Coupled Coincidence and Coupled Common Fixed Point Theorem Satisfying Certain Rational Type Contractive Condition in Dislocated Quasi Metric Space

A Thesis Submitted to the Department of Mathematics in Partial Fulfillment for the Requirements of the Degree of Masters of Science in Mathematics

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Declaration

I, the undersigned declare that, this research paper entitled ”Coupled Coincidence and Coupled Common Fixed Point Theorem Satisfying Certain Rational Type Contractive Condition in Dislocated Quasi Metric Spaces” is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged.

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Abstract

The purpose of this research is to establish the existence and uniqueness of a coupled coincidence and coupled common fixed point theorem involving pairs of weakly compatible mappings satisfying certain rational type contractive condition in the setting of dislocated quasi metric spaces. Our result extends and generalizes several well-known comparable results in literature. We also provided an example in support of our main result. In this research undertaking, we followed analytical design, secondary source of data such as published articles, related works browsed from internet etc. were used. The study procedure we used was that of Jhade and Khan, (2014) and Mohammed *et al*., (2018).

Contents

Chapter 1

Introduction

1.1 Background of the study

Fixed point theory is an important tool in the study in functional analysis. It is also considered to be the key connection between pure and applied mathematics. Its application is not limited to various branches of mathematics but also in many fields such as, Economics, Biology, Chemistry, Physics, Statistics, Computer Science, engineering etc. The beginning of fixed point theory on complete metric space is related to the work of Polish mathematician Stefan Banach (Banach Contraction Principle), published in 1922. Banach Contraction Principle says that any contractive self-mapping on a complete metric space has a unique fixed point. This principle is one of a very powerful test for existence and uniqueness of the solution of considerable problems arising in mathematics. Because of its importance for mathematical theory, Banach Contraction Principle has been extended and generalized in many directions. Since then, a number of generalizations have been made by many researchers in their works. For instance, (Dass and Gupta, 1975) presented the generalized form of well-known Banach contraction principle in a metric space for some rational type contractive conditions.

The idea of metric space has also been generalized in different directions. Some of well-known and important generalizations of metric spaces are dislocated metric space, quasi-metric space, dislocated quasi-metric space, generalized quasi-metric space, b-metric space, cone metric space, cone b-metric space, etc.

Hitzler (2001) introduced the concept of dislocated metric space and also generalized famous Banach contraction principle in dislocated metric space. In such a space self-distance between point need not to be zero necessarily. Dislocated metric space play a vital role in logical programming, computer science, topology and electronic engineering etc. Zeyada *et al.* (2005) generalized the result of Hitzler (2001) in dislocated quasi-metric space. With the passage of time many papers have been published containing fixed point results for a single and a pair of mapping for different types of contraction conditions in dislocated quasi metric spaces we refers Aage and Salunk (2008); Rahman and Sarwar (2014); Zeyada *et al*. (2005). Bhaskar and Lakshmikantham (2006) initiated the concept of coupled fixed point for non-linear contractions in partially ordered metric spaces.

Lakshmikantham and Ciric (2009) proved coupled coincidence and coupled common fixed point theorems for nonlinear contractive mappings in partially ordered complete metric spaces. There are also a number of works in this line of research in different spaces, for example we refer Akcay and Alaca (2012); Fadil and Bin Ahmad (2010); Kumer and Vantish (2013). Mohammed *et al.* (2018) has proved the coupled fixed point result in the setting of dislocated quasi-metric spaces.

Inspired and motivated by the result of Mohammed *et al.* (2018) the purpose of this research was to extend and generalize their main theorem to coupled coincidence and coupled common fixed point theorem involving pairs of weakly compatible mappings satisfying certain rational type contractive condition in the setting of dislocated quasi metric space.

1.2 Statement of the Problem

In this study, we focused in establishing the existence of coupled coincidence and coupled common fixed point theorem involving pairs of weakly compatible mapping satisfying certain rational type contractive condition in the setting of dislocated quasi metric space.

1.3 Objectives of the study

1.3.1 General objective

The general objective of this research was to establish a coupled common fixed point theorem involving a pair of weakly compatible mappings satisfying certain rational type contractive condition in the setting of dislocated quasi metric spaces.

1.3.2 Specific objectives

This study has the following specific objectives

- To prove existence of coupled coincidence point and coupled common fixed points involving a pair of weakly compatible mappings satisfying certain rational type contractive condition in the setting of dislocated quasi metric spaces.
- To verify the uniqueness of the coupled coincidence and coupled common fixed points.
- To provide an example in support of the main result.

1.4 Significance of the study

The result of this study may have the following importance

- The outcome of this study may contribute to research activities in the study area.
- It may provide basic research skill to researcher.
- It may have application in studying the existence and uniqueness of solution of nonlinear integral equation.

1.5 Delimitation of the Study

This study was delimited to establishing and proving existence and uniqueness of coupled coincidence and coupled common fixed points involving a pair of weakly compatible mappings satisfying certain rational type contractive condition in the setting of dislocated quasi metric spaces.

Chapter 2

Review of Related Literature

Fixed point theory is very important in diverse disciplines of mathematics since it can be applied for solving various problems and it is one of the most dynamic research subjects in nonlinear analysis. In this area the first important and significant result was proved by Banach in 1922 for a contraction mapping in a complete metric spaces .Due to the importance generalization of Banachs contraction principle have been investigated heavily by many researchers (Sumit Chandok and Deepak Kumar, 2013). Consequently, a number of generalization of Banach Contraction Principles have appeared (Banach, 1922).

In the fixed point theory, contraction is one of the main tools to prove the existence and uniqueness of a fixed point Banachs contraction principle, which gives an answer on the existence and uniqueness of the solution of an operator, is used in all analysis. The advantage of topology in logic programming has come to be recognition (Hitler and Seda, 2000). Particularly topological methods are applied to obtain fixed point semantics for logic programs. Such considerations motivated the concept of dislocated metric space. Especially, Jungck and Rhoades (1998), gave a common fixed point theorem for commuting mappings in metric spaces which generalize Banachs contraction theorem as a generalization metric spaces and Hitler and Seda, (2000) introduced dislocated metric spaces. In 2005, Zeyada *et al.* introduced a new space called a dislocated quasi metric space which is more general than the well-known metric spaces. In 2016, Sarwar *et al.* investigate the existence of fixed point of theorem of contractive type mapping involving rational expression in context of dislocated quasi metric spaces. Aage and Salunk (2008) derived fixed point theorem in dislocated quasi-metric spaces, similarly Isufati (2010) proved some fixed point results for continuous contractive condition with rational type expression in the context of dislocated quasi metric spaces.

In fact, recently, the existence of coupled fixed point, coupled coincidence point, coupled common fixed point and common fixed for nonlinear maps with two variables have attracted more and more attention. In 2006, Bhasker and Lakshmikantham introduced the concept of the mixed monotone property and a coupled fixed point. They also established some coupled fixed point theorems for mappings that satisfy the mixed monotone property and gave some applications in the existence and uniqueness of a solution for a periodic boundary value problem. Because of their important role in the study of nonlinear differential equations, nonlinear integral equations and differential inclusions, a number of coupled fixed point theorems have been studied by many authors.

In 2009, Lakshmintham and Ciric extended the concept of mixed monotone property to a mixed g-monotone property and proved coupled coincidence point and coupled common fixed point results. Recently, Abbas *et al*., defined the concept of w-compatible mappings and obtained coupled coincidence point theorems for nonlinear contractive mappings in a cone metric space with a cone having nonempty interior.

Chapter 3

Methodology

This chapter contains study design, description of the research methodology, data collection procedures and data analysis process.

3.1 Study period and site

The study has been conducted from November 2017 to September 2018 in Jimma University under Mathematics department.

3.2 Study Design

In order to achieve the objectives stated, this study has employed analytical design.

3.3 Source of Information

This study mostly depended on document materials or secondary data. So, the available sources of information for the study were Books, published articles.

3.4 Mathematical Procedure of the Study

In this research under taking we followed the standard procedures used in the published works of Jhade and Khan, (2014) and Mohammed *et al*., (2018). The procedures are

1. Establishing a theorem.
2. Constructing sequences.
3. Show that sequences are Cauchy.
4. Proving the existence and uniqueness of coupled coincidence and coupled common fixed point of weakly compatible mappings satisfying certain contractive condition of rational type in the setting of dislocated quasi metric spaces.
5. Giving applicable example for supporting the main result.

Chapter 4

Preliminaries and Main Results

4.1 Preliminaries

Definition 4.1 *Let X be a non-empty set, $\Re^+$ be the set of non-negative real numbers and let $d : X \times X \to \Re^+$ be a function satisfying the conditions*

(i) $d(x,x) = 0$.

(ii) $d(x,y) = d(y,x) = 0 \Rightarrow x = y$.

(iii) $d(x,y) = d(y,x)$ *for all* $x, y \in X$.

(iv) $d(x,y) \leq d(x,z) + d(z,y)$ *for all* $x, y, z \in X$.

If d satisfy conditions from (i) to (iv), then it is called a metric on X.
If d satisfy conditions (ii) to (iv), then it is called a dislocated metric (d-metric) on X.
If d satisfy conditions (ii) and (iv) only, then it is called a dislocated quasi-metric (dq-metric) on X. In this case the pair (X,d) is called a dislocated quasi-metric space.
Every metric space is a dislocated metric space which is also a dislocated quasi metric space, but the converse is not true. The following example shows this fact.

Example 4.1 (Sarwar and Rahman, 2014) *Let $X = \Re^+$ and $d : X \times X \to \Re^+$ define by*

$$d(x,y) = max\{x,y\}.$$

Clearly (X,d) is a dislocated metric space but it is not a metric space.

Example 4.2 (Zeyada *et al.*, 2005) *Let $X = [0,1]$ and $d : X \times X \to \Re^+$ defined by*

$$d(x,y) = |x-y| + |x|.$$

Then (X,d) is a dislocated quasi metric space on X since the symmetric condition fails to hold, it is neither a dislocated metric nor a metric space on X.

The following definition can be seen in Zeyada *et al.*, (2005).

Definition 4.2 *A sequence $\{x_n\}$ in a dislocated quasi metric space (X,d) is said to converge to a point $x \in X$ if and only if*

$$\lim_{n\to\infty} d(x_n,x) = \lim_{n\to\infty} d(x,x_n) = 0.$$

In this case x is called a dislocated quasi limit (dq-limit) of the sequence $\{x_n\}$.

Definition 4.3 *A sequence $\{x_n\}$ in a dislocated quasi metric space (X,d) is called a Cauchy sequence if for every $\varepsilon > 0$, there exists a positive integer n_0 such that for $m,n > n_0$, we have $d(x_n,x_m) < \varepsilon$. That is, $\lim_{n,m\to\infty} d(x_n,x_m)$=0.*

Definition 4.4 *A dislocated quasi metric space is called complete if every Cauchy sequence converges to an element in the same metric space.*

The following definition can be seen in Banach, (1922).

Definition 4.5 *Let (X,d) be a metric space and $T : X \to X$ be a self-map, then T is said to be a contraction mapping if there exists a constant $k \in [0,1)$ called a contraction factor, such that*

$$d(Tx,Ty) \leq kd(x,y)$$

for all $x,y \in X$.

Definition 4.6 *Let X be a nonempty set and $T : X \to X$ a self-map. We say that x is a fixed point of T if Tx = x.*

Theorem 4.1 *Suppose (X,d) be a complete metric space and $T : X \to X$ a contraction, then T has a unique fixed point.*

Definition 4.7 (Bashkar and Lakshmikatham, (2006)) *An element $(x,y) \in X \times X$, where X is any non-empty set, is called a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x,y) = x$ and $F(y,x) = y$.*

The following definition can be seen in Lakshmikatham and Ciric, (2009).

Definition 4.8 (Coupled coincidence point of F and g) *An element $(x,y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if F(x, y) = g(x) and F(y, x) = g(y), and (gx, gy) is called coupled point of coincidence.*

Definition 4.9 (Coupled common fixed point) *An element $(x,y) \in X \times X$, where X is any non-empty set, is called a coupled common fixed point of the mappings $F : X \times X \to X$ and and $g : X \to X$ if $F(x,y) = g(x) = x$ and $F(y,x) = g(y) = y$.*

Definition 4.10 *The mappings $F : X \times X \to X$ and $g : X \to X$ are called commutative if $g(F(x,y)) = F(gx, gy)$ for all $x, y \in X$.*

Definition 4.11 (Weakly Compatible) *The mappings F : $X \times X \to X$ and g : $X \to X$ are called w-Compatible if g(F(x, y)) = F(gx, gy) and g(F(y, x)) = F(gy, gx) whenever gx = F(x, y) and gy = F(y, x).*

Theorem 4.2 (Mohammed *et al.*, (2018)) *Let (X,d) be a complete dislocated quasi-metric space and $T : X \to X$ be a continuous mapping satisfying the following rational contractive condition*

$$\begin{aligned} d[T(x,y),T(u,v)] \leq\ & a_1[d(x,u)+d(y,v)] \\ & + a_2[d(x,T(x,y))+d(u,T(u,v))] \\ & + a_3[d(x,T(u,v))+d(u,T(x,y))] \\ & + a_4\left[\frac{d(x,T(x,y))d(u,T(u,v))}{d(x,u)+d(y,v)}\right] \\ & + a_5\left[\frac{[d(x,u)+d(y,v)][d(x,T(x,y))+d(u,T(u,v))]}{1+d(x,u)+d(y,v)}\right] \\ & + a_6\left[\frac{d(x,T(x,y))+d(x,T(u,v))}{1+d(u,T(u,v))d(u,T(x,y))}\right] \end{aligned}$$

for all $x,y,u,v \in X$ and a_1,a_2,a_3,a_4,a_5, and a_6 are non-negative constants with $2(a_1+a_2+a_5)+4(a_3+a_6)+a_4 < 1$.
Then T has a unique coupled fixed point in $X \times X$.

Remark 4.1 *for real numbers a and b, If $a < b$ and $a > 0$. Then $\frac{ab}{1+a} < b$.*

4.2 Main Results

Theorem 4.3 *Let (X,d) be a dislocated quasi-metric space and $T : X \times X \to X$ and $g : X \to X$ be a continuous and commutative mappings satisfying the following rational type contractive condition*

$$\begin{aligned}
d[T(x,y),T(u,v)] \quad \leq \quad & a_1[d(gx,gu)+d(gy,gv)] \\
& +a_2[d(gx,T(x,y))+d(gu,T(u,v))] \\
& +a_3[d(gx,T(u,v))+d(gu,T(x,y))] \\
& +a_4\left[\frac{d(gx,T(x,y))d(gu,T(u,v))}{d(gx,gu)+d(gy,gv)}\right] \\
& +a_5\left[\frac{[d(gx,gu)+d(gy,gv)][d(gx,T(x,y))+d(gu,T(u,v))]}{1+d(gx,gu)+d(gy,gv)}\right] \\
& +a_6\left[\frac{d(gx,T(x,y))+d(gx,T(u,v))}{1+d(gu,T(u,v))d(gu,T(x,y))}\right] \\
& +a_7\left[\frac{d(gx,T(x,y))[1+d(gu,T(u,v))]}{1+d(gx,gu)+d(gu,T(u,v))}\right] \qquad (4.1)
\end{aligned}$$

where $x,y,u,v \in X$ and $a_1,a_2,a_3,a_4,a_5,a_6,a_7 \in \Re^+$ with $2(a_1+a_2+a_5)+4(a_3+a_6)+a_4+a_7 < 1$. In addition

(i) $T(X \times X) \subseteq g(X)$,

(ii) *$g(X)$ is a complete subspace of X*

Then T and g have a unique coupled coincidence point. More over, if T and g are weakly compatible. then T and g have unique coupled common fixed point the of the form (u,u).

Proof: Choose x_0 and $y_0 \in X$, set

$$gx_1 = T(x_0,y_0), gy_1 = T(y_0,x_0).$$

This can be done because $T(X \times X) \subseteq g(X)$. Continuing this process, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$gx_{n+1} = T(x_n,y_n) \text{ and } gy_{n+1} = T(y_n,x_n)$$

and then we have

$$d(gx_n, gx_{n+1}) = d[T(x_{n-1}, y_{n-1}), T(x_n, y_n)].$$

Using (4.1), we have

$$\begin{aligned}
d(gx_n, gx_{n+1}) \leq\ & a_1[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)] \\
& + a_2[d(gx_{n-1}, T(x_{n-1}, y_{n-1})) + d(gx_n, T(x_n, y_n))] \\
& + a_3[d(gx_{n-1}, T(x_n, y_n)) + d(gx_n, T(x_{n-1}, y_{n-1}))] \\
& + a_4\left[\frac{d(gx_{n-1}, T(x_{n-1}, y_{n-1}))d(gx_n, T(x_n, y_n))}{d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)}\right] \\
& + a_5\left[\frac{[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)][d(gx_{n-1}, T(x_{n-1}, y_{n-1})) + d(gx_n, T(x_n, y_n))]}{1 + d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)}\right] \\
& + a_6\left[\frac{d(gx_{n-1}, T(x_{n-1}, y_{n-1})) + d(gx_{n-1}, T(x_n, y_n))}{1 + d(gx_n, T(x_n, y_n))d(gx_n, T(x_{n-1}, y_{n-1}))}\right] \\
& + a_7\left[\frac{d(gx_{n-1}, T(x_{n-1}, y_{n-1}))[1 + d(gx_n, T(x_n, y_n))]}{1 + d(gx_{n-1}, gx_n) + d(gx_n, T(x_n, y_n))}\right].
\end{aligned}$$

By using the definitions of the sequences $\{gx_n\}$ and $\{gy_n\}$, we have

$$\begin{aligned}
d(gx_n, gx_{n+1}) \leq\ & a_1[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)] \\
& + a_2[d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})] \\
& + a_3[d(gx_{n-1}, gx_{n+1}) + d(gx_n, gx_n)] \\
& + a_4\left[\frac{d(gx_{n-1}, gx_n)d(gx_n, gx_{n+1})}{d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)}\right] \\
& + a_5\left[\frac{[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)][d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})]}{1 + d(gx_{n-1}, gx_n) + (gy_{n-1}, gy_n)]}\right] \\
& + a_6\left[\frac{d(gx_{n-1}, gx_n) + d(gx_{n-1}, gx_{n+1})}{1 + d(gx_n, gx_{n+1})d(gx_n, gx_n)}\right] \\
& + a_7\left[\frac{d(gx_{n-1}, gx_n)[1 + d(gx_n, gx_{n+1})]}{1 + d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})}\right].
\end{aligned}$$

Using the triangle inequality and the fact that $d(x,y) \geq 0$, we have

$$\begin{aligned}
d(gx_n, gx_{n+1}) \leq\ & a_1[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)] \\
& + a_2[d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})] \\
& + a_3[d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1}) + d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})] \\
& + a_4\left[\frac{d(gx_{n-1}, gx_n)d(gx_n, gx_{n+1})}{d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)}\right] \\
& + a_5\left[\frac{[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)][d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})]}{1 + d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)}\right] \\
& + a_6\left[\frac{d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1}) + d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})}{1 + d(gx_n, gx_{n+1})d(gx_n, gx_{n+1})}\right] \\
& + a_7\left[\frac{d(gx_{n-1}, gx_n)[d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})]}{1 + d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})}\right].
\end{aligned}$$

Using Remark 4.1, we have

$$\begin{aligned}
d(gx_n, gx_{n+1}) \leq\ & a_1[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)] \\
& + a_2[d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})] \\
& + a_3[d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1}) + d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})] \\
& + a_4 d(gx_n, gx_{n+1}) \\
& + a_5[d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})] \\
& + a_6[d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1}) + d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})] \\
& + a_7 d(gx_{n-1}, gx_n)
\end{aligned}$$

which implies that

$$\alpha d(gx_n, gx_{n+1}) \leq \beta d(gx_{n-1}, gx_n) + a_1 d(gy_{n-1}, gy_n)$$

where

$$\begin{aligned}
\alpha &= 1 - (a_2 + 2a_3 + a_4 + a_5 + 2a_6), \\
\beta &= a_1 + a_2 + 2a_3 + a_5 + 2a_6 + a_7.
\end{aligned}$$

It follows that

$$d(gx_n, gx_{n+1}) \leq \eta d(gx_{n-1}, gx_n) + \theta d(gy_{n-1}, gy_n). \quad (4.2)$$

where $\eta = \frac{\beta}{\alpha}$ and $\theta = \frac{a_1}{\alpha}$.

Similarly we can prove that

$$d(gy_n, gy_{n+1}) \leq \eta d(gy_{n-1}, gy_n) + \theta d(gx_{n-1}, gx_n). \quad (4.3)$$

Adding (4.2) and (4.3), we get

$$[d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})] \leq \lambda [d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)]. \quad (4.4)$$

where $\lambda = \eta + \theta$.
Similarly, we have

$$[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)] \leq \lambda [d(gx_{n-2}, gx_{n-1}) + d(gy_{n-2}, gy_{n-1})].$$

Proceeding this way inductively, we get

$$[d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})] \leq \lambda^n [d(gx_0, gx_1) + d(gy_0, gy_1)]. \quad (4.5)$$

Letting $n \to \infty$, we have $\lambda^n \to 0$ since $0 < \lambda < 1$ and

$$[d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})] \to 0.$$

So $d(gx_n, gx_{n+1}) \to 0$ and $d(gy_n, gy_{n+1}) \to 0$.
Now, we show that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$.

Let $m > n \geq 1$, it follows that

$$\begin{aligned} d(gx_n, gx_m) &\leq d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + d(gx_{n+2}, gx_{n+3}) \\ &\quad + \dots + d(gx_{m-1}, gx_m) \\ &\leq \lambda^n d(gx_0, gx_1) + \lambda^{n+1} d(gx_0, gx_1) + \lambda^{n+2} d(gx_0, gx_1) \\ &\quad + \dots + \lambda^{m-1} d(gx_0, gx_1) \\ &\leq \frac{\lambda^n}{1-\lambda} d(gx_0, gx_1). \end{aligned} \tag{4.6}$$

Similarly, we have

$$\begin{aligned} d(gy_n, gy_m) &\leq d(gy_n, gy_{n+1}) + d(gy_{n+1}, gy_{n+2}) + d(gy_{n+2}, gy_{n+3}) \\ &\quad + \dots + d(gy_{m-1}, gy_m) \\ &\leq \lambda^n d(gy_0, gy_1) + \lambda^{n+1} d(gy_0, gx_1) + \lambda^{n+2} d(gy_0, gy_1) \\ &\quad + \dots + \lambda^{m-1} d(gy_0, gy_1) \\ &\leq \frac{\lambda^n}{1-\lambda} d(gy_0, gy_1). \end{aligned} \tag{4.7}$$

Adding (4.6) and (4.7), we get

$$[d(gx_n, gx_m) + d(gy_n, gy_m)] \leq \frac{\lambda^n}{1-\lambda} [d(gx_0, gx_1) + d(gy_0, gy_1)].$$

$$[d(gx_n, gx_m) + d(gy_n, gy_m)] \to 0.$$

Since $\lambda < 1, \lambda^n \to 0$ as $n \to \infty$ and $d(gx_n, gx_m) \to 0$ which in turn implies that $d(gy_n, gy_m) \to 0$ as $n, m \to \infty$.
Thus, $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$.
On the other hand, since $g(X)$ is complete subspace, there exist $x, y \in g(X)$ satisfying that $\{gx_n\}$ and $\{gy_n\}$ converge to x and y respectively. Now, we prove that $T(x,y) = gx$ and $T(y,x) = gy$.
Since T and g are commuting, it follows that

$$ggx_{n+1} = g(T(x_n, y_n)) = T(gx_n, gy_n) \tag{4.8}$$

Using (4.8) and continuity of T and g, we have

$$\begin{aligned}
\lim_{n\to\infty} ggx_n &= \lim_{n\to\infty} T(gx_n, gy_n) \\
g\left(\lim_{n\to\infty} gx_n\right) &= T\left(\lim_{n\to\infty} gx_n, \lim_{n\to\infty} gy_n\right) \\
g(x) &= T(x,y).
\end{aligned}$$

Similarly, we can show that $g(y) = T(y,x)$.
Hence, (gx, gy) is coupled point of coincidence of T and g.
Now, we claim that (gx, gy) is the unique coupled point of coincidence of T and g.
Suppose not. So, we have another coupled point of coincidence say (gx_1, gy_1) where $(x_1, y_1) \in X^2$ with $gx_1 = T(x_1, y_1)$ and $gy_1 = T(y_1, x_1)$.
Using (4.1), we have

$$\begin{aligned}
d(gx, gx) &= d[T(x,y), T(x,y)] \\
&\leq a_1[d(gx,gx) + d(gy,gy)] \\
&\quad + a_2[d(gx,gx) + d(gx,gx)] \\
&\quad + a_3[d(gx,gx) + d(gx,gx)] \\
&\quad + a_4\left[\frac{d(gx,gx)d(gx,gx)}{d(gx,gx) + d(gy,gy)}\right] \\
&\quad + a_5\left[\frac{[d(gx,gx) + d(gy,gy)][d(gx,gx) + d(gx,gx)]}{1 + d(gx,gx) + d(gy,gy)}\right] \\
&\quad + a_6\left[\frac{d(gx,gx) + d(gx,gx)}{1 + d(gx,gx)d(gx,gx)}\right] \\
&\quad + a_7\left[\frac{d(gx,gx)d(gx,gx)}{1 + d(gx,gx) + d(gx,gx)}\right].
\end{aligned}$$

Using Remark 4.1, we have

$$\begin{aligned} d(gx,gx) \leq\ & a_1[d(gx,gx)+d(gy,gy)] \\ & +a_2[d(gx,gx)+d(gx,gx)] \\ & +a_3[d(gx,gx)+d(gx,gx)] \\ & +a_4 d(gx,gx) \\ & +a_5[d(gx,gx)+d(gx,gx)] \\ & +a_6[d(gx,gx)+d(gx,gx)] \\ & +a_7 d(gx,gx). \end{aligned}$$

It follows that

$$d(gx,gx) \leq \phi d(gx,gx)+a_1 d(gy,gy) \tag{4.9}$$

where $\phi = a_1+2a_2+2a_3+a_4+2a_5+2a_6+a_7$.
Similarly

$$d(gy,gy) \leq \phi d(gy,gy)+a_1 d(gx,gx). \tag{4.10}$$

Adding (4.9) and (4.10), we get

$$[d(gx,gx)+d(gy,gy)] \leq \psi[d(gx,gx)+d(gy,gy)].$$

where $\psi = \phi + a_1$.
This is possible only when $d(gx,gx)+d(gy,gy) = 0$ since $\psi < 1$ which implies that $d(gx,gx) = 0$ and $d(gy,gy) = 0$.
Similarly $d(gx_1,gx_1) = 0$ and $d(gy_1,gy_1) = 0$.

Now, we consider

$$\begin{aligned}
d(gx,gx_1) &= d[T(x,y),T(x_1,y_1)] \\
&\leq a_1[d(gx,gx_1)+d(gy,gy_1)] \\
&\quad +a_2[d(gx,T(x,y))+d(gx_1,T(x_1,y_1))] \\
&\quad +a_3[d(gx,T(x_1,y_1))+d(gx_1,T(x,y))] \\
&\quad +a_4\left[\frac{d(gx,T(x,y))d(gx_1,T(x_1,y_1))}{d(gx,gx_1)+d(gy,gy_1)}\right] \\
&\quad +a_5\left[\frac{[d(gx,gx_1)+d(gy,gy_1)][d(gx,T(x,y))+d(gx_1,T(x_1,y_1))]}{1+d(gx,gx_1)+d(gy,gy_1)}\right] \\
&\quad +a_6\left[\frac{d(gx,T(x,y))+d(gx,T(x_1,y_1))}{1+d(gx_1,T(x_1,y_1))d(gx_1,T(x_1,y_1))}\right] \\
&\quad +a_7\left[\frac{d(gx,T(x,y))d(gx_1,T(x_1,y_1))}{1+d(gx,gx_1)+d(gx_1,T(x_1,y_1))}\right].
\end{aligned}$$

Using the fact that $gx=T(x,y)$ and $gx_1=T(x_1,y_1)$, we have

$$\begin{aligned}
d(gx,gx_1) &\leq a_1[d(gx,gx_1)+d(gy,gy_1)] \\
&\quad +a_2[d(gx,gx)+d(gx_1,gx_1)] \\
&\quad +a_3[d(gx,gx_1)+d(gx_1,gx)] \\
&\quad +a_4\left[\frac{d(gx,gx)d(gx_1,gx_1)}{d(gx,gx_1)+d(gy,gy_1)}\right] \\
&\quad +a_5\left[\frac{[d(gx,gx_1)+d(gy,gy_1)][d(gx,gx)+d(gx_1,gx_1)]}{1+d(gx,gx_1)+d(gy,gy_1)}\right] \\
&\quad +a_6\left[\frac{d(gx,gx)+d(gx,gx_1)}{1+d(gx_1,gx_1)d(gx_1,gx_1)}\right] \\
&\quad +a_7\left[\frac{d(gx,gx)d(gx_1,gx_1)}{1+d(gx,gx_1)+d(gx_1,gx_1)}\right].
\end{aligned}$$

Thus, we have

$$\begin{aligned}
d(gx,gx_1) &\leq (a_1+a_3+a_6)d(gx,gx_1)+a_1d(gy,gy_1) \\
&\quad +(a_3+a_6)d(gx_1,gx) \\
(1-(a_1+a_3+a_6))d(gx,gx_1) &\leq a_1d(gy,gy_1)+(a_3+a_6)d(gx_1,gx). \qquad (4.11)
\end{aligned}$$

Similarly

$$(1-(a_1+a_3+a_6))d(gy,gy_1) \leq a_1 d(gx,gx_1)+(a_3+a_6)d(gy_1,gy). \tag{4.12}$$

Adding (4.11) and (4.12) and then simplifying, we get

$$[d(gx,gx_1)+d(gy,gy_1)] \leq \omega[d(gx_1,gx)+d(gy_1,gy)] \tag{4.13}$$

where $\omega = \left[\frac{a_3+a_6}{1-(2a_1+a_3+a_6)}\right]$.

Similarly

$$[d(gx_1,gx)+d(gy_1,gy)] \leq \omega[d(gx,gx_1)+d(gy,gy_1)]. \tag{4.14}$$

Adding (4.13) and (4.14), we get

$$\begin{aligned}[d(gx_1,gx)+d(gy_1,gy)+d(gx,gx_1)+d(gy,gy_1)] \quad &\leq \quad \omega[d(gx_1,gx)+d(gy_1,gy)+d(gx,gx_1) \\ &\quad +d(gy,gy_1)]. \end{aligned} \tag{4.15}$$

So, $[d(gx_1,gx)+d(gy_1,gy)+d(gx,gx_1)+d(gy,gy_1)] = 0$ since $\omega < 1$.
It follows that
$d(gx_1,gx) = 0, d(gy_1,gy) = 0, d(gx,gx_1) = 0$, and $d(gy,gy_1) = 0$.
It follows that $gx_1 = gx$ and $gy_1 = gy$ so that $(gx,gy) = (gx_1,gy_1)$.
Thus, (gx,gy) is the unique coupled point of coincidence of T and g.

Next, we show that $gx = gy$.

$$
\begin{aligned}
d(gx,gy) &= d[T(x,y),T(y,x)] \\
&\leq a_1[d(gx,gy)+d(gy,gx)] \\
&\quad +a_2[d(gx,T(x,y))+d(gy,T(y,x))] \\
&\quad +a_3[d(gx,T(y,x))+d(gy,T(x,y))] \\
&\quad +a_4\left[\frac{d(gx,T(x,y))d(gy,T(y,x))}{d(gx,gy)+d(gy,gx)}\right] \\
&\quad +a_5\left[\frac{[d(gx,gy)+d(gy,gx)][d(gx,T(x,y))+d(gy,T(y,x))]}{1+d(gx,gy)+d(gy,gx)}\right] \\
&\quad +a_6\left[\frac{d(gx,T(x,y))+d(gx,T(y,x))}{1+d(gy,T(y,x))d(gy,T(y,x))}\right] \\
&\quad +a_7\left[\frac{d(gx,T(x,y))d(gy,T(y,x))}{1+d(gx,gy)+d(gy,T(y,x))}\right].
\end{aligned}
$$

Using (4.1) and the fact that $gx = T(x,y)$ and $gy = T(y,x)$, we have

$$
\begin{aligned}
d(gx,gy) &\leq a_1[d(gx,gy)+d(gy,gx)] \\
&\quad +a_2[d(gx,gx)+d(gy,gy)] \\
&\quad +a_3[d(gx,gy)+d(gy,gx)] \\
&\quad +a_4\left[\frac{d(gx,gx)d(gy,gy)}{d(gx,gy)+d(gy,gx)}\right] \\
&\quad +a_5\left[\frac{[d(gx,gy)+d(gy,gx)][d(gx,gx)+d(gy,gy)]}{1+d(gx,gy)+d(gy,gx)}\right] \\
&\quad +a_6\left[\frac{d(gx,gx)+d(gx,gy)}{1+d(gy,gy)d(gy,gy)}\right] \\
&\quad +a_7\left[\frac{d(gx,gx)d(gy,gy)}{1+d(gx,gy)+d(gy,gy)}\right].
\end{aligned}
$$

Thus, we have

$$
d(gx,gy) \leq \sigma d(gy,gx) \tag{4.16}
$$

where $\sigma = \left[\frac{a_1+a_3}{1-(a_1+a_3+a_6)}\right]$.

Similarly, we can show that

$$d(gy,gx) \quad \leq \quad \sigma d(gx,gy). \tag{4.17}$$

Adding (4.16) and (4.17), we have

$$[d(gx,gy)+d(gy,gx)] \leq \sigma[d(gx,gy)+d(gy,gx)].$$

Since $\sigma < 1$, the above inequality is only possible if

$$d(gx,gy) = d(gy,gx) = 0.$$

That is, $gx = gy$.
Now, we show that T and g have coupled common fixed point.
Now, let $gx = u$, then we have that $u = gx = T(x,y)$.
Since T and g are weakly compatible, then we have
$gu = g(gx) = gT(x,y) = T(gx,gy) = T(u,u)$ since $gx = gy$.
Hence (gu,gu) is a coupled point of coincidence and (u,u) is a coupled coincidence point of T and g.
The uniqueness of coupled point of coincidence implies that
$gu = u = gx = gy$. Therefore $T(u,u) = gu = u$.
That is (u,u) is a coupled common fixed point of T and g.
Finally, we show the uniqueness of a coupled common fixed point of T and g.
let $(u_1,u_1) \in X^2$ be another coupled common fixed point of T and g.
That is, $u_1 = gu_1 = T(u_1,u_1)$.
Hence (gu,gu) and (gu_1,gu_1) are two coupled points of coincidence of T and g.
The uniqueness of coupled point of coincidence implies that
$gu = gu_1$ and so $T(u_1,u_1) = u_1 = u$.
Hence (u,u) is the unique coupled common fixed point of T and g. □

Corollary 4.4 *Let (X,d) be a complete dislocated quasi-metric space. $T : X\times X \to X$ be a continuous mapping satisfying the following contractive condition of rational type*

$$\begin{aligned} d[T(x,y),T(u,v)] \leq\ & a_1[d(x,u)+d(y,v)] \\ & +a_2[d(x,T(x,y))+d(u,T(u,v))] \\ & +a_3[d(x,T(u,v))+d(u,T(x,y))] \\ & +a_4\left[\frac{d(x,T(x,y))d(u,T(u,v))}{d(x,u)+d(y,v)}\right] \\ & +a_5\left[\frac{[d(x,u)+d(y,v)][d(x,T(x,y))+d(u,T(u,v))]}{1+d[(x,u),(y,v)]}\right] \\ & +a_6\left[\frac{d(x,T(x,y))+d(x,T(u,v))}{1+d(u,T(u,v))d(u,T(u,v))}\right] \end{aligned}$$

for all $x,y,u,v \in X$ and a_1,a_2,a_3,a_4,a_5, and a_6 are non-negative constants with $2(a_1+a_2+a_5)+4(a_3+a_6)+a_4 < 1$.
Then T has a unique coupled fixed point in $X\times X$.

Proof: It follows from Theorem 4.3 by taking $g = I$ (the identity map on X) and $a_7 = 0$. □

Corollary 4.5 *Let (X,d) be a complete dislocated quasi-metric space. $T : X\times X \to X$ be a continuous mapping satisfying the following contractive condition of rational type*

$$\begin{aligned} d[T(x,y),T(u,v)] \leq\ & a_1[d(x,u)+d(y,v)] \\ & +a_2[d(x,T(x,y))+d(u,T(u,v))] \\ & +a_3[d(x,T(u,v))+d(u,T(x,y))] \\ & +a_4\left[\frac{d(x,T(x,y))d(u,T(u,v))}{d(x,u)+d(y,v)}\right] \\ & +a_5\left[\frac{[d(x,u)+d(y,v)][d(x,T(x,y))+d(u,T(u,v))]}{1+d[(x,u),(y,v)]}\right] \end{aligned}$$

for all $x,y,u,v \in X$ and a_1,a_2,a_3,a_4, and a_5 are non-negative constants with

$2(a_1+a_2+a_5)+4a_3+a_4<1$. *Then T has a unique coupled fixed point in $X\times X$.*

Proof: It follows from Theorem 4.3 by taking $g=I$ (the identity map on X) and $a_6=a_7=0$. $\square$

Corollary 4.6 *Let (X,d) be a complete dislocated quasi-metric space. $T:X\times X\to X$ be a continuous mapping satisfying the following contractive condition of rational type*

$$\begin{aligned} d[T(x,y),T(u,v)] \leq\ & a_1[d(x,u)+d(y,v)] \\ & +a_2[d(x,T(x,y))+d(u,T(u,v))] \\ & +a_3[d(x,T(u,v))+d(u,T(x,y))] \\ & +a_4\left[\frac{d(x,T(x,y))d(u,T(u,v))}{d(x,u)+d(y,v)}\right] \end{aligned}$$

for all $x,y,u,v\in X$ and a_1,a_2,a_3, and a_4 are non-negative constants with
$2(a_1+a_2)+4a_3+a_4<1$.
Then T has a unique coupled fixed point in $X\times X$.

Proof: It follows from Theorem 4.3 by taking $g=I$ (the identity map on X) and $a_5=a_6=a_7=0$. $\square$

Corollary 4.7 *Let (X,d) be a complete dislocated quasi-metric space. $T:X\times X\to X$ be a continuous mapping satisfying the following contractive condition*

$$\begin{aligned} d[T(x,y),T(u,v)] \leq\ & a_1[d(x,u)+d(y,v)] \\ & +a_2[d(x,T(x,y))+d(u,T(u,v))] \\ & +a_3[d(x,T(u,v))+d(u,T(x,y))] \end{aligned}$$

for all $x,y,u,v\in X$ and a_1,a_2, and a_3 are non-negative constants with
$2(a_1+a_2)+4a_3<1$.
Then T has a unique coupled fixed point in $X\times X$.

Proof: It follows from Theorem 4.3 by taking $g = I$ (the identity map on X) and $a_4 = a_5 = a_6 = a_7 = 0$. $\square$

Corollary 4.8 *Let (X,d) be a complete dislocated quasi-metric space.*
$T : X \times X \to X$ be a continuous mapping satisfying the following contractive condition

$$\begin{aligned} d[T(x,y),T(u,v)] &\leq a_1[d(x,u)+d(y,v)] \\ &\quad + a_2[d(x,T(x,y))+d(u,T(u,v))] \end{aligned}$$

for all $x,y,u,v \in X$ and a_1 and a_2 are non-negative constants with
$2(a_1+a_2) < 1$.
Then T has a unique coupled fixed point in $X \times X$.

Proof: It follows from Theorem 4.3 by taking $g = I$ (the identity map on X) and $a_3 = a_4 = a_5 = a_6 = a_7 = 0$. $\square$

Corollary 4.9 *Let (X,d) be a complete dislocated quasi-metric space.*
$T : X \times X \to X$ be a continuous mapping satisfying the following contractive condition

$$d[T(x,y),T(u,v)] \leq a_1[d(x,u)+d(y,v)]$$

for all $x,y,u,v \in X$ and a_1 is non-negative constants with $2a_1 < 1$.
Then T has a unique coupled fixed point in $X \times X$.

Proof: It follows from Theorem 4.3 by taking $g = I$ (the identity map on X) and $a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = 0$. $\square$

The following example supports our main theorem.

Example 4.3 *Let $X = [0,1)$ and $d : X \times X \to \Re^+$ be defined by*

$$d(x,y) = |x-y| + |y|$$

for all $x,y \in X$. Then (X,d) is dq-metric space.

We define the functions $T : X \times X \to X$ and $g : X \to X$ by

$$gx = \begin{cases} \frac{1}{3}x & \text{if } 0 \le x < \frac{9}{10} \\ \frac{3}{10} & \text{if } \frac{9}{10} \le x < 1 \end{cases}$$

and

$$T(x,y) = \begin{cases} \frac{x+y}{27} & \text{if } 0 \le x,y < \frac{9}{10} \\ \frac{1}{30}y & \text{if } \frac{9}{10} \le x < 1 \text{ and } 0 \le y < \frac{9}{10} \\ \frac{1}{30}x & \text{if } \frac{9}{10} < y < 1 \text{ and } 0 \le x < \frac{9}{10} \\ \frac{1}{15} & \text{if } \frac{9}{10} \le x < 1 \text{ and } \frac{9}{10} \le y < 1. \end{cases}$$

Clearly T and g are continuous, $T(X \times X) \subseteq g(X)$, and $g(X)$ is a complete subspace of X since

$$T(X \times X) = [0, \frac{1}{15}] \subseteq g(X) = [0, \frac{3}{10}].$$

***Case 1**: $0 \le x,y < \frac{9}{10}$*

$$\begin{aligned} d[T(x,y),T(u,v)] &= d\left(\frac{x+y}{27}, \frac{u+v}{27}\right) \\ &= \left|\frac{x+y}{27} - \frac{u+v}{27}\right| + \left|\frac{u+v}{27}\right| \\ &= \left|\frac{x}{27} + \frac{y}{27} - \frac{u}{27} - \frac{v}{27}\right| + \left|\frac{u}{27} + \frac{v}{27}\right| \\ &\le \left|\frac{x}{27} - \frac{u}{27}\right| + \left|\frac{y}{27} - \frac{v}{27}\right| + \left|\frac{u}{27}\right| + \left|\frac{v}{27}\right| \\ &= \frac{1}{9}\left[\left(\left|\frac{x}{3} - \frac{u}{3}\right| + \left|\frac{u}{3}\right|\right) + \left(\left|\frac{y}{3} - \frac{v}{3}\right| + \left|\frac{v}{3}\right|\right)\right] \\ &\le \frac{1}{9}[d(gx,gu) + d(gy,gv)] \\ &\le \frac{2}{9}[d(gx,gu) + d(gy,gv)]. \end{aligned}$$

Case 2: $\frac{9}{10} \leq x < 1$ *and* $0 \leq y < \frac{9}{10}$

$$\begin{aligned}
d[T(x,y),T(u,v)] &= d\left(\frac{1}{30}y, \frac{1}{30}v\right) \\
&= \left|\frac{1}{30}y - \frac{1}{30}v\right| + \left|\frac{1}{30}v\right| \\
&= \frac{1}{9}\left(\left|\frac{3}{10}y - \frac{3}{10}v\right| + \left|\frac{3}{10}v\right|\right) \\
&= \frac{1}{9}d(gy,gv) \\
&\leq \frac{1}{9}[d(gx,gu) + d(gy,gv)] \\
&\leq \frac{2}{9}[d(gx,gu) + d(gy,gv)].
\end{aligned}$$

Case 3: $\frac{9}{10} < y < 1$ *and* $0 \leq x < \frac{9}{10}$

$$\begin{aligned}
d[T(x,y),T(u,v)] &= d\left(\frac{1}{30}x, \frac{1}{30}u\right) \\
&= \left|\frac{1}{30}x - \frac{1}{30}u\right| + \left|\frac{1}{30}u\right| \\
&= \frac{1}{9}\left(\left|\frac{3}{10}x - \frac{3}{10}u\right| + \left|\frac{3}{10}u\right|\right) \\
&= \frac{1}{9}d(gx,gu) \\
&\leq \frac{1}{9}[d(gx,gu) + d(gy,gv)] \\
&\leq \frac{2}{9}[d(gx,gu) + d(gy,gv)].
\end{aligned}$$

Case 4: $\frac{9}{10} \leq x < 1 and \frac{9}{10} \leq y < 1$

$$
\begin{aligned}
d[T(x,y),T(u,v)] &= d\left(\frac{1}{15},\frac{1}{15}\right) \\
&= \frac{2}{9}\left(\left|\frac{3}{10}-\frac{3}{10}\right|+\left|\frac{3}{10}\right|\right) \\
&= \frac{2}{9}d(gx,gu) \\
&\leq \frac{2}{9}[d(gx,gu)+d(gy,gv)].
\end{aligned}
$$

It follows that

$$
\begin{aligned}
d[T(x,y),T(u,v)] \leq\ & a_1[d(gx,gu)+d(gy,gv)] \\
& +a_2[d(gx,T(x,y))+d(gu,T(u,v))] \\
& +a_3[d(gx,T(u,v))+d(gu,T(x,y))] \\
& +a_4\left[\frac{d(gx,T(x,y))d(gu,T(u,v))}{d(gx,gu)+d(gy,gv)}\right] \\
& +a_5\left[\frac{[d(gx,gu)+d(gy,gv)][d(gx,T(x,y))+d(gu,T(u,v))]}{1+d(gx,gu)+d(gy,gv)}\right] \\
& +a_6\left[\frac{d(gx,T(x,y))+d(gx,T(u,v))}{1+d(gu,T(u,v))d(gu,T(x,y))}\right] \\
& +a_7\left[\frac{d(gx,T(x,y))[1+d(gu,T(u,v))]}{1+d(gx,gu)+d(gu,T(u,v))}\right]
\end{aligned}
$$

where $x,y,u,v \in X$ and $a_1 = \frac{2}{9}, a_2 = \frac{1}{120}, a_3 = \frac{1}{64}, a_4 = \frac{1}{80}, a_5 = \frac{1}{100}, a_6 = \frac{1}{128}, a_7 = \frac{1}{32}$ since $2(a_1+a_2+a_5)+4(a_3+a_6)+a_4+a_7 = \frac{472}{763} < 1$.

Hence all the conditions of Theorem 4.3 are satisfied.

Hence, T and g have unique coupled point of coincidence and unique coupled common fixed point which are $(g0,g0)$ and $(0,0)$ respectively. This is due to the fact that

$$gT(0,0) = T(g0,g0) = T(0,0) = 0.$$

Also T and g are commuting and weakly compatible at $(0,0)$.

Chapter 5

Conclusion and Future Scope

In 2018, M. Mohammed established the existence of coupled fixed point for mapping satisfying certain rational type contraction condition in a complete dislocated quasi metric space. In this thesis, we have explored the properties of dislocated quasi-metric spaces and also discuss the difference between metric space and dislocated metric space. We established and proved existence of coupled coincidence point and existence and uniqueness of coupled common fixed point theorem for a pair of maps T and g in the setting of dislocated quasi metric spaces. Also we provided an example in support of our main result. Our work extended coupled fixed point result to common coupled fixed point result. Our result extends and generalizes several well-known comparable results in literature.

There are several published results related to existence of fixed points of self-maps defined on dislocated quasi metric space. There are also few results related to the existence of coupled common fixed points for a pair or more maps in this space. The researcher believes the search for the existence of coupled coincidence point and coupled common fixed points of maps satisfying different contractive conditions in dislocated quasi metric space is an active area of study. So, the forthcoming postgraduate students of Department of Mathematics and any researcher can exploit this opportunity and conduct their research work in this area.

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