.

3. Proving the existence and uniqueness of coupled coincidence and coupled common fixed point of (ψ, ϕ) contraction type T-coupling in metric spaces.

4. Giving applicable example for supporting the main result.

Chapter 4

Preliminaries and Main Results

4.1 Preliminaries

Definition 4.1 *Let X be a nonempty set and $T : X \to X$ a self-map. We say that x is a fixed point of T if $Tx = x$.*

Definition 4.2 *A sequence $\{x_n\}$ in a metric space (X,d) is said to converge to a point $x \in X$ if and only if $\lim_{n\to\infty} d(x_n,x) = 0$.*

Definition 4.3 *A sequence $\{x_n\}$ in a metric space (X,d) is called a Cauchy sequence if $\lim_{n,m\to\infty} d(x_n,x_m) = 0$. Furthermore, a metric space (X,d) is called complete if every Cauchy sequence $\{x_n\}$ in (X,d) converges to a point $x \in X$.*

Lemma 4.1 *Let $\{x_n\}$ and $\{y_n\}$ be sequences such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$ in a metric space (X,d). If $d(x_n,y_n) \to 0$ as $n \to \infty$, then $x = y$.*

Definition 4.4 *(Choudhury et al., 2017). Let A and B be two non-empty subsets of a complete metric space (X,d). A coupling $F : X \times X \to X$ is called a Banach type coupling with respect to A and B if it satisfies the following inequality:*

$$d(F(x,y),F(u,v)) \leq \tfrac{k}{2}[d(x,u)+d(y,v)]$$

where $x,v \in A, y,u \in B$, and $k \in [0,1)$.

Definition 4.5 *(Bhaskar & Lakshmikantham, 2006). Let X be a non-empty set. An element $(x,y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x,y) = x$ and $F(y,x) = y$.*

Definition 4.6 *(Choudhury & Maity, 2014). An element* $(x,y) \in X \times X$ *where X is any nonempty set, is called a strong coupled fixed point of the mapping* $F : X \times X \to X$ *if (x, y) is the coupled fixed point and x = y that is, F(x, x) = x.*

Definition 4.7 *(Kirk et al., 2003). Let A and B be two non-empty subsets of a given set X. Any function* $T : X \to X$ *is said to be cyclic (with respect to A and B) if* $T(A) \subset B$ *and* $T(B) \subset A$.

Definition 4.8 *(Choudhury et al., 2017). Let* (X,d) *be a metric space A and B be two non-empty subsets of* X*. Then a function* $F : X \times X \to X$ *is said to be a coupling with respect to A and B if* $F(x,y) \in B$ *and* $F(y,x) \in A$ *where* $x \in A$ *and* $y \in B$.

Theorem 4.2 *(Rashid & Khan, 2018). Let A and B be two non-empty closed subsets of a complete metric space* (X,d)*. Let* $F : X \times X \to X$ *be Banach type coupling with respect to A and B. Then* $A \cap B \neq \emptyset$ *and F has a unique strong coupled fixed point in* $A \cap B$.

Definition 4.9 *(Lakshmikantham & Ciric, 2019). An element* $(x,y) \in X \times X$ *is called a coupled coincidence point of the mappings* $F : X \times X \to X$ *and* $g : X \to X$ *if F(x, y) = g(x) and F(y, x) = g(y).*

Definition 4.10 *(Rashid & Khan, 2018). An element* $(x,y) \in X \times X$ *is called a strong coupled coincidence point of the mappings* $F : X \times X \to X$ *and* $g : X \to X$ *if* $x = y$*. That is, F(x, x) = g(x).*

Definition 4.11 *A function* $\psi : [0,\infty) \to [0,\infty)$ *is called an altering distance function, if the following properties are satisfied:*

(i) ψ *is monotonically non-deceasing and continuous.*

(ii) $\psi(t) = 0$ *if and only if* $t = 0$.

Definition 4.12 *(Rashid & Khan, 2018). Let A and B be two non-empty subsets of a metric space* (X,d) *and* ψ, ϕ *are two altering distance functions. Then a coupling* $F : X \times X \to X$ *is said to be* (ψ,ϕ)*-contraction type coupling with respect to A and B if it satisfies the following inequality:*
$\psi(d(F(x,y),F(u,v))) \leq \psi(max\{d(x, u), d(y, v)\}) - \phi(max\{d(x, u), d(y, v)\})$
for any x, $v \in A$ *and y,* $u \in B$.

Definition 4.13 *(Rashid & Khan, 2018). Let A and B be any two non-empty subsets of a metric space (X,d) and T : $X \to X$ be a self map on X.*
Then T is said to be SCC-Map with respect to A and B, if

(i) *T(A) $\subseteq$ A and T(B) $\subseteq$ B,*

(ii) *T(A) and T(B) are closed in X.*

Theorem 4.3 *(Rashid & Khan, 2018).Let A and B be two non-empty closed subsets of a complete metric space (X,d) and F : $X \times X \to X$ is a (ψ,ϕ)-contraction type coupling (with respect to A and B). That is there exists altering distance functions ψ, ϕ such that*
$\psi(d(F(x,y),F(u,v))) \leq \psi\ (max\{d(x, u), d(y, v)\})$ - $\phi(max\{d(x, u), d(y, v)\})$
for any x, v $\in$ A and y, u $\in$ B. Then

(i) $A \cap B \neq \emptyset$.

(ii) *F has a unique strong coupled fixed point in $A \cap B$.*

Definition 4.14 *(Rao et al., 2013). The mappings F : $X \times X \to X$ and g : $X \to X$ are called weakly Compatible if g(F(x, y)) = F(gx, gy) and g(F(y, x)) = F(gy, gx) whenever gx = F(x, y) and gy = F(y, x).*

Definition 4.15 ((ψ, ϕ) Contraction type T-Coupling) *Let A and B be any two non-empty closed subsets of a complete metric space (X,d), ψ, ϕ are two altering distance functions, and T : $X \to X$ is SCC-Map on X with respect to A and B. Then a coupling F: $X \times X \to X$ is said to be (ψ,ϕ)-contraction type T-coupling with respect to A and B if*

$$\psi(d(F(x,y),F(u,v))) \leq \psi(max\{d(Tx,Tu),d(Ty,Tv)\}) - \phi(max\{d(Tx,Tu),d(Ty,Tv)\}) \tag{4.1}$$

for any $x,v \in A$ and $y,u \in B$.

4.2 Main Results

Inspired by the works of Rashid and Khan, (2018) and Aydi *et al.*, (2017), we have established coupled coincidence and coupled common fixed point theorem and we have also showed the existence and uniqueness of coupled coincidence and coupled common fixed points of (ψ, ϕ)-contraction type T-coupling in metric spaces as follows:

Theorem 4.4 *Let A and B be any two non-empty closed subsets of a complete metric space* (X,d) *and* $T : X \to X$ *is SCC-Map on X (with respect to A and B).*
Let F: $X \times X \to X$ *be* (ψ, ϕ)*-contraction type T-coupling (with respect to A and B) if there exist altering distance functions* ψ, ϕ *such that*

$$\psi(d(F(x,y),F(u,v))) \leq \psi(max\{d(Tx,Tu),d(Ty,Tv)\}) - \phi(max\{d(Tx,Tu),d(Ty,Tv)\}) \tag{4.2}$$

for any $x, v \in A$ *and* $y, u \in B$*, then*

(i) $T(A) \cap T(B) \neq \emptyset$

(ii) F and T have a coupled coincidence point in $A \times B$*.*

(iii) If F and T are weakly compatible, then F and T have a unique common coupled common fixed point in $A \times B$*.*

Proof: Since A and B are non-empty subsets of X and F is (ψ, ϕ)-contraction type-T coupling with respect to A and B, then for $x_0 \in A$ and $y_0 \in B$, we define the sequence $\{x_n\}$ and $\{y_n\}$ in A and B respectively such that,

$$Tx_{n+1} = F(y_n, x_n) \text{ and } Ty_{n+1} = F(x_n, y_n). \tag{4.3}$$

If for some $n, Tx_{n+1} = Ty_n$ and $Ty_{n+1} = Tx_n,$ then using (4.3), we have $Tx_n = Ty_{n+1} = F(x_n, y_n)$ and $Ty_n = Tx_{n+1} = F(y_n, x_n)$. This show that (x_n, y_n) is a coupled coincidence point of F and T. So we are done in this case. Thus we assume that $Tx_n \neq Ty_{n+1}$ or $Ty_n \neq Tx_{n+1}$ for all $n \geq 0$.

Let us define a sequence $\{D_n\}$ by

$$D_n = max\{d(Tx_{n+1}, Ty_n), d(Ty_{n+1}, Tx_n)\}. \tag{4.4}$$

Then, we have $\{D_n\} \subseteq [0, \infty)$ for all $n \in \mathbb{N}$. Now using (4.2) and (4.3) and the fact that $x_n \in A$ and $y_n \in B$ for all n, we have

$$\begin{aligned} \psi(d(Tx_n, Ty_{n+1})) &= \psi[d(F(y_{n-1}, x_{n-1}), F(x_n, y_n))] \\ &= \psi[d(F(x_n, y_n), F(y_{n-1}, x_{n-1}))] \\ &\leq \psi[max\{d(Tx_n, Ty_{n-1}), d(Ty_n, Tx_{n-1})\}] - \\ &\quad \phi[max\{d(Tx_n, Ty_{n-1}), d(Ty_n, Tx_{n-1})\}]. \end{aligned} \tag{4.5}$$

Using the properties of ϕ, we have
$\psi(d(Tx_n, Ty_{n+1})) \leq \psi(max\{d(Tx_n, Ty_{n-1}), d(Ty_n, Tx_{n-1})\})$.
Again using the properties of ψ, we get

$$d(Tx_n, Ty_{n+1}) \leq max\{d(Tx_n, Ty_{n-1}), d(Ty_n, Tx_{n-1})\}. \tag{4.6}$$

Now using (4.2) and (4.3) and the fact that $x_n \in A$ and $y_n \in B$ for all n, we have

$$\begin{aligned} \psi(d(Ty_n, Tx_{n+1})) &= \psi[d(F(x_{n-1}, y_{n-1}), F(y_n, x_n))] \\ &\leq \psi[max\{d(Tx_{n-1}, Ty_n), d(Ty_{n-1}, Tx_n)\}] - \\ &\quad \phi[max\{d(Tx_{n-1}, Ty_n), d(Ty_{n-1}, Tx_n)\}]. \end{aligned} \tag{4.7}$$

Now using the properties of ψ and ϕ, we get

$$d(Ty_n, Tx_{n+1}) \leq max\{d(Tx_{n-1}, Ty_n), d(Ty_{n-1}, Tx_n)\}. \tag{4.8}$$

By using (4.6) and (4.8), we get
$max\{d(Ty_n, Tx_{n+1}), d(Ty_{n+1}, Tx_n)\} \leq max\{d(Tx_n, Ty_{n-1}), d(Ty_n, Tx_{n-1})\}$.
That is,

$$max\{d(Tx_{n+1}, Ty_n), d(Ty_{n+1}, Tx_n)\} \leq max\{d(Tx_n, Ty_{n-1}), d(Ty_n, Tx_{n-1})\} \tag{4.9}$$

From (4.4) and (4.9), we have $D_n \leq D_{n-1}$ for all $n \geq 1$.
Therefore $\{D_n\}$ is monotonically decreasing sequence of non-negative real numbers.
There exists $r \geq 0$ such that, $\lim_{n\to\infty} D_n = r$. That is,

$$\lim_{n\to\infty}\{d(Tx_{n+1}, Ty_n), d(Ty_{n+1}, Tx_n)\} = r. \tag{4.10}$$

Suppose $r > 0$.
Since $\psi : [0,\infty) \to [0,\infty)$ is non-decreasing, then for all $a, b \in [0,\infty)$, we have

$$max\{\psi(a), \psi(b)\} = \psi(max\{a,b\}). \tag{4.11}$$

Now using (4.5), (4.9), and (4.11), we get

$$\begin{aligned}
\psi[max\{d(Tx_n, Ty_{n+1}), d(Ty_n, Tx_{n+1})\}] &= max[\psi\{d(Tx_n, Ty_{n+1}), d(Ty_n, Tx_{n+1})\}] \\
&\leq \psi[max\{d(Tx_n, Ty_{n-1}), d(Ty_n, Tx_{n-1})\}] - \\
&\quad \phi[max\{d(Tx_n, Ty_{n-1}), d(Ty_n, Tx_{n-1})\}].
\end{aligned}$$

Letting $n \to \infty$ in the above inequality, using (4.10) and continuities of ψ and ϕ, we have $\psi(r) \leq \psi(r) - \phi(r) < \psi(r)$ which is a contradiction. Hence $\phi(r) = 0$ since ϕ is an altering distance function. So $r = 0$. Hence $\lim_{n\to\infty} D_n = 0$.
That is $\lim_{n\to\infty} max\{d(Tx_n, Ty_{n+1}), d(Ty_n, Tx_{n+1})\} = 0$. Thus

$$\lim_{n\to\infty} d(Tx_n, Ty_{n+1}) = 0 \text{ and } \lim_{n\to\infty} d(Ty_n, Tx_{n+1}) = 0. \tag{4.12}$$

Now we define a sequence $\{R_n\}$ by $R_n = d(Tx_n, Ty_n)$ and show that $R_n \to 0$ as $n \to \infty$. By using (4.2) and (4.3), we get

$$\begin{aligned}
\psi(R_n) &= \psi(d(Tx_n, Ty_n)) \\
&= \psi(d(F(y_{n-1}, x_{n-1}), F(x_{n-1}, y_{n-1})) \\
&\leq \psi(max(\{d(Ty_{n-1}, Tx_{n-1})\})) - \phi(max(\{d(Ty_{n-1}, Tx_{n-1})\})).
\end{aligned} \tag{4.13}$$

By properties of ψ and ϕ, we have $R_n \leq d(Tx_{n-1}, Ty_{n-1}) = R_{n-1}$.
That is, $R_n \leq R_{n-1}$ for all $n \geq 1$. Thus $\{R_n\}$ is monotone decreasing sequence of non-negative real numbers. There exist $s \geq 0$. Suppose $s > 0$ such that,

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} d(Tx_n, Ty_n) = s. \tag{4.14}$$

Taking $n \to \infty$ in (4.13), using continuities of ψ and ϕ,
we have $\psi(s) \leq \psi(s) - \phi(s) < \psi(s)$ which is a contradiction. Hence $\phi(s) = 0$, but since ϕ is an altering distance function, we have $s = 0$. That is,

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} d(Tx_n, Ty_n) = 0. \tag{4.15}$$

Now using the triangle inequality on (4.12) and (4.15), we have

$$\lim_{n\to\infty} d(Tx_n, Tx_{n+1}) \leq \lim_{n\to\infty} d(Tx_n, Ty_n) + \lim_{n\to\infty} d(Ty_n, Tx_{n+1}) = 0 \tag{4.16}$$

$$\lim_{n\to\infty} d(Ty_n, Ty_{n+1}) \leq \lim_{n\to\infty} d(Ty_n, Tx_n) + \lim_{n\to\infty} d(Tx_n, Ty_{n+1}) = 0. \tag{4.17}$$

Now we prove that the sequences $\{Tx_n\}$ and $\{Ty_n\}$ are Cauchy sequences in $T(A)$ and $T(B)$ respectively. If possible, let $\{Tx_n\}$ or $\{Ty_n\}$ is not a Cauchy sequence. Then there exist $\varepsilon > 0$ and a sequence of positive integer $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, with $n(k) > m(k) > k$, we have

$$g_k = max\{d(Tx_{m(k)}, Tx_{n(k)}), d(Ty_{m(k)}, Ty_{n(k)})\} \geq \varepsilon. \tag{4.18}$$

And further, corresponding to m_k, we can choose n_k in such away that k is the smallest integer with $n(k) \geq m(k) > k$ and satisfying (4.18), then

$$max\{d(Tx_{m(k)}, Tx_{n(k)-1}), d(Ty_{m(k)}, Ty_{n(k)-1})\} < \varepsilon. \tag{4.19}$$

Now we show that:
$d(Ty_{n(k)}, Tx_{m(k)+1}) \leq max\{d(Tx_{m(k)}, Ty_{n(k)-1}), d(Ty_{m(k)}, Tx_{n(k)-1})\}$.

By using (4.2) and (4.3), we get:

$$\begin{aligned}\psi[d(Ty_{n(k)},Tx_{m(k)+1})] &= \psi[d(F(x_{n(k)-1},y_{n(k)-1}),F(y_{m(k)},x_{m(k)}))]\\ &\leq \psi[max\{d(Tx_{n(k)-1},Ty_{m(k)}),d(Ty_{n(k)-1},Tx_{m(k)})\}]-\\ &\quad \phi[max\{d(Tx_{n(k)-1},Ty_{m(k)}),d(Ty_{n(k)-1},Tx_{m(k)})\}].\end{aligned}$$

Using properties of ψ and ϕ, we have

$$d(Ty_{n(k)},Tx_{m(k)+1}) \leq max\{d(Tx_{n(k)-1},Ty_{m(k)}),d(Ty_{n(k)-1},Tx_{m(k)})\}. \tag{4.20}$$

Similarly we can show by the same steps that

$$d(Tx_{n(k)},Ty_{m(k)+1}) \leq max\{d(Ty_{n(k)-1},Tx_{m(k)}),d(Tx_{n(k)-1},Ty_{m(k)})\}. \tag{4.21}$$

From (4.20) and (4.21), we have

$$max\{d(Ty_{n(k)},Tx_{m(k)+1}),d(Tx_{n(k)},Ty_{m(k)+1})\} \leq \lambda \tag{4.22}$$

where $\lambda = max\{d(Tx_{m(k)},Ty_{n(k)-1}),d(Ty_{m(k)},Tx_{n(k)-1})\}$.
It is a fact that for $a,b,c \in \Re^+, max\{a+c,b+c\} = c+max\{a,b\}$.
Therefore by the triangle inequality on (4.19) and the above fact, we have

$$\begin{aligned}\lambda &= max\{d(Tx_{m(k)},Ty_{n(k)-1}),d(Ty_{m(k)},Tx_{n(k)-1})\}\\ &\leq max\{d(Tx_{m(k)},Tx_{n(k)-1})+d(Tx_{n(k)-1},Ty_{n(k)-1}),d(Ty_{m(k)},Ty_{n(k)-1})+\\ &\quad d(Ty_{n(k)-1},Tx_{n(k)-1})\}\\ &= d(Tx_{n(k)-1},Ty_{n(k)-1})+max\{d(Tx_{m(k)},Tx_{n(k)-1}),d(Ty_{m(k)},Ty_{n(k)-1})\}\\ &< d(Tx_{n(k)-1},Ty_{n(k)-1})+\varepsilon. \end{aligned}\tag{4.23}$$

Thus from (4.22) and (4.23), we get

$$max\{d(Ty_{n(k)},Tx_{m(k)+1}),d(Tx_{n(k)},Ty_{m(k)+1})\} < d(Tx_{n(k)-1},Ty_{n(k)-1})+\varepsilon \tag{4.24}$$

Now again by the triangle inequality, we have

$$\begin{aligned} d(Tx_{n(k)},Tx_{m(k)}) &\leq d(Tx_{n(k)},Ty_{n(k)})+d(Ty_{n(k)},Tx_{m(k)+1})+ \\ &\quad d(Tx_{m(k)+1},Tx_{m(k)}). \qquad (4.25) \\ d(Ty_{n(k)},Ty_{m(k)}) &\leq d(Ty_{n(k)},Tx_{n(k)})+d(Tx_{n(k)},Ty_{m(k)+1})+ \\ &\quad d(Ty_{m(k)+1},Ty_{m(k)}). \qquad (4.26) \end{aligned}$$

From (4.18), (4.24), (4.25), and (4.26), we get

$$\begin{aligned} \varepsilon \leq g_k &= max\{d(Tx_{n(k)},Tx_{m(k)}),d(Ty_{n(k)},Ty_{m(k)})\} \\ &\leq d(Tx_{n(k)},Ty_{n(k)})+max\{d(Tx_{m(k)},Tx_{m(k)+1}),d(Ty_{m(k)},Ty_{m(k)+1})\}+ \\ &\quad max\{d(Ty_{n(k)},Tx_{m(k)+1}),d(Tx_{n(k)},Ty_{m(k)+1})\} \\ &< d(Tx_{n(k)},Ty_{n(k)})+max\{d(Tx_{m(k)},Tx_{m(k)+1}),d(Ty_{m(k)},Ty_{m(k)+1})\}+ \\ &\quad d(Tx_{n(k)-1},Ty_{n(k)-1})+\varepsilon. \qquad (4.27) \end{aligned}$$

Taking $k \to \infty$ in (4.27) and using (4.15), (4.16), (4.17), and (4.18), we have $\varepsilon < \varepsilon$, which is a contradiction.
Hence $\{Tx_n\}$ and $\{Ty_n\}$ are Cauchy sequences in $T(A)$ and $T(B)$ respectively.
Since $T(A)$ and $T(B)$ are closed subset of a complete metric space X
$\{Tx_n\}$ and $\{Ty_n\}$ are convergent in $T(A)$ and $T(B)$ respectively.
Thus, there exist $r \in T(A)$ and $s \in T(B)$ such that,

$$Tx_n \to r \text{ and } Ty_n \to s \text{ as } n \to \infty. \qquad (4.28)$$

From (4.15), we have

$$d(Tx_n,Ty_n) \to 0 \text{ as } n \to \infty. \qquad (4.29)$$

Therefore, from (4.28) and (4.29), we have

$$s = r. \qquad (4.30)$$

As $r \in T(A)$ and $s \in T(B)$ we have $s = r \in T(A) \cap T(B)$.
This proves part (i) i.e., $T(A) \cap T(B) \neq \emptyset$.

Now since $r \in T(A)$ and $s \in T(B)$, there exist $a \in A$ and $b \in B$ such that $r = T(a)$ and $s = T(b)$.
From (4.28) and (4.30), we have

$$Tx_n \to T(a), Ty_n \to T(b) \tag{4.31}$$

$$T(a) = T(b). \tag{4.32}$$

Now by (4.2), (4.3), (4.31), and (4.32) and the triangle inequality we have
$d(r,F(a,b)) \leq d(r,Ty_{n+1}) + d(Ty_{n+1},F(a,b))$.
Letting $n \to \infty$, we get

$$d(r,F(a,b)) \leq \lim_{n\to\infty} d(Ty_{n+1},F(a,b)).$$

It follows that

$$\begin{aligned}
\psi(d(r,F(a,b))) &\leq \lim_{n\to\infty} \psi(d(F(x_n,y_n),F(a,b))) \\
&\leq \lim_{n\to\infty} \psi(max\{d(Tx_n,Ta),d(Ty_n,T(b))\}) - \\
&\quad \lim_{n\to\infty} \phi(max\{d(Tx_n,T(a)),d(Ty_n,T(b))\}) \\
&= \psi(max\{d(r,T(a)),d(s,T(b))\}) - \\
&\quad \phi(max\{d(r,T(a)),d(s,T(b))\}) \\
&\leq \psi(max\{d(r,T(a)),d(s,T(b))\}).
\end{aligned}$$

Similarly $\psi(d(s,F(b,a))) \leq \psi(max\{d(s,T(b)),d(r,T(a))\})$.
Since

$$\begin{aligned}
\psi(max\{d(r,F(a,b)),d(s,F(b,a))\} &= max\{\psi(d(r,F(a,b))),\psi(d(s,F(b,a)))\} \\
&\leq \psi(max\{d(s,T(b)),d(r,T(a))\}) = 0. \\
max\{d(r,F(a,b)),d(s,F(b,a))\} &= 0.
\end{aligned}$$

So that $F(a,b) = r$ and $F(b,a) = s$.
Hence $F(a,b) = T(a) = r$ and $F(b,a) = T(b) = s$.
Therefore $(a,b) \in A \times B$ is the coupled coincidence point of F and T.

Now we show that the coupled coincidence point is unique.
Let (a',b') be another coupled coincidence point of F and T.
So we will prove that $T(a) = T(a')$ and $T(b) = T(b')$. The proof is as follows.
Suppose $T(a) \neq T(a')$.using (4.2)

$$\begin{aligned}
\psi(d(T(a),T(a'))) &= \psi(d(F(a,b),F(a',b'))) \\
&\leq \psi(max\{d(T(a),T(a')),d(T(b),T(b'))\}) - \\
&\quad \phi(max\{d(T(a),T(a')),d(T(b),T(b'))\}) \\
&= \psi(max\{d(T(a),T(a')),d(T(a),T(a'))\}) - \\
&\quad \phi(max\{d(T(a),T(a')),d(T(a),T(a'))\}) \\
&= \psi(d(T(a),T(a'))) - \phi(d(T(a),T(a'))) \\
&< \psi(d(T(a),T(a')))
\end{aligned}$$

which is a contradiction. Hence $\phi(d(T(a),T(a'))) = 0$ (since ϕ is an altering distance function) which in turn implies that $d(T(a),T(a')) = 0$. Hence $T(a) = T(a')$.
Similarly,Suppose $T(b) \neq T(b')$.using(4.2)

$$\begin{aligned}
\psi(d(T(b),T(b'))) &= \psi(d(F(b,a),F(b',a'))) \\
&\leq \psi(max\{d(T(b),T(b')),d(T(a),T(a'))\}) - \\
&\quad \phi(max\{d(T(b),T(b')),d(T(a),T(a'))\}) \\
&= \psi(max\{d(T(b),T(b')),d(T(b),T(b'))\}) - \\
&\quad \phi(max\{d(T(b),T(b')),d(T(b),T(b'))\}) \\
&= \psi(d(T(b),T(b'))) - \phi(d(T(b),T(b'))) \\
&< \psi(d(T(b),T(b'))).
\end{aligned}$$

So that $\phi(d(T(b),T(b'))) = 0$ (since ϕ is an altering distance function) which in turn implies that $d(T(b),T(b')) = 0$. Hence $T(b) = T(b')$.
Hence the coupled coincidence point of F and T is unique.
Using (4.32) $T(a) = T(b)$.
Thus $(T(a),T(a))$ is a unique coupled point of coincidence of the mappings F and T with respect to A and B.

Now we show that F and T have a unique coupled common fixed point. For this let $T(a)=z$, then, we have $z=T(a)=F(a,a)$.
By the w-compatibility of F and T, we have

$$Tz=T(T(a))=T(F(a,a))=F(T(a),T(a))=F(z,z).$$

Thus $(T(z),T(z))$ is coupled point of coincidence of F and T.
By the uniqueness of coupled point of coincidence of F and T we have $T(z)=T(a)$.
Thus, we obtain $z=T(z)=F(z,z)$.
Therefore (z,z) is the unique coupled common fixed point of F and T. □

Remark 4.1 *If we take $T=I$ (the identity map) and A and B be any two non-empty closed subsets of a complete metric space, then Theorem 4.4 will be reduced to Theorem 4.3 of Rashid and Khan, (2018).*

The following is an example which supports our main result.
Example: Let $X=[0,5]$and d be the usual metric defined on X by

$$d(x,y)=|x-y|.$$

Let $A=\{1\}$ and $B=\{1,2\}$. Then A and B are closed subsets of X.
We define $F: X\times X\to X$ by $F(x,y)=min\{x,y\}$, for all $x,y\in X$.
Let $T: X\to X$ be defined by

$$T(x)=\begin{cases}1 & \text{if } 0\leq x<2\\ 2 & \text{if } 2\leq x\leq 5.\end{cases}$$

Also we define $\psi,\phi:[0,\infty)\to[0,\infty)$ by $\phi(t)=t^2$ and $\psi(t)=t^3$.
Clearly ψ and ϕ are altering distances functions.
$T(A)=\{1\}$ and $T(B)=\{1,2\}$.
So, $T(A)$ and $T(B)$ are closed subsets of a complete metric space $X=[0,5]$.
We note that $T: X\to X$ is a SCC-Map.

Now we show that F is T-coupling with respect to A and B as $T(A) \cap B = \{1\}$ and $T(B) \cap A = \{1\}$.
So, for all $x \in A$ and $y \in B$ we have $F(x,y) = 1 \in B$ and $F(y,x) = 1 \in A$
i.e., $F(x,y) \in T(A) \cap B$ and $F(y,x) \in T(B) \cap A$ which show that F is T-coupling with respect to A and B.
Now it remains to prove that F is (ψ, ϕ)-contraction type T-coupling w.r.t. A and B.
Let $x, v \in A$ and $y, u \in B$ i.e., $x = 1$ and $y = 1, 2$. Four cases will arise for y and u.
Case (i): $x = v = 1$ and $y = u = 1$.
Case (ii): $x = v = 1$ and $y = 1, u = 2$.
Case (iii): $x = v = 1$ and $y = 2, u = 1$.
Case (iv): $x = v = 1$ and $y = u = 2$.
Case (i). When $x = v = 1$ and $y = u = 1$, we have $F(x,y) = F(1,1) = 1$, $F(u,v) = F(1,1) = 1$, $T(x) = T(y) = T(u) = T(v) = T(1) = 1$, $d(1,1) = 0$, and

$$\begin{aligned}
\psi(d(F(x,y),F(u,v))) &\leq \psi(max\{d(T(x),T(u)),d(T(y),T(v))\}) \\
&\quad - \phi(max\{d(T(x),T(u)),d(T(y),T(v))\}), \\
\psi(0) &\leq \psi(max\{0,0\}) - \phi(max\{0,0\}) \\
0 &\leq \psi(0) - \phi(0) = 0
\end{aligned}$$

which proves case (i).
For **case (ii)**. When $x = v = 1$ and $y = 1, u = 2$, we have $F(x,y) = F(1,1) = 1$, $F(u,v) = F(2,1) = 1, T(x) = T(y) = T(v) = T(1) = 1$,
$T(u) = T(2) = 2, d(1,1) = 0$, $d(1,2) = 1$, and

$$\begin{aligned}
\psi(d(F(x,y),F(u,v))) &\leq \psi(max\{d(T(x),T(u)),d(T(y),T(v))\}) - \\
&\quad \phi(max\{d(T(x),T(u)),d(T(y),T(v))\}). \\
\psi(0) &\leq \psi(max\{1,0\}) - \phi(max\{1,0\}) \\
0 &\leq \psi(1) - \phi(1) = 0
\end{aligned}$$

which proves case (ii). **case (iii)** . When $x = v = 1$ and $y = 2, u = 1$, we have $F(x,y) = F(1,2) = 1, F(u,v) = F(1,1) = 1, T(x) = T(u) = T(v) = 1, T(y) = T(2) = 2, d(1,1) = 0$,

$d(2,1) = 1$, and

$$\begin{aligned}
\psi(d(F(x,y),F(u,v))) &\leq \psi(max\{d(T(x),T(u)),d(T(y),T(v))\}) - \\
&\quad \phi(max\{d(T(x),T(u)),d(T(y),T(v))\}) \\
\psi(0) &\leq \psi(max\{0,1\}) - \phi(max\{0,1\}) \\
0 &\leq \psi(1) - \phi(1) = 0
\end{aligned}$$

which proves case (iii).
For **case (iv)** . When $x = v = 1$ and $y = u = 2$, we have
$F(x,y) = F(1,2) = 1, F(u,v) = F(2,1) = 1, T(x) = T(v) = 1,$
$T(y) = T(u) = T(2) = 2, d(1,1) = 0, d(1,2) = d(2,1) = 1$, and

$$\begin{aligned}
\psi(d(F(x,y),F(u,v))) &\leq \psi(max\{d(T(x),T(u)),d(T(y),T(v))\}) - \\
&\quad \phi(max\{d(T(x),T(u)),d(T(y),T(v))\}). \\
\psi(0) &\leq \psi(max\{1,1\}) - \phi(max\{1,1\}) \\
0 &\leq \psi(1) - \phi(1) = 0
\end{aligned}$$

which proves case (iv).
From the cases (i)-(iv) F and T satisfy all the conditions of Theorem 4.4.
Thus F and T have a strong coupled fixed points in $A \cap B$.
Clearly $T(A) \cap T(B) = \{1\} \neq \emptyset$.
1 is the unique strong coupled coincidence point and (1, 1) is the unique coupled common fixed point of F and T in $A \cap B$ as $T(1) = F(1,1) = min\{1,1\} = 1$.

Chapter 5

Conclusion and Future Scope

Rashid & Khan (2018) established and proved theorem of coupled coincidence Point of (ψ, ϕ)-contraction type coupling in metric spaces. In this thesis, we established and proved existence of coupled coincidence point and existence and uniqueness of coupled common fixed point theorem for (ψ, ϕ)-contraction type T-coupling in metric spaces. Where ψ and ϕ are two altering distance function and T is SCC-Map. We also provided an example in support of our main result. Our work extended coupled coincidence point of (ψ, ϕ)-contraction type coupling in metric spaces to coupled coincidence and coupled common fixed points of (ψ, ϕ)-contraction type T-coupling in metric spaces. Our result extend and generalized comparable results in the literatures.

Fixed point theory is one of the active and vigorous areas of research in mathematics and other sciences. There are several published results related to existence and uniqueness of coupled coincidence point and coupled common fixed point theorem for (ψ, ϕ)-contraction type coupling in metric spaces. So, it is recommend to postgraduate students and other interested researchers to exploit this opportunity and conduct their research work by setting different coupled fixed point theorems on certain contraction type coupling in metric spaces.

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