COUPLED FIXED POINT RESULTS OF φ- CONTRACTIVE TYPE COUPLING IN METRIC SPACES

A Thesis Submitted to the Department of Mathematics, Jimma University in Partial Fulfillment for the Requirements of the Degree of Masters of Science in Mathematics

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DECLARATION

I the undersigned declare that, this thesis entitled " coupled fixed point results of φ-contraction type coupling in metric spaces'' is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged.

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ABSTRACT

In this research work, based on the recent work of Choudhury *et al.*, (2017), we established a coupled fixed point theorem of Chatterjea φ-contraction type coupling in metric spaces and proved the existence and uniqueness of strong coupled fixed point in metric spaces. An example is also provided in support of our main result. The procedure that we followed was the standard procedures used in the published works of (Choudhury *et al*., 2017) and (Aydi, Barakat, Felhi, and Radenovic, 2017). This study was conducted from September 2017 G.C. to June 2018 G.C.

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CHAPTER ONE

INTRODUCTION

1.1. Background of the study

Fixed-point theory is an important tool in the study of nonlinear analysis. It is one of the most powerful and popular tools of modern mathematics and considered to be the key connection between pure and applied mathematics. It also serves as a bridge between analysis and topology. It is applicable in different fields like economics, physical sciences, such as Chemistry and Physics. It is widely applicable in solving differential equations and almost all engineering fields. (Banach, 1922; Beg and Butt, 2013. Marasi, (2016) proved the existence and multiplicity of solutions for nonlinear fractional differential equations.

The study of fixed-points of mappings satisfying certain contractive conditions has been at the center of rigorous research activity (Malhotra and Bansal, 2015).

In 1922, the famous Banach contraction principle which states that if (X, d) is a complete metric space and $T: X \to X$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$ where $k \in [0,1)$, then T has a unique fixed point in X $\left(\text{Banach } (1922)\right)$.

Another category of contraction which is separate from Banach contraction was proposed by Kannan (1968) who established fixed point theorem in which the map under consideration need not be continuous. Mappings belonging to this category are known as Kannan type maps.

Let (X, d) be a complete metric space and $T: X \to X$. If there exists $k \in \left[0, \frac{1}{2}\right)$ such that $d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)]$ for all $x, y \in X$, then T has a unique fixed point in X (Kannan, 1968).

Kannan type mappings, its generalizations and extension in various spaces have been considered in a large number of work.

(Chatterjea,1972) established a new theorem which states as follows:

Let (X, d) be a complete metric space and $T: X \to X$. If there exists $k \in \left[0, \frac{1}{2}\right)$ such that

$d(Tx,Ty) \leq k[d(x,Ty) + d(y,Tx)]$ for all $x, y \in X$, then T has a unique fixed point in X (Chatterjea, 1972).

As (Rohades, 1977) comparison of definition of contractive mappings showed that the inequalities in Banach, Kannan, and Chatterjea maps are independent.

Banach Contraction principle has many applications and is extended by several authors.

Zamfirescu (1979) established the following theorem which is a generalization of Banach contraction principle (Banach, 1922), Kannan's theorem (Kannan,1968) and Chatterjea's theorem (Chatterjea,1972) .

(Zamfirescu, 1979). Let (X,d) be a complete metric space and $T: X \to X$. If for all $x, y \in X$ and some $\alpha \in [0,1)$, $\beta, \gamma \in \left[0, \frac{1}{2}\right)$ satisfies at least one of the following

i. $d(Tx,Ty) \leq \alpha d(x,y)$
ii. $d(Tx,Ty) \leq \beta[d(x,Tx) + d(y,Ty)]$
iii. $d(Tx,Ty) \leq \gamma[d(x,Ty) + d(y,Tx)]$.

Then T has a unique fixed point in X.

The concept of coupled fixed point and the study of coupled fixed point theorems appeared for the first time (Opoitsev and Khurode, 1984; Opoitsev, 1975a ; Opoitsev, 1975b), the topic expanded with the work of (Guo and Lakishmikantham,1987), where the monotone iterations technique is exploited.

Several years later, the theory of coupled fixed points in the setting of an ordered metric spaces and under some contractive type conditions on the operator was re-considered by (Bhaskar and Lakshmikantham, 2006).

Coupled fixed point theorems have a large share in the recent development of a fixed point theory (Samet and Vetro, 2011).

(Abbas *et al*., 2015) generalized coupled common fixed point results in partially ordered metric spaces.

(Shatanaw *et al*,. 2012) studied coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces.

Recently (Choudhury *et al*., 2017) introduced the concept of couplings which are actually coupled cyclic mappings with respect to two given subsets of a metric space. They proved the existence of strong coupled fixed point for a coupling with respect to non-empty subsets of a complete metric space.

(Choudhury *et al.*, (2017) have recently extended the idea of Banach and Chatterjea contraction and combining it with the concept of coupling they define the Banach and Chatterjea type coupling and proved some strong unique coupled fixed point results in complete metric spaces.

(Aydi, *et al*., 2017) introduced coupled fixed point of Banach nonlinear contractive type couplings in metric spaces. They proved the existence of strong unique coupled fixed point for nonlinear contractive type couplings in complete metric spaces.

Inspired and motivated by the works of (Choudhury *et al*., 2017) and (Aydi, *et al*., 2017), we established a coupled fixed point theorem of Chatterjea φ-contraction type coupling in metric spaces. We also provided an example in support of our main result.

1.2 Statement of the problem

This study focused on establishing coupled fixed point theorem of Chatterjea φ-contraction type coupling in metric spaces and proving the existence and uniqueness of strong coupled fixed point of Chatterjea φ-contraction type coupling in complete metric spaces.

1.3 Objectives of the study

1.3.1 General objective

The main objective of this study was to establish coupled fixed points theorem of Chatterjea φ-contraction type coupling in metric spaces.

1.3.2 Specific objectives

This study has the following specific objectives:

1) To prove the existence of strong coupled fixed point.
2) To show the uniqueness of strong coupled fixed point.
3) To verify the applicability of the results obtained using specific examples.

1.4. Significance of the study

The results of this study may have the following importance.

- The outcome of this study may contribute to research activities in the study area.
- It may provide basic research skills to the researcher.
- It may be applied in solving application problems in different branches of mathematics.

1.5 Delimitation of the study

This paper is devoted to give an answer to an open problem presented by (Choudhury *et al*., 2017) concerning the investigation of coupled fixed point and couplings related properties for couplings satisfying other types of inequalities as well.

CHAPTER TWO

LITERATURE REVIEW

The theoretical framework of metric fixed point theory has been an active research field and the contraction mapping principle is one the most important theorems in functional analysis. Many authors have devoted their attention to generalizing metric spaces and the contraction mapping principle.

The family of contractive mappings in metric spaces has already been studied in the literature since long time.

The concept of coupled fixed point and the study of coupled fixed point problems appeared for the first time (Opoitsev and Khurode, 1984; Opoitsev, 1975a ;).

Several years later, the theory of coupled fixed points in the setting of an ordered metric space and under some contractive type conditions on the operator was re-considered by (Bhaskar and Lakshmikantham ., 2006).

For other results on coupled fixed theory we refer (Samet and Vetro, 2011; Lakshmikantham,. and ´Ciri´c., 2009. Coupled fixed point theorems for nonlinear contractions in partially ordered G-metric spaces is generalized by (Aydi *et al*., 2011).

Nashine *et al*., (2012) studied about cyclic generalized contractions and fixed point results with applications to an integral equation .

Definition:- (Cyclic coupled Kannan type contraction). Let A and B be two nonempty subsets of a metric space (X, d). We call a mapping $F: X×X \rightarrow X$ a cyclic coupled Kannan type contraction with respect to A and B if F is cyclic with respect to A and B satisfying, for some $k \in [0, 1/2)$, the inequality

$d\,(F\,(x,y), F\,(u, v)) \leq k\,[d\,(x, F\,(x, y)) + d\,(u, F\,(u, v))]$ where $x, v \in A$, $y, u \in B$.

(Aghajani, A. and Arab, R. 2013) studied fixed points of θ-φ C-contractive mappings in partially ordered b-metric spaces and application to quadratic integral equations.

Recently (Choudhury *et al*., 2017) introduced the concept of couplings which are actually coupled cyclic mappings with respect to two given subsets of a metric space. They proved the existence of strong coupled fixed point for a coupling with respect to non-empty subsets of a complete metric space. Two types of couplings are defined on metric spaces, namely, Banach type and Chatterjea type couplings were considered. These are actually coupled cyclic mappings with respect to two given non-empty subsets of a metric space, they establish the existence and uniqueness of strong coupled fixed points for both types of couplings.

Based on a recent work of (Aydi., 2017) introduce the coupled fixed point of Banach nonlinear contractive type couplings in metric spaces. They proved the existence of strong coupled fixed point for nonlinear contractive type couplings in complete metric spaces.

CHAPTER THREE

METHODOLOGY

3.1 Study period and site

The study was conducted from September 2017 to June 2018 in Jimma University at the Department of Mathematics.

3.2 Study design

In order to achieve the objective of the study we employed analytical design.

3.3 Source of Information

In this study secondary data such as, different mathematics books related to the study area, published articles related to the topic and internet sources were used.

3.4 Mathematical Procedure of the Study

In this study we followed the standard procedures used in the published work of (Choudhury *et al.*, 2017), (Rashid and Khan., 2017) and (Hassen Aydi, *et al*., 2017).

The procedures are:

- Constructing sequences and showing that the constructed sequences are Cauchy.
- Proving the existence of strong unique coupled fixed point.
- Giving applicable example in support of the main results.

CHAPTER FOUR

DISCUSSION AND RESULTS

4.1 Preliminaries

(Poincare, 1866). Let X be a nonempty set and $T: X \to X$ a selfmap. We say that x is a fixed point of T if $T(x) = x$. We denote the set of fixed point of T by Fix(T).

Definition 1.1: (Kirk *et al*., 2003). An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F: X \times X \to X$, if $x = F(x, y)$ and $y = F(y, x)$.

Definition 1.2: (Choudhury and Maitya., 2014). Let A and B be two nonempty subsets of a given set X. Any function $f: X \to X$ is said to be cyclic mapping with respect to A and B, if

$f(A) \subset B$ and $f(B) \subset A$.

Very recently, (Choudhury *et al*., 2017) introduced the concept of couplings between two non-empty subsets in a metric space by extending the idea behind a cyclic mapping.

Definition 1.3: (Choudhury *et al*., 2017). Let (X, d) be a metric space and let A and B be two nonempty subsets of X. A coupling with respect to A and B is a function $F: X \times X \to X$ such that

$F(x, y) \in B$ and $F(y, x) \in A$ whenever $x \in A$ and $y \in B$.

Definition 1.4: (Choudhury *et al*., 2017). Let X be a nonempty set. An element $(x, y) \in X \times X$ is said to be a strong coupled fixed point of the mapping $F: X \times X \to X$ if (x, y) is a coupled fixed point and $x = y$, that is, if $x = F(x, x)$.

(Choudhury *et al.*, 2017) extending the idea of Banach and Chatterjea contraction and combining it with the concept of coupling defined Banach and Chatterjea type coupling and theorems in connection to these couplings in metric spaces as follows:

Definition 1.5: (Choudhury *et al.*, 2017). Let A and B be two nonempty subsets of a complete metric space (X, d). A coupling $F: X \times X \to X$ is called a Banach type coupling with respect to A and B if it satisfies the following inequality:

$d\big(F(x, y), F(u, v)\big) \leq \frac{1}{2} k[d(x, u) + d(y, v)]$ where $x, v \in A, y, u \in B$ and $k \in [0,1)$.

Theorem 1.1: (Choudhury *et al.*, 2017). Let A and B be two nonempty closed subsets of a complete metric space (X, d). Let $F: X \times X \to X$ be a Banach type coupling with respect to A and B. Then $A \cap B \neq \emptyset$ and F has a unique strong coupled fixed point in $A \cap B$.

Definition 1.6: (Choudhury *et al.*, 2017). Let A and B be two nonempty subsets of a complete metric space (X, d). A coupling $F: X \times X \to X$ is called a Chatterjea type coupling with respect to A and B if it satisfies the following inequality:

$$d\big(F(x,y), F(u,v)\big) \leq k\big[d\big(x, F(u,v)\big) + d\big(u, F(x,y)\big)\big]$$

where $x, v \in A, y, u \in B$ and $k \in [0, \frac{1}{2})$.

Theorem 1.2: (Choudhury *et al.*, 2017). Let *A* and *B* be two nonempty closed subsets of a complete metric space (X, d). Let $F: X \times X \to X$ be a Chatterjea type coupling with respect to A and B. Then $A \cap B \neq \emptyset$ and F has a unique strong coupled fixed point in $A \cap B$.

4.2 Main Results

Definition 2.1 A function $\varphi: [0,\infty) \to [0,\infty)$is called an altering distance function, if the following properties are satisfied

i, φ is monotone increasing and continuous,

ii, $\varphi(t) = 0$ if and only if $t = 0$.

In this research paper we denote the class of functions $\varphi: [0,\infty) \to [0,\infty)$ satisfying

i) $\varphi(0) = 0$;

ii) $\varphi(t) < t$ for all $t > 0$;

iii) $\lim_{r\to t^+} \varphi(r) < t$ for all $t > 0$

by Φ.

Lemma 2.1:- (Aydi *et al*., 2013) (A Boyd- Wong type coupling)

Let $\varphi \in \Phi$ and $\{U_u\}$ be a given sequence such that $U_u \to 0^+$ as $n \to \infty$. Then $\varphi(U_u) \to 0^+$ as $n \to \infty$. Also $\varphi(0) = 0$.

Now, we introduce the following:

Definition 2.2. Let A and B be two nonempty subsets of a metric space (X, d).

We say that $F: X \times X \to X$ is a Chatterjea $\varphi -$ contraction type coupling with respect to A and B if there exists $\varphi \in \Phi$ such that

$$d(F(x,y), F(u,v)) \leq \varphi[max\{d(x, F(u,v)), d(u, F(x,y))\}] \qquad (1)$$

for any $x, v \in A \; and \; y, u \in B$ and $\varphi \in \Phi$.

Theorem 2.1. Let A and B be two nonempty closed subsets of a complete metric space (X, d). Let $F: X \times X \to X$ be a Chatterjea $\varphi -$ contraction type coupling with respectto A and B. Then $A \cap B \neq \emptyset$ and F has a unique strong coupled fixed point in $A \cap B$.

Proof. Let $x_0 \in A$ and $y_0 \in B$ be any two arbitrary elements of X. Choose

$x_1 = F(y_0, x_0)$ and $y_1 = F(x_0, y_0)$.

Since F is coupling with respect to A and B, we have $x_1 \in A$ and $y_1 \in B$. Continuing this process, we construct two sequences $\{x_n\}$ and $\{y_n\}$ as

$x_{n+1} = F(y_n, x_n)$ and $y_{n+1} = F(x_n, y_n)$ for all $n \in N$.

Clearly, $x_n \in A$ and $y_n \in B$ for all $n \geq 0$. Now by using (1), we have

$$
\begin{aligned}
d(x_{n+1}, y_{n+2}) &= d\big(F(y_n, x_n), F(x_{n+1}, y_{n+1})\big) \\
&\leq \varphi\big[\max\big\{d\big(y_n, F(x_{n+1}, y_{n+1})\big), d\big(x_{n+1}, F(y_n, x_n)\big)\big\}\big] \\
&= \varphi[\max\{d(y_n, x_{n+1}), d(x_{n+1}, y_n\ )] \\
&= \varphi(d(y_n, x_{n+1})).
\end{aligned}
$$

Thus, we have

$d(x_{n+1}, y_{n+2}) \leq \varphi(d(y_n, x_{n+1}))$ (2)

and

$$
\begin{aligned}
d(y_{n+1}, x_{n+2}) &= d\big(F(x_n, y_n), F(y_{n+1}, x_{n+1})\big) \\
&\leq \varphi\big[\max\big\{d\big(x_n, F(y_{n+1}, x_{n+1})\big), d\big(y_{n+1}, F(x_n, y_n)\big)\big\}\big] \\
&= \varphi\ [\max\{d(x_n, y_{n+1}), d(y_{n+1}, x_n)\}] \\
&= \varphi(d(x_n, y_{n+1})).
\end{aligned}
$$

Thus, we have

$d(y_{n+1}, x_{n+2}) \leq \varphi(d(x_n, y_{n+1}))$. (3)

By using (2) and (3), we have

$\max\{d(x_{n+1}, y_{n+2}), d(y_{n+1}, x_{n+2})\} \leq \varphi\{\max\{(d(x_n, y_{n+1}), d(y_n, x_{n+1})\}\})$. (4)

Let $\lambda_n = \max\{(d(x_n, y_{n+1}), d(y_n, x_{n+1})\})$then from (4) we have

$\lambda_{n+1} \leq \varphi(\lambda_n)$ for all *n*. (5)

Suppose that $\lambda_n = 0$ for some n. It follows that

$y_n = x_{n+1} = F(y_n, x_n), \quad x_n = y_{n+1} = F(x_n, y_n)$.

From (1), we have

$$
\begin{aligned}
d(x_n, y_n) = d(y_{n+1}, x_{n+1}) &= d\big(F(x_n, y_n), F(y_n, x_n)\big) \\
&\leq \varphi\big[\max\big\{d\big(x_n, F(y_n, x_n)\big), d\big(y_n, F(x_n, y_n)\big)\big\}\big] \\
&= \varphi[\max\{d(x_n, y_n), d(y_n, x_n)\}] \\
&= \varphi\big(d(x_n, y_n)\big).
\end{aligned}
$$

Which implies that $d(x_n, y_n) = 0$ that is $x_n = y_n$. Consequently, x_n is a strong coupled fixed point of F.

Now, suppose that $\lambda_n > 0$ for all *n*. From (5), we have

$\lambda_{n+1} \leq \varphi(\lambda_n) < \lambda_n$. (6)

The sequence $\{\lambda_n\}$is a monotone decreasing sequence of nonnegative real numbers, so there exists $a \geq 0$ such that

$\lim_{n\to\infty}\lambda_n = a^+$.

From (6), we have

$\lim_{n\to\infty}\varphi(\lambda_n) = a^+$. (7)

We show $a = 0$.

Suppose that $a > 0$. By (7) and $\lim_{r\to t^+}\varphi(r) < t for\ all\ t > 0$, we get

$$a = \lim_{n\to\infty}\varphi(\lambda_n) = \lim_{\lambda_n\to a^+}\varphi(\lambda_n) < a,$$

which is a contradiction. So $a = 0$, that is,

$\lim_{n\to\infty} d(x_n, y_{n+1}) = \lim_{n\to\infty} d(y_n, x_{n+1}) = 0.$ (8)

Again, by (1), we have

$$\begin{aligned} d(x_{n+1}, y_{n+1}) &= d\big(F(y_n, x_n), F(x_n, y_n)\big) \\ &\leq \varphi\big[\max\{d\big(y_n, F(x_n, y_n)\big), d\big(x_n, F(y_n, x_n)\big)\}\big] \\ &= \varphi(\max\{d(y_n, x_n), d(x_n, y_n)\}) = \varphi\big(d(x_n, y_n)\big). \end{aligned} \tag{9}$$

Let $\beta_n = d(x_n, y_n)$, then from (9) we have

$$\beta_{n+1} \leq \varphi(\beta_n). \tag{10}$$

Suppose that $\beta_n = 0$ for some $p \in N$, which gives $x_p = y_p$ and so $x_{p+1} = y_{p+1}$.

By induction, we have

$d(x_n, y_n) = 0$ for all $n \geq p$.

Thus $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Now, suppose that $\beta_n > 0$ for all n. From (10), we have

$\beta_{n+1} \leq \varphi(\beta_n) < \beta_n$.

The sequence $\{\beta_n\}$ is monotone decreasing sequence of nonnegative real numbers and bounded below, so there exists $z \geq 0$ such that

$\lim_{n\to\infty}\beta_n = z^+$.

Now we show $z = 0$.

Suppose that $z > 0$. By (7) and $\lim_{r \to t^+} \varphi(r) < t$ for all $t > 0$,we get

$$z = \lim_{n \to \infty} \varphi(\beta_n) \leq \lim_{\lambda_n \to a^+} \varphi(\beta_n) < z,$$

which is a contradiction, so $z = 0$. In both cases, we get

$$\lim_{n \to \infty} d(x_n, y_n) = 0. \tag{11}$$

By using (8) and (11), we obtain

$$\lim_{n \to \infty} d(x_n, x_{n+1}) \leq \lim_{n \to \infty} d(x_n, y_n) + \lim_{n \to \infty} d(y_n, x_{n+1}) = 0. \tag{12}$$

Similarly,

$$\lim_{n \to \infty} d(y_n, y_{n+1}) \leq \lim_{n \to \infty} d(y_n, x_n) + \lim_{n \to \infty} d(x_n, y_{n+1}) = 0. \tag{13}$$

From (12) and (13), have

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = \lim_{n \to \infty} d(y_n, y_{n+1}) = 0. \tag{14}$$

Now we shall show $\{x_n\}$and $\{y_n\}$ are Cauchy sequences in *A* and *B* respectively.

We argue by contradiction. Suppose that $\{x_n\}$ or $\{y_n\}$ is not a Cauchy sequence, then there exists $\varepsilon > 0$, for which we can find subsequences of integers $\{n_k\}$ and $\{m_k\}$ with

$n_k > m_k > k$ such that

$$\max\{d(x_{n_k}, x_{m_k}), d(y_{n_k}, y_{m_k})\} \geq \varepsilon. \tag{15}$$

Further, corresponding to m_k, we can choose m_k in such a way that it is the smallest

integer with$n_k > m_k$ and satisfying (15). Then

$$\max\{d(x_{n_{k-1}}, y_{m_k}), d(y_{n_{k-1}}, x_{m_k})\} < \varepsilon. \tag{16}$$

Using the triangle inequality and condition (1), we have

$$d(x_{n_k}, x_{m_k}) \leq d(x_{n_k}, y_{n_k}) + d(y_{n_k}, x_{m_{k+1}}) + d(x_{m_{k+1}}, x_{m_k})$$

$$= d\left(x_{n_k}, y_{n_k}\right) + d\left(x_{m_{k+1}}, x_{m_k}\right) + d\left(F(x_{n_{k-1}}, y_{n_{k-1}}), F\left(y_{m_k}, x_{m_k}\right)\right)$$

$$\leq d\left(x_{n_k}, y_{n_k}\right) + d\left(x_{m_{k+1}}, x_{m_k}\right)$$

$$+\varphi\left[\max\left\{d\left(x_{n_{k-1}}, F\left(y_{m_k}, x_{m_k}\right)\right), d\left(y_{m_k}, \left(F(x_{n_{k-1}}, y_{n_{k+1}})\right)\right)\right\}\right]$$

$$= d\left(x_{n_k}, y_{n_k}\right) + d\left(x_{m_{k+1}}, x_{m_k}\right) + \varphi\left[\max\left\{d\left(y_{n_{k-1}}, y_{m_k}\right), d\left(y_{m_k}, x_{n_{k-1}}\right)\right\}\right]$$

$$= d\left(x_{n_k}, y_{n_k}\right) + d\left(x_{m_{k+1}}, x_{m_k}\right) + \varphi\left(d\left(x_{n_{k-1}}, y_{m_k}\right)\right). \tag{17}$$

Again

$$d\left(y_{n_k}, y_{m_k}\right) \leq d\left(y_{n_k}, x_{n_k}\right) + d\left(x_{n_k}, y_{m_{k+1}}\right) + d\left(y_{m_{k+1}}, y_{m_k}\right)$$

$$= d\left(y_{n_k}, x_{n_k}\right) + d\left(y_{m_{k+1}}, y_{m_k}\right) + d\left(F(y_{n_{k-1}}, x_{n_{k-1}}), F\left(x_{m_k}, y_{m_k}\right)\right)$$

$$\leq d\left(y_{n_k}, x_{n_k}\right) + d\left(y_{m_{k+1}}, y_{m_k}\right)$$

$$+\varphi\left[\max\left\{d\left(y_{n_{k-1}}, F\left(x_{m_k}, y_{m_k}\right)\right), d\left(x_{m_k}, \left(F(y_{n_{k-1}}, x_{n_{k+1}})\right)\right)\right\}\right]$$

$$= d\left(y_{n_k}, x_{n_k}\right) + d\left(y_{m_{k+1}}, y_{m_k}\right) + \varphi\left[\max\left\{d\left(y_{n_{k-1}}, x_{m_k}\right), d\left(x_{m_k}, y_{n_{k-1}}\right)\right\}\right]$$

$$= d\left(y_{n_k}, x_{n_k}\right) + d\left(y_{m_{k+1}}, y_{m_k}\right) + \varphi\left(d\left(y_{n_{k-1}}, x_{m_k}\right)\right). \tag{18}$$

Using (16), (17) and (18) together with the fact that φ is nondecreasing,

$$\varepsilon \leq \max\left\{d\left(x_{n_k}, x_{m_k}\right), d\left(y_{n_k}, y_{m_k}\right)\right\}$$

$$\leq d\left(x_{n_k}, y_{n_k}\right) + \max\left\{d\left(x_{m_k}, x_{m_{k+1}}\right), d\left(y_{m_k}, y_{m_{k+1}}\right)\right\}$$

$$+\varphi[\max\left\{d\left(x_{n_{k-1}}, y_{m_k}\right), d\left(y_{n_{k-1}}, x_{m_k}\right)\right\}]$$

$$\leq d\left(x_{n_k}, y_{n_k}\right) + \max\left\{d\left(x_{m_{k+1}}, x_{m_k}\right), d\left(y_{m_{k+1}}, y_{m_k}\right)\right\} + \varphi(\varepsilon).$$

Passing to limit as $k \to \infty$ in the above inequality, and using (11), (14) and condition (iii) on φ, we obtain

$$\varepsilon \leq \varphi(\varepsilon) < \varepsilon.$$

This is a contradiction. This implies that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in A and B respectively. Since A and B are closed subsets of the complete metric space (X, d) there exist $x \in A \; and \; y \in B$ such that

$$d(x, x) = \lim_{n\to\infty} d((x_n, x) = \lim_{n,m\to\infty} d((x_n, x_m) = 0 \tag{19}$$

and

$$d(y, y) = \lim_{n\to\infty} d(y_n, y) = \lim_{n,m\to\infty} d(y_n, y_m) = 0. \tag{20}$$

From (11), (19) and (20) together with Lemma 2.1, we get

$$\lim_{n,m\to\infty} d(x_n, y_m) = d(x, y) = 0.$$

Hence $x = y$, and so $x \in A \cap B$. So that $A \cap B \neq \emptyset$.

Now, we show that x is a strong coupled fixed point in $A \cap B$. From (1)

$$\begin{aligned} d\big(x, F(x,x)\big) = d\big(x, F(x,y)\big) &\le d(x, x_{n+1}) + d\big(x_{n+1}, F(x,y)\big) \\ &= d(x, x_{n+1}) + d\big(F(x,y), F(y_n, x_n)\big) \\ &\le d(x, x_{n+1}) + \varphi(\max\{d\big(x, F(y_n, x_n)\big), d\big(y_n, F(x,y)\big) \\ &= d(x, x_{n+1}) + \varphi(\max\{d(x, y_n), d(y_n, x) \\ &= d(x, x_{n+1}) + \varphi\big(d(x, y_n)\big). \end{aligned} \tag{21}$$

Since $\lim_{n\to\infty} d(x, y_n) = 0^+$ by Lemma 2.1, we have

$$\lim_{n\to\infty} \varphi\big(d(x, y_n)\big) = 0^+.$$

Passing to limit as $n \to \infty$in (21), we obtain

$$d\big(x, F(x,x)\big) \le \varphi(d(x, y_n) = 0.$$

It follows that $x = F(x, x)$.

Now we show the strong coupled fixed point of F is unique.

Assume that F has two strong coupled fixed points u, v in $A \cap B$ that is,

$u = F(u, u) \; and \; v = F(v, v)$ with $u, v \in A \cap B$. Suppose that $u \neq v$, By (1),we have

$$d(u,v) = d\big(F(u,u), F(v,v)\big) \le \varphi\big(d(u,v)\big) < d(u,v)$$

which is a contradiction. This implies that $u = v$, i.e., the strong coupled fixed point of F is unique.

We state the following consequences from Theorem 2.3.

Corollary 2.1. Let A and B be two nonempty closed subsets of a complete metric space (X, d) and $F: X \times X \to X$ be such that

$$d\big(F(x,y), F(u,v)\big) \le kmax\big\{d\big(x, F(u,v)\big), d\big(u, F(x,y)\big)\big\}.$$

where $x, v \in A, y, u \in B$ and $k \in [0, \frac{1}{2})$. Then $A \cap B \neq \emptyset$ and F has a unique strong coupled fixed point in $A \cap B$.

Corollary 2.2. Let A and B be two nonempty closed subsets of a complete metric space (X, d) and $F: X \times X \to X$ be such that

$$d\big(F(x,y), F(u,v)\big) \le \frac{k}{2}\big[d\big(x, F(u,v)\big) + d\big(u, F(x,y)\big)\big]$$

where $x, v \in A, y, u \in B$ and $k \in [0, 1)$. Then $A \cap B \neq \emptyset$ and F has a unique strong coupled fixed point in $A \cap B$.

The following example is in support of Theorem 2.1

Example:- Let $X = [-4,4]$ and $d \; on \; X \times X$ be defined by

$d(x,y) = |x - y|$ for all $x, y \in X$. Let $A = [-1,1] \; and \; B = [0,3]$. Let us define

$F: X \times X \to X$ by

$$F(x,y) = \begin{cases} 1 \; if \; -1 \le x \le 1, y \le 1 \\ \dfrac{x+y}{24}, \; elsewhere \end{cases} . \qquad (*1)$$

Now we show $F: X \times X \to X$ is coupling with respect to A and B.

Let $x \in A \; and \; y \in B$. Here two cases arise for y,

Case (i): $0 \le y \le 1$.

Case (ii): $1 < y \leq 3$.

For case (i) i.e. $x, y \in A$, by using (*1), we have

$$F(x, y) = 1 \in B \ and \ F(y, x) = 1 \in A.$$

For case (ii) i.e. $x \in A \ and \ 1 < y \leq 3$, by using (*1), we have

$$F(x,y) = \frac{x+y}{24}$$

$$0 < F(x, y) \leq \frac{1}{6}, \qquad F(x, y) \in B$$

$$0 < F(y, x) \leq \frac{1}{6}, \qquad F(y, x) \in A.$$

Thus in both the cases considered above we get that *F* is a coupling with respect to *A* and *B*.

Now we consider a function $\varphi: [0, \infty) \to [0, \infty)$ given by

$$\varphi(t) = \begin{cases} \frac{2}{3}t, & 0 \leq t \leq \frac{22}{13} \\ \frac{22}{13}, & t > \frac{22}{13} \end{cases}. \tag{*2}$$

Now we show that F is Chaterjea φ-contraction type coupling with respect to *A* and *B*.

Let $x, v \in A \ and \ y, u \in B$, three cases for y and u.

Case (i): when both $y, u \in A$, i.e. $0 \leq y \leq 1 \ and \ 0 \leq u \leq 1$.

Case (ii): when $x, y, v \in A \ and \ 1 < u \leq 3$.

Case (iii): when $x, v \in A \ and \ y < 1, 1 < u \leq 3$.

We consider each case separately.

Case (i): when both $y, u \in A$, i.e. $0 \leq y \leq 1 \ and \ u \leq 1$, we have from (*2).

$F(x, y) = F(u, v) = 1$.

So $d\big(x, F(u, v)\big) = d\big(u, F(x, y)\big) = 0$.

Thus $\max\{ d\big(x, F(u, v)\big), d\big(u, F(x, y)\big)\} = 0$.

Using (*2) in above, we get

$\varphi\left(\max\left\{ d\left(x, F(u,v)\right), d\left(u, F(x,y)\right)\right\}\right) = \varphi(0) = 0.$ (*3)

Also for $, y, u, v \in A$, we have from (*1)

$$F(x,y) = F(u,v) = 1$$

$\Rightarrow d\left(F(x,y), F(u,v)\right) = 0.$ (*4)

Thus from (*3) and (*4), we get

$d\left(F(x,y), F(u,v)\right) = \varphi\left(\max\left\{ d\left(x, F(u,v)\right), d\left(u, F(x,y)\right)\right\}\right).$

Case (ii), when $x, y, v \in A \; and \; 1 < u \leq 3.$

Without loss of generality we assume $x, y, v \in A \; and \; u > 1.$

Now we have $F(x,y) = 1 \; and \; 0 < F(u,v) \leq \frac{1}{6}.$

Thus $\frac{5}{6} \leq d\left(x, F(u,v)\right) < 1 \; and \; 0 < d\left(u, F(x,y)\right) \leq 2.$

So $\max\{d\left(x, F(u,v)\right), d\left(u, F(x,y)\right)\} = 2.$

Using (*2) in above, we get

$\varphi\left(\max\left\{ d\left(x, F(u,v)\right), d\left(u, F(x,y)\right)\right\}\right) = \varphi(2) = \frac{22}{13}.$ (*5)

Also for $x, y, v \in A \; and \; 1 < u \leq 3$, we have

$F(x,y) = 1 \; and \; 0 < F(u,v) \leq \frac{1}{6}.$ (*6)

From (*3)

$d\left(F(x,y), F(u,v)\right) < 1 - 0 = 0 < \frac{22}{13}.$ (*7)

Thus from (*5) and (*7), we have

$$d\left(F(x,y), F(u,v)\right) < \varphi\left(\max\left\{ d\left(x, F(u,v)\right), d\left(u, F(x,y)\right)\right\}\right).$$

Case (iii): $v \in A \; and \; y < 1, 1 < u \leq 3$, we have

$0 < F(x,y) \leq \frac{1}{6} \; and \; 0 < F(u,v) \leq \frac{1}{6}.$

So $\frac{5}{6} \leq d\big(x, F(u,v)\big) < 1 \;\; and \; 1 < d\big(u, F(x,y)\big) \leq \frac{17}{6}$.

Thus $\max\big\{d\big(x, F(u,v)\big), d\big(u, F(x,y)\big)\big\} = \frac{17}{6}$.

Using (*2) in above, we get

$$\varphi\big(\max\big\{\, d\big(x, F(u,v)\big), d\big(u, F(x,y)\big)\big\}\big) = \varphi\left(\frac{17}{6}\right) = \frac{22}{13}. \qquad (*8)$$

Also for $x, v \in A \; and \;\; y > 1, 1 < u \leq 3$, we have

$$0 < F(x,y) \leq \frac{1}{6} \; and \; 0 < F(u,v) \leq \frac{1}{6} \qquad (*9)$$

From (*9)

$$d\big(F(x,y), F(u,v)\big) < \frac{1}{6} - 0 = 0 < \frac{22}{13}\,. \qquad (*10)$$

Thus from (*8) and (*10), we have

$$d\big(F(x,y), F(u,v)\big) < \varphi\big(\max\big\{\, d\big(x, F(u,v)\big), d\big(u, F(x,y)\big)\big\}\big).$$

Thus from all the cases we considered above F is Chatterjea φ-contraction type coupling with respect to A and B, F has a unique strong coupled fixed point in $A \cap B$.

Actually $1 \in A \cap B$ is the unique strong coupled fixed point of F. That is , $F(1,1) = 1$.

CHAPTEE FIVE

CONCLUSION AND FUTURE SCOPE

5.1. Conclusion

(Chadhury *et al*.,2017) established and proved a strong unique coupled fixed point for both Banach and Chatterjea type couplings with respect to two nonempty subsets A and B of a complete metric space X.

In this thesis, we established and proved the existence and uniqueness of a strong coupled fixed point for Chatterjea φ-contraction type coupling on complete metric spaces.

We provided an example in support of our main finding. Our result extend and generalize the work of (Chadhury *et al*., 2017) which is coupled fixed point result of Chatterjea type coupling.

5.2. Future Scope

Fixed point theory is one the active and vigorous area of research in mathematics and other sciences. There are several published results related to existence of strong coupled fixed point and unique fixed point theorem for both continuous and discontinuous maps satisfying some contractive condition in metric spaces. So it is recommend to the forthcoming postgraduate students and other researchers to exploit this opportunity and conduct their research work by setting different coupled fixed point theorems on certain contraction type coupling inequalities.

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