# A COUPLED COINCIDENCE AND COUPLED COMMON FIXED POINT THEOREM IN COMPLETE QUASI - b- METRIC SPACES

**A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS, COLLEGE OF NATURAL SCIENCES, JIMMA UNIVERSITY IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS**

**BY:**

**ALEMU ABUTE**

**UNDER SUPERVISSION OF:**

**KIDANE KOYAS ( PhD)**

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**Jimma, Ethiopia**

## DECLARATION

I undersigned declare that, this thesis entitled "A coupled coincidence and coupled common fixed point theorem in a complete quasi- b-metric space" is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged.

Name: Alemu Abute

Signature: ---------------------------

Date: ---------------------------------

The work has been done under the supervision of:

Name**:** Kidane Koyas (PhD)

Signature: ---------------------------

Date: ---------------------------------

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# ABSTRACT

The purpose of this study was to establish the existence of coupled coincidence and existence and uniqueness of coupled common fixed point theorem in partially ordered complete quasi-b-metric spaces. In this study analytical design has been employed and secondary sources of data such as Journals, Internet etc were used. The procedure that we followed was the standard procedures used in the published work of Bota *et al*. and Lakshmikantham and Ciric. We proved our established theorem for the existence of coupled coincidence point and existence and uniqueness of coupled common fixed point in the setting of partially ordered quasi-b-metric spaces and we also provided examples in support of our main result. Our work extended coupled fixed point to coupled common fixed point result.

# CHAPTER ONE:

# INTRODUCTION

## 1.1 Background of the study

**Notation**: Throughout this thesis, we denote by

$\mathbb{R} =$ The set of all real numbers,

$\mathbb{R}^+ = [0, +\infty)$,

$\mathbb{N} =$ The set of all natural numbers.

Let $X$ be a non empty set. A map $T: X \to X$ is said to be a self-map of $X$. We say that $x \in X$ is a fixed point of $T$ if $Tx = x$ and we denote the set of fixed points of T by Fix $(T)$.

**Example** Let $X$ be the set of real numbers then the fixed points of $T: X \to X$ defined by

$Tx = \frac{x^2+12}{7}$ are 3 and 4.

Let $(X, d)$ be a metric space. A self-map $T: X \to X$ is said to be a contraction map, if there exists $k \epsilon [0,1)$ such that

$$d(Tx, Ty) \leq kd(x, y), \text{ for all } x, y \in X.$$

The theory of fixed point is one of the most powerful and popular tools of modern mathematics. Its use is not only confined to pure and applied mathematics but also it serves as a bridge between analysis and topology and also to examine the quantitative problems involving certain maps and space structures required in various areas such as: economics, chemistry, biology, computer science, engineering and others (Banach, 1922; Beg and Butt, 2013; Chandok *et al.*, 2015).

The study of fixed point of maps satisfying certain contractive conditions has been at the center of rigorous research activity (Malhotra and Bansal, 2015). Due to its beautiful assertion and successful way of solving the implicit function existence theorem, the existence of a solution for a differential equation with initial condition, fixed-point theory caught the attention of scholars and it promotes people's inspirations towards in-depth and extensive research.

The Polish mathematician Banach (1922) celebrated the original theorem on metric fixed point theory which is known as Banach Contraction Principle theorem that can be seen as follows.

**Theorem 1.1.1** (Banach, 1922) Let $(X, d)$ be a complete metric space and $T: X \longrightarrow X$

a contraction map, that is, a map satisfying

$$d(Tx, Ty) \leq kd(x, y), for\ x, y \epsilon X, \tag{1.1}$$

where $0 \leq k < 1$ is a constant. Then $T$ has a unique fixed point $p$ in $X$.

"This pioneer result of Banach can be considered as a revolution in fixed point theory and hence in non-linear functional analysis and it has been used, generalized, extended and improved in various ways by several Mathematicians, Scientists, Economists for single valued and multi valued maps under different contractive conditions and various spaces."

Kannan (1968) proved a fixed point theorem for the map not necessarily continuous.

**Theorem 1.1.2** (Kannan, 1968) Let $(X, d)$ be a complete metric space and $T: X \rightarrow X$ be a self - map satisfying the inequality

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)], \tag{1.2}$$

where $k \in [0, \frac{1}{2})$ and for all $x,\ y \in X$. Then $T$ has a unique fixed point.

Maps satisfying inequality (1.2) are called Kannan type mappings.

Further, Chatterjea (1972) introduced a new concept which is different from that of Banach (1922) and Kannan (1968) for contraction type map and gave a new direction to the study of fixed point theory as follows:

**Theorem 1.1.3** (Chatterjea, 1972) If $T: X \rightarrow X$ where $(X,\ d)$ is a complete metric space, satisfies the inequality

$$d(Tx, Ty) \leq k[d(x, Ty) + d(y, Tx)], \tag{1.3}$$

where $k \in [0, \frac{1}{2})$ and for all $x, y \in X$. Then $T$ has a unique fixed point.

Maps satisfying inequality (1.3) is called Chatterjea type mapping.

In1977, Rhoades showed that the works of Banach, Kannan and Chatterjea are independent. For instance, Banach contraction map is continuous, but Kannan type maps need not be continuous except at the fixed point.

Zamfirscu (1979) established the following theorem which is a generalization of Banach contraction principle (Banach, 1922), Kannan's theorem (Kannan, 1968) and Chatterjea's theorem (Chatterjea, 1972).

**Theorem 1.1.4** (Zamfirescu, 1979) Let $(X, d)$ be a complete metric space and if a map $T: X \to X$ for all $x, y \in X$ and some $\alpha \in [0, 1)$, $\beta, \gamma \in [0, \frac{1}{2})$ satisfies at least one of

(i) $d(Tx, Ty) \leq \alpha d(x, y)$,

(ii) $d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)]$, (1.4)

(iii) $d(Tx, Ty) \leq \gamma[d(x, Ty) + d(y, Tx)]$.

Then $T$ has a fixed point in $X$.

Maps satisfying at least one of the above inequalities are called Zamfirscu maps.

On the other hand we can see so many generalizations of metric spaces. Some of such generalizations are as follows: cone metric space (Huang and Zhang, 2007), dislocated metric space (Hitzler, 2001), quasi-metric space (Wilson, 1931), dislocated quasi-metric space (Zeyada *et al*., 2005). Czerwik (1993) introduced b-metric space as a generalization of metric space. Finally, many other generalized b-metric spaces such as quasi-b-metric spaces introduced by Shah and Hussain (2012), b-metric like spaces (Alghamdi *et al*., 2013), quasi-b-metric like spaces (Zhu *et al*., 2014), quasi-partial b-metric spaces (Gupta and Gautam, 2015) were introduced.

Jungck (1976) proved common fixed point results in metric spaces for a pair of commuting maps. In 1987, mixed monotone maps were introduced by Guo and Lakshmikantham. Lakshmikantham and Ciric (2009) introduced the concept of commuting maps in the context of coupled fixed points and proved coupled coincidence and coupled common fixed points in partially ordered metric spaces. Choudhury and Kundu (2010) generalized the concept of commuting maps in the context of coupled fixed points by introducing compatible maps and established the existence of coupled coincidence points in partially ordered metric spaces.

Bota *et al*. (2015) have recently proved some coupled fixed point theorem for mixed monotone maps in complete b-metric space. The result in Bota *et al*. (2015) extended some results in

(Urs, 2013).

We can see this result from the following theorem:

**Theorem 1.1.5** (Bota *et al*., 2015) Let $(X, d)$ be a complete b-metric spaces with $s \geq 1$ and $T: X \times X \to X$ a continuous map with the mixed monotone property on $X \times X$. Assume the following conditions are satisfied:

i. there exists $k \in [0, \frac{1}{s})$ such that

$$d\big(T(x,y), T(u,v)\big) \leq \frac{k}{2}[d(x,u) + d(y,v)], \ for\ all\ x \succcurlyeq u, y \preccurlyeq v\ ; \tag{1.5}$$

ii. there exist $x_0, y_0 \in X$ such that $x_0 \preccurlyeq T(x_0, y_0)$ and $y_0 \succcurlyeq T(y_0, x_0)$. Then there exist $x, y \in X$ such that $x = T(x, y)$ and $y = T(y, x)$.

Motivated and inspired by the result of Bota *et al*. (2015) the researcher has tried to extend this result to a pair of maps $F: X \times X \to X$ and $g: X \to X$ in the setting of quasi-b-metric spaces. We also provided an example in support of our main result

## 1.1 Statement of the problem

This study focused on establishing and proving the existence of coupled coincidence point and existence and uniqueness of coupled common fixed point theorem in complete quasi b-metric space verifying the main results by providing supportive examples.

## 1.3 Objectives of the study

### 1.3.1 General objective

The main objective of this study was to establish and prove existence of coupled coincidence point and existence and uniqueness of coupled common fixed point theorem in complete quasi b-metric spaces and to provide an example in support of the main result.

### 1.3.2 Specific objectives

This study has the following specific objectives:

1. To prove existence of coupled coincidence and coupled common fixed point in complete quasi -b-metric spaces.
2. To show the uniqueness of coupled common fixed point in complete quasi- b-metric spaces.
3. To provide an example in support of the main result of this study.

## 1.4 Significance of the study

The result of this study may have the following importance:

- Help the researcher to develop scientific research writing skill and scientific communication in mathematics
- Provide some background information for other researchers who have interest in this area.
- Have application in studying the existence of a unique solution to a nonlinear integral equation.

## 1.5 Delimitation of the study

This study focuses only on proving the existence and uniqueness of coupled coincidence and coupled common fixed point in complete quasi b-metric spaces.

# CHAPTER TWO:

# LITERATURE REVIEW

It is well known that the theoretical framework of metric fixed point theory has been an active research field and the contraction mapping principle is one of the most important theorems in functional analysis. Many authors have devoted their attention to generalizing metric spaces and the contraction mapping principle. Some problems, particularly, the problem of the convergence of measurable functions with respect to measure leads Czerwik (1993) to the generalization of metric space and introduced the concept of b-metric space and proved Banach's contraction theorem in so called b- metric space. After Czerwik (1993) many papers have been published containing fixed point results on b-metric spaces.

Cone metric space was introduced by (Huang and Zhang, 2007), while studying cone metric spaces Khamsi (2010) re-introduced the b-metric to which he gave the name of metric type space. Several papers have been published in metric type spaces which contain fixed point result for single valued and multi valued functions (Ahmed, 2012; Jovanovic *et al*., 2010; Kir and Kizitune, 2013).

Abbas and Rhoads (2009) obtained some common fixed point theorems for non-commuting maps without continuity satisfying different contractive conditions in the setting of generalized metric spaces.

Alghamdi *et al*. (2013) introduced the notion of b-metric like space which generalized the notion of b-metric space, where they proved some new fixed point result in b-metric like space. In 2013, Shukla introduced the concept of partial b-metric space and gave some fixed point results and examples in such a space.

In 2012, Shah and Hussain introduced the concept of quasi-b-metric spaces and established some fixed point theorems in quasi-b-metric spaces.

In fact, recently, the existence of coupled fixed points, coupled coincidence points, coupled common fixed points and common fixed points for nonlinear maps with two variables has attracted more and more attention. For example Bhaskar and Lakshmikantham (2006) investigated some coupled fixed point theorem for maps satisfying mixed monotone property, and they also discussed an application of their result by investigating the existence and uniqueness of the solution for periodic boundary value problems. Sabetghadam *et al.* (2009) extended some results in (Bhaskar and Lakshimkantham, 2006) to cone metric spaces; Lakshimkantham and Ciric (2009) proved several coupled coincidence and coupled common fixed point theorems for nonlinear contractive maps in partially ordered complete metric spaces and they introduced the concept of mixed $g$ - monotone maps and proved coupled coincidence and coupled common fixed point theorems for commuting maps which extended the theorems due to Bhaskar and Lakshimkantham(2006).

Karapinar (2010) extended Some results of (Lakshimkantham and Ciric, 2009) to cone metric spaces; Successively, Choudhury and Kundu (2010) introduced the notion of compatibility of maps in partially ordered metric spaces and used this notion to establish a coupled coincidence point result which extended the works of Bhaskar and Lakshmikantham (2006) and Lakshmikantham and Ciric (2009).  In 2010, Sedghi *et al.* proved a coupled fixed point theorem for contractive maps in complete fuzzy metric spaces.

Ding *et al*. (2012) established some coupled coincidence and coupled common fixed point theorem in partially ordered metric spaces under some generalized contractive condition.

# CHAPTER THREE:

# METHODOLGY OF THE STUDY

## 3.1 Study area and period

The study was conducted in Jimma University at the Department of Mathematics from September 2016 to June 2017 G.C. Conceptually the study focused on the existence of a coupled coincidence and existence and uniqueness of coupled common fixed point in complete partially ordered quasi-b-metric spaces .

## 3.2 Study design

This study employed analytical design.

## 3.3 Source of information

The relevant sources of information for this study were published articles, journals and related study results from internet.

## 3.4 Mathematical procedures

In this research work the mathematical procedures that the researcher followed were the standard procedures used in the published work of Bota *et al*. (2015) and

Lakshmikantham and Ciric (2009).

These procedures were:

- Constructing Cauchy sequences in the setting of complete quasi-b-metric spaces.
- Working for coupled coincidence and coupled common fixed point by using the limit of constructed Cauchy sequences.

# CHAPTER FOUR:
# DISCUSSION AND RESULTS

## 4.1 Preliminaries

**Definition 4.1.1** (Wilson, 1931) A quasi-metric space is a pair $(X, d)$ where $d: X \times X \to \mathbb{R}^+$ satisfies the following conditions for all $x, y, z \in X$;

(i) $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

**Example 4.1.2** (Dung, 2014) Let $X = \mathbb{R}$ and $d: X \times X \to \mathbb{R}^+$ be a function defined by

$$d(x, y) = \begin{cases} x - y & \text{if } x \geq y, \\ 1 & \text{if } x < y. \end{cases}$$

for all $x, y \in X$ is a quasi metric, but it is not a metric on $X$.

Proof: It is clear that $d(x, y) = 0$ if and only if $x = y$.

For all $x, y, z \in X$, we consider the following two cases

**Case (i):** $x \geq y$, we have $d(x, y) = x - y$.

If $z < y$, then $d(x, z) = x - z$ and $d(z, y) = 1$.

If $y \leq z < x$, then $d(x, z) = x - z$ and $d(z, y) = z - y$.

If $x \leq z$, then $d(x, z) = 1$ and $d(z, y) = z - y$.

So we have $d(x, y) \leq d(x, z) + d(z, y)$.

**Case ( ii):** $x < y$. We have $d(x, y) = 1$.

If $z < x$, then $d(x, z) = x - z$ and $d(z, y) = 1$.

If $x \leq z < y$, then $d(x, z) = 1$ and $d(z, y) = 1$.

If $y \leq z$, then $d(x, z) = 1$ and $d(z, y) = z - y$.

So we have $d(x, y) \leq d(x, z) + d(z, y)$

By the above, $d$ is a quasi-metric on $\mathbb{R}$.

Since $d(0,2) = 1 \neq d(2,0) = 2, d$ is not symmetric. Therefore $d$ is not a metric on$\mathbb{R}$.

**Definition 4.1.3** (Czerwik, 1993) Let $X$ be a non-empty set and s ≥ 1 be a given real number. A function $d: X \times X \to \mathbb{R}^{+}$is called a b-metric provided that for all $x, y, z \in X$,

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair $(X, d)$ is called a b-metric space. It is clear that definition of b-metric space is an extension of the usual metric space. Clearly any metric space is a b-metric space with $s = 1$.

The following example shows that a b-metric space is a real generalization of metric space.

**Example 4.1.4** (Czerwik, 1993) let $X = \mathbb{R}$ together with the mapping, $d(x, y) = (x - y)^2$ for all $x, y \in X$ is a b-metric space with $s = 2$, but $d$ is not metric on $\mathbb{R}$.

**Proof:** We need to show that $d$ satisfies definition 4.1.3.

$(i)$ $d(x, y) = (x - y)^2 = \left(-1(y - x)\right)^2 = (-1)^2(y - x)^2 = 0 \Leftrightarrow x = y$. Hence $(i)$ is satisfied.

$(ii)$ $d(x, y) = (x - y)^2 = \left(-1(y - x)\right)^2 = (-1)^2(y - x)^2 = (y - x)^2 = d(y, x)$ for all $x, y \in X$. Hence, $(ii)$ is satisfied.

$(iii)$ Let $x, y, z \in X$, $d(x, z) = (x - y + y - z)^2$. Now let, $u = x - y$ and $v = y - z$, so $x - z = x - y + y - z = u + v$.

The convexity of the function $f(x) = x^2$, implies that $\left(\frac{a+b}{2}\right)^2 \leq \frac{1}{2}[a^2 + b^2]$.

Hence, $d(x, z) = (x - y + y - z)^2 = (u + v)^2 \leq 2[(u)^2 + (v)^2]$

$$= 2[(x - y)^2 + (y - z)^2]$$

$$= 2[d(x,y) + d(y,z)].$$

Hence all conditions of definition (4.1.3) are satisfied. Therefore, $(X, d)$ is a b-metric space with $s = 2$.

To show $d$ is not metric let $x = 2,\ y = 3,\ z = 4,$ with $s = 1$

$d(x,y) = d(2,3) = (2-3)^2 = 1,$

$d(x,z) = d(2,4) = (2-4)^2 = 4,$

$d(y,z) = d(3,4) = (3-4)^2 = 1.$

Since $4 = d(x,z) > 2 = d(x,y) + d(y,z)$, triangle inequality is not satisfied. Therefore $d$ is not metric.

**Definition 4.1.5 (**Shah and Hussain, 2012) Let $X$ be a non-empty set. A real valued function $d: X \times X \to \mathbb{R}^+$ is said to be a quasi-b-metric on $X$ with the constant $s \geq 1$ if the following conditions are satisfied.

(i) $d(x,\ y) = 0$ if and only if $x = y$,

(ii) $d(x,\ y) \leq s[d(x,z) + d(z,y)]$.

The pair $(X,\ d)$ is called a quasi-b-metric space. Observe that if $s = 1$, then the ordinary triangle inequality in quasi-metric space is satisfied, however it does not hold true when $s > 1$. Thus the class of quasi b-metric spaces is effectively larger than that of ordinary quasi-metric spaces. That is, every quasi-metric space is a quasi-b-metric space but the converse need not be true. This idea is explained by the following examples.

**Example 4.1.6** (Shah and Hussain, 2012) Let $X = C[0,\ 1]$ be the set of all continuous real valued functions defined on $[0,1]$.

Define $d: X \times X \to \mathbb{R}^+$ by

$$d(f,\ g) = \begin{cases} \int_0^1 [g(t) - f(t)]^3 dt,\ if\ f \leq g, \\ \int_0^1 [f(t) - g(t)]^3 dt,\ if\ f \geq g\,. \end{cases}$$

It holds true that, $d(f,\ g) \geq 0$ for all $f, g \in X,$ and $d(f,\ g) = 0$ if and only if $f = g$.

Also we have $d(f, g) = d(g, f\ ) = 0$ if and only if $f = g$, so that $d$ is not symmetric.

Since $d$ is not symmetric as shown above, hence $d$ is not b-metric.

Now we show $d$ is not quasi metric.

Let $f(t) = 2t,\ g(t) = 5t$ and $h(t) = 6t$ for $t \in [0,\ 1]$. Then

$d(f,\ \ h) = 16,\ d(f,\ \ g) = \frac{27}{4},\ \ \ d(g,\ \ h) = \frac{1}{4}.$ That is,

$d(f,\ \ h) > d(f,\ \ g) + d(g,\ \ h)$. That is,

$d(f, h) = 16 \nleq \frac{27}{4} + \frac{1}{4} = d(f, g) + d(g, h).$

This implies for $s = 1$ the triangle inequality is not satisfied. Therefore $d$ is not quasi metric on $X$.

From the above discussion it follows that $(X, d)$ is a quasi-b-metric space which is not a quasi-metric, b-metric and metric space.

**Remark 4.1.7** "A quasi-metric space is a quasi-b-metric space with a constant $s = 1$. Therefore the class of quasi-b-metric spaces is larger than that of the classes of metric spaces, quasi metric and b-metric spaces."

**Definition 4.1.8** A partially ordered set (Poset) is a system $(X,\ \preccurlyeq)$ where $X$ is non-empty set and $\preccurlyeq$ is a binary relation of $X$ satisfying for all $x, y, z \in X$:

$(i)$ $x \preccurlyeq x$ (Reflexivity: every element is related to itself),

$(ii)$ if $x \preccurlyeq y$ and $y \preccurlyeq x$ then $x = y$ (antsymmetry),

$(iii)$ If $x \preccurlyeq y$ and $y \preccurlyeq z$ then $x \preccurlyeq z$ (transitivity).

A set with a partial order is called a partially ordered set.

**Example 4.1.9**

$(i)$ If $X$ is any set $(P(X), \subseteq)$ is a partially ordered set. Where $P(X)$ is the power set of $X$.

$(ii)$ On the set of natural numbers $N$, define $m \preccurlyeq n$ if $m$ divides $n$ then $(N, \preccurlyeq)$ is a partially ordered set.

**Example 4.1.10** Let $X$ be a non-empty set. Then $(X, d, \preccurlyeq)$ is called partially ordered metric spaces if:

$(i)$ $(X, d)$ is a metric space and

$(ii)$ $(X, \preccurlyeq)$ is a partially ordered set.

**Definition 4.1.11 (**Bahaskar and Lakshmikantham, 2006) Let $(X, \preccurlyeq)$ be a partially ordered set and $F: X \times X \to X$. The mapping $F$ is said to have the mixed monotone property if $F$ is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument, That is, for any $x, y \in X$,

$x_1, x_2 \in X$, $x_1 \preccurlyeq x_2 \Rightarrow F(x_1, y) \preccurlyeq F(x_2, y)$,

and

$y_1, y_2 \in X$, $y_1 \preccurlyeq y_2 \Longrightarrow F(x, y_1) \succcurlyeq F(x, y_2)$.

**Definition 4.1.12 (**Lakshmikantham and Ciric, 2009) Let $(X, \preccurlyeq)$ be a partially ordered set and $F: X \times X \to X$ and $g: X \to X$ be two maps. We say that $F$ has the mixed g-monotone property if $F$ is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument, That is , if

for all $x_1, x_2 \in X$, $gx_1 \preccurlyeq gx_2$ implies $F(x_1, y) \preccurlyeq F(x_2, y)$, for any $y \in X$,

and

for all $y_1, y_2 \in X$, $gy_1 \preccurlyeq gy_2$ implies $F(x, y_1) \succcurlyeq F(x, y_2)$, for any $x \in X$.

**Definition 4.1.13** (Lakshmikantham and Ciric, 2009) An element $(x,\ y) \in X \times X$ is called a coupled coincidence point of the mappings $F: X \times X \to X$ and $g: X \to X$ if $F(x,\ y) = g(x)$ and $F(y,\ x) = g(y)$.

**Definition 4.1.14** (Bhaskar and Lakshmikantham, 2006) Let $(X, d)$ be a metric space and $F: X \times X \to X$ is a map. A point $(x, y) \in X \times X$ is called a coupled fixed point of $F$ if

$x = F(x, y)$ and $y = F(y, x)$.

**Definition 4.1.15** (Lakshmikantham and Ciric, 2009) An element $(x,\ y) \in X \times X$ is called a coupled common fixed point of the maps $F: X \times X \to X$ and $g: X \to X$ if $x = gx = F(x, y)$ and $y = gy = F(y, x)$.

**Definition 4.1.16** (Lakshmikantham and Ciric, 2009) Let $X$ be a non-empty set and $F: X \times X \to X$ and $g: X \to X$, we say $F$ and $g$ are commutative if

$$g\big(F(x, y)\big) = F(gx, gy) \text{ for all } x, y \in X.$$

**Definition 4.1.17** (Choudhury and Kundu, 2010) The maps $F$ and $g$ where $F: X \times X \to X$ and $g: X \to X$ are said to be compatible if

$\lim_{n\to\infty} d\Big(g\big(F(x_n,\ y_n)\big),\ F(gx_n,\ gy_n)\Big) = 0$ and

$\lim_{\to\infty} \Big(g\big(F(y_n,\ x_n)\big),\ F(gy_n,\ gx_n)\Big) = 0$, whenever $\{x_n\}$ and $\{y_n\}$ are sequences in $X$ such that,

$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n = x$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n = y$ for some $x, y \in X$ are satisfied.

**Proposition 4.1.18** (Shah and Hussain, 2012) Every convergent sequences in a quasi-b-metric space is a quasi-b-Cauchy.

Proof: Let $\{x_n\}$ be a sequence which converges to some $x$ in a quasi $-$ b $-$ metric space, and $\varepsilon > 0$, then there exist $n_0 \in \mathbb{N}$ with $d(x_n, x) < \frac{\varepsilon}{2s}$ for all $n \geq n_0$.

For $m, n \geq n_0$, we obtain

$$d(x_n,\ x_m) \leq s[d(x_n,\ x) + d(x,\ x_m)] < s\left[\frac{\varepsilon}{2s} + \frac{\varepsilon}{2s}\right] = \varepsilon.$$

Hence, $\{x_n\}$ is a quasi-b-Cauchy.

**Proposition 4.1.19** (Shah and Hussain, 2012) Let $(X,\ d)$ be a quasi-b-metric space, $\{x_n\}$ be a sequence in $X$, and $x \in X$. The sequence $x_n$ converges to $x$ if and only if

$\lim_{n\to\infty} d(x_n,\ \ x) = \lim_{n\to\infty} d(x,\ \ x_n) = 0.$

In this case $x$ is called the limit of $\{x_n\}$ and we write $x_n \to x$.

**Proposition 4.1.20** (Shah and Hussain, 2012) Limit of a convergent sequences in a quasi-b-metric space is unique.

Proof: Let $x$ and $y$ be limits of the sequence $\{x_n\}$.

By the triangle inequality of definition of quasi-b-metric space it follows that:

$d(x,y) \leq s[d(x,\ x_n) + d(x_n,y)].$

By definition of convergent sequence $[d(x,\ x_n) + d(x_n,\ \ y)] \to 0$ as $n \to \infty$.

Hence $d(x,y) = 0$ and by condition (i) of definition of quasi-b-metric space $x = y$. Therefore the limit of a convergent sequence in a quasi b-metric space is unique.

**Definition 4.1.21** (Hussain et al., 2016) Let $(X,d)$ be a quasi-b-metric space and $\{x_n\}$ be a sequence in $X$. We say that $\{x_n\}$ is Cauchy sequence if and only if for every $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that

$$d(x_n,\ x_m) < \varepsilon \text{ for all } m \geq n \geq N.$$

**Definition 4.1.22** (Hussain et al., 2016) Let $(X,d)$ be a quasi-b-metric space. We say that $(X,\ d)$ is complete if and only if each Cauchy sequence in $X$ is convergent.

## 4.2 Main Result

**Theorem 4.2.1** Let $(X, \preccurlyeq, d)$ be a partially ordered complete quasi-b-metric space with $s \geq 1$ is a real number. Let $F: X \times X \to X$ and $g: X \to X$ be two maps such that $F$ is continuous map with the mixed g-monotone property on $X \times X$. Assume that the following conditions are satisfied.

i) there exists $k \in [0, \frac{1}{s})$ such that

$$d(F(x,y), F(u,v)) \leq \frac{k}{2}[d(gx, gu) + d(gy, gv)] \tag{4.1}$$

for all $x, y, u, v \in X$ for which $gx \succcurlyeq gu$ and $gy \preccurlyeq gv$.

ii) $F(X \times X) \subseteq gX$, $g$ is continuous and compatible with $F$.

iii) There exist $x_0, y_0 \in X$ such that

$gx_0 \preccurlyeq F(x_0, y_0)$ and $gy_0 \succcurlyeq F(y_0, x_0)$, then there exist $x, y \in X$ such that

$gx = F(x,y)$ and $gy = F(y,x)$. That is, $F$ and $g$ have a coupled coincidence point.

If, in addition, for every $(x,y)$, $(x^*, y^*) \in X \times X$ there exists $(u,v) \in X \times X$ such that $(F(u,v), F(v,u))$ is comparable to $(F(x,y), F(y,x))$ and$(F(x^*, y^*), F(y^*, x^*))$, then $F$ and $g$ have a unique coupled common fixed point. That is, there exists a unique $(x,y) \in X \times X$ such that

$x = gx = F(x,y)$ and $y = gy = F(y,x)$.

**Proof:** Let $x_0, y_0 \in X$ such that $g(x_0) \preccurlyeq F(x_0, y_0)$ and $gy_0 \succcurlyeq F(y_0, x_0)$. since $F(X \times X) \subseteq gX$, we can choose $(x_1, y_1) \in X \times X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$, then

$gx_0 \preccurlyeq F(x_0, y_0) = gx_1$ and $gy_0 \succcurlyeq F(y_0, x_0) = gy_1$. Going on this way, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

$$g(x_{n+1}) = F(x_n, y_n) \text{ and } gy_{n+1} = F(y_n, x_n), \text{for all } n \geq 0 \tag{4.2}$$

Now we prove that

$$gx_n \preccurlyeq gx_{n+1} \text{ and } gy_n \succcurlyeq gy_{n+1}, \text{ for all } n \geq 0 \tag{4.3}$$

Let n = 0. Then, we have $gx_0 \preccurlyeq F(x_0, y_0) = gx_1$ and $gy_0 \succcurlyeq F(y_0, x_0) = gy_1$. Thus (4.3) holds for $n = 0$.

Suppose (4.3) holds for some $n > 0$. By using the mixed $g$-monotone property of $F,$ we have

$gx_{n+1} = F(x_n, y_n) \preccurlyeq F(x_{n+1}, y_n) \preccurlyeq F(x_{n+1}, y_{n+1}) = gx_{n+2}$ and

$gy_{n+1} = F(y_n, x_n) \succcurlyeq F(y_{n+1}, x_n) \succcurlyeq F(y_{n+1}, x_{n+1}) = gy_{n+2}$.

Hence (4.3) is true for $n + 1$. Thus, by the mathematical induction, (4.3) follows.

Therefore,

$gx_0 \preccurlyeq gx_1 \preccurlyeq gx_2 \preccurlyeq \cdots \preccurlyeq gx_n \preccurlyeq gx_{n+1} \preccurlyeq \cdots$ .

and

$gy_0 \succcurlyeq gy_1 \succcurlyeq gy_2 \succcurlyeq \cdots \succcurlyeq gy_n \succcurlyeq gy_{n+1}$ ...

Now we consider the following cases

<u>Case (i):</u> Suppose that $gx_n = gx_{n+1}$ and $gy_n = gy_{n+1}$ for some n.

That is,

$gx_{n+1} = F(x_n, y_n) = gx_n$, and $gy_{n+1} = F(y_n, x_n) = gy_n$.

Hence $(x_n, y_n)$ is a coupled coincidence point of $F$ and $g$.

<u>Case (ii):</u> Suppose that $gx_n \neq gx_{n+1}$ or $gy_n \neq gy_{n+1}$ for all n.

From (4.3) we have

Using $gx_n \succcurlyeq gx_{n-1}$ and $gy_n \preccurlyeq gy_{n-1}$, for all $n = 0,1,2$ ... from (4.3) and (4.1), (4.2)

we have

$$d(gx_n, gx_{n+1}) = d\left(F(x_{n-1}, y_{n-1}), F(x_n, y_n)\right)$$

$$\leq \frac{k}{2}[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)]$$

$$= \frac{k}{2}\left[d\big(F(x_{n-2}, y_{n-2}), F(x_{n-1}, y_{n-1})\big) + d\big(F(y_{n-2}, x_{n-2}), F(y_{n-1}, x_{n-1})\big)\right]$$

$$\leq \frac{k}{2}\left[\frac{k}{2}[d(gx_{n-2}, gx_{n-1}) + d(gy_{n-2}, gy_{n-1}) + d(gy_{n-2}, gy_{n-1}) + d(gx_{n-2}, gx_{n-1})]\right]$$

$$= \frac{k^2}{2}\,[d(gx_{n-2}, gx_{n-1}) + d(gy_{n-2}, gy_{n-1})]$$

$$= \frac{k^2}{2}\left[d\big(F(x_{n-3}, y_{n-3}), F(x_{n-2}, y_{n-2})) + d(F(y_{n-3}, x_{n-3}), F(y_{n-2}, x_{n-2})\big)\right]$$

$$\leq \frac{k^3}{2}[d(gx_{n-3}, gx_{n-2}) + d(gy_{n-3}, gy_{n-2})]$$

$$\vdots$$

$$= \frac{k^n}{2}[d(gx_0,\ gx_1) + d(gy_0,\ gy_1)]. \tag{4.4}$$

Since $gx_n \succcurlyeq gx_{n-1}, gy_n \preccurlyeq gy_{n-1}$ and using ineq. (4.1) and (4.2) we have

$$d(gx_{n+1}, gx_n) = d\left(F(x_n, y_n), F(x_{n-1}, y_{n-1})\right)$$

$$\leq \frac{k}{2}[d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})]$$

$$= \frac{k}{2}\left[d\big(F(x_{n-1}, y_{n-1}), F(x_{n-2}, y_{n-2})\big) + d\big(F(y_{n-1}, x_{n-1}), F(y_{n-2}, x_{n-2})\big)\right]$$

$$\leq \frac{k}{2}\left[\frac{k}{2}[d(gx_{n-1}, gx_{n-2}) + d(gy_{n-1}, gy_{n-2}) + d(gy_{n-1}, gy_{n-2}) + d(gx_{n-1}, gx_{n-2})]\right]$$

$$= \frac{k^2}{2}\,[d(gx_{n-1}, gx_{n-2}) + d(gy_{n-1}, gy_{n-2})]$$

$$= \frac{k^2}{2}\left[d\big(F(x_{n-2}, y_{n-2}), F(x_{n-3}, y_{n-3})) + d(F(y_{n-2}, x_{n-2}), F(y_{n-3}, x_{n-3})\big)\right]$$

$$\leq \frac{k^3}{2}[d(gx_{n-2}, gx_{n-3}) + d(gy_{n-2}, gy_{n-3})]$$

$$\vdots$$

$$= \frac{k^n}{2}[d(gx_1,\ gx_0) + d(gy_1,\ gy_0)]. \tag{4.5}$$

Again,

From ineq.(4.3) since $gy_{n-1} \succcurlyeq gy_n,\ gx_{n-1} \preccurlyeq gx_n$ for all $n \geq 0$ and using ineq. (4.1) and (4.2) we have

$$\begin{aligned}
d(gy_n, gy_{n+1}) &= d\big(F(y_{n-1},\ x_{n-1}), F(y_n,\ x_n)\big) \\
&\leq \frac{k}{2}\ [d(gy_{n-1},\ gy_n) + d(gx_{n-1}, gx_n)] \\
&= \frac{k}{2}\ \big[d\big(F(y_{n-2},\ x_{n-2}),\ F(y_{n-1}, x_{n-1})\big) + d\big(F(x_{n-2}, y_{n-2}),\ F(x_{n-1}, y_{n-1})\big)\big] \\
&\leq \frac{k}{2}\left[\frac{k}{2}\ [d(gy_{n-2},\ gy_{n-1}) + d(gx_{n-2},\ gx_{n-1})] + \frac{k}{2}[d(gx_{n-2},\ gx_{n-1}) + d(gy_{n-2},\ gy_{n-1})]\right] \\
&= \frac{k^2}{2}\ [d(gy_{n-2},\ gy_{n-1}) + \ d(gx_{n-2},\ gx_{n-1})] \\
&= \frac{k^2}{2}\begin{bmatrix} d\big(F(y_{n-3},\ x_{n-3}),\ F(y_{n-2},\ x_{n-2})\big) \\ +d\big(F(x_{n-3},\ y_{n-3}),\ F(x_{n-2},\ y_{n-2})\big)\end{bmatrix} \\
&\leq \frac{k^3}{2}\ [d(gy_{n-3},\ gy_{n-2}) + d(gx_{n-3},\ gx_{n-2})] \\
&\ \ \vdots \\
&\leq \frac{k^n}{2}\ [d(gy_0,\ gy_1) + d(gx_0,\ gx_1)].
\end{aligned} \tag{4.6}$$

Similarly,

$$\begin{aligned}
d(gy_{n+1}, gy_n) &= d\big(F(y_n,\ x_n), F(y_{n-1},\ x_{n-1})\big) \\
&\leq \frac{k}{2}\ [d(gy_n,\ gy_{n-1}) + d(gx_n, gx_{n-1})] \\
&= \frac{k}{2}\ \big[d\big(F(y_{n-1},\ x_{n-1}),\ F(y_{n-2}, x_{n-2})\big) + d\big(F(x_{n-1}, y_{n-1}),\ F(x_{n-2}, y_{n-2})\big)\big]
\end{aligned}$$

$$\leq \frac{k}{2}\left[\frac{k}{2}\,[d(gy_{n-1},\ gy_{n-2}) + d(gx_{n-1},\ gx_{n-2})] + \frac{k}{2}[d(gx_{n-1},\ gx_{n-2}) + d(gy_{n-1},\ gy_{n-2})]\right]$$

$$= \frac{k^2}{2}\,[d(gy_{n-1},\ gy_{n-2}) + d(gx_{n-1},\ gx_{n-2})]$$

$$= \frac{k^2}{2}\begin{bmatrix} d\big(F(y_{n-2},\ x_{n-2}),\ F(y_{n-3},\ x_{n-3})\big) \\ +d\big(F(x_{n-2},\ y_{n-2}),\ F(x_{n-3},\ y_{n-3})\big)\end{bmatrix}$$

$$\leq \frac{k^3}{2}\,[d(gy_{n-2},\ gy_{n-3}) + d(gx_{n-2},\ gx_{n-3})]$$

$$\vdots$$

$$\leq \frac{k^n}{2}\,[d(gy_1,\ gy_0) + d(gx_1,\ gx_0)]. \tag{4.7}$$

Since by ineq.(4.3) $gx_{n+1} \succcurlyeq gx_n\ and\ gy_{n+1} \preccurlyeq gy_n$ for all $n \geq 0$ and using ineq.(4.1), (4.2), (4.4) and (4.6) we have

$$d(gx_{n+1}, gx_{n+2}) = d\big(F(x_n, y_n), F(x_{n+1}, y_{n+1})\big)$$

$$\leq \frac{k}{2}\,[d(gx_n,\ gx_{n+1}) + d(gy_n,\ gy_{n+1})]$$

$$\leq \frac{k}{2}\left[\frac{k^n}{2}\,\big[2[d(gx_0, gx_1) + d(gy_0, gy_1)]\big]\right]$$

This implies

$$d(gx_{n+1}, gx_{n+2}) \leq \frac{k^{n+1}}{2}\,[d(gx_0,\ gx_1) + d(gy_0,\ gy_1)]. \tag{4.8}$$

Again, since by ineq.(4.3) $gx_{n+1} \succcurlyeq gx_n$ and $gy_{n+1} \preccurlyeq gy_n, for\ all\ n \geq 0$ and using inequ. (4.1), (4.2), (4.5) and (4.7) we have,

$$d(gx_{n+2}, gx_{n+1}) = d\big(F(x_{n+1}, y_{n+1}), F(x_n, y_n)\big)$$

$$\leq \frac{k}{2}\,[d(gx_{n+1},\ gx_n) + d(gy_{n+1},\ gy_n)]$$

$$\leq \frac{k}{2}\left[\frac{k^n}{2}\ \left[2[d(gx_1, gx_0) + d(gy_1, gy_0)]\right]\right]$$

This implies

$$d(gx_{n+2}, gx_{n+1}) \leq \frac{k^{n+1}}{2}\ [d(gx_1,\ gx_0) + d(gy_1,\ gy_0)]. \tag{4.9}$$

Similarly,

Since from eq. (4.3) $gx_{n+1} \succcurlyeq gx_n,\ gy_{n+1} \preccurlyeq gy_n$ and using ineq. (4.1), (4.2), (4.4) and (4.6) we have

$$d(gy_{n+1}, gy_{n+2}) = d\big(F(y_n, x_n),\ F(y_{n+1}, x_{n+1})\big)$$

$$\leq \frac{k}{2}[d(gy_n, gy_{n+1}\ ) + d(gx_n, gx_{n+1})$$

$$\leq\ \frac{k}{2}[\frac{k^n}{2}[2[d(gy_0, gy_1) + d(gx_0, gx_1)]]]$$

$$= \frac{k^{n+1}}{2}\ [d(gy_0, gy_1) + d(gx_0, gx_1)]. \tag{4.10}$$

Similarly, using inqu.(4.5) and (4.7) we have

$$d(gy_{n+2}, gy_{n+1}) = d(F(y_{n+1}, x_{n+1}), F(y_n, x_n))$$

$$\leq\ \frac{k}{2}[d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)]$$

$$d(gy_{n+2}, gy_{n+1})\ \ \leq \frac{k^{n+1}}{2}[d(gy_1, gy_0) + d(gx_1, gx_0)]. \tag{4.11}$$

Now for $m, n \in \mathbb{N}$ with $m \geq n,$ using triangle inequality in the definition of quasi-b-metric space, (4.4) and (4.8) we have

$$d(gx_n, gx_m) \leq sd(gx_n, gx_{n+1}) + s^2 d(gx_{n+1}, gx_{n+2}) + \cdots + s^{m-n} d(gx_{m-1}, gx_m)$$

$$\leq s\,\frac{k^n}{2}[\ d(gx_0,\ gx_1) + d(gy_0,\ gy_1)] + s^2\,\frac{k^{n+1}}{2}[d(gx_0,\ gx_1) + d(gy_0,\ gy_1)]$$

$$+s^3\ \frac{k^{n+2}}{2}\ [d(gx_0,\ gx_1) + d(gy_0,\ gy_1)] + \cdots$$

$$\leq s\ \frac{k^n}{2}(1+(sk)+(sk)^2+\ldots)[d(gx_0,\ gx_1)+d(gy_0,\ gy_1)]$$

$$= s\frac{k^n}{2}\left(\frac{1}{1-sk}\right)[d(gx_0,\ gx_1)+d(gy_0,\ gy_1)],\ sk<1. \qquad (4.12)$$

Taking limit $m,n \to \infty$ in (4.12) we have

$$\lim_{m,n\to\infty} d(gx_n, gx_m) = 0$$

Again for $m,n \in \mathbb{N}$ with $m \geq n$, by triangle inequality in the definition of quasi b-metric space and using (4.5) and (4.9) we have

$$d(gx_m,\ \ gx_n) \leq\ s^{m-n}d(gx_m, gx_{m-1}) + s^{m-n-1}d(gx_{m-1}, gx_{m-2})$$

$$+s^{m-n-2}d(gx_{m-2}, gx_{m-3}) + \cdots + s^2 d(gx_{n+2}, gx_{n+1}) + sd(gx_{n+1}, gx_n)$$

$$\leq\ [s^{m-n}k^{m-1} + s^{m-n-1}k^{m-2} + s^{m-n-2}k^{m-3} + \cdots + s^2k^{n+1} + sk^n]$$

$$[d(gx_1, gx_0) + d(gy_1, gy_0)]$$

$$= \begin{bmatrix} (sk)^{m-n}k^{n-1} + (sk)^{m-n-1}k^{n-1} + (sk)^{m-n-2}k^{n-1} + \cdots + \\ (sk)^2k^{n-1} + (sk)k^{n-1} \end{bmatrix}$$

$$[d(gx_1, gx_0) + d(gy_1, gy_0)]$$

$$\leq\ [k^{n-1} + k^{n-1} + k^{n-1} + \cdots + k^{n-1} + k^{n-1}][d(gx_1, gx_0) + d(gy_1, gy_0\ )]$$

$$=\ [(k)^{n-1}(m-n-1)][d(gx_1, gx_0) + d(gy_1, gy_0\ )]$$

$$\leq\ \ k^{n-1}\beta[d(gx_1, gx_0) + d(gy_1, gy_0\ )]\ \text{,for some } \beta > m-n-1. \qquad (4.13)$$

Taking n$\to \infty$ in inq. (4.13) we have

$\lim_{m,n\to\infty} d(gx_m,\ \ gx_n) \to 0.$

Therefore $\{gx_n\}$ is a Cauchy sequence.

Similarly, using inqu. (4.6), (4.10) and triangle inequality we have

$$d(gy_n, gy_m) \leq s[d(gy_n, gy_{n+1}) + d(gy_{n+1}, gy_{n+2})]$$

$$\leq s[d(gy_n, gy_{n+1})] + s^2[d(gy_{n+1}, gy_{n+2})]\ + \cdots + s^{m-n}[d(gy_{m-1},\ \ gy_m)]$$

$$\leq s\ \frac{k^n}{2}[d(gy_0, gy_1) + d(gx_0,\ gy_1)] + s^2\ \frac{k^{n+1}}{2}\ [d(gy_0,\ gy_1) + d(gx_0,\ gx_1)] + \cdots]$$

$$\leq s\ \frac{k^n}{2}[\ 1 + sk + (sk)^2 + (sk)^3 \ldots][d(gy_0,\ gy_1) + d(gx_0,\ gx_1)$$

$$= s\ \frac{k^n}{2}\ \frac{1}{1-sk}\ [d(gy_0,\ gy_1) + d(gx_0,\ gx_1)]. \qquad (4.14)$$

Taking limit $m, n \to \infty$ in inq. (4.14) we have

$\lim_{n\to\infty} d(\ gy_n\ , gy_m) = 0.$

Also by using inq. (4.7), (4.11) and triangle inequality we have

$$\begin{aligned} d(gy_m, gy_n) \leq\ & s^{m-n}\big(d(gy_m, gy_{m-1})\big) + s^{m-n-1}\big((gy_{m-1}, gy_{m-2})\big) + \cdots \\ & +\ s^2 d(gy_{n+2}, gy_{n+1}) + s\big((gy_{n+1}, gy_n)\big) \\ \leq\ & [s^{m-n}k^{m-1} + s^{m-n-1}k^{m-2} + s^{m-n-2}k^{m-3} + \cdots + s^2 k^{n+1}] \\ & [d(gy_1, gy_0) + d(gx_1, gx_0)] \\ =\ & \begin{bmatrix} (sk)^{m-n}k^{n-1} + (sk)^{m-n-1}k^{n-1} + (sk)^{m-n-2}k^{n-1} \\ + \cdots + (sk)^2 k^{n-1} + (sk)k^{n-1} \end{bmatrix} \\ & [d(gy_1, gy_0) + d(gx_1, gx_0)] \\ \leq\ & [k^{n-1} + k^{n-1} + k^{n-1} + \cdots + k^{n-1} + k^{n-1}][d(gy_1, gy_0) + d(gx_1, gx_0)] \\ =\ & [(k)^{n-1}(m-n-1)][d(gy_1, gy_0) + d(gx_1, gx_0)] \\ \leq\ & k^{n-1}\alpha[d(gy_1, gy_0) + d(gx_1, gx_0)]\ \text{, for some } \alpha \geq (m-n-1). \qquad (4.15) \end{aligned}$$

Since $k \in [0, \frac{1}{s})$, Taking limit $m, n \to \infty$ in inq. (4.15) we have

$d(gy_m, gy_n) = 0$ .

Therefore $\{gy_n\}$ is also a Cauchy sequence.

Since $X$ is complete, we get that $\{gx_n\}$ and $\{gy_n\}$ are convergent to some $x \in X$ and $y \in X$ respectively. That is,

$$x = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} F(x_n,\ y_n) \text{ and } y = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} F(y_n,\ x_n). \tag{4.16}$$

Since $F$ and $g$ are continuous and compatible maps, we have by (4.16)

$$\lim_{n\to\infty} d\left(g\big(F(x_n,\ y_n)\big),\ F(gx_n,\ gy_n)\right) = 0 \tag{4.17}$$

and

$$\lim_{n\to\infty} d\left(g\big(F(y_n, x_n)\big),\ F(gy_n, gx_n)\right) = 0. \tag{4.18}$$

Next we prove $gx = F(x, y)$ and $gy = F(y, x)$

For all $n \geq 0$, By the triangle inequality we have,

$$d\big(gx,\ \ F(gx_n, gy_n)\big) \leq s[\, d\left(gx,\ \ g\big(F(x_n, y_n)\big)\right) + d\left(g\big(F(x_n, y_n)\big),\ \ F(gx_n, gy_n)\right). \tag{4.19}$$

Taking the limit as $n \to \infty$ in inq.(4.19) using first condition in the definition of quasi-b-metric, eq.(4.16), eq.(4.17) and the fact that $F$ and $g$ are continuous, we have

$$d\big(gx, F(x, y)\big) = 0$$

This implies

$$gx = F(x, y). \tag{4.20}$$

Similarly, using triangle inequality we have

$$d\big(gy,\ \ F(gy_n, gx_n)\big) \leq s[\, d\left(gy,\ \ g\big(F(x_n, y_n)\big)\right) + d\left(g\big(F(y_n, x_n)\big),\ \ F(gx_n, gy_n)\right)]. \tag{4.21}$$

Taking the limit as $n \to \infty$ in inq. (4.21), using first condition in the definition of quasi-b-metric, eq.(4.16), (4.18) and the fact that $F$ and $g$ are continuous, we have

$$d\big(gy, F(y, x)\big) = 0.$$

This implies

$$gy = F(y, x). \tag{4.22}$$

Thus we proved that $F$ and $g$ have a coupled coincidence point. From what we have proved above the set of coupled coincidence points of $F$ and $g$ is non-empty.

Suppose that $(x, y)$ and $(x^*,\ y^*)$ are two coupled coincidence points of maps $F$ and $g$, that is

$$g(x) = F(x, y),\ \ g(y) = F(y, x),\ \ gx^* = F(x^*, y^*),\ \ gy^* = F(y^*, x^*),$$

Now, we show that

$$gx = gx^* \text{ and } gy = gy^* \tag{4.23}$$

Let $(u, v) \in X \times X$ such that $\big(F(u,\ v),\ F(v,\ u)\big)$ is comparable with $\big(F(x,\ y),\ F(y,\ x)\big)$ and $\big(F(x^*,\ y^*),\ F(y^*,\ x^*)\big)$.

Now, we construct sequences $\{u_n\}$ and$\{v_n\}$ defined by

$u_0 = u,\quad v_0 = v$ and choose $u_1,\ v_1 \in X$ so that $g(u_1) = F(u_0,\ v_0)$ and $g(v_1) = F(v_0,\ u_0)$.

Then, similarly as inq.(4.2), we can inductively define sequences $\{g(u_n)\}$ and $\{g(v_n)\}$ such that

$$g(u_{n+1}) = F(u_n,\ v_n) \text{ and } g(v_{n+1}) = F(v_n, u_n). \tag{4.24}$$

Further we set $x_0 = x,\ y_0 = y,\ x_0{}^* = x^*,\ \ y_0{}^* = y^*$ and we define the sequences $\{gx_n\}$, $\{gy_n\}$ and $\{gx_n{}^*\}$, $\{gy_n{}^*\}$ by

$gx_{n+1} = F(x_n,\ y_n),\ \ gy_{n+1} = F(y_n,\ x_n)$ and $gx_{n+1}{}^* = F(x_n{}^*,\ y_n{}^*)$,

$gy_{n+1}{}^* = F(y_n{}^*,\ x_n{}^*)$.

Since $\big(F(x,\ y),\ F(y,\ x)\big) = (gx_1,\ gy_1) = (gx,\ gy)$

and

$\big(F(u,\ v), F(v, u)\big) = \big(g(u_1),\ g(v_1)\big)$ are comparable. That is,$(gx, gy)$ and $(gu_1, gv_1)$ are comparable.

Without loss of generality we assume that

$$gx \succcurlyeq gu_1 \text{ and } gy \preccurlyeq gv_1. \tag{4.25}$$

Now, we prove that $\left(g(x),\ g(y)\right)$ and $\left(g(u_n),\ g(v_n)\right)$ are comparable, that is

$$g(x) \succcurlyeq g(u_n) \text{ and } g(y) \preccurlyeq g(v_n) \text{ for all } n \geq 1. \tag{4.26}$$

From (4.25) above (4.26) is true for $n = 1$.

Assume that (4.26) is true for some n.

Now, $gu_{n+1} = F(u_n, v_n) \succcurlyeq F(x, y) = gx$ and $gv_{n+1} = F(v_n, u_n) \succcurlyeq F(y, x) = gy$.

Thus by mathematical induction (4.26) is true for all $n \geq 0$.

Since $gx \succcurlyeq gu_n$ and $gy \preccurlyeq gv_n$, from (4.1) we have

$$\begin{aligned}
d\left(g(x), g(u_{n+1})\right) &= d\left(F(x,y),\ F(u_n,\ v_n)\right) \\
&\leq \frac{k}{2}[d(gx,\ gu_n) + d(gy,\ gv_n)] \\
&= \frac{k}{2}\left[d\left(F(x,\ y),\ F(u_{n-1},\ v_{n-1})\right) + d\left(F(y,\ x),\ F(v_{n-1},\ u_{n-1})\right)\right] \\
&\leq \frac{k}{2}\left[\frac{k}{2}[d(gx,\ gu_{n-1}) + d(gy,\ gv_{n-1}) + d(gy,\ gv_{n-1}) + d(gx,\ gu_{n-1})]\right] \\
&= \frac{k^2}{2}[d(gx,\ gu_{n-1}) + d(gy,\ v_{n-1})] \\
&\ \ \vdots \\
&\leq \frac{k^n}{2}[d(gx, gu_1) + d(gy, gv_1)].
\end{aligned} \tag{4.27}$$

Similarly, since $gx \succcurlyeq gu_n$ and $gy \preccurlyeq gv_n$ and from inq. (4.1) we have

$$\begin{aligned}
d(gy, gv_{n+1}) &= d\left(F(y,x),\ F(v_n,\ u_n)\right) \\
&\leq \frac{k}{2}[d(gy, gv_n) + d(gx, gu_n)] \\
&= \frac{k}{2}\left[d\left(F(y,x),\ F(v_{n-1},\ u_{n-1})\right) + d\left(F(x,y),\ F(u_{n-1},\ v_{n-1})\right)\right] \\
&\leq \frac{k^2}{2}[d(gy,\ gv_{n-1}) + d(gx,\ gu_{n-1})]
\end{aligned}$$

$$\vdots$$

$$\leq \frac{k^n}{2}\ [d(gy,\ gv_1) + d(gx,\ gu_1)]. \tag{4.28}$$

Adding inq. (4.27) and inq. (4.28), we get

$$d\big(gx, gu_{n+1}) + d(gy, g\ v_{n+1})\big)\ \leq k^n[d(gx, gu_1) + d(gy, gv_1)], \tag{4.29}$$

for each $n \geq 1$.

Taking limit as $n \to \infty$ in inq. (4.29) and since $k \in \left[0,\ \frac{1}{s}\right)$, we have

$$\lim_{n\to\infty} d(gx, gu_{n+1}) = 0 \text{ and } \lim_{n\to\infty} d(gy, gv_{n+1}) = 0. \tag{4.30}$$

Now again we show that

$$d(gu_{n+1},\ \ gx^*) \to 0 \text{ as } n \to \infty \tag{4.31}$$

and

$$d(gv_{n+1},\ \ gy^*) \to 0 \text{ as } n \to \infty. \tag{4.32}$$

Since $gu_n \preccurlyeq gx^*$ and $gv_n \succcurlyeq gy^*$, From (4.1) we have

$$d(gu_{n+1},\ \ gx^*) = d\big(F(u_n,\ \ v_n),\ \ F(x^*,\ \ y^*)\big)$$

$$\leq \frac{k}{2}[d(gu_n,\ gx^*) + d(gv_n,\ gy^*)]$$

$$\leq \frac{k}{2}\left[\frac{k}{2}[d(gu_{n-1},\ gx^*) + d(gv_{n-1},\ gy^*) + d(gv_{n-1},\ gy^*) + d(gu_{n-1},\ gx^*)]\right]$$

$$= \frac{k^2}{2}[d(gu_{n-1},\ gx^*) + d(gv_{n-1},\ gy^*)]$$

$$\vdots$$

$$\leq \frac{k^n}{2}[d(gu_1,\ gx^*) + d(gv_1,\ gy^*)]. \tag{4.33}$$

Similarly,

$$d(gv_{n+1},\ gy^*) = d\big(F(v_n,\ u_n),\ F(y^*,\ x^*)\big)$$

$$\le \frac{k}{2}[d(gv_n,\ gy^*) + d(gu_n,\ gx^*)]$$

$$= \frac{k}{2}\left[d\big(F(v_{n-1},\ u_{n-1}),\ F(y^*,\ x^*)\big) + d\big(F(u_{n-1},\ v_{n-1}),\ F(x^*,\ y^*)\big)\right]$$

$$\le \frac{k}{2}\left[\frac{k}{2}[d(gv_{n-1},\ gy^*) + d(gu_{n-1},\ gx^*) + d(gu_{n-1},\ gx^*) + d(gv_{n-1},\ gy^*)]\right]$$

$$= \frac{k^2}{2}[d(gu_{n-1},\ gx^*) + d(gv_{n-1},\ gy^*)]$$

$$\vdots$$

$$\le \frac{k^n}{2}[d(gu_1,\ gx^*) + d(gv_1,\ gy^*)]. \tag{4.34}$$

Adding (4.33) and (4.34) we get

$$d(gu_{n+1},\ gx^*) + d(gv_{n+1},\ gy^*) \le k^n[d(gu_1,\ gx^*) + d(gv_1,\ gy^*)]. \tag{4.35}$$

Since $k \in [0,\ \frac{1}{s})$, from inq. (4.35) we have

$$\lim_{n\to\infty} d(gu_{n+1},\ gx^*) = 0 \text{ and } \lim_{n\to\infty} d(gv_{n+1},\ gy^*) = 0. \tag{4.36}$$

By triangle inequality and using eq. (4.30) and eq. (4.36) we have

$$d(gx,\ gx^*) \le s[d(gx,\ gu_{n+1}) + d(gu_{n+1},\ gx^*)] \to 0 \ as\ n \to \infty.$$

This implies

$d(gx, gx^*) \le 0.$ But $d(gx, gx^*) \ge 0.$

That is,

$d(gx, gx^*) = 0.$

Therefore by $(i)$ in the definition of quasi b-metric we have

$gx = gx^*.$

Similarly we have

$d(gy,\ gy^*)\ \leq s[d(gy,\ gv_{n+1}) + d(gv_{n+1},\ gy^*)] \to 0$ as $n \to \infty$.

This implies

$d(gy, gy^*) \leq 0$. But, $d(gy, gy^*) \geq 0$.

Thus, $d(gy, gy^*) = 0$.

Therefore $gy = gy^*$.

Thus we proved eq. (4.23).

Since $gx\ = F(x,\ y)$ and $gy = F(y,\ x)$, by compatabilty of $F$ and $g$ we have

$$g(gx) = g\big(F(x,\ y)\big) = F(gx,\ gy) \text{ and } g(gy)\ = g\big(F(y,\ x)\big)\ = F(gy,\ gx). \tag{4.37}$$

We denote $gx\ = z,\quad gy\ = w$. Then from eq. (4.37) we have

$$gz = F(z,\ w) \text{ and } gw\ = F(w,\ z). \tag{4.38}$$

Thus $(z,\ w)$ is a coupled coincidence point. Then from eq. (4.23) with $x^* = z$ and $y^* = w$ it follows $g(z) = g(x)$ and $g(w) = g(y)$, that is,

$$g(z) = z \text{ and } g(w) = w. \tag{4.39}$$

From eq. (4.38) and eq. (4.39) we have

$z = gz\ = F(z,\ w)$ and $w = gw = F(w,\ z)$.

Therefore $(z,\ w)$ is a coupled common fixed-point of $F$ and $g$.

Now we prove uniqueness of coupled common fixed point of maps $F$ and $g$.

Assume that $(p,\ q)$ is another coupled common fixed-point. Then by eq. (4.23) we have $p = g(p) = g(z) = z$ and $q = g(q) = g(w) = w$. Thus $F$ and $g$ have a unique coupled common fixed-point.

**Corollary 4.2.2** Let $(X, d)$ be a partially ordered complete quasi-b-metric space with $s \geq 1$ is real number. Let $F: X \times X \rightarrow X$ be a continuous mapping with mixed monotone property on $X \times X$. Assume that the following conditions are satisfied.

$i)$ There exists $k \in [0, \frac{1}{s})$ such that

$$d(F(x, y), F(u, v) \leq \frac{k}{2}\ [d(x, u) + d(y, v)] \tag{4.40}$$

For all $x, y, u, v \in X$ for which $x \succcurlyeq u,\ \ y \preccurlyeq v$.

$ii)$ there exist $x_0,\ y_0 \in X$ such that

$x_0 \preccurlyeq F(x_0, y_0)$ and $y_0 \succcurlyeq F(y_0, x_0)$,

then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

If, In addition to the conditions above for every $(x, y)$, $(x^*, y^*) \in X \times X$ there exists $(u, v)$ in $X \times X$ such that $(u,\ v)$ is comparable to $(x, y)$ and $(x^*,\ y^*)$ then $F$ has a unique coupled fixed point.

**Proof:** It follows by taking $g = I_x(I_x = \text{Identity map on } X)$ in Theorem (4.2.1).

**Remark 4.2.3** Since every metric space is a quasi- b-metric space the work of Lakshimkantham and Ciric [Lakshimkantham and Ciric (2009). Coupled fixed point Theorems for nonlinear contractions in partially ordered metric space] follows as corollary to Theorem (4.2.1).

**Remark 4.2.4** Since every b-metric space is quasi b-metric space Theorem (1.1.5) follows as corollary to Corollary (4.2.2).

**Remark 4.2.5** Since every metric space is quasi-b-metric space the work of Bhaskar and Lakshimkantham [Bhaskar,T.G and Lakshimkantham, V. (2006). Fixed point Theorems in partially ordered metric spaces and applications. *Nonlinear Anal. TMA* 65(7): 1379-1393.] follows as corollary to corollary (4.2.2).

Now we give an example in support of theorem (4.2.1).

**Example 4.2.6** Let $X = \{0, 1, 3\}$ with a quasi-b-metric $d: X \times X \to [0, \infty)$ defined by

$$d(x,\ y) = \begin{cases} 0 & \text{if and only if } x = y, \\ 1 & \text{if } x < y, \\ \left|x - \frac{y}{2}\right|^2 & \text{if } x > y. \end{cases}$$

Note that $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$.

Also $d(x, y) = d(y, x)$ if and only if $x = y$. so that $d$ is not symmetric. Let $x = 3, y = 1,\ z = 0$.

Then

$$d(x, y) = d(3, 1) = \left|3 - \frac{1}{2}\right|^2 = \frac{25}{4}$$

$$d(x, z) = d(3, 0) = \left|3 - \frac{0}{2}\right|^2 = 9$$

$d(y, z) = d(1, 0) = \left|1 - \frac{0}{2}\right|^2 = 1$, so that the usual triangle inequality is not satisfied.

However if $p \in (0, 1]$, we have

$$d(x, z) \leq 2^{\frac{1}{p}}[d(x, y) + d(y, z)].$$

Since $s = 2^{\frac{1}{p}} \geq 2$ for $p \in (0, 1]$, so $d$ is a quasi b-metric on $X$.

We consider the relation $' \preccurlyeq '$ on $X$ as follows:

For any $x, y \in X,\ x \preccurlyeq y \Leftrightarrow x = y$ or $\{x, y \in \{0, 1, 3\}\ and\ x \leq y\}$ where $\leq$ is the usual ordering. Hence we have

$\preccurlyeq := \{(0,0), (1,1), (3,3), (0,1), (0,3), (1,3)\}$.

Clearly $(X, \preccurlyeq, d)$ is a partially ordered complete quasi-b-metric space with constant $s \geq 2$.

We set $A = \{(0,3), (0,1), (0,0), (1,3), (1,1), (1,0)\}$

$B = \{(3,3), (3,1), (3,0)\}$

We define self maps $F: X \times X \to X$ and $g: X \to X$ by

$$F(x,y) = \begin{cases} 0 \; if \; (x,y) \in A \\ 1 \; if \; (x,y) \in B \end{cases}$$

$g(0) = 0, \;\; g(1) = 1, \;\; g(3) = 3.$

Since,

$g(0) = 0 \preccurlyeq 1 = g(1) \Rightarrow F(0,0) = 0 \preccurlyeq 0 = F(1,0),$

$g(0) = 0 \preccurlyeq 1 = g(1) \Rightarrow F(0,1) = 0 \preccurlyeq 0 = F(1,1),$

$g(0) = 0 \preccurlyeq 1 = g(1) \Rightarrow F(0,3) = 0 \preccurlyeq 0 = F(1,3),$

$g(0) = 0 \preccurlyeq 1 = g(1) \Rightarrow F(0,1) = 0 \preccurlyeq F(0,0) = 0,$

$g(0) = 0 \preccurlyeq 3 = g(3) \Rightarrow F(0,0) = 0 \preccurlyeq F(3,0) = 1,$

$g(0) = 0 \preccurlyeq 3 = g(3) \Rightarrow F(0,1) = 0 \preccurlyeq F(3,1) = 1,$

$g(0) = 0 \preccurlyeq 3 = g(3) \Rightarrow F(0,3) = 0 \preccurlyeq F(3,3) = 1,$

$g(0) = 0 \preccurlyeq 3 = g(3) \Rightarrow \; F(0,3) = 0 \preccurlyeq F(0,0) = 0,$

$g(0) = 0 \preccurlyeq 3 = g(3) \Rightarrow \quad F(3,3) = 1 \preccurlyeq F(3, \; 0) = 1.$

$g(1) = 1 \preccurlyeq 3 = g(3) \Rightarrow F(1, \;\; 0) = 0 \preccurlyeq F(3, \;\; 0) = 1,$

$g(1) = 1 \leq 3 = g(3) \Rightarrow \; F(1, \; 1) = 0 \preccurlyeq F(3, \; 1) = 1,$

$g(1) = 1 \leq 3 = g(3) \Rightarrow F(1, \; 3) = 0 \preccurlyeq F(3, \; 3) = 1,$

$g(1) = 1 \leq 3 = g(3) \Rightarrow F(1,3) = 0 \preccurlyeq F(1,0) = 0.$

Therefore $F$ has mixed $g$-monotone property and also we observe that $F$ and $g$ are continuous, $F(X \times X) \subset g(X)$ and $g$ commutes with $F$, so they are compatible.

By choosing $x_0 = 0$ and $y_0 = 3$ we have

$g(x_0) = 0 \preccurlyeq F(x_0, \; y_0) = 0$ and $g(y_0)$= 3 $\succcurlyeq F(3, \; 0) = 1.$

The set of comparable elements in

$X \times X = \{(x, y), (u, v)/gx \succcurlyeq gu,\ \ gy \preccurlyeq gv\}$

= {((0,0),(0,0)), ((1,0),(0,0)), ((0,1), (0,1)), ((0,0), (0,1)), ((1,0),(0,1)), ((1,1),(0,1)),

((3,0), (0,1)), ((3,1), (0,1)), ((0,0), (0,3)), ((0,1),(0,3)), ((0,3),(0,3)), ((1,0),(0,3)),

((1,1),(0,3)), ((1,3),(0,3)), ((3,0),(0,3)), ((3,1),(0,3)), ((3,3),(0,3)), ((1,0),(1,0)),

((3,0),(1,0)), ((1,0),(1,3)), ((1,1),(1,3)), ((1,3),(1,3)), ((3,0),(1,3)), ((3,1),(1,3)),

((3,3),(1,3)), ((3,0),(3,0)), ((3,0),(3,1)), ((3,1),(3,1)), ((3,0),(3,3)), ((3,1),(3,3)),

((3,3),(3,3)), ((1,0),(1,1)), ((1,1),(1,1)), ((3,0),(1,1)), ((3,1),(1,1)), ((3,0),(0,0))}.

In the following, we show that inequality (4.1) holds for any comparable elements $(x,\ y)$ and $(u,\ v) \in X \times X$ with $gx \succcurlyeq gu,\ \ gy \preccurlyeq gv$ for $s = 2$ and $k = \frac{1}{3}$.

**<u>Case (i):</u>** Let $(x, y) = (3, 0)$ and $(u, v) = (0, 0)$ or $(x, y) = (3,1)$ and $(u, v) = (0,1)$ or

$(x, y) = (3, 3)$ and $(u, v) = (0, 3)$

$(a)\ d\big(F(x, y), F(u, v)\big) = d(1, 0) = \left|1 - \frac{0}{2}\right|^2 = 1,$

$(b)\big(d(gx, gu) + d(gy, gv)\big) = 9.$

This implies

$d\big(F(x, y), F(u, v)\big) = 1 \leq \frac{1}{3}\frac{1}{2}9 = \frac{3}{2} = \frac{k}{2}[d(gx, gu) + d(gy, gv)].$

**<u>Case (ii):</u>** Let $(x, y) = (3, 0)$ and $(u, v) = (0, 1)$ or $(x, y) = (3,0)$ and $(u, v) = (0,3)$

or $(x, y) = (3, 1)$ and $(u, v) = (0, 3)$.

$(a)\ d\big(F(x, y), F(u, v)\big) = d(1, 0) = \left|1 - \frac{0}{2}\right|^2 = 1,$

$(b)\ d(gx, gu) + d(gy, gv) = 9 + 1 = 10.$

$$d\big(F(x,y),F(u,v)\big)=1\leq\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)10=\frac{10}{3}=\frac{k}{2}[d(gx,gu)+d(gy,gv)].$$

**Case (iii):** Let $(x,y)=(3,0)$ and $(u,v)=(1,3)$ or $(x,y)=(3,1)$ and $(u,v)=(1,3)$ or $(x,y)=(3,0)$ and $(u,v)=(1,1)$ we have

$(a)\ d\big(F(x,y),F(u,v)\big)=d\big(F(3,0),F(0,3)\big)=d(1,0)=1,$

$(b)\ d(gx,gu)+d(gy,gv)=\frac{25}{4}+1=\frac{29}{4}.$

This implies

$$d\big(F(x,y),F(u,v)\big)=1\leq\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{29}{4}\right)=\frac{29}{24}=\frac{k}{2}[d(gx,gu)+d(gy,gv)].$$

.**Case (iv):** Let $(x,y)=(3,0)$ and $(u,v)=(1,0)$ or $(x,y)=(3,\ 3)$ and $(u,v)=(1,3)$ or $(x,y)=(3,1)$ and $(u,v)=(1,1)$ we have

$(a)\ d(F(x,y),F(u,v))=d(1,0)=1.$

$(b) d(gx,gu)+d(gy,gv)=\frac{25}{4}$

This implies

$$d\big(F(x,y\ ),F(u,v)\big)=1\leq\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{25}{4}\right)=\frac{25}{24}=\frac{k}{2}[d(gx,gu)+d(gy,gv)].$$

For the remaining comparable elements of $X\times X$ we have $d\big(F(x,y),F(u,v)\big)=0$ so the inequality (4.1) holds trivially.

From cases $(i)-(iv)$ $F$ and $g$ satisfy the inequality (4.1) of theorem (4.2.1) in our main result. Hence all hypotheses of theorem (4.2.1) are satisfied. So, $(0,0)$ is a coupled coincidence point of maps $F$ and $g$.

In fact, $(0,0)$ is a unique coupled common fixed point of $F$ and $g$.

The following is also an example in support of Theorem (4.2.1).

**Example 4.2.4** Let $X = [0, 1]$ with usual partial order $'\preccurlyeq'$ and define $d: X \times X \to [0, \infty)$ by

$$d(x, y) = \begin{cases} 0 \text{ if and only if } x = y; \\ 1 \quad \text{if } x < y \\ (x - \frac{y}{2})^2 \quad \text{if } x > y \end{cases}$$

Clearly $d$ is a quasi-b-metric on $X$ with $s \geq 2$,

and $(X, \preccurlyeq, d)$ is a partially ordered complete quasi-b-metric space.

We define $g: X \to X$ by

$gx = x^2$, for all $x \in X$ and

$F: X \times X \to X$ by

$$F(x, y) = \begin{cases} \frac{1}{8}(x^2 - y^2), \ if \ x, y \in [0, 1], x \geq y; \\ 0 \quad if \ x < y \end{cases}$$

It is clear that $F(X \times X) \subseteq g(X)$.

Now, we verify that $F$ and $g$ are compatible.

Let $\{x_n\}$ and $\{y_n\}$ be two sequences in $X$ such that,

$\lim_{n \to \infty} F(x_n, y_n) = r = \lim_{n \to \infty} gx_n$ and $\lim_{n \to \infty} F(y_n, x_n) = t = \lim_{n \to \infty} gy_n$.

Then obviously, $r = 0$ and $t = 0$.

Now for all $n \geq 0$,

$gx_n = {x_n}^2$, $gy_n = {y_n}^2$

$$F(x_n, y_n) = \begin{cases} \frac{{x_n}^2 - {y_n}^2}{8} & \text{if } x_n \geq y_n, \\ 0 & \text{if } x_n < y_n. \end{cases} \text{ and } F(y_n, x_n) = \begin{cases} \frac{{y_n}^2 - {x_n}^2}{8} & \text{if } y_n \geq x_n, \\ 0 & \text{if } y_n < x_n. \end{cases}$$

Then it follows that

$d\left(g\big(F(x_n, y_n)\big),\ \ F(gx_n, gy_n)\right) \to 0\ as\ n \to \infty$ and

$d\left(g\big(F(y_n, x_n)\big),\ \ F(gy_n, gx_n)\right) \to 0\ as\ n \to \infty.$

Hence, the maps $F$and $g$ are compatible in $X$.

Now, we verify that $F$ has mixed $g$-monotone property.

**Suppose** $x_1,\ x_2 \in X$ such that $gx_1 \preccurlyeq gx_2$, that is, ${x_1}^2 \preccurlyeq {x_2}^2$ and

$y_1, y_2 \in X$ such that $gy_1 \preccurlyeq gy_2 \Rightarrow {y_1}^2 \preccurlyeq {y_2}^2$.

Here we consider the following cases.

**<u>Case (i)</u>**: $x_1, x_2 \geq y$ and $y_1, y_2 < x$ for all $x, y \in X$.

$$x_1,\ x_2 \in X,\ \ gx_1 \preccurlyeq gx_2\ \Rightarrow {x_1}^2 \preccurlyeq {x_2}^2 \Rightarrow F(x_1, y_1) = \frac{1}{8}({x_1}^2 - y^2) \preccurlyeq \frac{1}{8}({x_2}^2 - y^2) = F(x_2, y)$$

and

$$y_1, y_2 \in X,\ \ gy_1 \preccurlyeq gy_2 \Rightarrow {y_1}^2 \preccurlyeq {y_2}^2 \Rightarrow F(x, y_1) = \tfrac{1}{8}(x^2 - {y_1}^2) \succcurlyeq \tfrac{1}{8}(x^2 - {y_2}^2) = F(x, y_2).$$

**<u>Case (ii)</u>** $x_1, x_2 < y$ and $x < y_1, y_2$. in this case the result holds trivially.

From cases (i) and ( ii) we see that $F$ has mixed $g$-monotone property.

By taking $x_0 = 0$ and $y_0 = \frac{1}{2}$ we have $g(x_0) = 0 \leq 0 = F\left(0, \frac{1}{2}\right) = F(x_0, y_0)$ and

$g(y_0) = g\left(\frac{1}{2}\right) = \frac{1}{4} \geq \frac{1}{32} = \frac{1}{8}[\left(\frac{1}{2}\right)^2 - 0^2] = F\left(\frac{1}{2}, 0\right) = F(y_0, x_0)$. Thus our initial condition is satisfied.

Next we verify the inequality (4.1) in our main result of theorem (4.2.1). We take $x, y, u, v \in X$, such that $gx \geq gu$ and $gy \leq gv$, that is, $x^2 \geq u^2$ and $y^2 \leq v^2$.

Now, we consider the following cases for $k = \frac{1}{3}$ and s = 2

**Case ( i):** $x \geq y$ and $u < v$

$$d\big(F(x,y),F(u,v)\big) = d\left(\frac{1}{8}(x^2-y^2),0\right)$$

$$= \left(\tfrac{1}{8}(x^2-y^2)-\tfrac{0}{2}\right)^2$$

$$= \tfrac{1}{64}(x^2-y^2)^2$$

$$= \tfrac{1}{64}(x^2-y^2+u^2-u^2)^2$$

$$\leq \tfrac{1}{64}(x^2-y^2+x^2-u^2)^2,\ x^2 \geq u^2$$

$$= \tfrac{1}{64}(2x^2-u^2-y^2)^2$$

$$\leq \tfrac{1}{64}[(2x^2-u^2)^2]$$

$$\leq \tfrac{1}{6}\left[\tfrac{1}{4}(2x^2-u^2)^2+1\right]$$

$$= \tfrac{k}{2}[d(gx,gu)+d(gy,gv)].$$

**Case (ii):** $x \geq y$ and $u \geq v$

$$d\big(F(x,y),F(u,v)\big) = d\left(\tfrac{1}{8}(x^2-y^2),\tfrac{1}{8}(u^2-v^2)\right)$$

$$= \left(\tfrac{1}{8}(x^2-y^2)-\left(\tfrac{1}{8}\left(\tfrac{u^2-v^2}{2}\right)\right)\right)^2$$

$$= \left(\tfrac{2(x^2-y^2)-(u^2-v^2)}{16}\right)^2$$

$$= \tfrac{1}{256}[(2x^2-2y^2-u^2+v^2)^2]$$

$$= \tfrac{1}{256}[\big(2x^2-u^2-(2y^2-v^2)\big)^2]$$

$$\leq \frac{1}{256}[(2x^2-u^2)^2+1],\ x,y,u,v \in [0,1]$$

$$\leq \frac{1}{6}[\frac{1}{4}(2x^2-u^2)^2+1]$$

$$= \frac{k}{2}[d(gx,gu)+d(gy,gv).$$

**<u>Case (iii):</u>** $x < y$ and $u < v$

$$d\big(F(x,y),F(u,v)\big) = d(0,0) = 0 \leq \frac{k}{2}[d(gx,gu)+d(gy,gv)]\ .$$

**<u>Case ( iv):</u>** $x < y$ and $u \geq v$

$$d\big(F(x,y),F(u,v)\big) = d\left(0,\ \frac{1}{8}(u^2-v^2)\right) = \left(0-\frac{1}{8}\left(\frac{u^2-v^2}{2}\right)\right)^2$$

$$= \frac{1}{256}(u^2-v^2)^2$$

$$= \frac{1}{256}(u^2+x^2-x^2-v^2)^2$$

$$\leq \frac{1}{256}(x^2+x^2-u^2+y^2-v^2)^2 \text{ (since } x^2 \geq u^2)$$

$$= \frac{1}{256}[\big(2x^2-u^2-(v^2-y^2)\big)^2]$$

$$\leq \frac{1}{256}[(2x^2-u^2)^2+(v^2-y^2)^2]$$

$$= \frac{1}{256}[(x^2-u^2)-(y^2-v^2)]^2$$

$$\leq \frac{1}{256}[(x^2-u^2)^2+(y^2-v^2)^2]$$

$$\leq \frac{1}{256}[(2x^2-u^2)^2+1] \text{ (since } (y^2-v^2)^2 \leq 1)$$

$$\leq \frac{1}{6}\left[\frac{1}{4}(2x^2-u^2)^2+1\right]$$

$$= \frac{k}{2}[d(gx,gu)+d(gy,gv)]\ .$$

From cases (i) - (iv) $F\ and\ g$ satisfied the Inequality (4.1) of Theorem (4.2.1).

Hence all the hypotheses of Theorem (4.2.1) are satisfied. (0,0) is a coupled coincidence point of maps $F$ and $g$.

In fact, $(0, 0)$ is the unique coupled common fixed point of maps $F$ and $g$.

# CHAPTER FIVE:
# CONCLUSION AND FUTURE SCOPE

## 5.1 CONCLUSION

Bota *et al.* established and proved some coupled fixed point theorem for a map with mixed monotone property satisfying contractive condition (1.5) in a partially ordered complete b-metric spaces.

In this thesis, we established and proved existence of coupled coincidence point and existence and uniqueness of coupled common fixed point theorem for a pair of maps $F$ and $g$ where $F$ satisfies property in the setting of partially ordered complete quasi-b-metric spaces. Also we provided examples in support of our main result. Our work extended coupled fixed point result to coupled common fixed point result.

## 5.2 FUTURE SCOPE

Fixed point theory is one of active and vigorous area of research in mathematics and other sciences. There are several published results related to existence of coupled coincidence point and coupled common fixed point theorem for a pair of maps satisfying some contractive conditions in metric spaces, b-metric spaces and other spaces rather than quasi-b-metric spaces.

The researcher believes that the search for the existence of coupled coincidence point and existence and uniqueness of coupled common fixed point for a pair of maps satisfying some contractive condition in partially ordered complete quasi-b-metric space is an attractive area of study. So, we recommend to the forthcoming postgraduate students or any other interested researchers can exploit this opportunity and conduct their research work in this area.

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