DISTRIBUTIONAL SOLUTIONS OF SINGULARLY PERTURBED

TWO POINT BOUBDARY VALUE PROBLEM

A Thesis Submitted to the Department of Mathematics, Jimma University in Partial Fulfillment for the Requirements of the Degree of Masters of Science (M.Sc.) in Mathematics.

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DECLARATION

Here, I submit a research entitled "**Distributional Solutions of Singularly Perturbed Two-Point Boundary Value Problem**" for the award of degree of Master of Science in Mathematics. I, the undersigned declare that, this study is original and it has not been submitted to any institution elsewhere for the award of any academic degree or the like, where other sources of information have been used, they have been acknowledged.

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Abstract

In this thesis, distributional solution of singularly perturbed two point boundary value problem is presented. In order to achieve this goal, some important terminologies related to distribution are defined together with their properties. Homogeneous solution to singularly perturbed two point boundary value problem under consideration is described and then Green's function was constructed in the sense of distribution to get the particular solution using convolution or without applying convolution. To verify the applicability of the method, three numerical examples were considered and solved. Using the developed method, problems with known exact solution is solved and it agrees with existing exact solution. Furthermore, using the developed method, problems with unknown exact solution is also solved. Finally, MATLAB simulation was implemented for various values of perturbation parameter in order to see the effect of this parameter and the nature of the layer created due to this parameter.

Key words: Distribution, Distributional solution, singularly perturbed problem, two pint boundary value problem, Green's function, Convolution.

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CHAPTER ONE

1. INTRODUCTION

1.1 Background of the Study

Distributions or Generalized functions are objects that generalize the classical notion of functions in mathematical analysis. Generalized functions make it possible to differentiate functions whose derivatives do not exist in the classical sense. The mathematical concept of distribution originates from physics. It was first used by O. Heaviside, a British engineer, in his theory of symbolic calculus and then by P.A. M. Dirac around 1920 in his research on quantum mechanics, in which he introduced the delta -function. The foundations of the mathematical theory of distributions were laid by S.L. Sobolev in 1936, while in the 1950s L.Schwartz gave a systematic account of the theory. The theory of distributions has numerous applications and is extensively used in mathematics, physics and engineering (Gerrit, 2013).

There has recently been a significant increase in the number of topics for which generalized functions have been found to be very effective tools. Familiarity with the basic concepts of this theory has become indispensable for students in applied mathematics, physics, and engineering. Methods based on generalized functions not only help us to solve unsolved problems but also enable us to recover known solutions in a very simple fashion (Kanwal, 1983). Methods of the distribution theory have been used in the several important areas, such as theoretical and mathematical physics, theory of differential equations, functional analysis and etc.

One of the important subjects in applied mathematics is the theory of singular perturbation problem. The mathematical model for this kind of problem usually arise in the form of either ordinary differential equations (ODE) or partial differential equations (PDE) in which the highest order derivative is multiplied by a small positive parameter (Tikhonov, 1952). Prandtl (1905) was a pioneer to emphasize the significance of singular problems and the necessity of their appearance as mathematical models. He pointed out the importance of the subject while he was developing the theory of the boundary layer in hydrodynamics in 1904.

In 2017, Kamsing *et al,* collaboratively studied distributional solution of nth order Bessel equation and obtained solutions in the form of infinite series of the Dirac delta functions and its derivatives. In 2017, Tohru and Kenichi studied about particular solutions of inhomogeneous differential equations with polynomial coefficients in terms of the Green's function and obtained in the framework of distribution theory. In 2018, Marat *et al* studied a singularly perturbed differential equation with piecewise constant argument of generalized type and approximate solution of the problem in distribution sense. In 1982, Wiener has studied various differential equations like second order Bessel equation, Legendre equation, Leguerre's equation, confluent Hypergeometric equations and etc. with singular coefficients and obtained their distributional solutions. In 1987, Littlejohn and Kanwal studied the distributional solutions to the Hypergeometric differential equation and obtained solutions in the form of infinite series of the Dirac delta functions and its derivatives.

Another motivation for studying solutions of the form of infinite series of the Dirac delta functions and its derivatives to ordinary differential equations comes from the works of several scholars like (Morton and krall, 1978; Krall, 1981; Cooke and Wiener, 1984; Littlejohn, 1984; Wiener and Cooke, 1990; Wiener *et al*., 1991; Hernandez-Urena and Estrada, 1995). These researchers had collectively shown that weight distributions for a certain class of orthogonal polynomials have the form of infinite series of the Dirac delta functions and its derivatives and simultaneously satisfy a system of ordinary differential equations.

The study of distributional solutions of differential equations plays a pivotal role in different field of study. Because once we know the solution of certain differential equation it can be easily interpreted in a physical sense to have meaningful message. Singularly perturbed problems have wide range applications in various field of applied mathematics such as fluid mechanics, elasticity, quantum mechanics, optimal control, chemical reaction, aerodynamics, reaction diffusion process, geophysics, and many other areas (Phaneendra *et al.*, 2011). Equations of this type typically exhibit solutions with layers, that is, the domain of the differential equations contains narrow regions where the solution derivatives are extremely large. The numerical treatment of such problems was investigated by different scholars. For example, In 2015, Gemechis File *et al*, studied Fitted-Stable Finite Difference Method for Singularly Perturbed

Two Point Boundary Value Problems. They obtained numerical results and compared with exact solutions. The error bound and convergence of the proposed method has also been established.

However, distributional solutions of singularly perturbed problems were not yet investigated in the existing literature. Being motivated by the applicability of singularly perturbed problems pointed out earlier and distributional solutions- a powerful tool to find a solution of differential equations discontinuity involving discontinuities which are not possible in the classical sense. It then seems relevant to look for distributional solutions to singularly perturbed two point boundary value problem.

Therefore, the present study focuses on distributional solutions of singularly perturbed two point boundary value problem of the form (Phaneendra *et al.*, 2011)

$$-\varepsilon y''(x)+\frac{k}{x}y'(x)+b(x)y(x)=f(x), \qquad 0<x<1\,, \tag{1.1}$$

with boundary conditions

$$y(0)=\alpha \quad \text{and} \quad y(1)=\beta$$

where $0<\varepsilon \ll 1, b(x), f(x)$ bounded continuous functions in $(0,1)$ and α, β, k are finite constants.

1.2 Statement of the Problem

Recently there has been considerable interest in problems concerning the existence of solutions to differential equations in various spaces of generalized functions. Many important areas in theoretical and mathematical physics, theory of partial differential equations, quantum electrodynamics, operational calculus, and functional analysis use the methods of distributions theory. However, for ordinary differential equations, research in this direction is insufficiently developed and remains restricted to isolated results for some second order equations or special higher-order systems. Consequently, this study focuses mainly on the following problems.

- Homogeneous solution of singularly perturbed two point boundary value problem represented by eq. (1.1),
- Constructing Green's function in the sense of distribution for the particular solution of singularly perturbed two point boundary value problem represented by eq. (1.1),
- Layers of the singularly perturbed two point boundary value problem using different numbers via simulation by MATLAB.

1.3 Objectives of the Study

1.3.1 General Objective

The general objective of this study is to investigate distributional or weak solutions of singularly perturbed two point boundary value problem given by eq.(1.1).

1.3.2 Specific objectives

The specific objectives of the study are to:

- Determine homogeneous solution of singularly perturbed two point boundary value problem represented by eq. (1.1),
- Construct Green's function in the sense of distribution for the particular solution of singularly perturbed two point boundary value problem represented by eq. (1.1),
- Detect Layers (if any) of the singularly perturbed two point boundary value problem using different numbers via simulation by MATLAB.

1.4 Significance of the Study

The output of this research can be used as a bench mark for those interested in this area to establish distributional solutions of different mathematical equations in Physics, Engineering and other related field of study whose classical solutions are difficult due singularities in the coefficients of the equation. Furthermore, it provides opportunities for those working in numerical analysis to compare their numerical solution with this solution.

1.5 Delimitation of the Study

This study is delimited to distributional or weak solution of singularly perturbed two point boundary value problem represented by eq. (1.1).

CHAPTER TWO

2. LITERATURE REVIEW

In the mathematics of the nineteenth century, aspects of generalized function theory appeared, for example in the definition of the Green's function, in the Laplace transform, and in Riemann's theory of trigonometric series, which were not necessarily the Fourier series of an integrable function. Differential equations appear in several forms. One has ordinary differential equations and partial differential equations, equations with constant coefficients and with variable coefficients. Equations with constant coefficients are relatively well understood. If the coefficients are variable, much less is known.

Distribution theory stems from the intention to apply the technologies of functional analysis to studying partial differential equations. The series of smooth functions cannot be differentiated term wise in general, which diminishes the scope of applications of analysis to differential equations. Today the concept of generalized derivative occupies a central place in distribution theory. Derivation is now treated as the operator that acts on the non-smooth functions according to the same integral laws as the procedure of taking the classical derivative. It is exactly this approach that was pursued steadily by Sobolev. It turned out that each distribution possesses derivatives of all orders; every series of distributions may be differentiated term wise. However, many “traditionally divergent” Fourier series admit presentations by explicit formulas. Mathematics has acquired additional fantastic degrees of freedom, which makes immortal the name of Sobolev as a pioneer of the calculus of the twentieth century (Kutateladze, 2008).

In 1950 the first volume of Theorie des Dis-tributiones was published in Paris, while Sobolev’s book Applications of Functional Analysis in Mathematical Physics was printed in Leningrad. In 1962 the Siberian Division of the Academy of Sciences of the USSR reprinted the book, while in 1963 it was translated into English by the American Mathematical Society. The second edition of the Schwartz book was published in 1966, slightly enriched with a generalized version of the de Rham currents. Curiously, Schwartz left the historical overview practically the same as in the introduction to the first edition.

The new methods of distribution theory turned out so powerful as to enable mathematicians to write down, in explicit form, the general solution of an arbitrary partial differential equation with constant coefficients. In fact, everything reduces to existence of fundamental solutions; i.e., to the case of the Dirac delta function on the right-hand side of the equation under consideration. Leray was one of the most prominent French mathematicians of the twentieth century. He was awarded with the Lomonosov Gold Medal together with Sobolev in 1988 by reviewing the contributions of Sobolev from 1930 to 1955. Distribution theory is now well developed due to the theory of topological vector spaces and their duality as well as the concept of tempered distribution which is one of the important achievements of Schwartz which enabled him to construct the beautiful theory of the Fourier transform for distributions

The applications of distribution theory in all areas of mathematics, theoretical physics, and numerical analysis remind of the dense forest hiding the tree whose seeds it has grown from. The generalized functions are “ideal elements” that complete the classical function spaces in much the same way as the real numbers complete the set of rational numbers.

Although the study distributional theory plays a significant role in various field of study there is no sufficient research in this direction. Therefore, the central goal of this study is the present distributional solutions of singularly perturbed two point boundary value problem given by equation (1.1).

CHAPTER THREE

METHODOLOGY

3.1. Study Area and Period

The study was conducted in Jimma University under the department of Mathematics from September, 2018 to October, 2019 G.C.

3.2. Study Design

This study employed mixed-design (documentary review design and experimental design) on distributional solutions given by equation (1.1).

3.3. Source of Information

The relevant sources of information for this study were books, published articles & related studies from internet.

3.4. Mathematical Procedures

This study was conducted based on the following procedures

1. Defining the Problem,
2. Finding solutions of the associated homogeneous equation (1.1),
3. Constructing Green's function in the sense of distribution,
4. Verifying the method via numerical examples,
5. Investigating the solution properties by taking into account different values for the perturbation parameter,
6. Identifying layers for different values of the perturbation parameter and the coefficients via simulation by MATLAB.

CHAPTER FOUR

DISCUSSION AND RESULTS

4.1 Preliminaries

Definition 4.1.1: A function $f(x)$ defined on an open set $U \subset R^n$ is said to have compact support if $f(x)=0$ for x in the complement of a compact subset of U. In particular, if $U = R^n$, then f has compact support if there is a positive constant, C such that $f(x)=0$ $for |x| \geq C$.

Definition 4.1.2: A function $f(x)$ is a test function if $f(x)$ has compact support and, in addition, $f(x)$ is infinitely differentiable on U. We use the notation $f \in C_0^\infty(U)$ or $f \in D(U)$ to indicate that $f(x)$ is a test function on U.

For instance, the function $$T(x) = \begin{cases} k\exp(\frac{-1}{1-|x|^2}), \text{if } |x|<1 \\ 0, \quad \text{if } \quad |x| \geq 1 \end{cases} \quad , x \in R^n$$

where the constant K is chosen such that $\int_{R^n} T(x)dx = 1$, is a test function on R^n.

Definition 4.1.3: The spaces of infinitely differentiable functions with compact support in Ω is defined as $D(\Omega) := \{ f : \Omega \to \mathbb{C};\ f \in C^\infty(\Omega) \text{ and supp}(f) \text{ is compact in } \Omega\} = C_c^\infty(\Omega)$. This space is called a space of test functions. The set of test functions, the supports of which are contained in the given region, is denoted by $D(\Omega)$.

Properties of Test Functions in $D(\Omega)$:

1. If ϕ_1 and ϕ_2 are in D, then so is $c_1\phi_1 + \phi_2 c_2$, where c_1 and c_2 are real numbers. Thus D is a linear space.
2. If $\phi \in D$, then so is $D^k\phi$.
3. For C^∞ function f(x) and $\phi \in D$, $f\phi \in D$.
4. Multiplication by a function: Let $\varphi \in D$, then $f\varphi \in D$.

Definition 4.1.4: A continuous linear functional on the space of test functions $D(\Omega)$ is called a distribution $D'(\Omega)$. Distribution in $D'(\Omega)$ is a class of continuous linear functional that maps a set of test function in $D(\Omega)$ into the (complex) numbers.

That is, A functional $f : D(\Omega) \to \mathbb{C}$ such that

1. $\langle f, \varphi \rangle \in \mathbb{C}$

2. $\langle f, c_1\varphi + c_2\varphi \rangle = \langle f, c_1\varphi \rangle + \langle f, c_2\varphi \rangle$

3. $\lim_{n\to\infty} \langle f, \varphi_n \rangle = \left\langle f, \lim_{n\to\infty} \varphi_n \right\rangle$, where $\varphi_1, \varphi_2, ..., \varphi_n \in D(\Omega)$, and c_1 and c_2 are constants, is called a distribution (generalized function).

4.1.1 Delta function

Definition 4.1.5: The Dirac delta function is defined by

$$\delta(x) = \begin{cases} 0, \text{if } x \neq 0 \\ \infty, \text{if } x = 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x)dx = 1$$

Properties of Dirac delta function:

1. The delta function satisfies the following scaling property for a non-zero scalar c:

$$\int_{-\infty}^{\infty} \delta(cx)dx = \int_{-\infty}^{\infty} \delta(u)\frac{du}{|c|} = \frac{1}{|c|}. \quad \text{So,} \quad \delta(cx) = \frac{\delta(x)}{|c|}$$

2. Dirac delta function is symmetric: $\delta(-x) = \delta(x)$

3. Suppose f(x) is a sufficiently smooth function continuous at the origin. Then

$$\langle \mathrm{f}(x),\ \delta(x) \rangle = f(0)$$

4. The Dirac delta function has a distributional derivative defined by $\langle \delta', \varphi \rangle = -\langle \delta, \varphi' \rangle$

4.1.2 Heaviside Function:

Definition 4.1.6: The Heaviside function $H(x)$ is defined to be equal to zero for every negative value of x and to unity for every positive value of x; that is

$$H(x) = \begin{cases} 1, x > 0 \\ 0, x < 0 \end{cases}$$

Derivative of Heaviside function

Let $\varphi \in \mathrm{D(R)}$,

$$H(x) = \begin{cases} 1, x > 0 \\ 0, x < 0 \end{cases} \quad \text{then,}$$

$$\begin{aligned}
\langle \mathrm{H}', \varphi\rangle &= -\langle \mathrm{H}, \varphi'\rangle \\
&= -\int_{-\infty}^{\infty} \mathrm{H(x)}\varphi'(\mathrm{x})dx \\
&= -\int_{0}^{\infty} \varphi'(\mathrm{x})dx \\
&= -\left[\varphi(\mathrm{x})\right]_0^{\infty} \\
&= -\left[\varphi(\infty) - \varphi(0)\right] \\
&= \varphi(0) \\
&= \langle \delta, \varphi\rangle
\end{aligned}$$

So, $H' = \delta$. That is, the derivative of Heaviside function is a delta function.

Theorem: Let a function $f(x)$ be n times continuously differentiable; then

$$f(x)\delta^{(n)}(x) = (-1)^n f^{(n)}(0)\delta(x) + (-1)^{n-1} n f^{(n-1)}(0)\delta'(x) + (-1)^{n-2}\frac{n(n-1)}{2!} f^{(n-2)}(0)\delta''(x) + \ldots + f(0)\delta^{(n)}(x).$$

Proof: First we show its validity with the help of a test function.

$$\begin{aligned}
\int_{-\infty}^{\infty}\left(f(x)\delta^{(n)}(x)\right)\phi(x)dx &= \int_{-\infty}^{\infty}\left(f(x)\phi(x)\right)\delta^{(n)}(x)dx \\
&= \left[\left(f(x)\phi(x)\right)\delta^{(n-1)}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\left(f(x)\phi(x)\right)'\delta^{(n-1)}(x)dx \\
&= -\int_{-\infty}^{\infty}\left(f(x)\phi(x)\right)'\delta^{(n-1)}(x)dx \\
&= -\left[\left[\left(f(x)\phi(x)\right)\delta^{(n-2)}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\left(f(x)\phi(x)\right)''\delta^{(n-2)}(x)dx\right] \\
&= (-1)^2\int_{-\infty}^{\infty}\left(f(x)\phi(x)\right)''\delta^{(n-2)}(x)dx \\
&= -\left[\left[\left(f(x)\phi(x)\right)\delta^{(n-3)}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\left(f(x)\phi(x)\right)'''\delta^{(n-3)}(x)dx\right] \\
&= (-1)^3\int_{-\infty}^{\infty}\left(f(x)\phi(x)\right)'''\delta^{(n-3)}(x)dx \\
&\quad . \\
&\quad . \\
&\quad . \\
&= (-1)^n\int_{-\infty}^{\infty}\left(f(x)\phi(x)\right)^{(n)}\delta(x)dx
\end{aligned}$$

Substitution of the formula

$$\left(f(x)\phi(x)\right)^{(n)} = f^{(n)}(x)\phi(x) + nf^{(n-1)}(x)\phi'(x) + \frac{n(n-1)}{2!} f^{(n-2)}(x)\phi''(x) + \ldots + f(x)\phi^{(n)}(x),$$

in the preceding relation and the application of the shifting property yields:

$$\int_{-\infty}^{\infty}\left(f(x)\delta^{(n)}(x)\right)\phi(x)dx = (-1)^{(n)}\left(f^{(n)}(0)\phi(0) + nf^{(n-1)}(0)\phi'(0) + \frac{n(n-1)}{2!} f^{(n-2)}(0)\phi''(0) + \ldots + f(0)\phi^{(n)}(0)\right).$$

Corollary:

$$\left(f(x)H(x)\right)^{(n)} = f^{(n)}(x)H(x) + f^{(n-1)}(0)\delta(x) + f^{(n-2)}(0)\delta'(x) + \ldots + f(0)\delta^{(n-1)}(x).$$

Proof: Observe that

$$\left(f(x)H(x)\right)^{(n)} = f^{(n)}(x)H(x) + nf^{(n-1)}(x)H'(x) + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!} f^{(n)}(x)H^{(r)}(x) + \ldots + f(x)H^{(n)}(x)$$

As $H^{(r)}(x) = \delta^{(r-1)}(x)$ and by theorem above,

$$f(x)\delta^{(n-1)}(x) = (-1)^{n-1}nf^{(n-1)}(0)\delta'(x) + (-1)^{n-2}\frac{n(n-1)}{2!} f^{(n-2)}(0)\delta''(x) + \ldots + f(0)\delta^{(n)}(x).$$

We get,

$$\left(f(x)H(x)\right)^{(n)} = f^{(n)}(x)H(x) + nf^{(n-1)}(0)\delta(x) + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!} f^{(n)}(x)\delta^{(r-1)}(x) + \ldots + (-1)^{n-1}nf^{(n-1)}(0)\delta'(x)$$
$$+ (-1)^{n-2}\frac{n(n-1)}{2!} f^{(n-2)}(0)\delta''(x) + \ldots + f(0)\delta^{(n)}(x).$$

Therefore,

$$\left(f(x)H(x)\right)^{(n)} = f^{(n)}(x)H(x) + f^{(n-1)}(0)\delta(x) + f^{(n-2)}(0)\delta'(x) + \ldots + f(0)\delta^{(n-1)}(x).$$

4.1.3 Green's Function:

Definition 4.1.7: A Green's function $G(x,\xi)$ for any Boundary Value Problem satisfies the equation

$$LG(x,\xi) = \delta(x-\xi) \;,$$

for some operator L_x with homogeneous Boundary Conditions, i.e. it is the solution corresponding to the data $\{\delta(x-\xi)=0\}$. Thus, $G(x,\xi)$ is the response under homogeneous Boundary Conditions to a forcing function consisting of a concentrated unit impulse (or inhomogeneity) at $x=\xi$.

A fundamental solution of the differential equation is the solution of

$$(L_x G(x,y))(x,y) = \delta(x-y),\; x,y \in R^d$$, in the distributional sense.

In general, the Green's function $G(x,\xi)$ associated with the non-homogeneous equation $Ly = f(x)$ satisfies the differential equation

$LG(x,\xi) = \delta(x-\xi)$:

Once $G(x,\xi)$ is known, then the solution to $Ly = f(x)$ is given by

$$y(x) = \int_{-\infty}^{\infty} G(x,\xi) f(\xi) d\xi$$

Because,

$$\begin{aligned} Ly &= L\int_{-\infty}^{\infty} G(x,\xi) f(\xi) d\xi \\ &= \int_{-\infty}^{\infty} [LG(x,\xi)] f(\xi) d\xi \\ &= \int_{-\infty}^{\infty} \delta(x-\xi) f(\xi) d\xi \\ &= \int_{-\infty}^{\infty} \delta(\xi-x) f(x) d\xi \\ &= f(x)\int_{-\infty}^{\infty} \delta(\xi-x) d\xi \\ &= f(x) \end{aligned}$$

Note that this is true over any interval that contains x on which $f(x)$ is continuous.

Hence, Green's function of a differential equation is a fundamental solution satisfying the boundary conditions.

Properties Green's function:

For $\dfrac{d}{dx}\left[p(x)\dfrac{dy(x)}{dx}\right] + q(x)y(x) = f(x),$

A Green's function $G(x,\xi)$ satisfies the following properties.

1. It satisfies the homogeneous form the given differential equation.

   That is, for $x \neq \xi$, $LG(x,\xi) = 0$

2. The function $G(x,\xi)$ is continuous at $x = \xi$.

   That is, $\lim_{x\to\xi^-} G(x,\xi) = G(x=\xi) = \lim_{x\to\xi^+} G(x,\xi)$

3. The derivative of $G(x,\xi)$ with respect to ξ is discontinuous at $x = \xi$.

That is, jump discontinuity of $\frac{\partial G}{\partial x}$ at $x=\xi$: $G'(x,\xi^{+})-G'(x,\xi^{-})=\frac{1}{p(\xi)}$

4. $G(x,\xi)$, satisfies a given boundary condition of the problem

5. The function $G(x,\xi)$ is symmetric in its arguments: $G(x,\xi)=G(\xi,x)$

All the preliminaries are from (Kanwal, 1983)

4.2 Main Result

Consider singularly perturbed two point boundary value problem given by:

$$-\varepsilon y''(x)+\frac{k}{x}y'(x)+b(x)y(x)=f(x),\ 0<x<1 \tag{1.1}$$

$$y(0)=\alpha \quad \text{and} \quad y(1)=\beta$$

Case1: $k=0$ and $b(x)=p$, where p is an arbitrary constant.

Equation (1.1) becomes

$$-\varepsilon y''(x)+py(x)=f(x),\ 0<x<1 \tag{1.2}$$

$$y(0)=\alpha \quad \text{and} \quad y(1)=\beta$$

The general solution to Eq. (1.2) is given by:

$$y(x)=y_h(x)+y_{p'}(x) \tag{1.3}$$

where $y_h(x)$ is an homogeneous solution and $y_{p'}(x)$ is a particular solution of equation (1.2)

To find the Homogeneous Solution,

$$-\varepsilon y''(x)+py(x)=0,\ 0<x<1 \tag{1.4}$$

The solution to eq. (1.4) is assumed to be $y(x)=e^{mx}$, where m is a constant.

Auxiliary equation: $\varepsilon m^2-p=0$

$$m_1=\frac{\sqrt{p}}{\sqrt{\varepsilon}} \quad \text{or} \quad m_2=\frac{-\sqrt{p}}{\sqrt{\varepsilon}}$$

$$y_h(x) = c_1 e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}x} + c_2 e^{\frac{-\sqrt{p}}{\sqrt{\varepsilon}}x} \tag{1.5}$$

To find the particular solution,

$$-\varepsilon y''(x) + py(x) = \delta(x-\xi),\ \ 0 < x < 1 \quad \text{and} \quad 0 < \xi < 1 \tag{1.6}$$

$$y(0) = \alpha \quad \text{and} \quad y(1) = \beta$$

Since the non-homogeneous part of eq. (1.6) is delta function, we assume the form of the particular solution to be:

$$y_p(x) = G(x)H(x-\xi) \tag{1.7}$$

where $H(x-\xi)$ is the Heaviside function, $G(x)$ is unknown function.

$$y_p(x) = G(x)H(x-\xi)$$

$$y'_p(x) = G'(x)H(x-\xi) + G(x)H'(x-\xi)$$

$$= G'(x)H(x-\xi) + G(x)\delta(x-\xi)$$

$$= G'(x)H(x-\xi) + G(\xi)\delta(x-\xi)$$

$$y''_p(x) = G''(x)H(x-\xi) + G'(x)H'(x-\xi) + G(\xi)\delta'(x-\xi)$$

$$= G''(x)H(x-\xi) + G'(x)\delta(x-\xi) + G(\xi)\delta'(x-\xi)$$

$$= G''(x)H(x-\xi) + G'(\xi)\delta(x-\xi) + G(\xi)\delta'(x-\xi)$$

Plugging $y_p(x)$ and $y''_p(x)$ into eq. (1.6) gives

$$-\varepsilon\left[G''(x)H(x-\xi) + G'(\xi)\delta(x-\xi) + G(\xi)\delta'(x-\xi)\right] + pG(x)H(x-\xi) = \delta(x-\xi)$$

$$\left[-\varepsilon G''(x) + pG(x)\right]H(x-\xi) - \varepsilon G'(\xi)\delta(x-\xi) - \varepsilon G(\xi)\delta'(x-\xi) = \delta(x-\xi) \tag{1.8}$$

Then, comparing the corresponding coefficients of eq. (1.8),

$$\begin{cases} -\varepsilon G''(x) + pG(x) = 0 \\ -\varepsilon G'(\xi) = 1 \\ -\varepsilon G(\xi) = 0 \end{cases}$$

$$\begin{cases} -\varepsilon G''(x) + pG(x) = 0 \\ G'(\xi) = \dfrac{-1}{\varepsilon} \\ G(\xi) = 0 \end{cases} \tag{1.9}$$

The solution of eq. (1.9) is given by $G(x) = c_3 e^{m_1 x} + c_4 e^{m_2 x}$, where

$$m_1 = \frac{\sqrt{p}}{\sqrt{\varepsilon}} \quad \text{and} \quad m_2 = \frac{-\sqrt{p}}{\sqrt{\varepsilon}}$$

$$G(x) = c_3 e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}x} + c_4 e^{\frac{-\sqrt{p}}{\sqrt{\varepsilon}}x} \tag{1.10}$$

Using the initial conditions of eq. (1.9) we find for the constants c_3 and c_4:

$$G(\xi) = c_3 e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}\xi} + c_4 e^{\frac{-\sqrt{p}}{\sqrt{\varepsilon}}\xi} = 0$$

$$G'(\xi) = c_3 \frac{\sqrt{p}}{\sqrt{\varepsilon}} e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}\xi} - c_4 \frac{\sqrt{p}}{\sqrt{\varepsilon}} e^{\frac{-\sqrt{p}}{\sqrt{\varepsilon}}\xi} = \frac{-1}{\varepsilon}$$

Solving this system of equations gives:

$$\begin{cases} c_3 = \dfrac{-\sqrt{\varepsilon} e^{\frac{-\sqrt{p}}{\sqrt{\varepsilon}}\xi}}{2\varepsilon\sqrt{p}} \\ c_4 = \dfrac{\sqrt{\varepsilon}}{2\varepsilon\sqrt{p}} e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}\xi} \end{cases} \tag{1.11}$$

Substituting eq. (1.11) into eq. (1.10) we get

$$G(x) = \frac{\sqrt{\varepsilon}}{2\varepsilon\sqrt{p}} e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}(\xi - x)} - \frac{\sqrt{\varepsilon}}{2\varepsilon\sqrt{p}} e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}(x-\xi)} \tag{1.12}$$

Plugging eq. (1.12) into eq. (1.7) we get:

$$y_p(x) = \left[\frac{\sqrt{\varepsilon}}{2\varepsilon\sqrt{p}} e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}(\xi - x)} - \frac{\sqrt{\varepsilon}}{2\varepsilon\sqrt{p}} e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}(x-\xi)} \right] H(x-\xi)$$

$$y_p(x) = \begin{cases} 0, & x < \xi < 1 \\ \dfrac{\sqrt{\varepsilon}}{2\varepsilon\sqrt{p}} e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}(\xi - x)} - \dfrac{\sqrt{\varepsilon}}{2\varepsilon\sqrt{p}} e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}(x-\xi)}, & 0 < \xi < x \end{cases} \quad (1.13)$$

is the particular solution of eq. (1.6).

Since the non-homogeneous part of eq. (1.6) is Dirac delta function, this particular solution is the same with the Green's function of eq. (1.2)

$$G(x,\xi) = \begin{cases} 0, & x < \xi < 1 \\ \dfrac{\sqrt{\varepsilon}}{2\varepsilon\sqrt{p}} e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}(\xi - x)} - \dfrac{\sqrt{\varepsilon}}{2\varepsilon\sqrt{p}} e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}(x-\xi)} & , 0 < \xi < x \end{cases} \quad (1.14)$$

As a result, the particular solution to eq. (1.2) is given by:

$$\begin{aligned} y_{p'}(x) &= \int_0^1 G(x,\xi) f(\xi) d\xi \\ &= \int_0^x G(x,\xi) f(\xi) d\xi + \int_x^1 G(x,\xi) f(\xi) d\xi \end{aligned}$$

$$y_{p'}(x) = \begin{cases} 0, & x < \xi < 1 \\ \displaystyle\int_0^x \left[\frac{\sqrt{\varepsilon}}{2\varepsilon\sqrt{p}} e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}(\xi - x)} - \frac{\sqrt{\varepsilon}}{2\varepsilon\sqrt{p}} e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}(x-\xi)} \right] f(\xi) d\xi, & 0 < \xi < x \end{cases} \quad (1.15)$$

Plugging equations (1.5) and (1.15) into eq. (1.3) we get,

$$y(x)=c_1e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}x}+c_2e^{\frac{-\sqrt{p}}{\sqrt{\varepsilon}}x}+\begin{cases}0, & x<\xi<1 \\ \int_0^x\left[\frac{\sqrt{\varepsilon}}{2\varepsilon\sqrt{p}}e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}(\xi-x)}-\frac{\sqrt{\varepsilon}}{2\varepsilon\sqrt{p}}e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}(x-\xi)}\right]f(\xi)d\xi, & 0<\xi<x\end{cases} \tag{1.16}$$

Using the given boundary conditions in eq. (1.2) we can get c_1 and c_2 as follows.

$$y(0)=c_1+c_2+0=\alpha\,,$$

$$c_2=\alpha-c_1 \tag{1.17}$$

$$y(1)=c_1e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}}+c_2e^{\frac{-\sqrt{p}}{\sqrt{\varepsilon}}}+\frac{\sqrt{\varepsilon}}{2\varepsilon\sqrt{p}}\int_0^1\left[e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}(\xi-1)}-e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}(1-\xi)}\right]f(\xi)d\xi=\beta$$

$$c_1e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}}+c_2e^{\frac{-\sqrt{p}}{\sqrt{\varepsilon}}}=\beta-\frac{\sqrt{\varepsilon}}{2\varepsilon\sqrt{p}}\int_0^1\left[e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}(\xi-1)}-e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}(1-\xi)}\right]f(\xi)d\xi \tag{1.18}$$

Substituting eq. (1.17) into eq. (1.18) we get:

$$c_1e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}}+\left(\alpha-c_1\right)e^{\frac{-\sqrt{p}}{\sqrt{\varepsilon}}}=\beta-\frac{\sqrt{\varepsilon}}{2\varepsilon\sqrt{p}}\int_0^1\left[e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}(\xi-1)}-e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}(1-\xi)}\right]f(\xi)d\xi$$

$$c_1\frac{e^{\frac{2\sqrt{p}}{\sqrt{\varepsilon}}}-1}{e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}}}=\beta-\alpha e^{\frac{-\sqrt{p}}{\sqrt{\varepsilon}}}-\frac{\sqrt{\varepsilon}}{2\varepsilon\sqrt{p}}\int_0^1\left[e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}(\xi-1)}-e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}(1-\xi)}\right]f(\xi)d\xi$$

$$c_1=\left(\frac{e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}}}{e^{\frac{2\sqrt{p}}{\sqrt{\varepsilon}}}-1}\right)\left[\beta-\alpha e^{\frac{-\sqrt{p}}{\sqrt{\varepsilon}}}-\frac{\sqrt{\varepsilon}}{2\varepsilon\sqrt{p}}\int_0^1\left(e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}(\xi-1)}-e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}(1-\xi)}\right)f(\xi)d\xi\right] \tag{1.19}$$

Substituting eq. (1.19) into eq. (1.17) we get

$$c_2 = \alpha - \left(\frac{e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}}}{e^{\frac{2\sqrt{p}}{\sqrt{\varepsilon}}} - 1} \right) \left[\beta - \alpha e^{\frac{-\sqrt{p}}{\sqrt{\varepsilon}}} - \frac{\sqrt{\varepsilon}}{2\varepsilon\sqrt{p}} \int_0^1 \left(e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}(\xi-1)} - e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}(1-\xi)} \right) f(\xi) d\xi \right] \quad (1.20)$$

Therefore,

$$y(x) = c_1 e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}x} + c_2 e^{\frac{-\sqrt{p}}{\sqrt{\varepsilon}}x} + \begin{cases} 0, & x < \xi < 1 \\ \int_0^x \left[\frac{\sqrt{\varepsilon}}{2\varepsilon\sqrt{p}} e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}(\xi-x)} - \frac{\sqrt{\varepsilon}}{2\varepsilon\sqrt{p}} e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}(x-\xi)} \right] f(\xi) d\xi, & 0 < \xi < x \end{cases} \quad (1.21)$$

where c_1 and c_2 are given by equations (1.19) and (1.20) respectively.

As a result, a solution to eq. (1.2) which is reduced from eq. (1.1) is given by eq. (1.21)

Case 2: $k \neq 0$, $b(x) = 0$

Equation (1.1) is reduced to

$$-\varepsilon y''(x) + \frac{k}{x} y'(x) = f(x), \; 0 < x < 1 \quad (1.22)$$

$$y(0) = \alpha \quad \text{and} \quad y(1) = \beta$$

To find the homogeneous solution,

$$-\varepsilon y''(x) + \frac{k}{x} y'(x) = 0, \; 0 < x < 1 \quad (1.23)$$

$$\frac{y''(x)}{y'(x)} = \frac{k}{\varepsilon x}$$

$$\int \frac{y''(x)}{y'(x)} dx = \int \frac{k}{\varepsilon x} dx$$

$$\int \frac{y''(x)}{y'(x)} dx = \frac{k}{\varepsilon} \int \frac{1}{x} dx$$

$$\ln y'(x) = \frac{k}{\varepsilon}\ln x + \ln c_1$$

$$\ln y'(x) = \ln c_1 x^{\frac{k}{\varepsilon}}$$

$$\int y'(x)dx = \int c_1 x^{\frac{k}{\varepsilon}} dx + c_2$$

$$y_h(x) = c_1 \frac{\varepsilon}{k+\varepsilon} x^{\frac{k+\varepsilon}{\varepsilon}} + c_2 \tag{1.24}$$

To find the particular solution

$$-\varepsilon y''(x) + \frac{k}{x} y'(x) = \delta(x),\ \ 0 < x < 1 \tag{1.25}$$

Since the non-homogeneous part of eq. (1.25) is delta function, we can assume the form of the particular solution to be:

$$y_p(x) = G(x)H(x-\xi) \tag{1.26}$$

where $H(x-\xi)$ is the Heaviside function, $G(x)$ is unknown function.

$$y_p(x) = G(x)H(x-\xi)$$

$$\begin{aligned} y'_p(x) &= G'(x)H(x-\xi) + G(x)H'(x-\xi) \\ &= G'(x)H(x-\xi) + G(x)\delta(x-\xi) \\ &= G'(x)H(x-\xi) + G(\xi)\delta(x-\xi) \end{aligned}$$

$$\begin{aligned} y''_p(x) &= G''(x)H(x-\xi) + G'(x)H'(x-\xi) + G(\xi)\delta'(x-\xi) \\ &= G''(x)H(x-\xi) + G'(x)\delta(x-\xi) + G(\xi)\delta'(x-\xi) \\ &= G''(x)H(x-\xi) + G'(\xi)\delta(x-\xi) + G(\xi)\delta'(x-\xi) \end{aligned}$$

Plugging $y'_p(x)$ and $y''_p(x)$ into eq. (1.25) gives

$$-\varepsilon(G''(x)H(x-\xi) + G'(\xi)\delta(x-\xi) + G(\xi)\delta'(x-\xi)) + \frac{k}{x}(G'(x)H(x-\xi)$$
$$+G(\xi)\delta(x-\xi)) = \delta(x-\xi)$$

$$[-\varepsilon G''(x)+\frac{k}{x}G'(x)]H(x-\xi)+[-\varepsilon G'(\xi)+G(\xi)\frac{k}{\xi}]\delta(x-\xi)-\varepsilon G(\xi)\delta'(x-\xi))$$
$$=\delta(x-\xi) \tag{1.27}$$

Comparing the corresponding coefficients of equation (1.27) :

$$\begin{cases} -\varepsilon G''(x)+\frac{k}{x}G'(x)=0 \\ -\varepsilon G'(\xi)+G(\xi)\frac{k}{\xi}=1 \\ -\varepsilon G(\xi)=0 \end{cases}$$

$$\begin{cases} -\varepsilon G''(x)+\frac{k}{x}G'(x)=0 \\ G'(\xi)=\frac{-1}{\varepsilon} \\ G(\xi)=0 \end{cases} \tag{1.28}$$

$$-\varepsilon xG''(x)+kG'(x)=0$$

$$\frac{G''(x)}{G'(x)}=\frac{k}{\varepsilon x}$$
$$\int\frac{G''(x)}{G'(x)}dx=\int\frac{k}{\varepsilon x}dx$$
$$\ln G'(x)=\frac{k}{\varepsilon}\ln x+\ln c_3$$
$$=\ln(c_3x^{\frac{k}{\varepsilon}})$$
$$G'(x)=c_3x^{\frac{k}{\varepsilon}}$$
$$G(x)=c_3\int x^{\frac{k}{\varepsilon}}dx+c_4$$

$$G(x)=\frac{\varepsilon}{k+\varepsilon}c_3x^{\frac{k+\varepsilon}{\varepsilon}}+c_4 \tag{1.29}$$

Using the initial conditions of eq. (1.28) we calculate for the constant c_3 and c_4:

$$G'(\xi) = c_3 \xi^{\frac{k}{\varepsilon}} = \frac{-1}{\varepsilon} \quad \text{and} \quad G(\xi) = \frac{\varepsilon}{k+\varepsilon} c_3 \xi^{\frac{k+\varepsilon}{\varepsilon}} + c_4 = 0$$

$$c_3 = \frac{-1}{\varepsilon \xi^{\frac{k}{\varepsilon}}} \quad \text{and} \quad c_4 = \frac{\xi}{k+\varepsilon}$$

Hence, eq. (1.29) becomes

$$G(x) = \frac{-1}{(k+\varepsilon)\xi^{\frac{k}{\varepsilon}}} x^{\frac{k+\varepsilon}{\varepsilon}} + \frac{\xi}{k+\varepsilon} \tag{1.30}$$

Substituting eq. (1.30) into eq. (1.26) we obtain:

$$y_p(x) = \left[\frac{-1}{(k+\varepsilon)\xi^{\frac{k}{\varepsilon}}} x^{\frac{k+\varepsilon}{\varepsilon}} + \frac{\xi}{k+\varepsilon} \right] H(x-\xi$$

$$y_p(x) = \begin{cases} 0, & x < \xi < 1 \\ \dfrac{-1}{(k+\varepsilon)\xi^{\frac{k}{\varepsilon}}} x^{\frac{k+\varepsilon}{\varepsilon}} + \dfrac{\xi}{k+\varepsilon}, & 0 < \xi < x \end{cases} \tag{1.31}$$

is the particular solution of eq. (1.25).

Since the non-homogeneous part of eq. (1.25) is Dirac delta function, this particular solution is the same with the Green's function of eq. (1.22).

$$G(x,\xi) = \begin{cases} 0, & x < \xi < 1 \\ \dfrac{-1}{(k+\varepsilon)\xi^{\frac{k}{\varepsilon}}} x^{\frac{k+\varepsilon}{\varepsilon}} + \dfrac{\xi}{k+\varepsilon}, & 0 < \xi < x \end{cases} \tag{1.32}$$

$$y_{p'}(x)=\int_0^1 G(x,\xi)\,f(x,)dx$$
$$=\int_0^x G(x,\xi)\,f(x,)dx+\int_x^1 G(x,\xi)\,f(x,)dx$$

$$y_{p'}(x)=\begin{cases}0, & x<\xi<1\\ \displaystyle\int_0^x\left[\frac{-1}{(k+\varepsilon)\xi^{\frac{k}{\varepsilon}}}x^{\frac{k+\varepsilon}{\varepsilon}}+\frac{\xi}{k+\varepsilon}\right]f(\xi)d\xi\ , & 0<\xi<x\end{cases} \tag{1.33}$$

From equations (1.24) and (1.33) we get:

$$y(x)=y_h(x)+y_{p'}(x)$$

$$y(x)=c_1\frac{\varepsilon}{k+\varepsilon}x^{\frac{k+\varepsilon}{\varepsilon}}+c_2+\begin{cases}0, & x<\xi<1\\ \displaystyle\int_0^x\left[\frac{-1}{(k+\varepsilon)\xi^{\frac{k}{\varepsilon}}}x^{\frac{k+\varepsilon}{\varepsilon}}+\frac{\xi}{k+\varepsilon}\right]f(\xi)d\xi\,, & 0<\xi<x\end{cases} \tag{1.34}$$

Using the given boundary conditions in eq. (1.22) we solve for c_1 and c_2 as follows.

$$y(0)=c_1.0+c_2+0=\alpha\ ,\ \text{and}$$

$$y(1)=c_1\frac{\varepsilon}{k+\varepsilon}+c_2+\int_0^1\left[\frac{-1}{(k+\varepsilon)\xi^{\frac{k}{\varepsilon}}}+\frac{\xi}{k+\varepsilon}\right]f(\xi)d\xi=\beta$$

$$c_2=\alpha\ ,\ \text{and}\ c_1\frac{\varepsilon}{k+\varepsilon}+\alpha+\int_0^1\left[\frac{-1}{(k+\varepsilon)\xi^{\frac{k}{\varepsilon}}}+\frac{\xi}{k+\varepsilon}\right]f(\xi)d\xi=\beta$$

` $$c_2=\alpha\ ,\ \text{and}\ c_1\frac{\varepsilon}{k+\varepsilon}-\frac{1}{k+\varepsilon}\int_0^1\left[\xi^{\frac{-k}{\varepsilon}}-\xi\right]f(\xi)d\xi=\beta-\alpha$$

$$c_2 = \alpha\ , \quad \text{and}\quad c_1 \frac{\varepsilon}{k+\varepsilon} = \beta - \alpha + \frac{1}{k+\varepsilon}\int_0^1 \left[\xi^{\frac{-k}{\varepsilon}} - \xi \right] f(\xi) d\xi$$

$$\begin{cases} c_1 = (\beta - \alpha)\left(\dfrac{k+\varepsilon}{\varepsilon}\right) + \dfrac{1}{\varepsilon}\displaystyle\int_0^1 \left[\xi^{\frac{-k}{\varepsilon}} - \xi \right] f(\xi) d\xi \\ c_2 = \alpha \end{cases} \qquad (1.35)$$

Substituting eq. (1.35) into eq. (1.34) we obtain:

$$y(x) = \left[(\beta-\alpha)\left(\frac{k+\varepsilon}{\varepsilon}\right) + \frac{1}{\varepsilon}\int_0^1 \left(\xi^{\frac{-k}{\varepsilon}} - \xi \right) f(\xi) d\xi \right] \frac{\varepsilon}{k+\varepsilon} x^{\frac{k+\varepsilon}{\varepsilon}} + \alpha$$
$$+ \begin{cases} 0, & x < \xi < 1 \\ \displaystyle\int_0^x \left[\frac{-1}{(k+\varepsilon)\xi^{\frac{k}{\varepsilon}}} x^{\frac{k+\varepsilon}{\varepsilon}} + \frac{\xi}{k+\varepsilon} \right] f(\xi) d\xi, & 0 < \xi < x \end{cases}$$

$$y(x) = \left[\beta - \alpha + \frac{1}{k+\varepsilon}\int_0^1 \left(\xi^{\frac{-k}{\varepsilon}} - \xi \right) f(\xi) d\xi \right] x^{\frac{k+\varepsilon}{\varepsilon}} + \alpha$$
$$+ \begin{cases} 0, & x < \xi < 1 \\ \displaystyle\int_0^x \left[\frac{-1}{(k+\varepsilon)\xi^{\frac{k}{\varepsilon}}} x^{\frac{k+\varepsilon}{\varepsilon}} + \frac{\xi}{k+\varepsilon} \right] f(\xi) d\xi, & 0 < \xi < x \end{cases} \qquad (1.36)$$

As a result a solution to eq. (1.22) which is reduced from eq. (1.1) is given by eq. (1.36)

Case 3: $k \neq 0,\ b(x) = \dfrac{p}{x^2}$, where p is constant.

In this case eq. (1.1) is reduced to:

$$-\varepsilon y''(x) + \frac{k}{x} y'(x) + \frac{p}{x^2} y(x) = f(x), 0 < x < 1$$
$$y(0) = \alpha \text{ and } y(1) = \beta \qquad (1.37)$$

Eq. (1.37) can be written as:

$$-\varepsilon x^2 y''(x) + kxy'(x) + py(x) = x^2 f(x), 0 < x < 1$$
$$y(0) = \alpha \text{ and } y(1) = \beta \tag{1.38}$$

Eq. (1.38) is Cauchy Euler Equation.

Its solution is obtained as follows.

Associated homogenous part:

$$-\varepsilon x^2 y''(x) + kxy'(x) + py(x) = 0,\ 0 < x < 1 \tag{1.39}$$

The solution to eq. (1.39) is assumed to be

$$\left.\begin{aligned} y(x) &= x^m \\ y'(x) &= mx^{m-1} \\ y''(x) &= m(m-1)x^{m-2} \end{aligned}\right\} \tag{1.40}$$

Plugging eq. (1.40) into eq. (1.39) leads to:

$$-\varepsilon m(m-1)x^m + kmx^m + px^m = 0$$
$$\left[-\varepsilon m^2 + (k+\varepsilon)m + p\right]x^m = 0$$

$$-\varepsilon m^2 + (k+\varepsilon)m + p = 0 \tag{1.41}$$

While solving eq. (1.41), there are three cases.

Case i: Distinct real roots say m_1 and m_2.

Its corresponding solution is:

$$y_h(x) = c_1 x^{m_1} + c_2 x^{m_2} \tag{1.42}$$

c_1 and c_2 are arbitrary constants.

Case ii: Repeated real root say m.

Its corresponding solution is:

$$y_h(x) = c_1 x^{m_1} + c_2 x^{m_2} \ln x \tag{1.43}$$

c_1 and c_2 are arbitrary constants.

Case iii: Complex root of the form $m = \alpha_1 \pm \beta_1 i$

Its corresponding solution is:

$$y_h(x) = x^{\alpha_1}\left[c_1 \cos(\beta_1 \ln x) + c_2 \sin(\beta_1 \ln x)\right] \tag{1.44}$$

To find the particular solution,

$$-\varepsilon y''(x) + \frac{k}{x} y'(x) + \frac{p}{x^2} y(x) = \delta(x-\xi), \; 0 < x < 1 \quad \text{and} \quad 0 < \xi < 1 \tag{1.45}$$

$$y(0) = \alpha \quad \text{and} \quad y(1) = \beta$$

Since the non-homogeneous part of eq. (1.6) is delta function, we assume the form of the particular solution to be:

$$y_p(x) = G(x) H(x-\xi) \tag{1.46}$$

where $H(x-\xi)$ is the Heaviside function, $G(x)$ is unknown function.

$$y_p(x) = G(x) H(x-\xi)$$

$$y'_p(x) = G'(x) H(x-\xi) + G(x) H'(x-\xi)$$

$$= G'(x) H(x-\xi) + G(x) \delta(x-\xi)$$

$$= G'(x) H(x-\xi) + G(\xi) \delta(x-\xi)$$

$$y''_p(x) = G''(x) H(x-\xi) + G'(x) H'(x-\xi) + G(\xi) \delta'(x-\xi)$$

$$= G''(x) H(x-\xi) + G'(x) \delta(x-\xi) + G(\xi) \delta'(x-\xi)$$

$$= G''(x) H(x-\xi) + G'(\xi) \delta(x-\xi) + G(\xi) \delta'(x-\xi)$$

Plugging $y_p(x)$, $y'_p(x)$ and $y''_p(x)$ into eq. (1.45) gives

$$-\varepsilon\left[G''(x)H(x-\xi)+G'(\xi)\delta(x-\xi)+G(\xi)\delta'(x-\xi)\right]+\frac{k}{x}\left[G'(x)H(x-\xi)+G(\xi)\delta(x-\xi)\right]$$

$$+\frac{p}{x^2}G(x)H(x-\xi)=\delta(x-\xi)$$

$$\left[-\varepsilon G''(x)+\frac{k}{x}G'(x)+\frac{p}{x^2}G(x)\right]H(x-\xi)+\left[-\varepsilon G'(\xi)+\frac{k}{\xi}G(\xi)\right]\delta(x-\xi)$$
$$-\varepsilon G(\xi)\delta'(x-\xi)=\delta(x-\xi) \tag{1.47}$$

Then, comparing the corresponding coefficients of eq. (1.47),

$$\begin{cases} -\varepsilon G''(x)+\dfrac{k}{x}G'(x)+\dfrac{p}{x^2}G(x)=0 \\ -\varepsilon G'(\xi)+\dfrac{k}{\xi}G(\xi)=1 \\ -\varepsilon G(\xi)=0 \end{cases}$$

$$\begin{cases} -\varepsilon x^2 G''(x)+xG'(x)+pG(x)=0 \\ G'(\xi)=\dfrac{-1}{\varepsilon} \\ G(\xi)=0 \end{cases} \tag{1.48}$$

Eq. (1.48) is homogeneous Cauchy Euler Equation

The solution to eq. (1.48) is assumed to be

$$\left.\begin{aligned} G(x)&=x^m \\ G'(x)&=mx^{m-1} \\ G''(x)&=m(m-1)x^{m-2} \end{aligned}\right\} \tag{1.49}$$

Plugging eq. (1.49) into eq. (1.48) leads to:

$$-\varepsilon m(m-1)x^m + kmx^m + px^m = 0$$
$$x^m\left[-\varepsilon m^2 + (k+\varepsilon)m + p\right] = 0$$

$$-\varepsilon m^2 + (k+\varepsilon)m + p = 0 \tag{1.50}$$

While solving eq. (1.50), there are three cases which can be treated as equations (1.42), (1.43) and (1.44)

From eq. (1.50) we have,

$$-\varepsilon m^2 + (k+\varepsilon)m + p = 0$$
$$\varepsilon m^2 - (k+\varepsilon)m - p = 0$$
$$m = \frac{k+\varepsilon \pm \sqrt{(k+\varepsilon)^2 + 4\varepsilon p}}{2\varepsilon}$$

$$\begin{cases} m_1 = \dfrac{k+\varepsilon + \sqrt{(k+\varepsilon)^2 + 4\varepsilon p}}{2\varepsilon} \\ m_2 = \dfrac{k+\varepsilon - \sqrt{(k+\varepsilon)^2 + 4\varepsilon p}}{2\varepsilon} \end{cases} \tag{1.51}$$

For case i.

$$G(x) = c_3 x^{m_1} + c_4 x^{m_2} \tag{1.52}$$

Using the initial conditions of equation (1.48) we calculate for the constant c_3 and c_4:

$$G'(\xi) = \frac{-1}{\varepsilon} \quad \text{and} \quad G(\xi) = 0$$

$$G(\xi) = c_3\xi^{m_1} + c_4\xi^{m_2} = 0$$
$$G'(\xi) = c_3 m_1 \xi^{m_1-1} + c_4 m_2 \xi^{m_2-1} = \frac{-1}{\varepsilon}$$

$$c_4 = -c_3\xi^{m_1-m_2} \tag{1.53}$$

$$c_3 m_1 \xi^{m_1-1} + c_4 m_2 \xi^{m_2-1} = \frac{-1}{\varepsilon} \tag{1.54}$$

Plugging eq. (1.53) into eq. (1.54) we get:

$$c_3 m_1 \xi^{m_1-1} - c_3 m_2 \xi^{m_1-m_2} \xi^{m_2-1} = \frac{-1}{\varepsilon}$$

$$c_3 \left(m_1 \xi^{m_1-1} - m_2 \xi^{m_1-1} \right) = \frac{-1}{\varepsilon}$$

$$c_3 = \frac{-\xi^{1-m_1}}{\varepsilon(m_1 - m_2)} \tag{1.55}$$

Substituting eq. (1.55) into (1.53) we get:

$$c_4 = -\frac{-\xi^{1-m_1}}{\varepsilon(m_1 - m_2)} \xi^{m_1-m_2}$$

$$c_4 = \frac{\xi^{1-m_2}}{\varepsilon(m_1 - m_2)} \tag{1.56}$$

Substituting equations (1.55) and (1.56) into eq. (1.52) we get:

$$G(x) = \frac{-\xi^{1-m_1}}{\varepsilon(m_1 - m_2)} x^{m_1} + \frac{\xi^{1-m_2}}{\varepsilon(m_1 - m_2)} x^{m_2}$$

$$G(x) = \frac{1}{\varepsilon(m_1 - m_2)} \left(\xi^{1-m_2} x^{m_2} - \xi^{1-m_1} x^{m_1} \right) \tag{1.57}$$

Plugging eq. (1.57) into eq. (1.46) we get:

$$y_p(x) = \frac{1}{\varepsilon(m_1 - m_2)} \left(\xi^{1-m_2} x^{m_2} - \xi^{1-m_1} x^{m_1} \right) H(x - \xi)$$

$$y_p(x) = \begin{cases} 0, \quad x < \xi < 1 \\ \dfrac{1}{\varepsilon(m_1 - m_2)} \left(\xi^{1-m_2} x^{m_2} - \xi^{1-m_1} x^{m_1} \right), 0 < \xi < x \end{cases} \tag{1.58}$$

is a particular solution for eq. (1.45).

Since the non-homogeneous part of eq. (1.45) is Dirac delta function, this particular solution is the same with the Green's function of eq. (1.37).

$$G(x,\xi)=\begin{cases}0, & x<\xi<1\\ \dfrac{1}{\varepsilon(m_1-m_2)}\left(\xi^{1-m_2}x^{m_2}-\xi^{1-m_1}x^{m_1}\right), & 0<\xi<x\end{cases} \tag{1.59}$$

is a Green function satisfying eq. (1.37)

$$\begin{aligned} y_{p'}(x) &= \int_0^1 G(x,\xi)f(\xi)d\xi \\ &= \int_0^x G(x,\xi)f(\xi)d\xi + \int_x^1 G(x,\xi)f(\xi)d\xi \end{aligned}$$

$$y_{p'}(x)=\begin{cases}0, & x<\xi<1\\ \displaystyle\int_0^x \frac{1}{\varepsilon(m_1-m_2)}\left(\xi^{1-m_2}x^{m_2}-\xi^{1-m_1}x^{m_1}\right)f(\xi)d\xi, & 0<\xi<x\end{cases} \tag{1.60}$$

$$\begin{aligned} y(x) &= y_h(x)+y_{p'}(x) \\ &= c_1x^{m_1}+c_2x^{m_2}+\begin{cases}0, & x<\xi<1\\ \displaystyle\int_0^x \frac{1}{\varepsilon(m_1-m_2)}\left(\xi^{1-m_2}x^{m_2}-\xi^{1-m_1}x^{m_1}\right)f(\xi)d\xi, & 0<\xi<x\end{cases} \end{aligned} \tag{1.61}$$

In order to apply the given boundary conditions in eq. (1.2) we can consider the following cases.

1. If $m_1, m_2>0$, then $\alpha=0$. In this case we have infinite many solutions.
2. If $m_1, m_2<0$, then $\alpha=0$. In this case we have trivial solution for homogeneous part.
3. If $m_1>0$ and $m_2<0$, then $\alpha=0$. In this case we have unique solution.

Since we are interested in unique solution, let $m_1>0$ and $m_2<0$ then $\alpha=0$. This holds only if $c_2=0$

$$y(1)=c_1+c_2+\frac{1}{\varepsilon(m_1-m_2)}\int_0^1\left(\xi^{1-m_2}-\xi^{1-m_1}\right)f(\xi)d\xi=\beta$$

$$c_1 + 0 + \frac{1}{\varepsilon(m_1 - m_2)} \int_0^1 \left(\xi^{1-m_2} - \xi^{1-m_1} \right) f(\xi) d\xi = \beta$$

$$c_1 = \beta - \frac{1}{\varepsilon(m_1 - m_2)} \int_0^1 \left(\xi^{1-m_2} - \xi^{1-m_1} \right) f(\xi) d\xi$$

$$y(x) = c_1 x^{m_1} + \begin{cases} 0, & x < \xi < 1 \\ \int_0^x \frac{1}{\varepsilon(m_1 - m_2)} \left(\xi^{1-m_2} x^{m_2} - \xi^{1-m_1} x^{m_1} \right) f(\xi) d\xi, & 0 < \xi < x \end{cases}$$

$$c_1 = \beta - \frac{1}{\varepsilon(m_1 - m_2)} \int_0^1 \left(\xi^{1-m_2} - \xi^{1-m_1} \right) f(\xi) d\xi \tag{1.62}$$

As a result, a solution to eq. (1.37) which is reduced from eq. (1.1) and with homogeneous solution of eq. (1.42) is given by eq. (1.62)

4.3 Numerical Examples

Example 1: Consider singularly perturbed two point boundary value problem (Kanwal, 1983)

$$-\varepsilon y''(x) + \frac{4}{x} y'(x) = \delta(x - \xi),\ 0 < x < 1,\ 0 < \xi \ll 1$$
$$y(0) = 0 \ \text{ and } \ \mathrm{y}(1) = 1$$

Solution: $-\varepsilon y''(x) + \frac{4}{x} y'(x) = \delta(x - \xi)$

To find the Homogeneous Solution,

$$-\varepsilon y''(x) + \frac{4}{x} y'(x) = 0$$

$$y''(x) - \frac{4}{\varepsilon x} y'(x) = 0$$

$$\frac{y''(x)}{y'(x)} = \frac{4}{\varepsilon x}$$

$$\int \frac{y''(x)}{y'(x)} dx = \int \frac{4}{\varepsilon x} dx$$

$$\ln y'(x) = \frac{4}{\varepsilon}\ln x + \ln m$$

$$\ln y'(x) = \ln m x^{\frac{4}{\varepsilon}}$$

$$y'(x) = m x^{\frac{4}{\varepsilon}}$$

$$y_h(x) = m\int x^{\frac{4}{\varepsilon}}dx$$

$y_h(x) = m\dfrac{\varepsilon}{4+\varepsilon}x^{\frac{4+\varepsilon}{\varepsilon}} + d$, where m and d are arbitrary constants.

To find the particular solution,

Assume $y_p(x) = G(x)H(x-\xi)$ the particular solution,

where $H(x-\xi)$ is the Heaviside function, $G(x)$ is unknown function.

$$\begin{aligned} y'_p(x) &= G'(x)H(x-\xi) + G(x)H'(x-\xi) \\ &= G'(x)H(x-\xi) + G(\xi)\delta(x-\xi) \\ y''_p(x) &= G''(x)H(x-\xi) + G'(x)H'(x-\xi) + G(\xi)\delta'(x-\xi) \\ &= G''(x)H(x-\xi) + G'(\xi)\delta(x-\xi) + G(\xi)\delta'(x-\xi) \end{aligned}$$

Plugging $y_p'(x)$ and $y_p''(x)$ into the problem yields:

$$-\varepsilon[G''(x)H(x-\xi) + G'(\xi)\delta(x-\xi) + G(\xi)\delta'(x-\xi)] + \frac{4}{x}[G'(x)H(x-\xi)$$
$$+G(\xi)\delta(x-\xi)] = \delta(x-\xi)$$
$$[-\varepsilon G''(x) + \frac{4}{x}G'(x)]H(x-\xi) + [-\varepsilon G'(\xi) + \frac{4}{\xi}G(\xi)]\delta(x-\xi) - \varepsilon G(\xi)\delta'(x-\xi) = \delta(x-\xi)$$

Comparing the corresponding coefficients:

$$\begin{cases} -\varepsilon G''(x) + \dfrac{4}{x} G'(x) = 0 \\ -\varepsilon G'(\xi) + \dfrac{4}{\xi} G(\xi) = 1 \\ -\varepsilon G(\xi) = 0 \end{cases}$$

$$\begin{cases} G''(x) - \dfrac{4}{\varepsilon x} G'(x) = 0 \\ G'(\xi) = \dfrac{-1}{\varepsilon} \\ G(\xi) = 0 \end{cases}$$

Then, solving $\quad G''(x) - \dfrac{4}{\varepsilon x} G(x) = 0$

$$\begin{aligned} &G''(x) - \frac{4}{\varepsilon x} G'(x) = 0 \\ &\frac{G''(x)}{G'(x)} = \frac{4}{\varepsilon x} \\ &\int \frac{G''(x)}{G'(x)} dx = \int \frac{4}{\varepsilon x} dx \\ &\ln G'(x)) = \frac{4}{\varepsilon} \ln x + \ln A \quad , \end{aligned}$$

where A is arbitrary constant.

$$\begin{aligned} \ln(G'(x)) &= \ln(x^{\frac{4}{\varepsilon}}) + \ln A \\ &= \ln(Ax^{\frac{4}{\varepsilon}}) \\ G'(x) \quad &= Ax^{\frac{4}{\varepsilon}} \end{aligned}$$

$$\begin{aligned} G(x) &= A \int x^{\frac{4}{\varepsilon}} dx + c \\ G(x) &= \frac{\varepsilon A}{4+\varepsilon} x^{\frac{4+\varepsilon}{\varepsilon}} + c \end{aligned}$$

Applying the initial conditions $G(\xi)=0$ and $G'(\xi)=\frac{-1}{\xi}$

$$\frac{\varepsilon A}{4+\varepsilon}\xi^{\frac{4+\varepsilon}{\varepsilon}}+c=0 \quad \text{and} \quad A\xi^{\frac{4}{\varepsilon}}=\frac{-1}{\varepsilon}$$

$$c=\frac{-\varepsilon A}{4+\varepsilon}\xi^{\frac{4+\varepsilon}{\varepsilon}} \quad \text{and} \quad A=\frac{-\xi^{\frac{-4}{\varepsilon}}}{\varepsilon}$$

$$c=\frac{-\varepsilon\,\xi^{\frac{4+\varepsilon}{\varepsilon}}(-\xi^{\frac{-4}{\varepsilon}})}{(4+\varepsilon)\varepsilon}$$

$$c=\frac{\xi}{4+\varepsilon}$$

$$G(x)=\left(\frac{-\xi^{\frac{-4}{\varepsilon}}}{\varepsilon}\right)\frac{\varepsilon}{4+\varepsilon}x^{\frac{4+\varepsilon}{\varepsilon}}+\frac{\xi}{4+\varepsilon}$$

$$G(x)=\frac{-\xi^{\frac{-4}{\varepsilon}}}{4+\varepsilon}x^{\frac{4+\varepsilon}{\varepsilon}}+\frac{\xi}{4+\varepsilon}$$

$$G(x)=\frac{1}{4+\varepsilon}\left(\xi-\xi^{\frac{-4}{\varepsilon}}x^{\frac{4+\varepsilon}{\varepsilon}}\right)$$

$$y_p(x)=\frac{1}{4+\varepsilon}\left(\xi-\xi^{\frac{-4}{\varepsilon}}x^{\frac{4+\varepsilon}{\varepsilon}}\right)H(x-\xi)$$

$$y_p(x)=\begin{cases}0, & x<\xi<1\\ \frac{1}{4+\varepsilon}\left(\xi-\xi^{\frac{-4}{\varepsilon}}x^{\frac{4+\varepsilon}{\varepsilon}}\right), & 0<\xi<x\end{cases}$$ is the particular solution.

Hence, the general solution is:

$$y(x) = m\frac{\varepsilon}{4+\varepsilon}x^{\frac{4+\varepsilon}{\varepsilon}} + d + \begin{cases} 0, & x < \xi < 1 \\ \dfrac{1}{4+\varepsilon}\left(\xi - \xi^{\frac{-4}{\varepsilon}}x^{\frac{4+\varepsilon}{\varepsilon}}\right), & 0 < \xi < x \end{cases}$$

Using the boundary conditions of the given problem we solve for the constants.

$$y(0) = 0 + d + 0 = 0$$

$$d = 0$$

$$y(1) = m\frac{\varepsilon}{4+\varepsilon} + d + \frac{\xi - \xi^{\frac{-4}{\varepsilon}}}{4+\varepsilon} = 1$$

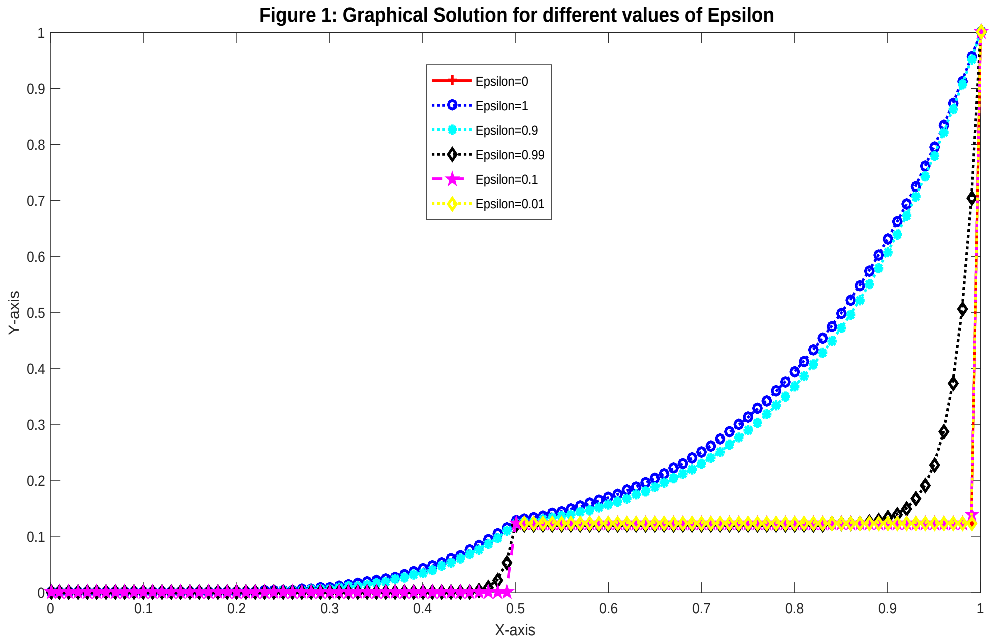
$$m\frac{\varepsilon}{4+\varepsilon} + \frac{\xi - \xi^{\frac{-4}{\varepsilon}}}{4+\varepsilon} = 1$$

$$m = \frac{4+\varepsilon}{\varepsilon} + \frac{\xi^{\frac{-4}{\varepsilon}} - \xi}{\varepsilon}$$

Hence, the general solution of the problem is:

$$y(x) = \left(\frac{4+\varepsilon}{\varepsilon} + \frac{\xi^{\frac{-4}{\varepsilon}} - \xi}{\varepsilon}\right)\frac{\varepsilon}{4+\varepsilon}x^{\frac{4+\varepsilon}{\varepsilon}} + \begin{cases} 0, & x < \xi < 1 \\ \dfrac{1}{4+\varepsilon}\left(\xi - \xi^{\frac{-4}{\varepsilon}}x^{\frac{4+\varepsilon}{\varepsilon}}\right), & 0 < \xi < x \end{cases}$$

$$= \left(1 + \frac{\xi^{\frac{-4}{\varepsilon}} - \xi}{4+\varepsilon}\right)x^{\frac{4+\varepsilon}{\varepsilon}} + \begin{cases} 0, & x < \xi < 1 \\ \dfrac{1}{4+\varepsilon}\left(\xi - \xi^{\frac{-4}{\varepsilon}}x^{\frac{4+\varepsilon}{\varepsilon}}\right), & 0 < \xi < x \end{cases}$$



Example 2: Consider the singularly perturbed two point boundary value problem (Li, 2008)

$$-\varepsilon y''(x) - \frac{1}{x} y'(x) - \frac{1}{x^2} y(x) = 2\varepsilon + 3 - \frac{2}{x}, \quad 0 < x < 1$$
$$y(0) = 0 = y(1)$$

Solution:

$$k = -1,\ b(x) = \frac{-1}{x^2},\ p = -1, \alpha = 0 = \beta$$

$$\begin{cases} m_1 = \dfrac{k + \varepsilon + \sqrt{(k+\varepsilon)^2 + 4\varepsilon p}}{2\varepsilon} = \dfrac{\varepsilon - 1 + \sqrt{(\varepsilon-1)^2 - 4\varepsilon}}{2\varepsilon} \\ m_2 = \dfrac{k + \varepsilon - \sqrt{(k+\varepsilon)^2 + 4\varepsilon p}}{2\varepsilon} = \dfrac{\varepsilon - 1 - \sqrt{(\varepsilon-1)^2 - 4\varepsilon}}{2\varepsilon} \end{cases}$$

$$y(x) = c_1 x^{m_1} + \int_0^x \frac{1}{\varepsilon(m_1 - m_2)} \left(\xi^{1-m_2} x^{m_2} - \xi^{1-m_1} x^{m_1}\right)\left(2\varepsilon + 3 - \frac{2}{\xi}\right)$$
$$c_1 = \beta - \frac{1}{\varepsilon(m_1 - m_2)} \int_0^1 \left(\xi^{1-m_2} - \xi^{1-m_1}\right) f(\xi) d\xi$$

$$\int_0^x \left(\xi^{1-m_2}x^{m_2} - \xi^{1-m_1}x^{m_1}\right)\left(2\varepsilon+3-\frac{2}{\xi}\right)d\xi$$

$$=\int_0^x \left(\xi^{1-m_2}x^{m_2}\right)(2\varepsilon+3)\,d\xi - \int_0^x \left(\xi^{1-m_2}x^{m_2}\right)\left(\frac{2}{\xi}\right)d\xi - \int_0^x \left(\xi^{1-m_1}x^{m_1}\right)(2\varepsilon+3)\,d\xi + \int_0^x \left(\xi^{1-m_1}x^{m_1}\right)\left(\frac{2}{\xi}\right)d\xi$$

Now integrating each,

$$\int_0^x \left(\xi^{1-m_2}\right)(2\varepsilon+3)x^{m_2}\,d\xi = (2\varepsilon+3)x^{m_2}\left[\frac{\xi^{2-m_2}}{2-m_2}\right]_{\xi=0}^{\xi=x} = (2\varepsilon+3)x^{m_2}\,\frac{x^{2-m_2}}{2-m_2} = \frac{(2\varepsilon+3)}{2-m_2}x^2$$

$$-\int_0^x \left(\xi^{1-m_2}x^{m_2}\right)2d\xi = -2x^{m_2}\int_0^x \left(\xi^{-m_2}\right)d\xi = -2x^{m_2}\left[\frac{\xi^{1-m_2}}{1-m_2}\right]_{\xi=0}^{\xi=x} = \frac{-2x}{1-m_2}$$

$$-\int_0^x \left(\xi^{1-m_1}x^{m_1}\right)(2\varepsilon+3)\,d\xi = -(2\varepsilon+3)x^{m_1}\left[\frac{\xi^{2-m_1}}{2-m_1}\right]_{\xi=0}^{\xi=x} = \frac{-(2\varepsilon+3)}{2-m_1}x^2$$

$$\int_0^x 2\left(\xi^{1-m_1}x^{m_1-1}\right)d\xi = 2x^{m_1}\int_0^x \left(\xi^{-m_1}\right)d\xi = 2x^{m_1}\left[\frac{\xi^{1-m_1}}{1-m_1}\right]_{\xi=0}^{\xi=x} = 2x^{m_1}\left[\frac{x^{1-m_1}}{1-m_1}\right] = \frac{2x}{1-m_1}$$

We have,

$$y(x) = c_1x^{m_1} + \frac{1}{\varepsilon(m_1-m_2)}\left[\begin{array}{l}\int_0^x \left(\xi^{1-m_2}x^{m_2}\right)(2\varepsilon+3)\,d\xi - \int_0^x \left(\xi^{1-m_2}x^{m_2}\right)\left(\frac{2}{x}\right)d\xi \\ -\int_0^x \left(\xi^{1-m_1}x^{m_1}\right)(2\varepsilon+3)\,d\xi + \int_0^x \left(\xi^{1-m_1}x^{m_1}\right)\left(\frac{2}{x}\right)d\xi\end{array}\right]$$

$$= c_1x^{m_1} + \frac{1}{\varepsilon(m_1-m_2)}\left[\frac{(2\varepsilon+3)}{2-m_2}x^2 - \frac{2x}{1-m_2} - \frac{(2\varepsilon+3)}{2-m_1}x^2 + \frac{2x}{1-m_1}\right]$$

Up on simplification we obtain,

$$y(x) = c_1x^{m_1} + \frac{2\varepsilon+3}{\varepsilon}\left(\frac{-1}{(2-m_2)(2-m_1)}\right)x^2 + \frac{2}{\varepsilon}\left(\frac{1}{(1-m_1)(1-m_2)}\right)x$$

Further,

$$
\begin{aligned}
(2-m_2)(2-m_1) &= \left(2-\frac{\varepsilon-1+\sqrt{(\varepsilon-1)^2-4\varepsilon}}{2\varepsilon}\right)\left(2-\frac{\varepsilon-1-\sqrt{(\varepsilon-1)^2-4\varepsilon}}{2\varepsilon}\right)\\
&= \left(\frac{4\varepsilon-\varepsilon+1-\sqrt{(\varepsilon-1)^2-4\varepsilon}}{2\varepsilon}\right)\left(\frac{4\varepsilon-\varepsilon+1+\sqrt{(\varepsilon-1)^2-4\varepsilon}}{2\varepsilon}\right)\\
&= \left(\frac{3\varepsilon+1-\sqrt{(\varepsilon-1)^2-4\varepsilon}}{2\varepsilon}\right)\left(\frac{3\varepsilon+1+\sqrt{(\varepsilon-1)^2-4\varepsilon}}{2\varepsilon}\right)\\
&= \left(\frac{9\varepsilon^2+6\varepsilon+1-\left[(\varepsilon-1)^2-4\varepsilon\right]}{4\varepsilon^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{9\varepsilon^2+6\varepsilon+1-\left[\varepsilon^2-2\varepsilon+1-4\varepsilon\right]}{4\varepsilon^2}\right)\\
&= \left(\frac{9\varepsilon^2+6\varepsilon+1-\left[\varepsilon^2-6\varepsilon+1\right]}{4\varepsilon^2}\right)\\
&= \left(\frac{9\varepsilon^2+6\varepsilon+1-\varepsilon^2+6\varepsilon-1}{4\varepsilon^2}\right)\\
&= \left(\frac{8\varepsilon^2+12\varepsilon}{4\varepsilon^2}\right)\\
&= \frac{2\varepsilon+3}{\varepsilon},
\end{aligned}
$$

$$
\begin{aligned}
(1-m_1)(1-m_2) &= \left(1-\frac{\varepsilon-1+\sqrt{(\varepsilon-1)^2-4\varepsilon}}{2\varepsilon}\right)\left(1-\frac{\varepsilon-1-\sqrt{(\varepsilon-1)^2-4\varepsilon}}{2\varepsilon}\right)\\
&= \left(\frac{2\varepsilon-\varepsilon+1-\sqrt{(\varepsilon-1)^2-4\varepsilon}}{2\varepsilon}\right)\left(\frac{2\varepsilon-\varepsilon+1-\sqrt{(\varepsilon-1)^2-4\varepsilon}}{2\varepsilon}\right)\\
&= \left(\frac{\varepsilon+1-\sqrt{(\varepsilon-1)^2-4\varepsilon}}{2\varepsilon}\right)\left(\frac{\varepsilon+1-\sqrt{(\varepsilon-1)^2-4\varepsilon}}{2\varepsilon}\right)\\
&= \left(\frac{\varepsilon^2+2\varepsilon+1-\left[\varepsilon^2-2\varepsilon+1-4\varepsilon\right]}{4\varepsilon^2}\right)\\
&= \left(\frac{\varepsilon^2+2\varepsilon+1-\left[\varepsilon^2-6\varepsilon+1\right]}{4\varepsilon^2}\right)\\
&= \left(\frac{8\varepsilon}{4\varepsilon^2}\right)\\
&= \frac{2}{\varepsilon}
\end{aligned}
$$

From Eq. (1.62),

$$
c_1 = \frac{-1}{\varepsilon(m_1-m_2)}\int_0^1\left(\xi^{1-m_2}-\xi^{1-m_1}\right)\left(2\varepsilon+3-\frac{2}{\xi}\right)d\xi
$$

$$
\begin{aligned}
&\int_0^1\left(\xi^{1-m_2}-\xi^{1-m_1}\right)\left(2\varepsilon+3-\frac{2}{\xi}\right)d\xi\\
&= \int_0^1\left(\xi^{1-m_2}\right)(2\varepsilon+3)\,d\xi-\int_0^1\left(\xi^{1-m_2}\right)\left(\frac{2}{\xi}\right)d\xi-\int_0^1\left(\xi^{1-m_1}\right)(2\varepsilon+3)\,d\xi+\int_0^1\left(\xi^{1-m_1}\right)\left(\frac{2}{\xi}\right)d\xi\\
&= (2\varepsilon+3)\int_0^1\left(\xi^{1-m_2}\right)d\xi-2\int_0^1\left(\xi^{-m_2}\right)d\xi-(2\varepsilon+3)\int_0^1\left(\xi^{1-m_1}\right)d\xi+2\int_0^1\left(\xi^{-m_1}\right)d\xi\\
&= \frac{2\varepsilon+3}{2-m_2}-\frac{2}{1-m_2}-\frac{2\varepsilon+3}{2-m_1}+\frac{2}{1-m_1}\\
&= (2\varepsilon+3)\left[\frac{1}{2-m_2}-\frac{1}{2-m_1}\right]+2\left[\frac{1}{1-m_1}-\frac{1}{1-m_2}\right]\\
&= (2\varepsilon+3)\left[\frac{m_2-m_1}{(2-m_2)(2-m_1)}\right]+2\left[\frac{m_1-m_2}{(1-m_1)(1-m_2)}\right]\\
&= 0
\end{aligned}
$$

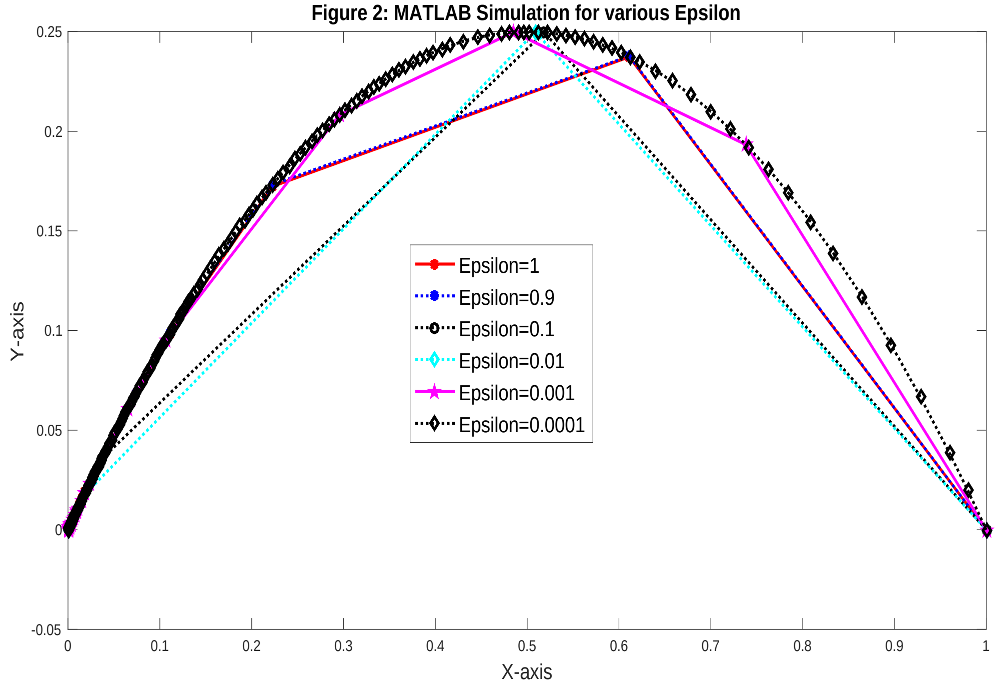
Therefore,

$$y(x) = c_1 x^{m_1} + \frac{2\varepsilon+3}{\varepsilon}\left(\frac{-1}{(2-m_2)(2-m_1)}\right)x^2 + \frac{2}{\varepsilon}\left(\frac{1}{(1-m_1)(1-m_2)}\right)x$$

$$\begin{aligned} y(x = c_1 x^{m_1} &+ \frac{2\varepsilon+3}{\varepsilon}\left(\frac{-1}{(2-m_2)(2-m_1)}\right)x^2 + \frac{2}{\varepsilon}\left(\frac{1}{(1-m_1)(1-m_2)}\right)x \\ &= 0 + \frac{2\varepsilon+3}{\varepsilon}\left(\frac{-1}{\frac{2\varepsilon+3}{\varepsilon}}\right)x^2 + \frac{2}{\varepsilon}\left(\frac{1}{\frac{2}{\varepsilon}}\right)x \\ &= -x^2 + x \\ &= x - x^2 \end{aligned}$$

This exact solution agrees with the existing exact solution in the literature.

Figure 2: MATLAB Simulation for various Epsilon



Example 3: Consider the singularly perturbed two point boundary value problem (Feyisa Edosa and Gemechis File, 2017)

$$-\varepsilon y''(x)+y(x)=1-3x\cos\pi x,\ 0<x<1$$
$$y(0)=0=y(1)$$

Solution:

The problem is of the form in case 1with $k=0,\ b(x)=p=1,\ \alpha=0=\beta$ and $f(x)=1-3x\cos\pi x$

Therefore, the solution of the problem is given by equation (1.21) .

$$y(x)=c_1e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}x}+c_2e^{\frac{-\sqrt{p}}{\sqrt{\varepsilon}}x}+\begin{cases}0, & x<\xi<1\\ \int\left[\frac{\sqrt{\varepsilon}}{2\varepsilon\sqrt{p}}e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}(\xi-x)}-\frac{\sqrt{\varepsilon}}{2\varepsilon\sqrt{p}}e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}(x-\xi)}\right][1-3\xi\cos\pi\xi]d\xi, & 0<\xi<x\end{cases}$$

$$c_1=\left(\frac{e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}}}{e^{\frac{2\sqrt{p}}{\sqrt{\varepsilon}}}-1}\right)\left[\beta-\alpha e^{\frac{-\sqrt{p}}{\sqrt{\varepsilon}}}-\frac{\sqrt{\varepsilon}}{2\varepsilon\sqrt{p}}\int_0^1\left(e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}(\xi-1)}-e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}(1-\xi)}\right)f(\xi)d\xi\right]$$

$$c_2=\alpha-\left(\frac{e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}}}{e^{\frac{2\sqrt{p}}{\sqrt{\varepsilon}}}-1}\right)\left[\beta-\alpha e^{\frac{-\sqrt{p}}{\sqrt{\varepsilon}}}-\frac{\sqrt{\varepsilon}}{2\varepsilon\sqrt{p}}\int_0^1\left(e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}(\xi-1)}-e^{\frac{\sqrt{p}}{\sqrt{\varepsilon}}(1-\xi)}\right)f(\xi)d\xi\right]$$

It follows that

$$y(x)=c_1e^{\frac{x}{\sqrt{\varepsilon}}}+c_2e^{\frac{-x}{\sqrt{\varepsilon}}}+\begin{cases}0, & x<\xi<1\\ \int_0^x\left[\frac{\sqrt{\varepsilon}}{2\varepsilon}e^{\frac{\xi-x}{\sqrt{\varepsilon}}}-\frac{\sqrt{\varepsilon}}{2\varepsilon}e^{\frac{x-\xi}{\sqrt{\varepsilon}}}\right][1-3\xi\cos\pi\xi]d\xi, & 0<\xi<x\end{cases}$$

$$c_1 = \left(\frac{e^{\frac{1}{\sqrt{\varepsilon}}}}{e^{\frac{2}{\sqrt{\varepsilon}}} - 1} \right) \left[-\frac{\sqrt{\varepsilon}}{2\varepsilon} \int_0^1 \left(e^{\frac{1}{\sqrt{\varepsilon}}(\xi-1)} - e^{\frac{1}{\sqrt{\varepsilon}}(1-\xi)} \right) (1 - 3\xi \cos \pi\xi)\, d\xi \right]$$

$$= -\left(\frac{e^{\frac{1}{\sqrt{\varepsilon}}}}{e^{\frac{2}{\sqrt{\varepsilon}}} - 1} \right) \left[\frac{\sqrt{\varepsilon}}{2\varepsilon} \int_0^1 \left(e^{\frac{\xi-1}{\sqrt{\varepsilon}}} - e^{\frac{1-\xi}{\sqrt{\varepsilon}}} \right) (1 - 3\xi \cos \pi\xi)\, d\xi \right]$$

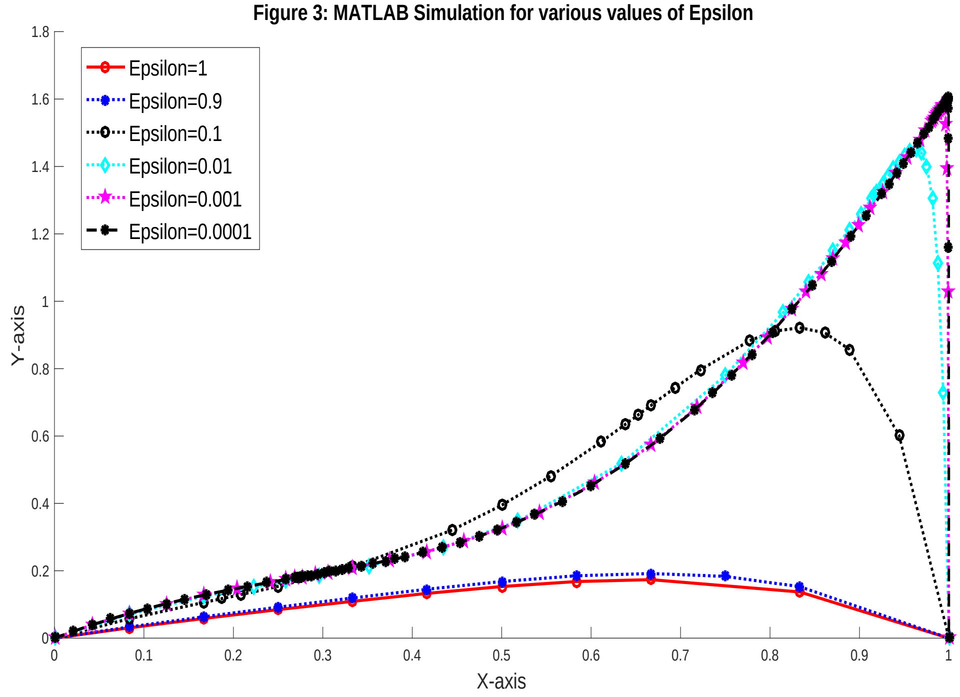
$$c_2 = -\left(\frac{e^{\frac{1}{\sqrt{\varepsilon}}}}{e^{\frac{2}{\sqrt{\varepsilon}}} - 1} \right) \left[-\frac{\sqrt{\varepsilon}}{2\varepsilon} \int_0^1 \left(e^{\frac{1}{\sqrt{\varepsilon}}(\xi-1)} - e^{\frac{1}{\sqrt{\varepsilon}}(1-\xi)} \right) (1 - 3\xi \cos \pi\xi)\, d\xi \right]$$

$$= \left(\frac{e^{\frac{1}{\sqrt{\varepsilon}}}}{e^{\frac{2}{\sqrt{\varepsilon}}} - 1} \right) \left[\frac{\sqrt{\varepsilon}}{2\varepsilon} \int_0^1 \left(e^{\frac{\xi-1}{\sqrt{\varepsilon}}} - e^{\frac{1-\xi}{\sqrt{\varepsilon}}} \right) (1 - 3\xi \cos \pi\xi)\, d\xi \right]$$

From these two equations we see that c_1 and c_2 are equal in magnitude and opposite in sign.

That is, $c_2 = -c_1$.

Figure 3: MATLAB Simulation for various values of Epsilon



4.4 Discussion

Figure 1 indicates that, the problem has right layer because there is rapid increment on the behavior of the solution on the right due to perturbation parameter. It also indicates that, all the solution curve occurs when the values of the perturbation parameter lies between one and zero. Figure 2 also depicts that the MATLAB simulation for various values of perturbation parameter. The shape of the graph agrees with the exact solution that obtained by the method developed in this thesis. Figure 3 also support this fact. Therefore, the method developed in this thesis agrees with the MATLAB simulation.

CHAPTER FIVE

5. CONCLUSION AND FUTURE WORK

5.1 Conclusion

In this thesis, distributional solution of singularly perturbed two point boundary value problem is presented. Firstly, some important terminologies like distribution, test function, Heaviside function, and Dirac delta function and its properties and Green's function and its properties are briefly explained. Secondly, homogeneous solution to singularly perturbed two point boundary value problem under consideration is described. Thirdly, Green's function was constructed in the sense of distribution to get the particular solution using convolution or without applying convolution. Fourthly, the general solution was set as the sum of homogeneous solution and particular solution. Then, applying the two boundary conditions, the two arbitrary constants were fixed in order to get solution free from arbitrary constants. Fifthly, in order to verify the applicability of the method three numerical examples were considered and solved. Finally, MATLAB simulation was implemented for various values of small perturbation parameter in order to see the effect of this parameter and the nature of the layer created due to this parameter. Using the developed method, problems with known exact solution is solved and it agrees with existing exact solution. Furthermore, using the developed method problems with unknown exact solution is also solved. The result of study indicates that Green's function method is a powerful tool to solve singularly perturbed two point boundary value problem. Therefore, the method developed in this thesis is reliable and promising to treat other related problems whose exact solution is unknown.

5.2 Future work

Based on the result of the study, the following points are open problems for others researchers interested on this area. First, one can investigate the distributional solution to the problem by considering other cases. Next, one can introduce another technique which is easier and simpler than the technique introduced in this thesis to get the distributional solution to the problem. Furthermore, solution to the problem in terms of infinite series of delta function is also another open problem. Moreover, the researcher cordially invites other researchers working on numerical analysis to compare the approximate solution obtained by numerical method with the exact solution obtained in this thesis.

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