A Coupled Fixed Point Theorem for Maps Satisfying Rational Type Contractive Condition in Dislocated Quasi b - Metric Spaces

A Thesis Submitted to Jimma University, Department of Mathematics in Partial Fulfillment for the Requirements of the Degree of Masters of Science in Mathematics

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Declaration

I, the undersigned declare that, this thesis entitled ’’**A Coupled Fixed Point Theorem for Maps Satisfying Rational Type Contractive Condition in Dislocated Quasi b - Metric Spaces**” is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged.
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Abstract

The aim of this study is to establish a coupled fixed point theorem for maps satisfying rational type contractive condition in the perspective of dislocated quasi b-metric space and prove the existence and uniqueness of a couple fixed point. Our result improves and generalizes comparable results in the literature.

Contents

Chapter 1

Introduction

1.1 Background of the study

The concept of metric space was introduced by Fréchet (1906), which is one of the benchmark of not only mathematics but also for several quantitative sciences. The most important result of fixed point theory is contraction mapping which was proved by the Polish Mathematician Banach (1922) called the Banach contraction mapping principle which states that: Let (X,d) be a complete metric space and let $T : X \to X$ be a contraction on X , that is, there exists a constant $k \in [0,1)$ such that $d(Tx,Ty) \leq kd(x,y)$ for all $x,y \in X$. Then T has a unique fixed point in X. This principle has been generalized by various authors by situating different type of contractive conditions either on the mappings or on the spaces Sharma (2018).

In a complete dislocated metric space the celebrated Banach contraction principle was introduced and generalized by (P.Hitzler & Seda, 2000; Hitzler, 2001). Since then, Zeyada et al. (2005) introduced the notion of dislocated quasi metric space for the first time. The most interesting property of this space was that self-distance need not to be necessarily zero. Then after, the idea of dislocated quasi b-metric space was presented by Rahman & Sarwar (2016).

The concept of coupled fixed point extended with the work of Guo & Lakshmikantham (1987) where the monotone iterations technique is exploited. Then after, again this concept was introduced by them for Partially ordered set.
Bhaskar & Lakshmikantham (2006) studied the existence and uniqueness of a coupled fixed point results in partially ordered metric space.They also introduced the concept of coupled fixed point and proved some fixed point theorems under certain contractive condition. Moreover, after the work of Bhaskar & Lakshmikantham (2006) coupled fixed point results were studied by many authors in different type of spaces (Al Muhiameed

et al., 2018). Recently, Mohammad et al. (2018) proved a coupled fixed point result in the setting of dislocated quasi-metric spaces.

Provoked by the result of Mohammad et al. (2018), the aim of this thesis work was to establish a coupled fixed point theorem for maps satisfying rational type contractive condition in the perspective of dislocated quasi b-metric spaces which extends, improves and generalizes comparable results in the existing literature. Moreover, we provided examples to support our main result.

1.2 Statement of the Problem

This study emphasized on establishing a coupled fixed point theorem for maps satisfying rational type contractive condition in the perspective of complete dislocated quasi b-metric space and proving its existence and uniqueness.

1.3 Objectives of the study

1.3.1 General objective

The main objective of this study was to establish a coupled fixed point theorem for maps satisfying rational type contractive condition in the perspective of complete dislocated quasi b-metric spaces.

1.3.2 Specific objectives

The specific objectives of the study were:

- To prove the existence of a coupled fixed point in a complete dislocated quasi b-metric spaces.
- To show the uniqueness of a coupled fixed point in a complete dislocated quasi b-metric spaces.
- To provide examples in support of the main result of the study.

1.4 Significance of the study

The result obtained from this study may have the following importance:

- It may give research skills for the researcher.

- It may serve as a reference material for other researchers who have interest to conduct a research work in this area.
- It may have applications in studying the existence and uniqueness of solution of non-linear integral equations.

1.5 Delimitation of the Study

This study was delimited only on establishing and proving the existence and uniqueness of a coupled fixed point for maps satisfying rational type contractive condition in the perspective of complete dislocated quasi b-metric spaces.

Chapter 2

2.1 Review of Related literatures

One of the most dynamic research topic in non-linear analysis is fixed point theory and this theory was proved by Banach (1922) for a contraction mapping in a complete metric space. Fixed point theory has gained impetus, due to its wide range of applicability to determine diverse problems emanating from the theory of non-linear differential equations, theory of non-linear integral equation, game theory, mathematical economics and so forth. This contraction principle assures the existence and uniqueness of fixed points of certain self-maps on metric spaces, and gives a constructive method to find those fixed points.

The concept of dislocated metric space was introduced by P.Hitzler & Seda (2000) and also they had been generalized the famous Banach contraction principle in this space. There result was generalized by Zeyada et al. (2005) in dislocated quasi-metric space and they also initiated the notion of complete dislocated quasi-metric space. In such spaces self-distance between points need not to be necessarily zero. Due to its application in various non-linear analysis this concept has been widely extended, improved and generalized in many different ways by various authors which employs relatively more general contractive conditions ensuring the existence and uniqueness of a fixed point (Jhade & Khan, 2015). Guo & Lakshmikantham (1987) extended the concept of coupled fixed point by the monotone iterations technique which is exploited for two variable contractive type mappings. Recently, Mohammad et al. (2018) established a coupled fixed point theorem in the setting of dislocated quasi metric space.

Provoked by the result of Mohammad et al. (2018), the aim of this thesis work was to establish a coupled fixed point theorem for maps satisfying rational type contractive condition in the perspective of dislocated quasi b-metric spaces which extends, improves and generalizes comparable results in the existing literature. Moreover, we provided examples to support our main result.

Chapter 3

Methodology

This chapter Consists study design, study site and period, source of information, and mathematical procedure.

3.1 Study period and site

The study was conducted at Jimma University under the department of mathematics from September 2018 to June 2019 G.C.

3.2 Study Design

In order to achieve the stated objectives we employed analytical design.

3.3 Source of Information

The relevant sources of information for this study were different books, published articles and journals.

3.4 Mathematical Procedures of the Study

In this study the procedures that we followed were the standard procedures used in the published work of Al Muhiameed et al. (2018) and Mohammad et al. (2018).
To achieve the stated objectives, the study followed the following procedures:

1. Establishing a theorem.

2. Constructing Sequences.

3. Showing the constructed sequences are Cauchy sequences.
4. Proving the existence of coupled fixed point.
5. Showing the uniqueness of coupled fixed point.
6. Providing supportive examples to validate our main result.
7. Driving corollaries from the main result.

Chapter 4

Preliminaries and Main Result

4.1 Preliminaries

Note: Throughout this thesis R^+ represents the set of non-negative real numbers and N represents the set of natural numbers.
First we remember some known definition and lemmas.

Definition 4.1 (Rahman & Sarwar, 2016) Let X be a non-empty set and $k \geq 1$ be any given real number. Let $d : X \times X \to [0, \infty)$ be a function satisfying the conditions

1. $d(x,y) = d(y,x) = 0 \Rightarrow x = y$.
2. $d(x,y) \leq k\left[d(x,z) + d(z,y)\right]$ for all $x,y,z \in X$.

Then d is known as dislocated quasi b-metric on X and the pair (X,d) is called a dislocated quasi b -metric space or in short (dq b) metric spaces.

Definition 4.2 (Rahman & Sarwar, 2016) A sequence $\{x_n\}$ in a dislocated quasi b-metric space (X,d) is said to converge to a point $x \in X$ if and only if

$$\lim_{n\to\infty} d(x_n,x) = 0 = \lim_{n\to\infty} d(x,x_n)$$

.

Definition 4.3 (Rahman & Sarwar, 2016) Let (X,d) be a dq b-metric space. Then a sequence $\{x_n\}$ is said be a Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbf{N}$ such that $d(x_n,x_m) < \varepsilon$ for all $n,m \geq n_0(\varepsilon)$ that is, $\lim\limits_{n,m\to\infty} d(x_n,x_m) = 0$.

Definition 4.4 (Rahman & Sarwar, 2016) A dislocated quasi b-metric space (X,d) is called complete if every Cauchy sequence $\{x_n\}$ in (X,d) converges to a point $x \in X$.

Definition 4.5 (Rahman & Sarwar, 2016) Let (X,d_1) and (Y,d_2) be two dislocated quasi b-metric spaces, then the mapping $T: X \to Y$ is said to be continuous if for each sequence $\{x_n\}$ which is convergent to x_0 in X, the sequence $\{Tx_n\}$ converges to Tx_0 in Y.

Lemma 4.1 (Rahman & Sarwar, 2016) Limit of a convergent sequence in dislocated quasi b-metric space is unique.

Definition 4.6 (Banach, 1922) Let (X,d) be a complete metric space and $T: X \to X$ be a self-map, then T is said to be a contraction mapping if there exist a constant $k \in [0,1)$, such that

$$d(Tx,Ty) \leq kd(x,y)$$

for all $x,y \in X$.

Definition 4.7 Let X be a nonempty set and $T: X \to X$ a self-map. We say that x is a fixed point of T if $Tx = x$.

Theorem 4.2 (Rahman, 2017) Let (X,d) be a complete dislocated quasi-b-metric space. Let $T: X \to X$ be a continuous contraction with $\lambda \in [0,1)$ and $0 \leq \lambda < \frac{1}{k}$ for $k \geq 1$. Then T has a unique fixed point in X.

Definition 4.8 (Bhaskar & Lakshmikantham, 2006) An element $(x,y) \in X \times X$, where X is any non-empty set, is called a coupled fixed point of the mapping $T: X \to X$ if $T(x,y) = x$ and $T(y,x) = y$.

Theorem 4.3 (Mohammad et al., 2018) Let (X,d) be a complete dislocated quasi-metric space and $T: X \to X$ be a continuous mapping satisfying the following rational type contractive conditions

$$\begin{aligned}
d[T(x,y),T(u,v)] \leq\ & a_1[d(x,u)+d(y,v)] \\
& + a_2[d(x,T(x,y))+d(u,T(u,v))] \\
& + a_3[d(x,T(u,v))+d(u,T(x,y))] \\
& + a_4\left[\frac{d(x,T(x,y))d(u,T(u,v))}{d(x,u)+d(y,v)}\right] \\
& + a_5\left[\frac{[d(x,u)+d(y,v)][d(x,T(x,y))+d(u,T(u,v))]}{1+d(x,u)+d(y,v)}\right] \\
& + a_6\left[\frac{d(x,T(x,y))+d(x,T(u,v))}{1+d(u,T(u,v))d(u,T(x,y))}\right]
\end{aligned}$$

for all $x,y,u,v \in X$ and a_1,a_2,a_3,a_4,a_5, and a_6 are non-negative constants with $2(a_1+a_2+a_5)+4(a_3+a_6)+a_4 < 1$.
Then T has a unique coupled fixed point in $X \times X$.

4.2 Main Results

Theorem 4.4 Let (X,d) be a complete dislocated quasi b-metric space with constant coefficient $k \geq 1$ and $T : X \times X \to X$ be a continuous map satisfying the following rational type contractive conditions

$$\begin{aligned} d[T(x,y),T(u,v)] \leq\ & a_1[d(x,u)+d(y,v)] \\ & + a_2[d(x,T(x,y))+d(u,T(u,v))] \\ & + a_3[d(x,T(u,v))+d(u,T(x,y))] \\ & + a_4\left[\frac{d(x,T(x,y))d(u,T(u,v))}{d(x,u)+d(y,v)}\right] \\ & + a_5\left[\frac{[d(x,u)+d(y,v)][d(x,T(x,y))+d(u,T(u,v))]}{1+d(x,u)+d(y,v)}\right] \\ & + a_6\left[\frac{d(u,T(x,y))+d(x,T(u,v))}{1+d(u,T(u,v))d(u,T(x,y))}\right] \\ & + a_7\left[\frac{[d(y,v)+d(x,T(x,y))]d(u,T(u,v))}{1+d(y,v)+d(x,T(x,y))}\right] \end{aligned} \tag{4.1}$$

for all $x,y,u,v \in X$ and a_1,a_2,a_3,a_4,a_5,a_6, and a_7 are non-negative constants with $2ka_1+(k+1)(a_2+a_5)+(2k^2+2k)(a_3+a_6)+a_4+a_7 < 1$. Then T has a unique coupled fixed point in $X \times X$.

Proof: Let x_0, $y_0 \in X$ be any two arbitrary points.
We can construct two sequences $\{x_n\}$ and $\{y_n\} \in X$ such that
$x_{n+1} = T(x_n,y_n)$ and $y_{n+1} = T(y_n,x_n)$ for $n = 0,1,2,...$
consider $d(x_n,x_{n+1}) = d[T(x_{n-1},y_{n-1}),T(x_n,y_n)]$.
Now from (4.1), we have

$$\begin{aligned} d(x_n,x_{n+1}) =\ & d[T(x_{n-1},y_{n-1}),T(x_n,y_n)] \\ \leq\ & a_1[d(x_{n-1},x_n)+d(y_{n-1},y_n)] \\ & + a_2[d(x_{n-1},T(x_{n-1},y_{n-1}))+d(x_n,T(x_n,y_n))] \\ & + a_3[d(x_{n-1},T(x_n,y_n))+d(x_n,T(x_{n-1},y_{n-1}))] \\ & + a_4\left[\frac{d(x_{n-1},T(x_{n-1},y_{n-1}))d(x_n,T(x_n,y_n))}{d(x_{n-1},x_n)+d(y_{n-1},y_n)}\right] \\ & + a_5\left[\frac{[d(x_{n-1},x_n)+d(y_{n-1},y_n)][d(x_{n-1},T(x_{n-1},y_{n-1}))+d(x_n,T(x_n,y_n))]}{1+d(x_{n-1},x_n)+d(y_{n-1},y_n)}\right] \\ & + a_6\left[\frac{d(x_n,T(x_{n-1},y_{n-1}))+d(x_{n-1},T(x_n,y_n))}{1+d(x_n,T(x_n,y_n))d(x_n,T(x_{n-1},y_{n-1}))}\right] \\ & + a_7\left[\frac{[d(y_{n-1},y_n)+d(x_{n-1},T(x_{n-1},y_{n-1}))]d(x_n,T(x_n,y_n))}{1+d(y_{n-1},y_n)+d(x_{n-1},T(x_{n-1},y_{n-1})}\right]. \end{aligned}$$

By using the definitions of the sequences $\{x_n\}$ and $\{y_n\}$, we have

$$\begin{aligned}
d(x_n,x_{n+1}) \leq\ & a_1[d(x_{n-1},x_n)+d(y_{n-1},y_n)] \\
& +a_2[d(x_{n-1},x_n)+d(x_n,x_{n+1})] \\
& +a_3[d(x_{n-1},x_{n+1})+d(x_n,x_n)] \\
& +a_4\left[\frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{d(x_{n-1},x_n)+d(y_{n-1},y_n)}\right] \\
& +a_5\left[\frac{[d(x_{n-1},x_n)+d(y_{n-1},y_n)][d(x_{n-1},x_n)+d(x_n,x_{n+1})]}{1+d(x_{n-1},x_n)+(y_{n-1},y_n)]}\right] \\
& +a_6\left[\frac{d(x_n,x_n)+d(x_{n-1},x_{n+1})}{1+d(x_n,x_{n+1})d(x_n,x_n)}\right] \\
& +a_7\left[\frac{[d(y_{n-1},y_n)+d(x_{n-1},x_n)]d(x_n,x_{n+1})}{1+d(y_{n-1},y_n)+d(x_{n-1},x_n)}\right].
\end{aligned}$$

Using the triangle inequality and the fact that $d(x,y)\geq 0$, we have

$$\begin{aligned}
d(x_n,x_{n+1}) =\ & d[T(x_{n-1},y_{n-1}),T(x_n,y_n)] \\
\leq\ & a_1[d(x_{n-1},x_n)+d(y_{n-1},y_n)] \\
& +a_2[d(x_{n-1},x_n)+d(x_n,x_{n+1})] \\
& +ka_3[d(x_{n-1},x_n)+d(x_n,x_{n+1})+d(x_{n-1},x_n)+d(x_n,x_{n+1})] \\
& +a_4\left[\frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{d(x_{n-1},x_n)+d(y_{n-1},y_n)}\right] \\
& +a_5\left[\frac{[d(x_{n-1},x_n)+d(y_{n-1},y_n)][d(x_{n-1},x_n)+d(x_n,x_{n+1})]}{1+d(x_{n-1},x_n)+d(y_{n-1},y_n)}\right] \\
& +ka_6\left[\frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})+d(x_{n-1},x_n)+d(x_n,x_{n+1})}{1+d(x_n,x_{n+1})d(x_n,x_{n+1})}\right] \\
& +a_7\left[\frac{[d(y_{n-1},y_n)+d(x_{n-1},x_n)]d(x_n,x_{n+1})}{1+d(y_{n-1},y_n)+d(x_{n-1},x_n)}\right].
\end{aligned}$$

Simplfying the above inequality, we have

$$\begin{aligned}
d(x_n,x_{n+1}) \leq\ & a_1[d(x_{n-1},x_n)+d(y_{n-1},y_n)]+a_2[d(x_{n-1},x_n)+d(x_n,x_{n+1})] \\
& +ka_3[d(x_{n-1},x_n)+d(x_n,x_{n+1})+d(x_{n-1},x_n)+d(x_n,x_{n+1})] \\
& +a_4d(x_n,x_{n+1})+a_5[d(x_{n-1},x_n)+d(x_n,x_{n+1})]+ka_6[d(x_{n-1},x_n) \\
& +d(x_n,x_{n+1})+d(x_{n-1},x_n)+d(x_n,x_{n+1})]+a_7d(x_n,x_{n+1}).
\end{aligned}$$

which implies that

$$\begin{aligned}d(x_n,x_{n+1}) &\le (a_2+2k(a_3+a_6)+a_4+a_5+a_7)d(x_n,x_{n+1})\\ &\quad +(a_1+a_2+2k(a_3+a_6)+a_5)d(x_{n-1},x_n)\\ &\quad +a_1 d(y_{n-1},y_n).\end{aligned}$$

Simplification yields

$$\begin{aligned}d(x_n,x_{n+1}) &\le \frac{a_1+a_2+2k(a_3+a_6)+a_5}{1-(a_2+2k(a_3+a_6)+a_4+a_5+a_7)}d(x_{n-1},x_n)\\ &\quad +\frac{a_1}{1-(a_2+2k(a_3+a_6)+a_4+a_5+a_7)}d(y_{n-1},y_n). \qquad (4.2)\end{aligned}$$

Proceeding similarly, we can show that

$$\begin{aligned}d(y_n,y_{n+1}) &\le \frac{a_1+a_2+2k(a_3+a_6)+a_5}{1-(a_2+2k(a_3+a_6)+a_4+a_5+a_7)}d(y_{n-1},y_n)\\ &\quad +\frac{a_1}{1-(a_2+2k(a_3+a_6)+a_4+a_5+a_7)}d(x_{n-1},x_n). \qquad (4.3)\end{aligned}$$

Adding (4.2) and (4.3), we obtain

$$[d(x_n,x_{n+1})+d(y_n,y_{n+1})] \le \frac{2a_1+a_2+2k(a_3+a_6)+a_5}{1-(a_2+2k(a_3+a_6)+a_4+a_5+a_7)}[d(x_{n-1},x_n)+d(y_{n-1},y_n)].$$

Since $2ka_1+(k+1)(a_2+a_5)+(2k^2+2k)(a_3+a_6)+a_4+a_7<1$.
with

$$\lambda=\frac{2a_1+a_2+2k(a_3+a_6)+a_5}{1-(a_2+2k(a_3+a_6)+a_4+a_5+a_7)}<1.$$

So, we have $k\lambda<1$
Thus, the above inequality becomes,

$$[d(x_n,x_{n+1})+d(y_n,y_{n+1})] \le \lambda[d(x_{n-1},x_n)+d(y_{n-1},y_n)].$$

Furthermore,

$$\begin{aligned}d(x_{n+1},x_{n+2}) &\le \frac{a_1+a_2+2k(a_3+a_6)+a_5}{1-(a_2+2k(a_3+a_6)+a_4+a_5+a_7)}d(x_n,x_{n+1})\\ &\quad +\frac{a_1}{1-(a_2+2k(a_3+a_6)+a_4+a_5+a_7)}d(y_n,y_{n+1}). \qquad (4.4)\end{aligned}$$

Similarly,

$$\begin{aligned} d(y_{n+1}, y_{n+2}) &\leq \frac{a_1 + a_2 + 2k(a_3 + a_6) + a_5}{1 - (a_2 + 2k(a_3 + a_6) + a_4 + a_5 + a_7)} d(y_n, y_{n+1}) \\ &+ \frac{a_1}{1 - (a_2 + 2k(a_3 + a_6) + a_4 + a_5 + a_7)} d(x_n, x_{n+1}). \end{aligned} \quad (4.5)$$

Adding (4.4) and (4.5), we obtain

$$\begin{aligned} [d(x_{n+1}, x_{n+2}) + d(y_{n+1}, y_{n+2})] &\leq \lambda [d(x_n, x_{n+1}) + d(y_n, y_{n+1})] \\ &= \lambda^2 [d(x_{n-1}, x_n) + d(y_{n-1}, y_n)]. \end{aligned}$$

Continuing this process inductively, we have

$$[d(x_n, x_{n+1}) + d(y_n, y_{n+1})] \leq \lambda^n [d(x_0, x_1) + d(y_0, y_1)].$$

Now, we show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X.
For non negative integers m and n with $m > n$, we have

$$\begin{aligned} [d(x_n, x_m) + d(y_n, y_m)] &\leq k[d(x_n, x_{n+1}) + d(x_{n+1}, x_m) + d(y_n, y_{n+1}) + d(y_{n+1}, y_m)] \\ &\leq k[d(x_n, x_{n+1}) + d(y_n, y_{n+1})] + k^2 [d(x_{n+1}, x_{n+2}) + (d(y_{n+1}, y_{n+2})] \\ &\quad + \ldots + k^{m-n} [d(x_{m-1}, x_m) + d(y_{m-1}, y_m)] \\ &\leq k\lambda^n [d(x_0, x_1) + d(y_0, y_1)] + k^2 \lambda^{n+1} [d(x_0, x_1) + d(y_0, y_1)] \\ &\quad + \ldots + k^{m-n} \lambda^{m-1} [d(x_0, x_1) + d(y_0, y_1)] \\ &= k\lambda^n (1 + k\lambda + (k\lambda)^2 + \ldots + (k\lambda)^{m-n-1}) [d(x_0, x_1) + d(y_0, y_1)] \\ &\leq \frac{k\lambda^n}{1 - k\lambda} [d(x_0, x_1) + d(y_0, y_1)] \to 0 \text{ as } n \to \infty \end{aligned} \quad (4.6)$$

It follows that

$$[d(x_n, x_m) + d(y_n, y_m)] \to 0 \quad as \quad n, m \to \infty.$$

Which in turn implies that $d(x_n, x_m) \to 0$ and $d(y_n, y_m) \to 0$ as $n, m \to \infty$.
Hence $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in a complete dislocated quasi b-metric space X.
As a result there must exist $(x, y) \in X \times X$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.
In addition, Since T is continuous we have

$$x = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} T(x_n, y_n) = T\left(\lim_{n \to \infty} x_n, \lim_{n \to \infty} y_n \right) = T(x, y).$$

Similarly,

$$y = \lim_{n \to \infty} y_{n+1} = \lim_{n \to \infty} T(y_n, x_n) = T\left(\lim_{n \to \infty} y_n, \lim_{n \to \infty} x_n \right) = T(y, x).$$

Therefore, $(x, y) \in X \times X$ is a coupled fixed point of T.

Uniqueness

Now, we show that $(x,y) \in X \times X$ is a unique coupled fixed point of T.
Suppose that T has another coupled fixed point say $(x^*,y^*) \neq (x,y)$
where $(x^*,y^*) \in X \times X$ with $x^* = T(x^*,y^*)$ and $y^* = T(y^*,x^*)$.
Then, by using (4.1), we have

$$
\begin{aligned}
d(x,x) &= d[T(x,y),T(x,y)] \\
&\leq a_1[d(x,x)+d(y,y)] \\
&\quad + a_2[d(x,T(x,y))+d(x,T(x,y))] \\
&\quad + a_3[d(x,T(x,y))+d(x,T(x,y))] \\
&\quad + a_4\left[\frac{d(x,T(x,y))d(x,T(x,y))}{d(x,x)+d(y,y)}\right] \\
&\quad + a_5\left[\frac{[d(x,x)+d(y,y)][d(x,T(x,y))+d(x,T(x,y))]}{1+d(x,x_)+d(y,y)}\right] \\
&\quad + a_6\left[\frac{d(x,T(x,y))+d(x,T(x,y))}{1+d(x,T(x,y))d(x,T(x,y))}\right] \\
&\quad + a_7\left[\frac{[d(y,y)+d(x,T(x,y))]d(x,T(x,y))}{1+d(y,y)+d(x,T(x,y))}\right].
\end{aligned}
$$

$$
\begin{aligned}
&= a_1[d(x,x)+d(y,y)] \\
&\quad + a_2[d(x,x)+d(x,x)] \\
&\quad + a_3[d(x,x)+d(x,x)] \\
&\quad + a_4\left[\frac{d(x,x)d(x,x)}{d(x,x)+d(y,y)}\right] \\
&\quad + a_5\left[\frac{[d(x,x)+d(y,y)][d(x,x)+d(x,x)]}{1+d(x,x_)+d(y,y)}\right] \\
&\quad + a_6\left[\frac{d(x,x)+d(x,x)}{1+d(x,x)d(x,x)}\right] \\
&\quad + a_7\left[\frac{[d(y,y)+d(x,x)]d(x,x)}{1+d(y,y)+d(x,x)}\right].
\end{aligned}
$$

$$
\begin{aligned}
&\leq a_1[d(x,x)+d(y,y)] + a_2[d(x,x)+d(x,x)] \\
&\quad + a_3[d(x,x)+d(x,x)] + a_4 d(x,x) \\
&\quad + a_5[d(x,x)+d(x,x)] + a_6[d(x,x)+d(x,x)] \\
&\quad + a_7 d(x,x).
\end{aligned}
$$

Then simplification yields

$$d(x,x) \leq \delta d(x,x) + a_1 d(y,y). \tag{4.7}$$

where $\delta = a_1 + 2(a_2 + a_3 + a_5 + a_6) + a_4 + a_7$.
Similarly, we can show that

$$d(y,y) \leq \delta d(y,y) + a_1 d(x,x). \tag{4.8}$$

Adding (4.7) and (4.8), we obtain

$$[d(x,x) + d(y,y)] \leq \mu[d(x,x) + d(y,y)].$$

where $\mu = \delta + a_1$.
Since, $\mu < 1$ hence the above inequality is possible if and only if $d(x,x) + d(y,y) = 0$ which implies that $d(x,x) = 0$ and $d(y,y) = 0$.
Similarly, $d(x^*,x^*) = 0$ and $d(y^*,y^*) = 0$.
Now, we consider

$$\begin{aligned}
d(x,x^*) &= d[T(x,y),T(x^*,y^*)] \\
&\leq a_1[d(x,x^*) + d(y,y^*)] \\
&\quad + a_2[d(x,T(x,y)) + d(x^*,T(x^*,y^*))] \\
&\quad + a_3[d(x,T(x^*,y^*)) + d(x^*,T(x,y))] \\
&\quad + a_4\left[\frac{d(x,T(x,y))d(x^*,T(x^*,y^*))}{d(x,x^*) + d(y,y^*)}\right] \\
&\quad + a_5\left[\frac{[d(x,x^*) + d(y,y^*)][d(x,T(x,y)) + d(x^*,T(x^*,y^*))]}{1 + d(x,x^*) + d(y,y^*)}\right] \\
&\quad + a_6\left[\frac{d(x^*,T(x,y)) + d(x,T(x^*,y^*))}{1 + d(x^*,T(x^*,y^*))d(x^*,T(x,y))}\right] \\
&\quad + a_7\left[\frac{d(y,y^*) + d(x,T(x,y))d(x^*,T(x^*,y^*))}{1 + d(y,y^*) + d(x^*,T(x^*,y^*))}\right].
\end{aligned}$$

In fact $T(x,y)=x$ and $T(x^*,y^*)=x^*$, then we have

$$\begin{aligned} d(x,x^*) &= a_1[d(x,x^*)+d(y,y^*)] \\ &\quad +a_2[d(x,x)+d(x^*,x^*)] \\ &\quad +a_3[d(x,x^*)+d(x^*,x)] \\ &\quad +a_4\left[\frac{d(x,x)d(x^*,x^*)}{d(x,x^*)+d(y,y^*)}\right] \\ &\quad +a_5\left[\frac{[d(x,x^*)+d(y,y^*)][d(x,x)+d(x^*,x^*)]}{1+d(x,x^*)+d(y,y^*)}\right] \\ &\quad +a_6\left[\frac{d(x^*,x)+d(x,x^*)}{1+d(x^*,x^*)d(x^*,x)}\right] \\ &\quad +a_7\left[\frac{[d(y,y^*)+d(x,x^*)]d(x^*,x^*)}{1+d(y,y^*)+d(x,x^*)}\right]. \end{aligned}$$

Since $d(x,x)=d(x^*,x^*)=0$, we have

$$\begin{aligned} d(x,x^*) &\leq (a_1+a_3+a_6)d(x,x^*)+(a_3+a_6)d(x^*,x)+a_1d(y,y^*) \\ \Big(1-(a_1+a_3+a_6)\Big)d(x,x^*) &\leq (a_3+a_6)d(x^*,x)+a_1d(y,y^*). \end{aligned} \tag{4.9}$$

By following similar procedure we can get

$$\Big(1-(a_1+a_3+a_6)\Big)d(y,y^*)\leq(a_3+a_6)d(y^*,y)+a_1d(x,x^*). \tag{4.10}$$

Adding (4.9) and (4.10) and then simplifying we obtain

$$[d(x,x^*)+d(y,y^*)] \leq \sigma[d(x^*,x)+d(y^*,y)]. \tag{4.11}$$

where

$$\sigma=\left[\frac{a_3+a_6}{1-(2a_1+a_3+a_6)}\right].$$

Similarly, we can get

$$[d(x^*,x)+d(y^*,y)]\leq\sigma[d(x,x^*)+d(y,y^*)]. \tag{4.12}$$

Adding (4.11) and (4.12), we get

$$\begin{aligned} [d(x,x^*)+d(y,y^*)+d(x^*,x)+d(y^*,y)] &\leq \sigma[d(x,x^*)+d(y,y^*) \\ &\quad +d(x^*,x)+d(y^*,y)]. \end{aligned}$$

since $\sigma<1$ so, the above inequality is possible if and only if $[d(x,x^*)+d(y,y^*)+d(x^*,x)+d(y^*,y)]=0$.

Which implies that
$d(x,x^*) = 0, d(y,y^*) = 0, d(x^*,x) = 0$, and $d(y^*,y) = 0$.
It follows that $x = x^*$ and $y = y^*$ such that $(x,y) = (x^*,y^*)$ which contradicts our assumption.
Therefore, (x,y) is a unique coupled fixed point of T in $X \times X$.
□ We deduce the following corollaries from Theorem 4.4

Corollary 4.5 Let (X,d) be a complete dislocated quasi b - metric space with constant coefficient $k \geq 1$ and $T : X \times X \to X$ be a continuous map satisfying the following rational type contractive condition

$$\begin{aligned} d[T(x,y),T(u,v)] \leq\ & a_1[d(x,u)+d(y,v)] \\ & + a_2[d(x,T(x,y))+d(u,T(u,v))] \\ & + a_3[d(x,T(u,v))+d(u,T(x,y))] \\ & + a_4\left[\frac{d(x,T(x,))d(u,T(u,v))}{d(x,u)+d(y,v)}\right] \\ & + a_5\left[\frac{[d(x,u)+d(y,v)][d(x,T(x,y))+d(u,T(u,v))]}{1+d[(x,u),(y,v)]}\right] \\ & + a_6\left[\frac{d(u,T(x,y))+d(x,T(u,v))}{1+d(u,T(u,v))d(u,T(u,v))}\right] \end{aligned}$$

for all $x,y,u,v \in X$ and a_1,a_2,a_3,a_4,a_5, and a_6 are non-negative constants with
$2ka_1 + (k+1)(a_2+a_5) + (2k^2+2k)(a_3+a_6) + a_4 < 1$.
Then T has a unique coupled fixed point in $X \times X$.

Proof: It follows from Theorem 4.4 by taking $a_7 = 0$. □

Remark 4.1 If we take $k = 1$ in Corollary 4.5, we get the result of Mohammad et al. (2018).
Thus our established theorem generalizes Theorem 4.3.

Corollary 4.6 Let (X,d) be a complete dislocated quasi b - metric space with constant coefficient $k \geq 1$ and $T : X \times X \to X$ be a continuous map satisfying the following rational type contractive conditions

$$\begin{aligned} d[T(x,y),T(u,v)] \leq\ & a_1[d(x,u)+d(y,v)] \\ & + a_2[d(x,T(x,y))+d(u,T(u,v))] \\ & + a_3[d(x,T(u,v))+d(u,T(x,y))] \\ & + a_4\left[\frac{d(x,T(x,y))d(u,T(u,v))}{d(x,u)+d(y,v)}\right] \\ & + a_5\left[\frac{[d(x,u)+d(y,v)][d(x,T(x,y))+d(u,T(u,v))]}{1+d[(x,u),(y,v)]}\right] \end{aligned}$$

for all $x,y,u,v \in X$ and a_1,a_2,a_3,a_4, and a_5 are non-negative constants with $2ka_1+(k+1)(a_2+a_5)+(2k^2+2k)a_3+a_4 < 1$. Then T has a unique coupled fixed point in $X \times X$.

Proof: It follows from Corollary 4.5 by taking $a_6 = 0$. □

Corollary 4.7 Let (X,d) be a complete dislocated quasi b- metric space with constant coefficient $k \geq 1$ and $T : X \times X \to X$ be a continuous map satisfying the following rational type contractive conditions

$$\begin{aligned} d[T(x,y),T(u,v)] \leq\ & a_1[d(x,u)+d(y,v)] \\ & +a_2[d(x,T(x,y))+d(u,T(u,v))] \\ & +a_3[d(x,T(u,v))+d(u,T(x,y))] \\ & +a_4\left[\frac{d(x,T(x,y))d(u,T(u,v))}{d(x,u)+d(y,v)}\right] \end{aligned}$$

for all $x,y,u,v \in X$ and a_1,a_2,a_3, and a_4 are non-negative constants with $2ka_1+(k+1)a_2+(2k^2+2k)a_3+a_4 < 1$.
Then T has a unique coupled fixed point in $X \times X$.

Proof: It follows from Corollary 4.6 by taking $a_5 = 0$. □

Corollary 4.8 Let (X,d) be a complete dislocated quasi b- metric space with constant coefficient $k \geq 1$ and $T : X \times X \to X$ be a continuous map satisfying the following rational type contractive conditions

$$\begin{aligned} d[T(x,y),T(u,v)] \leq\ & a_1[d(x,u)+d(y,v)] \\ & +a_2[d(x,T(x,y))+d(u,T(u,v))] \\ & +a_3[d(x,T(u,v))+d(u,T(x,y))] \end{aligned}$$

for all $x,y,u,v \in X$ and a_1,a_2, and a_3 are non-negative constants with $2ka_1+(k+1)a_2+(2k^2+2k)a_3 < 1$.
Then T has a unique coupled fixed point in $X \times X$.

Proof: It follows from Corollary 4.7 by taking $a_4 = 0$. □

Corollary 4.9 Let (X,d) be a complete dislocated quasi- b metric space with constant coefficient $k \geq 1$ and $T : X \times X \to X$ be a continuous mapping satisfying the following rational type contractive conditions

$$\begin{aligned} d[T(x,y),T(u,v)] \leq\ & a_1[d(x,u)+d(y,v)] \\ & +a_2[d(x,T(x,y))+d(u,T(u,v))] \end{aligned}$$

for all $x, y, u, v \in X$ and a_1 and a_2 are non-negative constants with
$2ka_1 + (k+1)a_2 < 1$.
Then T has a unique coupled fixed point in $X \times X$.

Proof: It follows from Corollary 4.8 by taking $a_3 = 0$. □

Corollary 4.10 Let (X, d) be a complete dislocated quasi-metric space with constant coefficient $k \geq 1$ and $T : X \times X \to X$ be a continuous mapping satisfying the following rational type contractive conditions

$$d[T(x,y), T(u,v)] \leq a_1[d(x,u) + d(y,v)]$$

for all $x, y, u, v \in X$ and a_1 is non-negative constants with $2ka_1 < 1$.
Then T has a unique coupled fixed point in $X \times X$.

Proof: It follows from Corollary 4.9 by taking $a_2 = 0$. □

Example 4.1 Let $X = [0, 1]$. Define $d : X \times X \to \Re^+$ by

$$d(x,y) = |2x + y|^2 + |2x - y|^2$$

for all $x, y \in X$. Then (X, d) is dqb-metric space with constant coefficient $k = 2$.
If we define a continuous map $T : X \times X \to X$ by

$$T(x,y) = \frac{xy}{10}$$

.
Since $|2xy + uv|^2 \leq |2x + u|^2 + |2y + v|^2, |2xy - uv|^2 < |2x - u|^2 + |2y - v|^2$ holds for all $x, y, u, v \in X$.
Then, we have

$$\begin{aligned}
d[T(x,y), T(u,v)] &= d\left(\frac{2xy}{10}, \frac{uv}{10}\right) \\
&= \left|\frac{2xy}{10} + \frac{uv}{10}\right|^2 + \left|\frac{2xy}{10} - \frac{uv}{10}\right|^2 \\
&= \frac{1}{100}\left(|2xy + uv|^2 + |2xy - uv|^2\right) \\
&\leq \frac{1}{100}\left(|2x + u|^2 + |2y + v|^2 + |2x - u|^2 + |2y - v|^2\right) \\
&= \frac{1}{10}\left[d(x,u) + d(y,v)\right].
\end{aligned}$$

It shows that

$$\begin{aligned}
d[T(x,y),T(u,v)] \leq\ & a_1[d(x,u)+d(y,v)] \\
&+a_2[d(x,T(x,y))+d(u,T(u,v))] \\
&+a_3[d(x,T(u,v))+d(u,T(x,y))] \\
&+a_4\left[\frac{d(x,T(x,y))d(u,T(u,v))}{d(x,u)+d(y,v)}\right] \\
&+a_5\left[\frac{[d(x,u)+d(y,v)][d(x,T(x,y))+d(u,T(u,v))]}{1+d(x,u)+d(y,v)}\right] \\
&+a_6\left[\frac{d(u,T(x,y))+d(x,T(u,v))}{1+d(u,T(u,v))d(u,T(x,y))}\right] \\
&+a_7\left[\frac{[d(y,v)+d(x,T(x,y))]d(u,T(u,v))}{1+d(y,v)+d(x,T(x,y))}\right]
\end{aligned}$$

where $x,y,u,v \in X$ and $a_1 = \frac{1}{10}, a_2 = \frac{1}{25}, a_3 = \frac{1}{64}, a_4 = \frac{1}{40}, a_5 = \frac{1}{30}, a_6 = \frac{1}{128}, a_7 = \frac{1}{75}$ since $2ka_1 + (k+1)(a_2+a_5) + (2k^2+2k)(a_3+a_6) + a_4 + a_7 = \frac{451}{480} < 1$.
Hence all the conditions of Theorem 4.4 are satisfied having $(0,0) \in X \times X$ as a unique coupled fixed point of T in $X \times X$.

Example 4.2 Suppose $X = [-1,1]$. Define the mapping $d : X \times X \to \Re^+$ by

$$d(x,y) = |x-y|^2 + 3|x| + 4|y|$$

for all $x,y \in X$. Then (X,d) is dqb-metric space with constant coefficient $k = 2$.

If we define a continuous map $T : X \times X \to X$ by

$$T(x,y) = \frac{xy}{12}$$

for each $x,y \in X$.
Since $|xy-uv| \leq |x-u| + |y-v|$, $|xy| \leq |x| + |y|$ and $2xy \leq x^2+y^2$ holds for all $x,y,u,v \in X$.

Then, we have

$$\begin{aligned}
d[T(x,y),T(u,v)] &= d\left(\frac{xy}{12},\frac{uv}{12}\right) \\
&\leq \left|\frac{xy}{12}-\frac{uv}{12}\right|^2+\frac{3}{12}|xy|+\frac{4}{12}|uv| \\
&\leq \frac{1}{144}|xy-uv|^2+\frac{3}{12}|xy|+\frac{4}{12}|uv| \\
&\leq \frac{1}{144}\left(2|x-u|^2+2|y-v|^2\right)+\frac{3}{12}\left(|x|+|y|\right)+\frac{4}{12}\left(|u|+|v|\right) \\
&\leq \frac{1}{12}\left(|x-u|^2+3|x|+4|u|\right)+\frac{1}{12}\left(|y-v|^2+3|y|+4|v|\right) \\
&= \frac{1}{12}\left[d(x,u)+d(y,v)\right].
\end{aligned}$$

It shows that

$$\begin{aligned}
d[T(x,y),T(u,v)] \leq\ & a_1[d(x,u)+d(y,v)] \\
&+a_2[d(x,T(x,y))+d(u,T(u,v))] \\
&+a_3[d(x,T(u,v))+d(u,T(x,y))] \\
&+a_4\left[\frac{d(x,T(x,y))d(u,T(u,v))}{d(x,u)+d(y,v)}\right] \\
&+a_5\left[\frac{[d(x,u)+d(y,v)][d(x,T(x,y))+d(u,T(u,v))]}{1+d(x,u)+d(y,v)}\right] \\
&+a_6\left[\frac{d(u,T(x,y))+d(x,T(u,v))}{1+d(u,T(u,v))d(u,T(x,y))}\right] \\
&+a_7\left[\frac{[d(y,v)+d(x,T(x,y))]d(u,T(u,v))}{1+d(y,v)+d(x,T(x,y))}\right]
\end{aligned}$$

where $x,y,u,v \in X$ and $a_1=\frac{1}{12}, a_2=\frac{1}{25}, a_3=\frac{1}{72}, a_4=\frac{1}{15}, a_5=\frac{1}{50}, a_6=\frac{1}{144}, a_7=\frac{1}{20}$ since $2ka_1+(k+1)(a_2+a_5)+(2k^2+2k)(a_3+a_6)+a_4+a_7=\frac{22}{25}<1$.
Hence all the conditions of Theorem 4.4 are satisfied having $(0,0)\in X\times X$ as a unique coupled fixed point of T in $X\times X$.

Chapter 5

Conclusion and Future Scope

Mohammad et al. (2018) proved the existence and uniqueness of a coupled fixed point result for maps satisfying certain rational type contractive condition in the setting of complete dislocated quasi metric space. In this thesis work, we established and proved the existence and uniqueness of a coupled fixed point result for maps satisfying rational type contractive condition in the perspective of complete dislocated quasi b- metric spaces.

Our established result generalizes and extends the result of Mohammad et al. (2018) and related results in the existing literature. Also, we provided examples in support of the main result.

The researcher believes that the search for the existence and uniqueness of coupled fixed point for maps satisfying different contractive conditions in dislocated quasi b- metric space is an active area of research.

As a result, any interested researchers can utilize this opportunity to conduct their thesis work in this area.

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