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Domain Decomposition Method for Solving Singular Perturbation Problems

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Abstract: In this paper, a domain decomposition method for solving singularly perturbed two-point boundary value problems is presented. By using a terminal point, the original problem is divided into inner and outer region problems. An implicit terminal boundary condition at the terminal point is determined. The outer region problem with the implicit boundary condition is solved and produces an explicit boundary condition for the inner region problem. Then, the modified inner region problem (using the stretching transformation) is solved as a two-point boundary value problem. We used fourth order stable central difference method to solve both the inner and outer region problems. The proposed method is iterative on the terminal point. To demonstrate the applicability of the method, we solved several linear and nonlinear singular perturbation problems.

Keywords: Singular perturbation problems; finite differences; terminal boundary condition; boundary layer.

1. Introduction

Singular perturbation problems containing a small parameter, ε multiplying to their highest derivative term arise in many fields such as, fluid mechanics, fluid dynamics, chemical reactor theory, elasticity etc and have received a significant amount of attention in past and recent years. The solution of these types of problems exhibits a multi scale characters. That is, there are a narrow region called boundary layer in which their solution changes rapidly and the outer region where solution changes smoothly. Thus, numerical treatment of such problems is not trivial because of the boundary layer behavior of their solutions. Pearson [16] was the first to attempt something like net adjustments in difference schemes while solving singular perturbation problems. His idea was to use a variable mesh width in applying finite difference scheme in the domain of interest. Besides, there are a wide variety of asymptotic expansion methods available for solving singular perturbation problems. However, it may be difficult to apply these asymptotic expansion methods as finding of the appropriate asymptotic expansions in the inner and outer regions is not routine exercises rather requires skill, insight, and experimentations. Moreover, the matching of the coefficients of the inner and outer solution expansions can also be a demanding process. Thus, the general motivation of this paper is to provide simpler and efficient computational techniques which helps to know the behavior of the solutions of the problems in the inner region, where the solution of the problem changes rapidly, of the boundary layer and to find the computational results at uniform mesh length. For detail discussion of

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solving singular perturbation problems by asymptotic expansion methods, one can refer to the books and high level monographs: O'Malley [13, 14], Nayfeh [11, 12], Cole and Kevorkian [4], Bender and Orszag [2], Eckhaus [5], Van Dyke [19], and Bellman [1].

Moreover, in the recent times many researchers have been trying to develop and present numerical methods for solving these problems. For instance, based on the asymptotic behavior of singular perturbation problems, Kadalbajoo and Patidar [7] and Kadalbajoo and Reddy [8, 9] have discussed numerical schemes for the solution of linear singularly perturbed two-point boundary value problems. Xie and et al [20] have presented a novel approach for solving parameterized singularly perturbed two-point boundary value problems. They have treated these problems by converting the original problem into non-singularly perturbed algebraic equation and a first order initial value problem by making use of the boundary layer correction technique and then easily apply conventional numerical method, Runge-Kutta method, to solve the initial value problems numerically. Geng [6] has presented reproducing kernel method (RKM) for solving a class of singularly perturbed boundary value problems by transforming the original problem in to a new boundary value problem whose solution does not change rapidly. RKM has the advantage that it can produce smooth approximate solutions, but it is difficult to apply the method for singularly perturbed boundary value problems without transforming using appropriate transformation. Padmaja and et al [15] have presented a nonstandard explicit method involving the reduction of order for solving singularly perturbed two point boundary value problems. The original problem is approximated by a pair of initial value problems. In order to know the behavior of the solution of the problems in the boundary layer region, these researchers solved the first initial value problem as outer region problem whose solution can be required in the second initial value problem which they considered it as an inner region problem and is modified using the stretching transformation. Prasad and Reddy [17] applied the Differential Quadrature Method (DQM) for finding the numerical solution of singularly perturbed two point boundary value problems with mixed condition. DQM is based on the approximation of the derivatives of the unknown functions involved in the differential equations at the mesh point of the solution domain and is an efficient discretization technique in solving boundary value problems using a considerably small number of grid points.

In the present paper, the domain decomposition method for singularly perturbed two point boundary value problems with the boundary layer at the left end of the interval is presented. Based on the decomposition of the domain into inner and outer regions, the method consists of the following steps: (i) the original problem is divided in to two problems, inner region and outer region problems; (ii) a terminal boundary condition in the implicit form is determined; (iii) then, the outer region problem with the implicit boundary condition is solved as a two-point boundary-value problem, and an explicit terminal boundary condition is obtained; (iv) the inner region problem is modified and solved as a two-point boundary value problem using the explicit terminal boundary condition. Finally, we combine the solutions of both the inner region and outer region problems to get the approximate solution of the original problem.

The present method is iterative on the terminal point. We repeat the process (numerical scheme) for various choices of the terminal point, until the solution profiles do not differ materially from iteration to iteration.

2. Description of the method

Consider a linear singularly perturbed two-point boundary value problem of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \quad 0 \le x \le 1 \tag{1}$$

with $y(0) = \alpha$ (2a)

and $y(1) = \beta$; (2b)

where ε is a small positive parameter $(0 < \varepsilon << 1)$ and α, β are known constants. We assume that *a(x), b(x)* and *f(x)* are sufficiently continuously differentiable functions in [0, 1]. Furthermore, we assume that $a(x) \ge M > 0$ throughout the interval [0, 1], where M is some positive constant. Under these assumptions, (1) has a unique solution $y(x)$ which in general, displays a boundary layer of width O(ε) at x=0 for small values of ε. (O'Malley [13] and Nayfeh [12]).

As mentioned above, we divide the original problem in to two regions: an inner region and outer region problem. Let x_p $(0 < x_p << 1)$ be the terminal point or width or thickness of the boundary layer (inner region), then the inner and outer region problems are defined on $0 \le x \le x_p$ and $x_p \le x \le 1$ respectively.

By using Taylor's expansion, we have

$$y(x - x_p) \approx y(x) - x_p y'(x) + \frac{{x_p}^2}{2} y''(x) \tag{3}$$

Using (3) in to (1), we get

$$2\varepsilon x_p y'(x) - 2\varepsilon y(x) + 2\varepsilon y(x - x_p) + x_p^2 a(x)y'(x) + x_p^2 b(x)y(x) = x_p^2 f(x) \tag{4}$$

Evaluating (4) at $x = x_p$, we get

$$c_1 y(x_p) + c_2 y'(x_p) = c_3 \tag{5}$$

where $c_1 = {x_p}^2 b(x_p) - 2\varepsilon$ (6a)

$$c_2 = x_p(2\varepsilon + x_p a(x_p)) \tag{6b}$$

$$c_3 = x_p^2 f(x_p) - 2\varepsilon y(0) \tag{6c}$$

which is in implicit form and is taken as the terminal boundary condition at $x = x_p$ (the terminal point).

Using the terminal boundary condition (5), which is in implicit form, we solve the outer region problem as a two point boundary value problem

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \quad x_p \le x \le 1 \tag{7}$$

with $c_1 y(x_p) + c_2 y'(x_p) = c_3$ (8a)

and $y(1) = \beta$ (8b)

We solve this two point boundary value problem and get solution $y(x)$ over $[x_p, 1]$.

From the solution $y(x)$ of the outer region problem (7)-(8) on the interval $x_p \le x \le 1$ we get the value of $y(x_p)$ which is the explicit terminal boundary condition and let us denote it by

$y(x_p) = \gamma$.

In order to solve the inner region problem, since the terminal point of the inner region is common to both the inner and outer regions, we can formulate the inner problem as a two-point boundary-value problem

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \quad 0 \le x \le x_p \tag{9}$$

with $y(0) = \alpha$ (10a)

and $y(x_p) = \gamma$; (10b)

we choose the transformation

$$t = \frac{x}{\varepsilon} \tag{11}$$

to form a new differential equation for the inner region solution. By using (11) we transform equation (9) with

$$y(x) = y(t\varepsilon) = Y(t) \tag{12a}$$

$$y'(x) = \frac{y'(t\varepsilon)}{\varepsilon} = \frac{Y'(t)}{\varepsilon} \tag{12b}$$

$$y''(x) = \frac{y''(t\varepsilon)}{\varepsilon^2} = \frac{Y''(t)}{\varepsilon^2} \tag{12c}$$

$$a(x) = a(t\varepsilon) = A(t) \tag{12d}$$

$$b(x) = b(t\varepsilon) = B(t) \tag{12e}$$

$$f(x) = f(t\varepsilon) = F(t) \tag{12f}$$

to the new inner region problem of the form:

$$Y''(t) + A(t)Y'(t) + \varepsilon B(t)Y(t) = \varepsilon F(t), \quad 0 \le t \le t_p \tag{13}$$

with $Y(0) = \alpha$ (14a)

and $Y(t_p) = y(x_p) = \gamma$ (14b)

where $t_p = \dfrac{x_p}{\varepsilon}$. We solve this new inner region problem (13)-(14) to obtain the solutions over the interval $0 \le t \le t_p$.

To solve the two-point boundary value problems given in equations (7)-(8) (outer region problem) and (13)-(14) (inner region problem), we make use of fourth order stable central difference method (Choo and Schultz [3]). In fact, any standard analytical or numerical method can be used. Finally, we combine the solutions of both the inner region defined on $0 \le x \le x_p$ and outer region defined on $x_p \le x \le 1$ problems to get the approximate solution of the original problem (1)-(2) over the interval $0 \le x \le 1$.

We repeat the process (numerical scheme) for various choices of x_p (the terminal point), until the solution profiles do not differ materially from iteration to iteration. For computational point of view, we use an absolute error criterion, namely

$$\left|y^{m+1}(x) - y^{m}(x)\right| \le \sigma \quad 0 \le x \le x_p \tag{15}$$

Where $y^{m}(x)$ = the solution for the m^{th} iterate of x_p and σ = the prescribed tolerance bound.

3. Fourth order stable central difference method

To set up the difference equation of the outer region problem (7)-(8) we divide $[x_p, 1]$ into N equal parts, each of length h, $x_p = x_0 < x_1 < x_2 < ... < x_N = 1$. Then, we have $x_i = x_p + ih$; $i = 0, 1, 2, ..., N$. For simplicity let $a(x_i) = a_i$, $b(x_i) = b_i$, $f(x_i) = f_i$, $y(x_p) = y_0$, $y(x_i) = y_i$, $y(x_i + h) = y_{i+1}$, $y(x_i - h) = y_{i-1}$, $y'(x_i) = y_i'$, $y''(x_i) = y_i''$, etc. By Taylor expansion, we obtain the following central difference formulas for y'' and y' at x assuming that y has continuous fourth derivatives on [0, 1].

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{h^2}{12} y^{(4)}(\xi) + R_1 \tag{16}$$

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6} y'''(\eta) + R_2 \tag{17}$$

Where $R_1 = {-2h^4 y^{(6)}(\xi)}\big/{6!}$ and $R_2 = {-h^4 y^{(5)}(\eta)}\big/{5!}$ for $\xi, \eta \in [x_i - h, x_i + h]$.

Substituting (16) and (17) into (7) we can write the central difference approximation of (7) in the form that includes all the $O(h^2)$ error terms as follows:

$$\left[\frac{\varepsilon}{h^2} - \frac{a_i}{2h}\right] y_{i-1} + \left[b_i - \frac{2\varepsilon}{h^2}\right] y_i + \left[\frac{\varepsilon}{h^2} + \frac{a_i}{2h}\right] y_{i+1} - \frac{h^2}{12}\left[2a_i y_i''' + \varepsilon y_i^{(4)}\right] + R = f_i \tag{18}$$

Where $R = \varepsilon R_1 + a_i R_2$. Now, from (1) we have

$$\varepsilon y_i'' = -a_i y_i' - b_i y_i + f_i \tag{19}$$

Differentiating both sides of (19) we get

$$\varepsilon y_i''' = -\left(a_i y_i'' + a_i' y_i' + b_i' y_i + b_i y_i'\right) + f_i \tag{20}$$

$$y_i''' = -\frac{1}{\varepsilon}\left[a_i y_i'' + (a_i' + b_i) y_i' + b_i' y_i\right] + \frac{f_i}{\varepsilon} \tag{21}$$

Differentiating both sides of (20) again we have

$$\varepsilon y_i^{(4)} = -\left[a_i y_i''' + (2a_i' + b_i) y_i'' + (a_i'' + 2b_i') y_i' + b_i'' y_i\right] + f_i'' \tag{22}$$

By making use of (21) and (22) into (18) for y_i''' and $y_i^{(4)}$ we obtain

$$\left(\frac{\varepsilon}{h^2}-\frac{a_i}{2h}\right)y_{i-1}+\left(b_i-\frac{2\varepsilon}{h^2}\right)y_i+\left(\frac{\varepsilon}{h^2}+\frac{a_i}{2h}\right)y_{i+1}+\frac{h^2}{12}\left(\frac{a_i^{\,2}}{\varepsilon}+2a_i'+b_i\right)y_i''$$
$$+\frac{h^2}{12}\left(\frac{a_ia_i'}{\varepsilon}+\frac{a_ib_i}{\varepsilon}+a_i''+2b_i'\right)y_i'+\frac{h^2}{12}\left(\frac{a_ib_i'}{\varepsilon}+b_i''\right)y_i+R=f_i+\frac{h^2}{12}\left(\frac{a_if_i'}{\varepsilon}+f_i''\right) \tag{23}$$

Now, approximating the converted error term, which has a stabilizing effect, in (23) by using the central difference formulas (16) and (17) for y_i'' and y_i' we obtain

$$\left(\frac{\varepsilon}{h^2}-\frac{a_i}{2h}\right)y_{i-1}+\left(b_i-\frac{2\varepsilon}{h^2}\right)y_i+\left(\frac{\varepsilon}{h^2}+\frac{a_i}{2h}\right)y_{i+1}+\frac{1}{12}\left(\frac{a_i^{\,2}}{\varepsilon}+2a_i'+b_i\right)y_{i+1}$$
$$-\frac{1}{6}\left(\frac{a_i^{\,2}}{\varepsilon}+2a_i'+b_i\right)y_i+\frac{1}{12}\left(\frac{a_i^{\,2}}{\varepsilon}+2a_i'+b_i\right)y_{i-1}+\frac{h}{24}\left(\frac{a_ia_i'}{\varepsilon}+\frac{a_ib_i}{\varepsilon}+a_i''+2b_i'\right)y_{i+1}$$
$$-\frac{h}{24}\left(\frac{a_ia_i'}{\varepsilon}+\frac{a_ib_i}{\varepsilon}+a_i''+2b_i'\right)y_{i-1}+\frac{h^2}{12}\left(\frac{a_ib_i'}{\varepsilon}+b_i''\right)y_i=f_i+\frac{h^2}{12}\left(\frac{a_if_i'}{\varepsilon}+f_i''\right)+\widetilde{R} \tag{24}$$

where $\widetilde{R}=\left(\frac{a_i^{\,2}}{\varepsilon}+2a_i'+b_i\right)\frac{h^4}{144}y_i^{(4)}+\left(\frac{a_ia_i'}{\varepsilon}+\frac{a_ib_i}{\varepsilon}+a_i''+2b_i'\right)\frac{h^4}{72}y_i'''-R$ is the truncation error and $R=\varepsilon R_1+a_iR_2=O\left(h^4\right)$.

Rearranging (24) we obtain the three term recurrence relation

$$E_iy_{i-1}-F_iy_i+G_iy_{i+1}=H_i,\quad \mathrm{i}=0,1,\ldots,\mathrm{N}-1 \tag{25}$$

Where

$$E_i=\frac{\varepsilon}{h^2}-\frac{a_i}{2h}+\frac{1}{12}\left(\frac{a_i^{\,2}}{\varepsilon}+2a_i'+b_i\right)-\frac{h}{24}\left(\frac{a_ia_i'}{\varepsilon}+\frac{a_ib_i}{\varepsilon}+a_i''+2b_i'\right)$$
$$F_i=\frac{2\varepsilon}{h^2}-b_i+\frac{1}{6}\left(\frac{a_i^{\,2}}{\varepsilon}+2a_i'+b_i\right)-\frac{h^2}{12}\left(\frac{a_ib_i'}{\varepsilon}+b_i''\right)$$
$$G_i=\frac{\varepsilon}{h^2}+\frac{a_i}{2h}+\frac{1}{12}\left(\frac{a_i^{\,2}}{\varepsilon}+2a_i'+b_i\right)+\frac{h}{24}\left(\frac{a_ia_i'}{\varepsilon}+\frac{a_ib_i}{\varepsilon}+a_i''+2b_i'\right) \tag{26}$$
$$H_i=f_i+\frac{h^2}{12}\left(\frac{a_if_i'}{\varepsilon}+f_i''\right)$$

Equation (25) gives a system of N equations with $N+1$ unknown y_{-1} to $\mathrm{y}_{\mathrm{N}-1}$. To eliminate the unknown y_{-1}, we make use of the implicit boundary condition (8a) and then by employing the second order central difference approximation in it, we get

$$y_{-1} = \frac{2hc_1}{c_2} y_0 + y_1 - \frac{2hc_3}{c_2} \tag{27}$$

Where c_1, c_2 and c_3 are defined in (6). Making use of (27) in the first equation of the recurrence relation (25) at $i = 0$, we get

$$-(F_0 + \frac{2hc_1}{c_2} E_0) y_0 + (E_0 + G_0) y_1 = H_0 + \frac{2hc_3}{c_2} E_0 \tag{28}$$

Now, equations (25) and (28) give N by N tri-diagonal system which can be easily solved by using Thomas Algorithm.

Similarly, to set up the difference equation of the inner region problem (13)-(14) we divide the interval $0 \le t \le t_p$ in to N subintervals of equal mesh length $h = \frac{t_p - 0}{N}$ with mesh points $0 = t_0 < t_1 < t_2, \ldots, < t_N = t_p$. Following the same procedures/steps (16)-(24), we obtain the three term recurrence relation

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \quad i = 1, 2, \ldots, N-1 \tag{29}$$

Where

$$\begin{aligned} E_i &= \frac{1}{h^2} + \frac{1}{12}\left(A_i^2 + 2A_i' + \varepsilon B_i\right) - \frac{A_i}{2h} - \frac{h}{24}\left(A_i A_i' + \varepsilon A_i B_i + A_i'' + 2\varepsilon B_i'\right) \\ F_i &= \frac{2}{h^2} + \frac{1}{6}\left(A_i^2 + 2A_i' + \varepsilon B_i\right) - \varepsilon B_i - \frac{h^2}{12}\left(\varepsilon A_i B_i' + \varepsilon B_i''\right) \\ G_i &= \frac{1}{h^2} + \frac{1}{12}\left(A_i^2 + 2A_i' + \varepsilon B_i\right) + \frac{A_i}{2h} + \frac{h}{24}\left(A_i A_i' + \varepsilon A_i B_i + A_i'' + 2\varepsilon B_i'\right) \\ H_i &= \varepsilon F_i + \frac{\varepsilon h^2}{12}\left(A_i F_i' + F_i''\right) \end{aligned} \tag{30}$$

To solve the tri diagonal system (29), we also used Thomas Algorithm.

4. Numerical examples

To demonstrate the applicability of the method, we have implemented it on three numerical examples.

Example 4.1.

Consider the singular perturbation problem from Kevorkian and Cole [10], Page 33, equations 2.3.26 and 2.3.27 with α=0.

$\varepsilon y''(x) + y'(x) = 0$; $0 \le x \le 1$ with y(0)=0 and y(1)=1.

Outer region problem:

$\varepsilon y''(x) + y'(x) = 0$, $x_p \le x \le 1$ with $c_1 y(x_p) + c_2 y'(x_p) = c_3$ and $y(1) = 1$

Using the transformation $t = \dfrac{x}{\varepsilon}$ and rescaling, we get inner region problem:

$Y''(t) + Y'(t) = 0$, $0 \le t \le t_p$, with $Y(0) = 0$ and $Y(t_p) = y(x_p) = \gamma$

The exact solution is given by $y(x) = \dfrac{1 - \exp(-x/\varepsilon)}{1 - \exp(-1/\varepsilon)}$.

Numerical results are presented in Table 1 and 2 for ε=10^{-3} and ε=10^{-4} respectively.

Table 1. Numerical results for example 4.1. ε=10^{-3}

x	y(x)	y(x)	y(x)	Exact Solution
	t_p=5	t_p=10	t_p=20	
0.0000	0.0000000	0.0000000	0.0000000	0.0000000
0.0005	0.3964216	0.3934893	0.3934622	0.3934693
0.0010	0.6368634	0.6321526	0.6321090	0.6321205
0.0025	0.9248022	0.9179614	0.9178982	0.9179150
0.0050	0.9933118	0.9933118	0.9932433	0.9932621
0.0100		0.9999947	0.9999361	1.0000000
0.0200			0.9999997	1.0000000
0.1000	1.0000000	1.0000000	1.0000000	1.0000000
0.2000	1.0000000	1.0000000	1.0000000	1.0000000
0.3000	1.0000000	1.0000000	1.0000000	1.0000000
0.4000	1.0000000	1.0000000	1.0000000	1.0000000
0.5000	1.0000000	1.0000000	1.0000000	1.0000000
0.6000	1.0000000	1.0000000	1.0000000	1.0000000
0.8000	1.0000000	1.0000000	1.0000000	1.0000000
0.9000	1.0000000	1.0000000	1.0000000	1.0000000
1.0000	1.0000000	1.0000000	1.0000000	1.0000000

Table 2. Numerical results for example 4.1. ε=10^{-4}

x	y(x)	y(x)	y(x)	Exact Solution
	t_p=5	t_p=10	t_p=20	
0.00000	0.0000000	0.0000000	0.0000000	0.0000000
0.00005	0.3964216	0.3934893	0.3934622	0.3934693
0.00010	0.6368634	0.6321526	0.6321090	0.6321205
0.00025	0.9248022	0.9179614	0.9178982	0.9179150
0.00050	1.0000000	0.9933118	0.9932433	0.9932621
0.00100		1.0000000	0.9999361	0.9999546
0.00200			1.0000000	1.0000000
0.10000	0.9994636	0.9994636	0.9994636	1.0000000
0.20000	0.9995232	0.9995232	0.9995232	1.0000000
0.30000	0.9995828	0.9995828	0.9995828	1.0000000
0.40000	0.9996424	0.9996424	0.9996424	1.0000000
0.50000	0.9997020	0.9997020	0.9997020	1.0000000
0.60000	0.9997616	0.9997616	0.9997616	1.0000000
0.80000	0.9998808	0.9998808	0.9998808	1.0000000
0.90000	0.9999404	0.9999404	0.9999404	1.0000000
1.00000	1.0000000	1.0000000	1.0000000	1.0000000

Example 4.2.

Consider the following singular perturbation problem from fluid dynamics for fluid of small viscosity, Reinhardt [18], Example 2.

$\varepsilon y''(x) + y'(x) = 1 + 2x$; $0 \le x \le 1$, with y(0)=0 and y(1)=1.

Outer region problem:

$\varepsilon y''(x) + y'(x) = 1 + 2x$, $x_p \le x \le 1$, with $c_1 y(x_p) + c_2 y'(x_p) = c_3$ and $y(1) = 1$

Using the transformation $t = \dfrac{x}{\varepsilon}$ and rescaling, we get the inner region problem:

$Y''(t) + Y'(t) = \varepsilon(1 + 2\varepsilon t)$, $0 \le t \le t_p$, with $Y(0) = 0$ and $Y(t_p) = y(x_p) = \gamma$

The exact solution is given by $y(x) = x(x + 1 - 2\varepsilon) + (2\varepsilon - 1)(\dfrac{1 - \exp(-x/\varepsilon)}{1 - \exp(-1/\varepsilon)})$.

Numerical results are presented in Table 3 and 4 for $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$ respectively.

Table 3. Numerical results for example 4.2. $\varepsilon = 10^{-3}$

x	y(x)	y(x)	y(x)	Exact Solution
	$t_p=5$	$t_p=10$	$t_p=20$	
0.0000	0.0000000	0.0000000	0.0000000	0.0000000
0.0005	-0.3988031	-0.3922527	-0.3908068	-0.3921831
0.0010	-0.6406894	-0.6301660	-0.6278433	-0.6298573
0.0025	-0.9303619	-0.9150810	-0.9117079	-0.9135779
0.0050	-1.0067350	-0.9901992	-0.9865495	-0.9862605
0.0075		-0.9963951	-0.9927238	-0.9899068
0.0100		-0.9972157	-0.9935324	-0.9878747
0.0200			-0.9965126	-0.9776400
0.1000	-0.8852087	-0.8852087	-0.8852087	-0.8882000
0.2000	-0.7554023	-0.7554023	-0.7554023	-0.7584000
0.3000	-0.6055970	-0.6055970	-0.6055970	-0.6086000
0.4000	-0.4357930	-0.4357930	-0.4357930	-0.4388000
0.6000	-0.0361892	-0.0361892	-0.0361892	-0.0392001
0.8000	0.4434079	0.4434079	0.4434079	0.4403999
0.9000	0.7132034	0.7132034	0.7132034	0.7102000
1.0000	1.0000000	1.0000000	1.0000000	1.0000000

Table 4. Numerical results for example 4.2. ε=10^{-4}

x	y(x) t_p=5	y(x) t_p=10	y(x) t_p=20	Exact Solution
0.00000	0.0000000	0.0000000	0.0000000	0.0000000
0.00005	-0.3981239	-0.3945281	-0.3923660	-0.3933406
0.00010	-0.6395984	-0.6338219	-0.6303481	-0.6318942
0.00025	-0.9287786	-0.9203904	-0.9153460	-0.9174814
0.00050	-1.0050240	-0.9959474	-0.9904884	-0.9925632
0.00075		-1.0021850	-0.9966897	-0.9984966
0.00100		-1.0030070	-0.9975108	-0.9987538
0.00200			-1.0004910	-0.9977964
0.10000	-0.8898201	-0.8898201	-0.8898201	-0.8898200
0.20000	-0.7598400	-0.7598400	-0.7598400	-0.7598400
0.30000	-0.6098603	-0.6098603	-0.6098603	-0.6098600
0.40000	-0.4398800	-0.4398800	-0.4398800	-0.4398800
0.60000	-0.0399200	-0.0399200	-0.0399200	-0.0399201
0.80000	0.4400399	0.4400399	0.4400399	0.4400399
0.90000	0.7100198	0.7100198	0.7100198	0.7100199
1.00000	1.0000000	1.0000000	1.0000000	1.0000000

Example 4.3.

Consider the following singular perturbation problem from Kevorkian and Cole [10], Page 33, equations 2.3.26 and 2.3.27 with α=-1/2.

$\varepsilon y''(x)+(1-\frac{x}{2})y'(x)-\frac{1}{2}y(x)=0$; $0\le x\le 1$, with y(0)=0 and y(1)=1.

Outer region problem:

$\varepsilon y''(x)+(1-\frac{x}{2})y'(x)-\frac{1}{2}y(x)=0$; $x_p\le x\le 1$, with $c_1 y(x_p)+c_2 y'(x_p)=c_3$ and $y(1)=1$.

Using the transformation $t=\frac{x}{\varepsilon}$ and rescaling, we get the inner region problem:

$Y''(t)+(1-\frac{\varepsilon t}{2})Y'(t)-\frac{\varepsilon}{2}Y(t)=0$, $0\le t\le t_p$, with $Y(0)=0$ and $Y(t_p)=y(x_p)=\gamma$

The exact solution is given by: $y(x)=\frac{1}{2-x}-\frac{1}{2}e^{-(x-x^2/4)/\varepsilon}$.

Numerical results are presented in Table 5 and 6 for ε=10^{-3} and ε=10^{-4} respectively.

Table 5. Numerical results for example 4.3. $\varepsilon=10^{-3}$

x	y(x)	y(x)	y(x)	Exact Solution
	$t_p=5$	$t_p=10$	$t_p=20$	
0.0000	0.0000000	0.0000000	0.0000000	0.0000000
0.0005	0.2086948	0.2003571	0.1982316	0.1968407
0.0010	0.3353204	0.3219237	0.3185086	0.3162644
0.0025	0.4873530	0.4678823	0.4629188	0.4595191
0.0050	0.5287614	0.5076362	0.5022516	0.4978630
0.0075		0.5111049	0.5056835	0.5016016
0.0100		0.5115069	0.5060741	0.5024893
0.0200			0.5087405	0.5050505
0.1000	0.5275981	0.5275981	0.5275981	0.5263158
0.2000	0.5568738	0.5568738	0.5568738	0.5555556
0.3000	0.5895875	0.5895875	0.5895875	0.5882353
0.4000	0.6263814	0.6263814	0.6263814	0.6250000
0.6000	0.7156940	0.7156940	0.7156940	0.7142857
0.8000	0.8346714	0.8346714	0.8346714	0.8333333
0.9000	0.9103107	0.9103107	0.9103107	0.9090909
1.0000	1.0000000	1.0000000	1.0000000	1.0000000

Table 6. Numerical results for example 4.3. $\varepsilon=10^{-4}$

x	y(x)	y(x)	y(x)	Exact Solution
	$t_p=5$	$t_p=10$	$t_p=20$	
0.00000	0.0000000	0.0000000	0.0000000	0.0000000
0.00005	0.2080094	0.1988881	0.1956313	0.1967453
0.00010	0.3341934	0.3195390	0.3143064	0.3160807
0.00025	0.4854656	0.4641780	0.4565767	0.4590136
0.00050	0.5258662	0.5028067	0.4945732	0.4967540
0.00075		0.5066160	0.4983189	0.4999107
0.00100		0.5075817	0.4992693	0.5002273
0.00200			0.5025655	0.5005005
0.10000	0.5263897	0.5263897	0.5263896	0.5263158
0.20000	0.5556296	0.5556296	0.5556295	0.5555555
0.30000	0.5883077	0.5883077	0.5883077	0.5882353
0.40000	0.6250740	0.6250740	0.6250739	0.6250000
0.60000	0.7143514	0.7143514	0.7143514	0.7142857
0.80000	0.8333839	0.8333839	0.8333839	0.8333333
0.90000	0.9091176	0.9091176	0.9091176	0.9090909
1.00000	1.0000000	1.0000000	1.0000000	1.0000000

5. Nonlinear problems

To solve nonlinear singular perturbation problems we have used the method of quasilinearization.

Example 5.1.

Consider the following singular perturbation problem from Bender and Orszag [2], page 463, equations 9.7.1.

$\varepsilon y''(x) + 2y'(x) + e^{y(x)} = 0$; $0 \le x \le 1$, with y(0)=0 and y(1)=0.

The linear form of this example is

$$\varepsilon y''(x) + 2y'(x) + \frac{2}{x+1} y(x) = \left(\frac{2}{x+1}\right)\left[\log_e\left(\frac{2}{x+1}\right) - 1\right]$$

Outer region problem:

$$\varepsilon y''(x) + 2y'(x) + \frac{2}{x+1} y(x) = \left(\frac{2}{x+1}\right)\left[\log_e\left(\frac{2}{x+1}\right) - 1\right]; \quad x_p \le x \le 1$$

with $c_1 y(x_p) + c_2 y'(x_p) = c_3$ and $y(1) = 0$.

Using the transformation $t = \dfrac{x}{\varepsilon}$ and rescaling, we get the inner region problem:

$$Y''(t) + Y'(t) + \frac{2\varepsilon}{\varepsilon t+1} Y(t) = \varepsilon\left(\frac{2}{\varepsilon t+1}\right)\left[\log\left(\frac{2}{\varepsilon t+1}\right) - 1\right], \quad 0 \le t \le t_p,$$

with $Y(0) = 0$ and $Y(t_p) = y(x_p) = \gamma$

We have chosen to use Bender and Orszag's uniformly valid approximation [[2], page 463, equation 9.7.6] for comparison,

$$y(x) = \log_e\left(\frac{2}{x+1}\right) - (\log_e 2)\, e^{-2x/\varepsilon}$$

For this example, we have boundary layer of thickness O(ε) at x=0. [cf. Bender and Orszag [2]].

Numerical results are presented in Table 7 and 8 for $\varepsilon=10^{-3}$ and $\varepsilon=10^{-4}$ respectively.

Table 7. Numerical results for example 5.1. $\varepsilon=10^{-3}$

x	y(x)	y(x)	y(x)	Exact Solution
	t_p=5	t_p=10	t_p=20	
0.0000	0.0000000	0.0000000	0.0000000	0.0000000
0.0005	0.4408946	0.4370730	0.4348807	0.4376527
0.0010	0.6028609	0.5976343	0.5946361	0.5983404
0.0025	0.6912875	0.6852906	0.6818503	0.6859799
0.0050	0.6935626	0.6875402	0.6840836	0.6881282
0.0075		0.6856030	0.6821464	0.6856750
0.0100		0.6836601	0.6802093	0.6831968
0.0200			0.6724607	0.6733446
0.1000	0.5971998	0.5971998	0.5971998	0.5978370
0.2000	0.5102048	0.5102048	0.5102048	0.5108256
0.3000	0.4301792	0.4301792	0.4301792	0.4307829
0.4000	0.3560878	0.3560878	0.3560878	0.3566749
0.6000	0.2225883	0.2225883	0.2225883	0.2231435
0.8000	0.1048342	0.1048342	0.1048342	0.1053605
0.9000	0.0507805	0.0507805	0.0507805	0.0512933
1.0000	1.0000000	1.0000000	1.0000000	1.0000000

Table 8. Numerical results for example 5.1. $\varepsilon=10^{-4}$

x	y(x)	y(x)	y(x)	Exact Solution
	$t_p=5$	$t_p=10$	$t_p=20$	
0.00000	0.0000000	0.0000000	0.0000000	0.0000000
0.00005	0.4397037	0.4385542	0.4366111	0.4381026
0.00010	0.6014532	0.5998808	0.5972227	0.5992399
0.00025	0.6908551	0.6890486	0.6859952	0.6882268
0.00050	<u>0.6954359</u>	0.6936173	0.6905436	0.6926158
0.00075		0.6937961	0.6907224	0.6923972
0.00100		<u>0.6939749</u>	0.6909012	0.6921477
0.00200			<u>0.6916165</u>	0.6911492
0.10000	0.5978820	0.5978820	0.5978820	0.5978370
0.20000	0.5108603	0.5108603	0.5108603	0.5108256
0.30000	0.4308094	0.4308094	0.4308094	0.4307829
0.40000	0.3566946	0.3566946	0.3566946	0.3566750
0.60000	0.2231531	0.2231531	0.2231531	0.2231436
0.80000	0.1053641	0.1053641	0.1053641	0.1053605
0.90000	0.0512948	0.0512948	0.0512948	0.0512933
1.00000	1.0000000	1.0000000	1.0000000	0.0000000

Example 5.2.

Let us consider the following singular perturbation problem from Kevorkian and Cole [10], page 56, equation 2.5.1.

$\varepsilon y''(x) + y(x)y'(x) - y(x) = 0;\ \ 0 \le x \le 1$ with y(0)= -1 and y(1)=3.9995

The linear problem concerned to this example is

$\varepsilon y''(x) + (x + 2.9995)y'(x) = x + 2.9995$

Outer region problem:

$\varepsilon y''(x) + (x + 2.9995)y'(x) = x + 2.9995;\ \ x_p \le x \le 1$

with $c_1 y(x_p) + c_2 y'(x_p) = c_3$ and $y(1) = 1$

Using the transformation $t = \dfrac{x}{\varepsilon}$ and rescaling, we get the inner region problem:

$Y''(t) + (\varepsilon t + 2.9995)Y'(t) = \varepsilon(\varepsilon t + 2.9995)$, $0 \le t \le t_p$, with $Y(0) = 0$ and $Y(t_p) = y(x_p) = \gamma$.

We have chosen to use the Kevorkian and Cole's uniformly valid approximation [10], pages 57 and 58, equations (2.5.5), (2.5.11) and (2.5.14) for comparison,

$y(x) = x + c_1 \tanh\left(\left(\dfrac{c_1}{2}\right)\left(\dfrac{x}{\varepsilon} + c_2\right)\right)$, where c_1=2.9995 and $c_2=(1/c_1)\log_e[(c_1-1)/(c_1+1)]$

For this example also we have a boundary layer of width O(ε) at x=0 [cf. Kevorkian and Cole [10], pages 56-66].

The numerical results are given in Table 9 and 10 for $\varepsilon=10^{-3}$ and 10^{-4} respectively.

Table 9. Numerical results for example 5.2. $\varepsilon=10^{-3}$

x	y(x) $t_p=5$	y(x) $t_p=10$	y(x) $t_p=20$	Exact Solution
0.0000	-1.0000000	-1.0000000	-1.0000000	-1.0000000
0.0005	2.1181140	2.1176840	2.1238030	1.1484590
0.0010	2.8140390	2.8135130	2.8209980	2.4569390
0.0025	3.0114610	3.0109090	3.0187820	2.9953620
0.0050	<u>3.0139090</u>	3.0133570	3.0212340	3.0044960
0.0075		3.0136260	3.0215020	3.0070000
0.0100		<u>3.0135220</u>	3.0213970	3.0095000
0.0200			<u>3.0220790</u>	3.0195000
0.1000	3.1005670	3.1005670	3.1005670	3.0995000
0.2000	3.2005610	3.2005600	3.2005600	3.1995000
0.3000	3.3005550	3.3005550	3.3005550	3.2995000
0.4000	3.4005500	3.4005490	3.4005490	3.3995000
0.6000	3.6005310	3.6005310	3.6005310	3.5995000
0.8000	3.8005190	3.8005190	3.8005190	3.7995000
0.9000	3.900508	3.9005080	3.9005080	3.8995000
1.0000	3.9995000	3.9995000	3.9995000	3.9995000

Table 10. Numerical results for example 5.2. $\varepsilon=10^{-4}$

x	y(x) $t_p=5$	y(x) $t_p=10$	y(x) $t_p=20$	Exact Solution
0.00000	-1.0000000	-1.0000000	-1.0000000	-1.0000000
0.00005	2.1153530	2.1095570	2.1085090	1.1480090
0.00010	2.8105100	2.8034220	2.8021390	2.4560400
0.00025	3.0072620	2.9998090	2.9984590	2.9931120
0.00050	<u>3.0081140</u>	3.0006620	2.9993110	2.9999960
0.00075		3.0016230	3.0002490	3.0002500
0.00100		<u>3.0032370</u>	3.0018640	3.0005000
0.00200			<u>3.0027990</u>	3.0015000
0.10000	3.1001470	3.1001470	3.1001470	3.0995000
0.20000	3.2000710	3.2000710	3.2000710	3.1995000
0.30000	3.3000090	3.3000090	3.3000090	3.2995000
0.40000	3.3999360	3.3999360	3.3999360	3.3995000
0.60000	3.5997990	3.5997990	3.5997990	3.5995000
0.80000	3.7996520	3.7996520	3.7996510	3.7995000
0.90000	3.8995750	3.8995750	3.8995750	3.8995000
1.00000	3.9995000	3.9995000	3.9995000	3.9995000

6. Discussion and conclusions

We have presented a domain decomposition method for solving singularly perturbed two-point boundary value problems. As mentioned the method is iterative on the terminal point x_p and the process is to be repeated for different values of x_p (the terminal point which is not unique), until the solution profile stabilizes in both the inner and outer regions. We have implemented the present method on three linear and two nonlinear problems with left-end boundary layer, by taking different values of ε. From the results presented in tables, it can be observed that the present method approximates the exact solution very well. The present method is simple, easy and efficient technique for solving singular perturbation problems. In fact, our method helps us to get good results and also to know the behavior of the solution in the boundary layer/inner region with $h \geq \varepsilon$ itself. Where as the existing numerical methods produce good results only for $h \leq \varepsilon$ which is very costly and time consuming. Further for $h \geq \varepsilon$ the existing methods fails to give good results and in fact produce oscillatory solutions. Thus, the present method provides an alternative technique to the conventional ways of solving singularly perturbed boundary value problems.

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