EIGHTH ORDER COMPACT FINITE DIFFERENCE METHOD FOR SINGULARLY PERTURBED ONE DIMENSIONAL REACTION DIFFUSION PROBLEMS

A Thesis Submitted to Department of Mathematics, Jimma University, for the Partial Fulfillment of the Requirements for the Degree of Masters of Science in Mathematics

(Numerical Analysis)

BY

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Jimma, Ethiopia

Declaration

I, undersigned declare that this thesis entitled "**Eighth Order Compact Finite Difference Method for Singularly Perturbed One Dimensional Reaction Diffusion Problems**" is original and it has not been submitted to any institution elsewhere for the award of any academic degree or like, where other sources of information have been used, they have been acknowledged.

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Feyisa Edosa

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Abstract

In this thesis, eighth order compact finite difference method has been presented for solving singularly perturbed one dimensional reaction diffusion problems. First, the given interval is discretized and the given differential equation is replaced by finite difference approximations. Then, the given differential equation is transformed to linear systems of algebraic equations and then using Taylor's series and central finite difference approximation, it is reduced to a three term recurrence relation which can be easily solved by using Thomas Algorithm. To validate the applicability of the proposed method three model examples with and without exact solution were considered and solved for different values of perturbation parameter and mesh sizes. Numerical experiments are carried out extensively to support the theoretical results using MATLAB software. The results have been presented in tables in terms of maximum absolute errors and also in graphs. The present method approximates the exact solution very well. Both the theoretical and computational rate of convergence has been established and observed to be in agreement. In a net shell the present method is simple and efficient than some of the methods reported in the literature.

CHAPTER ONE
INTRODUCTION

1.1. Background of the study

Numerical analysis is the branch of mathematics that deals with the computational methods which helps to find approximate solutions for difficult problems such as finding the roots of non-linear equations, integration involving complex expressions and solving differential equations for which analytical solutions does not exist.

The problems in which the highest order derivative term is multiplied by a small positive parameter ε where $0 < \varepsilon << 1$ are known to be Singularly Perturbed Problems and the parameter is known as the perturbation parameter. Singularly perturbed problems have always played a prominent role in the theory of differential equations and in their applications to the physical world. Ever since Prandtl's [26] work in the beginning of 20th century, singular perturbation techniques have been a traditional tool of fluid dynamics. These techniques entered into various other areas of application, where of course, the same terminology of 'boundary layer', 'interior layer', 'outer' and 'inner' was already in use.

The various applications of singularly perturbation problems are fluid dynamics, plasticity, chemical reactor theory, nuclear reactor theory, plasma physics, aerodynamics, meteorology, oceanography, rarefied gas dynamics, diffraction theory, reaction-diffusion process, non-equilibrium and other domains of the great world of fluid motion.

In the intensive development of science and technology, many practical problems, such as the mathematical boundary layer theory or approximation of solutions of various problems described by differential equations involving large or small parameters become more complex. In some problems, the perturbations are operative over a very narrow region across which the dependent variable undergoes very rapid changes. These problems depend on a small positive parameter ε, where $0 < \varepsilon << 1$ in such a way that the solution varies rapidly in some parts of the domain and varies slowly in some other parts of the domain. Typically, there are thin transition layers where the solution varies rapidly or jumps abruptly, while away from the layers the solution behaves regularly and varies slowly. If we apply the existing standard numerical methods for solving such types of problems large oscillations may arise and pollute the solution in the entire interval because of the boundary layer behavior.

For more than two decades, a great deal of research work on the qualitative and quantitative analysis of these problems both for ordinary and partial differential equations has been reported in the literature. But the major problem of obtaining accurate approximations to the solutions of these problems is still an open question. Classical numerical methods which have been known to be effective for solving most problems that arise in applications have failed when applied to singularly perturbed problems. As a result, this area has attracted a keen interest amongst mathematicians today.

Consequently, there are now a variety of methods for solving these kinds of problems. Basically, the problem of inaccuracy in results of singularly perturbed problems has been associated with the perturbation parameter. This perturbation parameter prevents us from obtaining satisfactory numerical solutions. Most of the classical numerical methods are not effective for solving such problems because, as the singular perturbation parameter tends to zero, the errors in the numerical solutions increase and often becomes comparable in magnitude to the exact solution Farrell [8].

Thus, more efficient and simpler computational techniques are required to solve singularly perturbed two-point boundary value problems. Sometimes, to find the exact solutions of the boundary value problems is too difficult, so we have to apply numerical methods. In addition to that, obtaining accurate and fast numerical solution of two-point boundary value problems is a great importance due to its wide application in scientific and engineering researches. As a result, many numerical methods intensively have been proposed to solve two-point boundary value problems such as finite difference, finite element and finite volume methods. Further, compact finite difference scheme in one dimension on a uniform step length found in Collatz [6] and Lele [18] which have been formulated on the first and second order derivatives concludes that it was high order accurate and also resolves shorter scales of the solution better than classical finite difference schemes which brings them closer to higher accuracy.

The fourth and sixth order compact finite difference methods for singularly perturbed one dimensional reaction diffusion of two boundary value problem was done by Fasika Wondimu who was worked under the supervision of my supervisor.

Therefore, the main objective of this study is to extend the previous work to the eighth order compact finite difference method for solving singularly perturbed one dimensional reaction diffusion problems of two-point boundary value problems.

1.2. Statement of the Problem

The subject of numerical analysis is concerned with devising methods for approximating in an efficient manner of the solutions to mathematical problems. The efficiency of the method depends up on the accuracy required and the ease with which it can be implemented. In practical situation the mathematical problem is derived from physical phenomena where some simplifying assumptions have been made to allow the mathematical representation to develop. The increasing desire for the numerical solutions to mathematical problems, which are more difficult or impossible to solve analytically, has become the present- day scientific research. This time it sounds more appropriate to find an approximate solution to a more complicated model. It is clear that numerical methods can give approximate solutions in an efficient manner, when ordinary analytic methods fail.

Obtaining accurate and fast numerical solutions for singularly perturbed reaction diffusion problem has a great importance due to its wide applications in scientific and engineering research. Owing to this, in this study we developed the eighth order compact finite difference method to find the solutions of singularly perturbed one dimensional reaction diffusion problems.

As a result, this study attempted to answer the following questions:

1. How do we formulate eighth order compact finite difference method for singularly perturbed one dimensional reaction diffusion problems?
2. To what extent the present method approximate the exact solution?
3. To what extent the proposed method converges?
4. What is the advantage of the proposed method over other numerical methods?

1.3. Objectives of the Study

1.3.1. General Objective

The general objective of this research is to present eighth order compact finite difference method for solving singularly perturbed one dimensional reaction diffusion problems.

1.3.2. Specific Objectives

The specific objectives of the study are:

- To formulate the numerical method for solving singularly perturbed one dimensional reaction diffusion problems.
- To test the extent to which the proposed method approximate the exact solution.
- To establish convergence of the method formulated.
- To describe the advantage of the present method over others.

1.4. Significance of the Study

The outcome of the study may:

- be used as a reference material for students, teachers and anyone who works on this area.
- provide significant contribution for scientific investigation in the area of applied mathematics.
- improve the application of numerical methods in different field of studies.

1.5. Delimitation of the Study

Singularly perturbed problems can be solved using different numerical techniques such as finite difference method (FDM), finite element method (FEM), finite volume method (FVM), B-Spline method, calculus of variation and so on. However, this study was delimited to eighth order compact finite difference method to solve one dimensional reaction diffusion equations of the form:

$$-\varepsilon y''(x) + g(x)y(x) = f(x), \qquad 0 \le x \le 1 \tag{1.1}$$

$$y(0) = \alpha \qquad y(1) = \beta \tag{1.2}$$

where α, β are constants, $0 < \varepsilon << 1$(perturbation parameter), f and g are sufficiently smooth functions and according to Khan and et al. [16], we can assume $g(x) \geq g > 0$, with uniform step length.

1.6. Definition of Key Terms

Compact finite difference method: is a finite difference method which employs a linear combination of three consecutive points of derivatives to approximate a linear combination of the same three consecutive values of function $y(x_j),\ j = i-1, i, i+1$

Boundary Value Problem: A problem, typically an ODE or a PDE, which has values assigned on the physical boundary of the domain in which the problem is specified, is called a boundary value problem (BVP).

Two-Point Boundary Value Problem: Let $f:\ lR^3 \to R$ be given function and $\alpha,\ \beta$ are given constants. The problem $y'' = f(x, y.y'),\ x \in (a,b)$,

$$y(a) = \alpha,\ y(b) = \beta$$

is called two-point boundary value problem.

CHAPTER TWO
REVIEW OF RELATED LITERATURE

2.1. Singular Perturbation Theory

Since Prandtl [26] work in the beginning of 20^{th} century, singular perturbation techniques have been a traditional tool of fluid dynamics, which has the same terminology of 'boundary layer', 'interior layer', 'outer' and 'inner' was already in use. As a term singular perturbation was first introduced by Friedrichs, et al. [9]. In Russia, mainly at Moscow State University, research activity on singular perturbations for ordinary differential equations, originated and developed and continues to be vigorously pursued even today Vasil'yeva [33]. An excellent survey of the historical development of singular perturbations is found in a recent book by O'Malley [24].

In Mathematics, more precisely in perturbation theory, a singular perturbation problem is a problem containing a small parameter ε that cannot be approximated by setting the parameter value to zero. This is in contrast to regular perturbation problems, for which an approximation can be obtained by simply setting the small parameter to zero. It means the solution cannot be uniformly approximated be an asymptotic expansion as $\varepsilon \to 0$.

The problems in which the highest order derivative term is multiplied by a small positive parameter are known to be Singularly Perturbed Problems and the parameter is known as the perturbation parameter. The various applications of singularly perturbation problems are fluid dynamics, plasticity, chemical reactor theory, nuclear reactor theory, plasma physics, aerodynamics, meteorology, oceanography, rarefied gas dynamics, diffraction theory, reaction-diffusion process, non-equilibrium and other domains of the great world of fluid motion.

In the intensive development of science and technology, many practical problems, such as the mathematical boundary layer theory or approximation of solutions of various problems described by differential equations involving large or small parameters become more complex. In some problems, the perturbations are operative over a very narrow region across which the dependent variable undergoes very rapid changes. These problems depend on a small positive parameter ε, where $0 < \varepsilon << 1$ in such a way that the solution varies rapidly in some parts of the domain and varies slowly in some other parts of the domain. Typically, there are thin transition layers where the solution varies rapidly or jumps abruptly, while away from the layers the solution behaves regularly and varies slowly. If we apply the existing standard numerical methods for solving such types of problems large oscillations may arise and pollute the solution in the entire interval because of the boundary layer behavior.

2.2. Two Boundary Value Problem

The numerical solution of a boundary value problem will be more difficult matter than the numerical solution of the corresponding initial-value problems. Hence, many scholars prefer to convert the second-order problem into first order problems.

Reddy and Chakravarthy [27] proposed method of reduction of order for solving singularly perturbed two point boundary value problems. The solution of the given two-point boundary value problem is numerically computed by solving two suitable initial-value problems easily deduced from the original problem through asymptotic expansion procedures. The method is very easy to implement and is tested on several linear and non-linear problems. They proposed an initial value technique for solving singularly perturbed two point boundary value problems.

Gasparo and Macconi [10-11], Natesan and Ramanujam [22] have studied initial-value technique for singularly perturbed boundary-value problems for second-order ordinary differential equations arising in chemical reactor theory. Error estimates for approximate solutions are obtained. The initial-value technique has been applied to solve various singularly perturbed boundary value problems for second-order ordinary differential equations subjects to Dirichlet-type boundary conditions.

To demonstrate the applicability of this method, it can be applied on several nonlinear examples with left-end boundary layer and right end layer. The survey paper by Kadalbajoo and Reddy [13], gives an erudite outline of the singular perturbation problems and their treatment starting from the fluid dynamical boundary layers. It remains as one of the most readable source on singular perturbations. This theory can be explained in detail on the monographs: Nayfeh [23], Kevorkian and Cole [15], Smith [29], Brauner et.al. [3] and Kato [14]. From the numerical results, the method seems accurate and solutions to problems with extremely thin boundary layers are obtained.

2.3. Singularly Perturbed Reaction Diffusion Problem

Clavero et al. [5], considered the finite difference hybrid scheme constructed by Natesan et al. [21] for obtaining uniformly convergent global solution and uniformly convergent normalized flux for singularly perturbed reaction diffusion equation under the consideration. The global solution is obtained from the numerical solution at mesh points of this scheme having almost second order uniform convergence at the nodal points when it is constructed on piecewise uniform Shishkin mesh. Using classical cubic spline Clavero et al. [5], defined and proved the normalized flux on the entire domain which is almost second order uniformly convergent in the whole domain.

Reshidinia et al. [28], used spline in compression to develop a class of methods which are second and fourth order convergent for singularly perturbed reaction-diffusion equation under the consideration. Natesan et al. [22], proposed a numerical scheme for singularly perturbed reaction diffusion equation under consideration which is a combination of the cubic splines and the classical central difference scheme with piecewise uniform Shishkin mesh which uniformly convergent of second order.

Kumar et al. [17], proposed a high order parameter robust finite difference method for singularly perturbed reaction diffusion equation of the form: $-\varepsilon y''(x)+g(x)y(x)=f(x),\quad 0<x<1$ with the boundary condition; $y(0)=0,\quad y(1)=0$, where, ε is small positive parameter such that $0<\varepsilon<<1$ and $g(x), f(x)$ are assumed to be sufficiently continuously differentiable functions such that $g(x)\geq\beta>0$. The problem is discretized using a suitable combination of upwind scheme and central difference scheme on generalized Shishkin mesh in which it is almost fourth order uniformly convergent in maximum norm with respect to perturbation parameter ε.

Bawa et al. [2], considered a one dimensional singularly perturbed reaction-diffusion equation of the above type. A modified Shishkin mesh is introduced and a higher order compact finite difference solution on this mesh is presented. Piece-wise cubic interpolants for both exact and discrete solution were formulated. The authors proved that the convergence in the sense that the convergence accuracy is the same for any value of the diffusion parameter ε. More precisely, the convergence order analysis contains two principle results. The first result states that the method is almost fourth order convergence. The second result states the normalized flux of the piece-wise cubic interpolant of the discrete solution approximates the exact solution by order three, almost everywhere and by the order four at mid-points of the mesh.

2.4. Numerical solution versus Analytical solution

The method of finding the exact solution of the differential equation by using calculus, trigonometry and other techniques is known as analytic methods because we used the analysis to figure it out. The exact solution is also referred to as a closed form solution or analytical solution. But this tends to work only for simple differential equations with simple coefficients. For higher order or non-linear differential equations with complex coefficient, it becomes very difficult to find exact solution. Therefore, we need numerical methods for solving the equations.

Even, classical numerical methods which have been known to be effective for solving most problems that arise in applications have failed when applied to singularly perturbed problems. Basically, the problem of inaccuracy results of singularly perturbed problems has been associated with the perturbation parameter. This perturbation parameter prevents us from obtaining satisfactory numerical solutions. Most of the classical numerical methods are not effective for solving such problems because, as the singular perturbation parameter tends to zero, the errors in the numerical solutions increase and often becomes comparable in magnitude to the exact solution Farrell [8]. Thus, more efficient and simpler computational techniques are required to solve singularly perturbed two-point boundary value problems. Sometimes, to find the exact solutions of the boundary value problems is too difficult, so we have to apply numerical methods

2.5. Finite Difference Method

Finite difference methods are one of the most widely used numerical schemes to solve differential equations. In finite difference methods, derivatives appearing in the differential equations are replaced by finite difference approximations obtained by Taylor series expansions at the grid points. This gives a large algebraic system of equations to be solved by different iterative techniques in place of the differential equation to give the solution value at the grid points and hence the solution is obtained at grid points. Some of the finite difference methods include forward difference method, backward difference method, central difference method, etc.

Vesna et al. [34], presented a numerical-asymptotic solution technique for solving singular perturbation problems. They constructed a division point which divides the initial interval in to two sub intervals, so that the layer belongs only to one of them. The reduced problem is used to get the solution at the terminal point. The inner region problem is solved as a two-point boundary value problem. Hu et al. [12], developed a discretization method for one-dimensional singular perturbation problems based on finite difference scheme. Its discretization error has a bound that is second order in the mesh size and uniform in the perturbation parameter.

2.6. Compact Finite Difference Method

Pinto, et.al [25], has analyzed the difference scheme of exponential type for solving non-linear singular perturbation problems. The compact finite difference schemes, introduced as far back as the 1930s, have been found simple ways of reaching the objectives of high accuracy and low computational cost. Compared with the others finite difference schemes of the same order, compact schemes have proved to be significantly more accurate with the added benefit of using smaller stencil sizes, which can be essential when treating the boundary conditions.

In the standard compact finite difference methods the formulation of the method for the approximation of first derivative includes the function and its odd derivatives. In a similar manner, the formula of the method for approximation of the second derivative includes the function and its even derivatives. But it is possible to derive another class of compact finite difference schemes that their formulation can be used to approximate the first and second derivatives simultaneously.

By using the sufficient conditions, which ensure the well conditioning of tri diagonal matrices, Mazzia and Trigiante [19] have developed methods for singularly perturbed two-point boundary value problems. A compact finite difference method for second order singular perturbation problems is presented by Mazzia and Trigiante [19]. It is based on a mesh selection strategy derived by using sufficient conditions which ensure the well conditioning of tri-diagonal matrices. Using the theory of n – widths, the solutions of singularly perturbed reaction diffusion problems is quantified by Stynes [30].

Chu and Fan [4] in 1998 presented a higher order compact scheme and showed that their method has better resolution characteristics than others. In this article, the idea of using both odd and even derivatives as unknowns in the formulation of a compact finite difference scheme is used to introduce a general class of highly accurate finite difference schemes of arbitrary order with the uniform and non-uniform grid points. It is shown that the solution of this scheme converges uniformly in ε to the exact solution.

CHAPTER THREE
METHODOLOGY

This chapter consists of methods and materials used to undertake the study. These are; study design, study site and period, source of information, procedure of the study and ethical consideration.

3.1. Study Area and Period

The study site of this research is at Jimma University and the period is from September, 2014 to September, 2015. Conceptually, the study was focused on eighth order compact finite difference scheme for singularly perturbed one dimensional reaction diffusion problems with Dirichlet boundary conditions.

3.2. Study Design

The study was employed mixed design i.e. documentary review and experimental design.

3.3. Source of Information

The study depends on various sources of information such as; books, journals and different related studies published/ unpublished. The experimental results were obtained by writing code using MATLAB software for the presented numerical method.

3.4. Procedures of the study

Important materials and data for the study were collected by means of documentary review and algorithm development. Hence, to attain the objective of the study, the following procedure was undertaken:

i. Defining the problem /formulating the method.
ii. Discretizing the domain /interval.
iii. Replacing the given equation by the finite difference approximation and obtaining systems of equations.
iv. Rewrite the resulting systems of equations in tri-diagonal form.
v. Writing a code for the problem by using MALAB language.
vi. Validating the schemes using numerical examples.

3.5. Ethical Consideration

In order to conduct this research, it was made appropriate communication with responsible officials of Jimma University to get a cooperation letter to concerned bodies for legal consent. Moreover, rules and regulations of the campus were kept.

CHAPTER FOUR

DESCRIPTION OF THE METHOD, RESULT AND DISCUSSION

4.1. Description of the Method

In this section, the description of eighth order compact finite difference method and its theoretical error analysis have been given.

Consider a uniform mesh with interval $[a,b]$ in which $a = x_0 < x_1 < x_2 < ... < x_n = b$ where $h = \frac{b-a}{n}$ and $x_i = x_0 + ih,\ i = 0,1,2,...n.$ Let $y_i = y(x_i)$ denotes the solution of problem (4.1) and (4.2) below and also $x = x_i,\ y_i^{(n)} = y^{(n)}(x_i),\ f_i = f(x_i)$ and $f_i^{(n)} = f^{(n)}(x_i)$ denote its n[th] derivative at $x = x_i$

$$-\varepsilon y''(x) + g(x)y(x) = f(x), \quad 0 \le x \le 1 \tag{4.1}$$

$$y(0) = \alpha, \quad y(1) = \beta \tag{4.2}$$

where $g(x) \ge g > 0$, α, β are constants, ε is small positive parameter, f and g are sufficient smooth functions.

By using Taylor Series expansion we obtain:

$$y_{i+1} = y_i + hy_i^{'} + \frac{h^2}{2!}y_i^{''} + \frac{h^3}{3!}y_i^{'''} + \frac{h^4}{4!}y_i^{(4)} + \frac{h^5}{5!}y_i^{(5)} + \frac{h^6}{6!}y_i^{(6)} + \frac{h^7}{7!}y_i^{(7)} + \frac{h^8}{8!}y_i^{(8)} + \frac{h^9}{9!}y_i^{(9)} + \frac{h^{10}}{10!}y_i^{(10)} + ... (4.3)$$

$$y_{i-1} = y_i - hy_i^{'} + \frac{h^2}{2!}y_i^{''} - \frac{h^3}{3!}y_i^{'''} + \frac{h^4}{4!}y_i^{(4)} - \frac{h^5}{5!}y_i^{(5)} + \frac{h^6}{6!}y_i^{(6)} - \frac{h^7}{7!}y_i^{(7)} + \frac{h^8}{8!}y_i^{(8)} - \frac{h^9}{9!}y_i^{(9)} + \frac{h^{10}}{10!}y_i^{(10)} - ... (4.4)$$

Subtracting Eq. (4.4) from Eq. (4.3), we get:

$$y_{i+1} - y_{i-1} = 2hy_i^{'} + \frac{h^3}{3}y_i^{'''} + \frac{2h^5}{5!}y_i^{(5)} + \frac{2h^7}{7!}y_i^{(7)} + \frac{2h^9}{9!}y_i^{(9)} + ... \tag{4.5}$$

Now, let us denote the second order central difference by $\delta_c^1 y_i$ of the first derivative of y_i and the standard second order central difference $\delta_c^2 y_i$ of the second derivative of y_i as below:

$$\delta_c^1 y_i = y_i^{'} = \frac{y_{i+1} - y_{i-1}}{2h} + \tau_1 \tag{4.6}$$

where $\tau_1 = -\frac{h^2}{6}y_i^{'''} + O(h^4)$

Again by adding Eqs.(4.3) and (4.4) we obtain:

$$\delta_c^2 y_i = y_i^{"} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + \tau_2 \tag{4.7}$$

where $\tau_2 = -\frac{h^2}{12} y_i^{(4)} + O(h^4)$

Using Eqs.(4.3), (4.4) and (4.7), we obtain:

$$\delta_c^2 y_i = y_i^{"} + \frac{h^2}{12} y_i^{(4)} + \frac{h^4}{360} y_i^{(6)} + \frac{h^6}{20160} y_i^{(8)} + \tau_3 \tag{4.8}$$

where $\tau_3 = \frac{2h^8}{10!} y_i^{(10)} + O(h^{10})$

To obtain the eighth order finite difference scheme, we can apply δ_c^2 to $y_i^{(6)}$ and we obtain:

$$y_i^{(8)} = \delta_c^2 y_i^{(6)} + \tau_4 \tag{4.9}$$

where $\tau_4 = \frac{h^2}{90} y_i^{(10)} + O(h^4)$

Now, at any point x_i Eq. (4.1) can be written as:

$$-y_i^{"} + u_i y_i = r_i \tag{4.10}$$

where $\mathrm{u_i} = \frac{g_i}{\varepsilon}$, $\mathrm{r_i} = \frac{f_i}{\varepsilon}$ and $g_i \geq g$ which is positive constant (by previous assumption).

Thus, differentiating Eq. (4.10) successively, we obtain:

$$y_i^{(4)} = u_i y_i^{"} - r_i^{"} \tag{4.11}$$

$$y_i^{(6)} = u_i y_i^{(4)} - r_i^{(4)} \tag{4.12}$$

Substituting Eqs.(4.11) and (4.12) into Eq. (4.8), we obtain:

$$\begin{aligned}\delta_c^2 y_i = &\left(1 + \frac{h^2 u_i}{12} + \frac{h^4 u_i^2}{360} + \frac{h^6 u_i^2}{20160}\delta_c^2\right) y_i^{"} - \left(\frac{h^2}{12} + \frac{h^4 u_i}{360} + \frac{h^6 u_i}{20160}\delta_c^2\right) r_i^{"} \\ &- \left(\frac{h^4}{360} + \frac{h^6}{20160}\delta_c^2\right) r_i^{(4)} + \tau_5\end{aligned} \tag{4.13}$$

where $\tau_5 = \frac{h^6}{20160}\tau_4 + \tau_3 = \frac{h^8}{907200} y_i^{(10)} + O(h^{10})$

Solving for $y_i^{''}$, we obtain:

$$y_i^{''} = \frac{\delta_c^2 y_i + \left(\left(\frac{h^2}{12} + \frac{h^4 u_i}{360} + \frac{h^6 u_i}{20160} \delta_c^2 \right) r_i^{''} + \left(\frac{h^4}{360} + \frac{h^6}{20160} \delta_c^2 \right) r_i^{(4)} + \tau_5 \right)}{1 + \frac{h^2 u_i}{12} + \frac{h^4 u_i^2}{360} + \frac{h^6 u_i^2}{20160} \delta_c^2} \tag{4.14}$$

Substituting Eq. (4.14) into Eq. (4.10), we obtain:

$$-\left(\frac{\delta_c^2 y_i + \left(\left(\frac{h^2}{12} + \frac{h^4 u_i}{360} + \frac{h^6 u_i}{20160} \delta_c^2 \right) r_i^{''} + \left(\frac{h^4}{360} + \frac{h^6}{20160} \delta_c^2 \right) r_i^{(4)} + \tau_5 \right)}{1 + \frac{h^2 u_i}{12} + \frac{h^4 u_i^2}{360} + \frac{h^6 u_i^2}{20160} \delta_c^2} \right) + u_i y_i = r_i$$

$$\begin{aligned} &-\delta_c^2 y_i - \left(\frac{h^2}{12} + \frac{h^4 u_i}{360} \right) r_i^{''} - \frac{h^6 u_i}{20160} \delta_c^2 r_i^{''} - \frac{h^4}{360} r_i^{(4)} - \frac{h^6}{20160} \delta_c^2 r_i^{(4)} - \tau_5 \\ &+ u_i \left(1 + \frac{h^2 u_i}{12} + \frac{h^4 u_i^2}{360} \right) y_i + \frac{h^6 u_i^3}{20160} \delta_c^2 y_i = \left(1 + \frac{h^2 u_i}{12} + \frac{h^4 u_i^2}{360} \right) r_i + \frac{h^6 u_i^2}{20160} \delta_c^2 r_i \end{aligned} \tag{4.15}$$

But from central difference and Eq. (4.7), we have the following conditions

$$\begin{aligned} &\delta_c^2 y_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \qquad && \delta_c^2 r_i = \frac{r_{i+1} - 2r_i + r_{i-1}}{h^2} \\ &\delta_c^2 r_i^{''} = \frac{r_{i+1}^{''} - 2r_i^{''} + r_{i-1}^{''}}{h^2} \qquad && \delta_c^2 r_i^{(4)} = \frac{r_{i+1}^{(4)} - 2r_i^{(4)} + r_{i-1}^{(4)}}{h^2} \end{aligned} \tag{4.16}$$

Substituting these four equations of Eq. (4.16) into Eq. (4.15) and by rearranging, we get:

$$\begin{aligned} &-\left(\frac{1}{h^2} - \frac{h^4 u_i^3}{20160} \right) y_{i-1} + \left(\frac{2}{h^2} + u_i \left(1 + \frac{h^2 u_i}{12} + \frac{3h^4 u_i^2}{1120} \right) \right) y_i - \left(\frac{1}{h^2} - \frac{h^4 u_i^3}{20160} \right) y_{i+1} \\ &= \frac{h^4 u_i^2}{20160} r_{i-1} + \left(1 + \frac{h^2 u_i}{12} + \frac{3h^4 u_i^2}{1120} \right) r_i + \frac{h^4 u_i^2}{20160} r_{i+1} + \frac{h^4 u_i}{20160} r_{i-1}^{''} \\ &+ \left(\frac{h^2}{12} + \frac{3h^4 u_i}{1120} \right) r_i^{''} + \frac{h^4 u_i}{20160} r_{i+1}^{''} + \frac{h^4}{20160} r_{i-1}^{(4)} + \frac{3h^4}{1120} r_i^{(4)} + \frac{h^4}{20160} r_{i+1}^{(4)} \end{aligned} \tag{4.17}$$

Eq. (4.17) can be written as a three recurrence relation of the form:

$$-E_i y_{i-1} + F_i y_i - G_i y_{i+1} = H_i \tag{4.18}$$

where;

$$E_i = \frac{1}{h^2} - \frac{h^4 u_i^3}{20160}$$

$$F_i = \frac{2}{h^2} + u_i\left(1 + \frac{h^2 u_i}{12} + \frac{3h^4 u_i^2}{1120}\right)$$

$$G_i = \frac{1}{h^2} - \frac{h^4 u_i^3}{20160}$$

$$H_i = \frac{h^4 u_i^2}{20160} r_{i-1} + \left(1 + \frac{h^2 u_i}{12} + \frac{3h^4 u_i^2}{1120}\right) r_i + \frac{h^4 u_i^2}{20160} r_{i+1} + \frac{h^4 u_i}{20160} r_{i-1}^{"} + \left(\frac{h^2}{12} + \frac{3h^4 u_i}{1120}\right) r_i^{"} + \frac{h^4 u_i}{20160} r_{i+1}^{"} + \frac{h^4}{20160} r_{i-1}^{(4)} + \frac{3h^4}{1120} r_i^{(4)} + \frac{h^4}{20160} r_{i+1}^{(4)}$$

4.2. Error Analysis

Writing the tri-diagonal system Eq. (4.18) in matrix vector form, we obtain:

$$AY = C \tag{4.19}$$

where $A = (m_{ij})$, $1 \le i,\ j \le N-1$ is a tri-diagonal matrix of order $N-1$, with

$$m_{ii+1} = -\frac{1}{h^2} + \frac{h^4 u_i^3}{20160}$$

$$m_{ii} = \frac{2}{h^2} + u_i + \frac{h^2 u_i^2}{12} + \frac{3h^4 u_i^3}{1120}$$

$$m_{ii-1} = -\frac{1}{h^2} + \frac{h^4 u_i^3}{20160}$$

and $C = (d_i)$ be a column vector with

$$d_i = \frac{h^4 u_i^2}{20160} r_{i-1} + \left(1 + \frac{h^2 u_i}{12} + \frac{3h^4 u_i^2}{1120}\right) r_i + \frac{h^4 u_i^2}{20160} r_{i+1} + \frac{h^4 u_i}{20160} r_{i-1}^{"} + \left(\frac{h^2}{12} + \frac{3h^4 u_i}{1120}\right) r_i^{"} + \frac{h^4 u_i}{20160} r_{i+1}^{"} + \frac{h^4}{20160} r_{i-1}^{(4)} + \frac{3h^4}{1120} r_i^{(4)} + \frac{h^4}{20160} r_{i+1}^{(4)}$$

for $i = 1, 2, ..., N$ with local truncation error given by:

$$\tau_i(h_i) = \frac{h^8}{907200} y_i^{(10)} + O(h^{10}) \tag{4.20}$$

We also have

$$A\overline{Y} - \tau(h) = C \tag{4.21}$$

where $\bar{Y} = (\bar{y}_0, \bar{y}_1, \bar{y}_2, ..., \bar{y}_N)^t$ denotes the exact solution and

$\tau(h) = (\tau_1(h_0), \tau_2(h_1), ..., \tau_N(h_N))^t$ is local truncation error.

From Eqs.(4.19) and (4.21), we obtain:

$$A(\bar{Y} - Y) = \tau(h) \tag{4.22}$$

Thus, we get an error equation

$$A\mathrm{E} = \tau(h) \tag{4.23}$$

where $\mathrm{E} = \bar{Y} - Y = (e_0, e_1, e_2, ..., e_N)$

Let S_i be the sum of elements of the i^{th} row of A, then we have:

For $i = 1$

$$S_i = \sum_{j=1}^{N-1} m_{1j}, \ = \frac{2}{h^2} + u_i + \frac{h^2 u_i^2}{12} + \frac{3h^4 u_i^3}{1120} - \frac{1}{h^2} + \frac{h^4 u_i^3}{20160}$$

Therefore,

$$S_1 = u_i + \frac{1}{h^2} + \frac{h^2 u_i^2}{12} + \frac{11h^4 u_i^3}{4032}$$

$$= B_1 + A_1 h^4$$

where $B_1 = u_i + \frac{1}{h^2} + \frac{h^2 u_i^2}{12}$ and $A_1 = \frac{11u_i^3}{4032}$

Therefore, $S_1 = B_1 + O(h^4)$, where $|B_1| = \min S_1$

For $i = 2, 3, ..., N-2$

$$S_i = \sum_{j=1}^{N-1} m_{i,j} \ = -\frac{1}{h^2} + \frac{h^4 u_i^3}{20160} + \frac{2}{h^2} + u_i + \frac{h^2 u_i^2}{12} + \frac{3h^4 u_i^3}{1120} - \frac{1}{h^2} + \frac{h^4 u_i^3}{20160}$$

$$= u_i + \frac{h^2 u_i^2}{12} + \frac{h^4 u_i^3}{360}$$

$$= B_i + A_0 h^4,$$

where $B_i = u_i + \frac{h^2 u_i^2}{12}$ and $A_0 = \frac{u_i^3}{360}$

Therefore,

$$S_i = B_i + O(h^4), \text{ where } |B_i| = \min S_i$$

For $i = N-1$

$$S_{N-1} = \sum_{j=1}^{N-1} m_{N-1,j} \quad = -\frac{1}{h^2} + \frac{h^4 u_i^3}{20160} + \frac{2}{h^2} + u_i + \frac{h^2 u_i^2}{12} + \frac{3h^4 u_i^3}{1120}$$

Therefore, $S_{N-1} = u_i + \frac{1}{h^2} + \frac{h^2 u_i^2}{12} + \frac{11 h^4 u_i^3}{4032}$

$$= B_1 + A_1 h^4$$

where $B_1 = u_i + \frac{1}{h^2} + \frac{h^2 u_i^2}{12}$ and $A_1 = \frac{11 u_i^3}{4032}$

Therefore, $S_{N-1} = B_1 + O(h^4)$, where $|B_1| = \min S_{N-1}$

From the above we have $B_i \le B_1$ which implies B_i is the minimum value.

Since $0 < \varepsilon << 1$, we can choose h, sufficiently small so that the matrix A is irreducible and monotonic Mohanty and Jha [20]. Then it follows that A^{-1} exists and its elements are non-negative.

Hence, from Eq. (4.23), we get:

$$\mathrm{E} = A^{-1}\tau(h) \tag{4.24}$$

And

$$\|\mathrm{E}\| = \|A^{-1}\| . \|\tau(h)\| \tag{4.25}$$

Let $\overline{m}_{k,i}$ be the (k,i) elements of A^{-1}. Since $\overline{m}_{k,i} \ge 0$, by the definition of multiplication of matrices with its inverses (from the theory of matrices) we have:

$$\sum_{i=1}^{N-1} \overline{m}_{k,i} \, S_i = 1, \; k = 1, 2, 3, \ldots, N-1 \tag{4.26}$$

Therefore, it follows that

$$\sum_{i=1}^{N-1} \overline{m}_{k,i} \le \frac{1}{\min\limits_{1 \le i \le N-1} S_i} = \frac{1}{|B_i|} \tag{4.27}$$

We define $\|A^{-1}\| = \max\limits_{1 \le i \le N-1} \sum_{i=1}^{N-1} |\overline{m}_{k,i}|$ and $\|\tau(h)\| = \max\limits_{1 \le i \le N-1} |\tau_i(h)|$

Therefore, from Eqs.(4.20), (4.23) and (4.27), we obtain:

$$e_j = \sum_{i=1}^{N-1} \overline{m}_{k,i}\, \tau_i(h),\; j = 1, 2, 3, ..., N-1$$

$$e_j \le \frac{1}{|B_i|}\,.\tau_i(h) = \frac{1}{|B_i|}\frac{h^8}{907200}\, y_i^{(10)}$$

Therefore, $e_j \le \dfrac{kh^8}{|B_i|},\; j = 1, 2, 3, ..., N-1$

where $k = \left(\dfrac{1}{907200}\right)\left|y_i^{(10)}\right|$, which is a constant independent of h

Therefore, $\|\mathrm{E}\| \le O(h^8)$.

This implies that the method gives an eighth order convergence for uniform mesh.

4.3. Thomas Algorithm

To solve the tri-diagonal system the description of the Discrete Invariant Impending Algorithm called Thomas Algorithm is presented as follows. Consider the scheme:

$$-E_i y_{i-1} + F_i y_i - G_i y_{i+1} = H_i \tag{4.28}$$

subject to the boundary condition:

$$y(0) = \alpha \text{ , and } y(1) = \beta \tag{4.29}$$

We set

$$y_i = W_i y_{i+1} + T_i\,,\quad i = N-1, N-2, ..., 2, 1 \tag{4.30}$$

where $W_i = W(x_i)$ and $T_i = T(x_i)$ which is to be determined.

Computing Eq. (4.30) at $x = x_{i-1}$, we obtain:

$$y_{i-1} = W_{i-1} y_i + T_{i-1} \tag{4.31}$$

Substituting Eq. (4.31) to (4.28) and comparing with Eq. (4.30) we obtain the recurrence relations:

$$W_i = \frac{G_i}{F_i - E_i W_{i-1}} \tag{4.32}$$

$$T_i = \frac{H_i + E_i T_{i-1}}{F_i - E_i W_{i-1}} \tag{4.33}$$

To solve these recurrence relations for $i = 2,3,..., N-1$ we need to find initial conditions for W_o and T_o. For this, we take $y_o = y(0) = W_o y_1 + T_o$. Choose $W_o = 0$, then the value $T_o = y(0) = \alpha$. With these initial values, we compute W_i and T_i for $i = 2,3,..., N-1$ from Eqs. (4.32) and (4.33) in forward process and obtaining y_i in backward process from Eqs.(4.29) and (4.30).

The conditions for the discrete invariant imbedding algorithm to be stable are, set Angel et al. [1], Elsgolt's et al. [7]:

$$E_i > 0, G_i > 0, F_i \geq E_i + G_i \text{ and } |E_i| \leq |G_i| \tag{4.34}$$

One can easily show that in this method Eq. (4.18) satisfies the conditions given in Eq. (4.34) and hence Thomas Algorithm is stable in this method.

4.4. Numerical Examples

In order to test the validity of the proposed method, we have considered the following model problems.

Example 4.1 Consider the singularly perturbed problem:

$$-\varepsilon y'' + y = x$$

with the boundary conditions: $y(0) = 1, \qquad y(1) = 1 + \exp\left(-\frac{1}{\sqrt{\varepsilon}}\right),$

The exact solution is given by:

$$y(x) = x + \exp\left(\frac{-x}{\sqrt{\varepsilon}}\right)$$

The numerical solutions in terms of maximum absolute errors are given in Table 4.1.

Example 4.2 Consider the singularly perturbed problem:

$$-\varepsilon y'' + y = -\cos^2(\pi x) - 2\varepsilon\pi^2 \cos(2\pi x), \quad 0 \leq x \leq 1$$

with the boundary conditions: $y(0) = 0 \; y(1) = 0$,

The exact solution is given by:

$$y(x) = \frac{e^{\left(\frac{x-1}{\sqrt{\varepsilon}}\right)} + e^{\left(\frac{-x}{\sqrt{\varepsilon}}\right)}}{1 + e^{\left(\frac{-1}{\sqrt{\varepsilon}}\right)}} - \cos^2(\pi x).$$

The maximum absolute errors of are given in Table 4.2.

Example 4.3 Consider the singularly perturbed problem:

$$-\varepsilon y'' + y = 1 - 3x\cos(\pi x)$$

with an boundary condition: $y(0) = y(1) = 0$

The exact solution of the problem is not known. The maximum absolute errors are tabulated in Table 4.3.

4.5. Numerical Results

Table 4.1: The maximum absolute errors $\|E\|$ for Example 4.1

ε	N=16	N=32	N=64	N=128	N=256
Our Method					
1/16	9.8908E-012	3.8192E-014	4.4298E-014	2.0206E-014	7.8404E-013
1/32	1.5796E-010	6.2705E-013	6.8834E-015	5.7732E-014	8.2823E-014
1/64	2.4930E-009	9.9758E-012	3.8691E-014	5.4956E-014	3.8081E-014
1/128	3.6637E-008	1.5803E-010	6.2672E-013	9.6589E-015	6.5503E-014
Rashidinia Method [28]					
1/16	2.96E-006	1.85E-007	1.15E-008	7.24E-010	4.56E-011
1/32	1.18E-005	7.54E-007	4.67E-008	2.96E-009	1.82E-010
1/64	4.74E-005	2.96E-006	1.86E-007	1.16E-008	7.30E-010
1/128	1.78E-004	1.18E-005	7.46E-007	4.67E-008	2.92E-009

Table 4.2: The maximum absolute errors $\|E\|$ for Example 4.2

ε	N=16	N=32	N=64	N=128	N=256
Our Method					
1/16	9.0031E-010	3.5116E-012	7.2164E-015	1.8596E-015	1.0325E-013
1/32	4.8094E-010	1.8783E-012	7.6467E-015	1.2657E-014	2.3148E-014
1/64	2.6577E-009	1.0625E-011	4.2411E-014	2.4425E-014	1.2323E-014
1/128	3.6756E-008	1.5861E-010	6.2550E-013	3.9413E-015	4.1855E-014
Rashidinia Method [28]					
1/16	4.07E-005	2.53E-006	1.58E-007	9.87E-009	6.17E-010
1/32	2.00E-005	1.24E-006	7.74E-008	4.83E-009	3.02E-010
1/64	5.45E-005	3.42E-006	2.14E-007	1.34E-008	8.39E-010
1/128	1.83E-004	1.22E-005	7.68E-007	4.81E-008	3.01E-009
Surla and Herceg and Cvekovic's Method [32]					
1/16	4.14E-003	1.02E-003	2.54E-004	6.35E-005	1.58E-005
1/32	3.68E-003	9.03E-004	5.61E-005	1.40E-005	3.50E-006
1/64	3.45E-003	8.40E-004	2.08E-004	5.20E-005	1.30E-005
1/128	3.43E-003	8.21E-004	2.03E-004	5.06E-005	1.26E-005
Surla and Vukoslavcevic's Method [31]					
1/16	1.20E-004	7.47E-006	4.67E-007	2.90E-008	4.39E-009
1/32	1.28E-004	8.00E-006	5.00E-007	3.14E-008	1.99E-009
1/64	1.60E-004	1.00E-005	6.26E-007	3.92E-008	2.31E-009
1/128	2.344E-004	1.47E-005	9.23E-007	5.77E-008	3.72E-009

The computational rate of convergence is obtained by using the double mesh principle given as below.

Let $Z_h = \max_j \left| y_j^h - y_j^{h/2} \right|, j = 0,1,\ldots, N-1$

where y_j^h is the computed solution on the mesh point $\{x_j\}_0^N$ at the nodal point of x_j for $x_j = x_{j-1} + h, j = 1,2,\ldots, N$ and $y_j^{h/2}$ is the computed solution at the nodal point x_j on the mesh $\{x_j\}_0^{2N}$ where $x_j = x_{j-1} + h/2$ for $j = 1(1)2N$

In the same case we can define $Z_{h/2}$ by replacing h by $h/2$ and N by $2N$ we obtain

$$Z_{h/2} = \max_j \left| y_j^{h/2} - y_j^{h/4} \right|, \; j = 0,1,2,..., N-1$$

The computed order of convergence is evaluated as

$$\text{Rate} = \frac{\log Z_h - \log Z_{h/2}}{\log(2)}$$

Also the maximum absolute error based on double mesh principle mesh principle is given by:

$E_i^N = \max_j \left| y_j^N - y_{2j}^{2N} \right|$, for $j = 0,1,2,..., N$ and $y_j^{h/2}$ denotes the value of y_i for mesh length $h/2$.

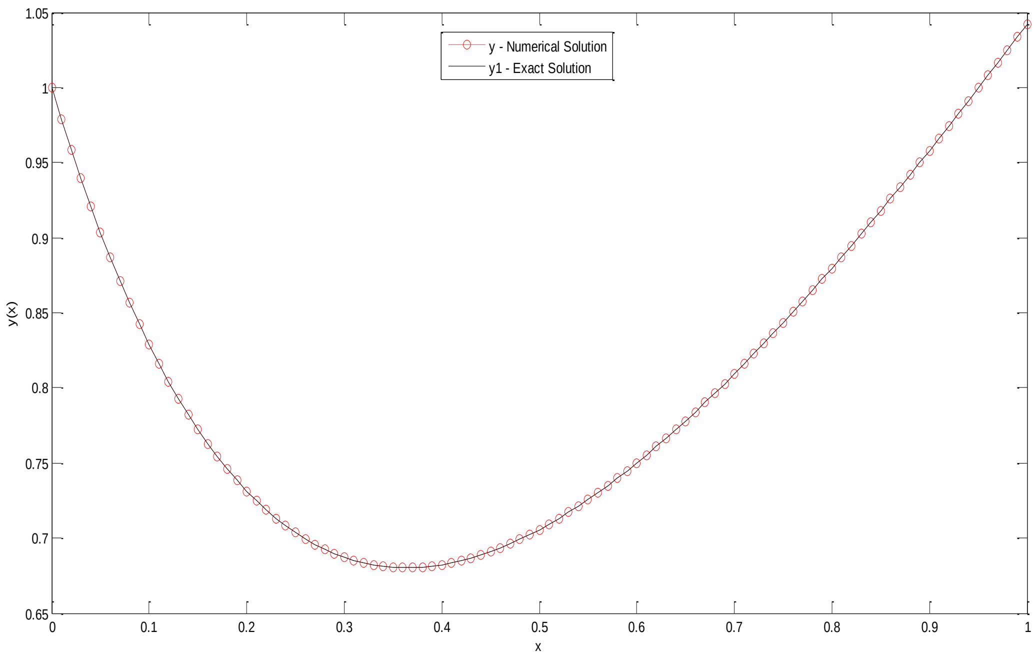
Table 4.3: Maximum Absolute Errors $\|E\|$ for Example 4.3

ε	N=16	N=32	N=64	N=128	N=256	N=512
2^{-4}	2.2121E-011	8.9040E-014	2.2204E-015	7.1054E-015	5.1936E-013	6.3960E-013
2^{-5}	5.2890E-010	2.0908E-012	1.0436E-014	7.9936E-015	3.8858E-014	6.4393E-014
2^{-6}	8.9703E-009	3.5893E-011	1.4033E-013	7.9936E-015	1.1768E-014	3.1686E-013
2^{-7}	1.3813E-007	5.9595E-010	2.3554E-012	1.2434E-014	1.6431E-014	3.7970E-014
2^{-8}	2.2537E-006	9.6555E-009	3.8634E-011	1.4255E-013	1.5676E-013	4.7518E-014
2^{-9}	3.0463E-005	1.4388E-007	6.200E-010	2.4603E-012	6.6169E-014	2.4025E-013

Table 4.4: Numerical rate of convergence for Example 4.1, 4.2 and 4.3 when $\varepsilon = 1/128$

	h	$h/2$	Z_h	$h/4$	$Z_{h/2}$	Rate
Example 4.1	2^{-4}	2^{-5}	3.6479E-008	2^{-6}	1.5740E-010	7.8565
	2^{-5}	2^{-6}	1.5740E-010	2^{-7}	6.1706E-013	7.9948
Example 4.2	2^{-4}	2^{-5}	3.6597E-008	2^{-6}	1.5798E-010	7.8558
	2^{-5}	2^{-6}	1.5798E-010	2^{-7}	6.2156E-013	7.9897
Example 4.3	2^{-4}	2^{-5}	1.3753E-007	2^{-6}	5.9359E-010	7.8561
	2^{-5}	2^{-6}	5.9359E-010	2^{-7}	2.3430E-012	7.9850

The following figures (figures 4.1-4.9) shows the numerical solutions obtained by the present method for $h > \varepsilon$, $h = \varepsilon$ and $h < \varepsilon$.



Figures 4.1: Numerical Solution of Example 4.1 for $\varepsilon = 0.1$ and $h = 0.01$

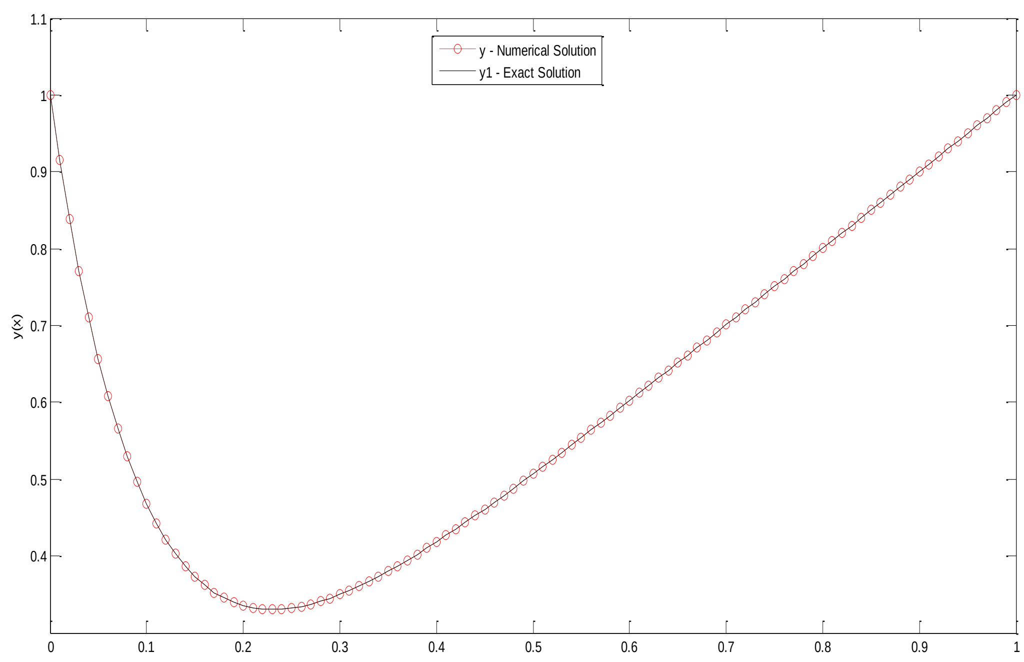


Figure 4.2: Numerical Solution of Example 4.1 for $\varepsilon = 0.01$ and $h = 0.01$

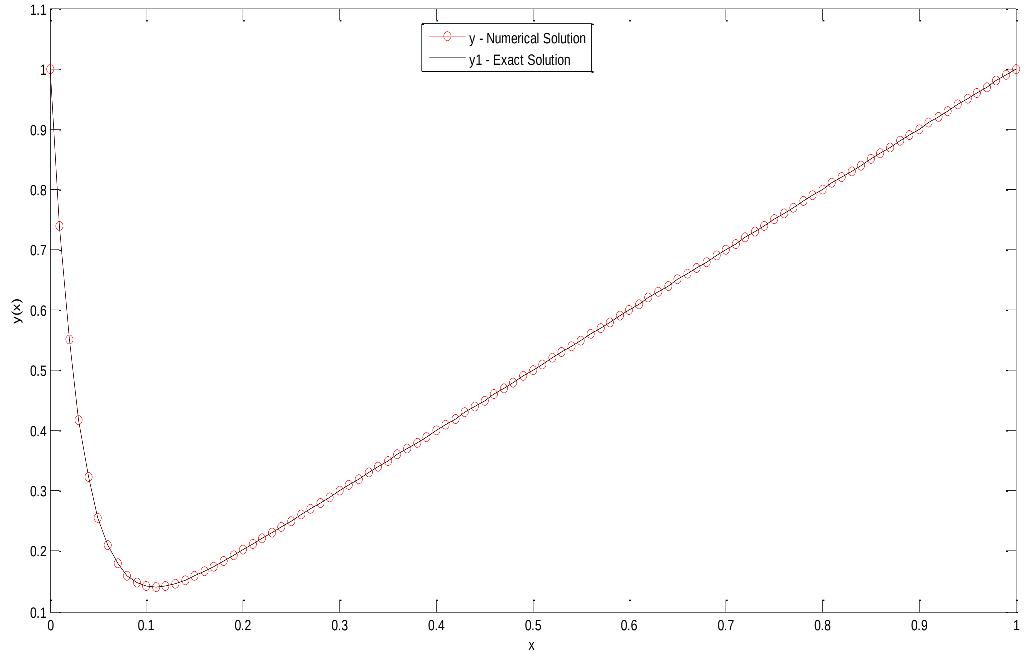


Figure 4.3: Numerical Solution of Example 4.1 for $\varepsilon = 0.001$ and $h = 0.01$

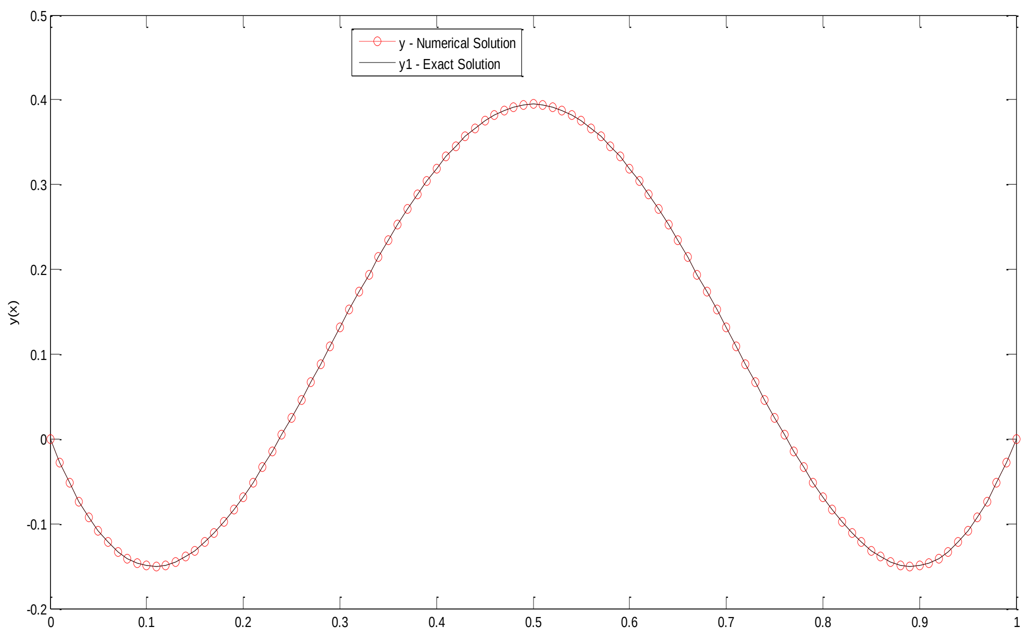


Figure 4.4: Numerical Solution of Example 4.2 for $\varepsilon = 0.1$ and $h = 0.01$

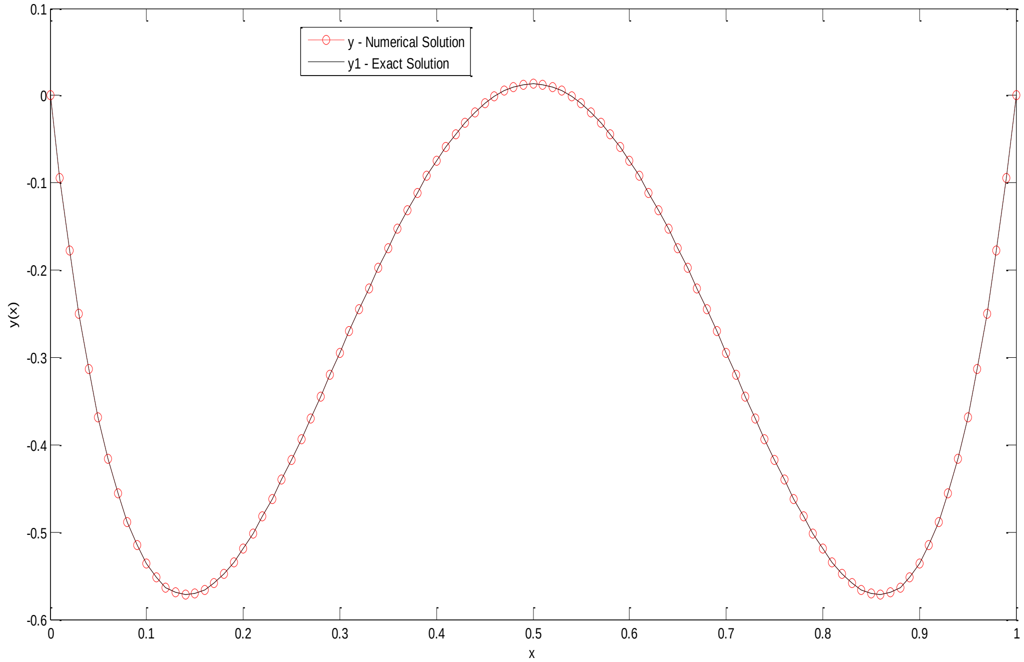


Figure 4.5: Numerical Solution of Example 4.2 for $\varepsilon = 0.01$ and $h = 0.01$

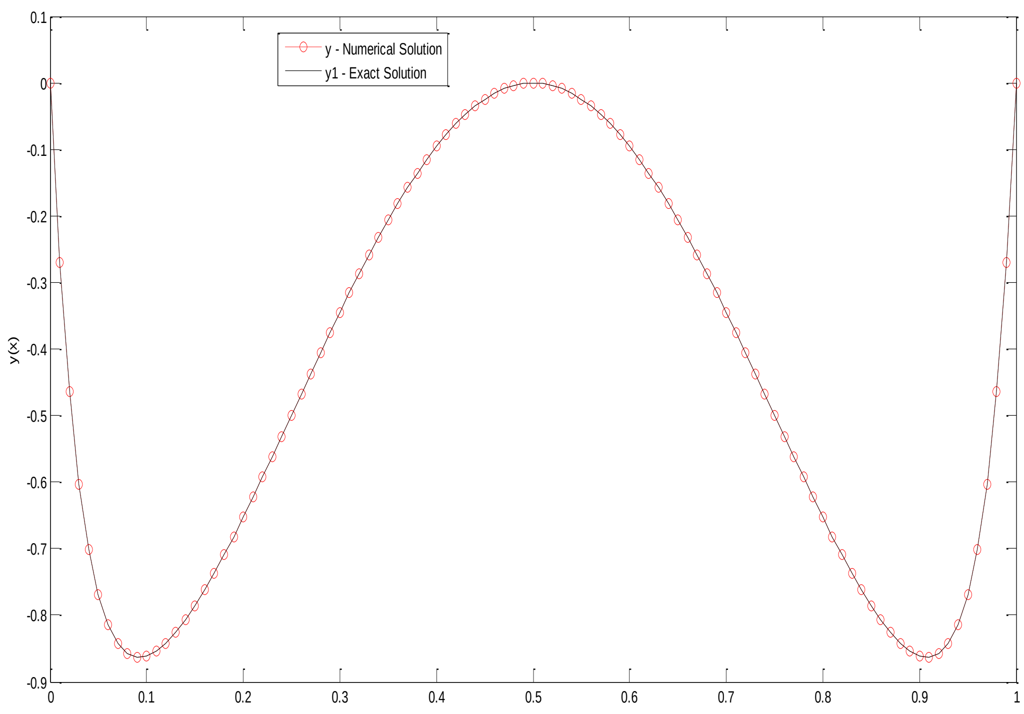


Figure 4.6: Numerical Solution of Example 4.2 for $\varepsilon = 0.001$ and $h = 0.01$

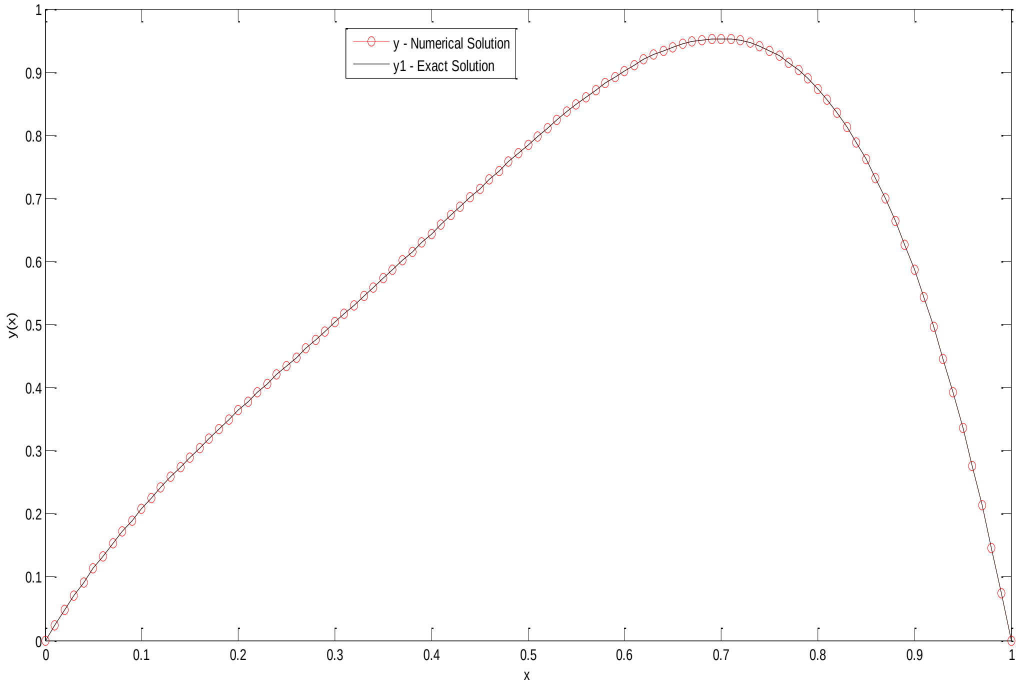


Figure 4.7: Numerical Solution of Example 4.3 for $\varepsilon = 0.1$ and $h = 0.01$

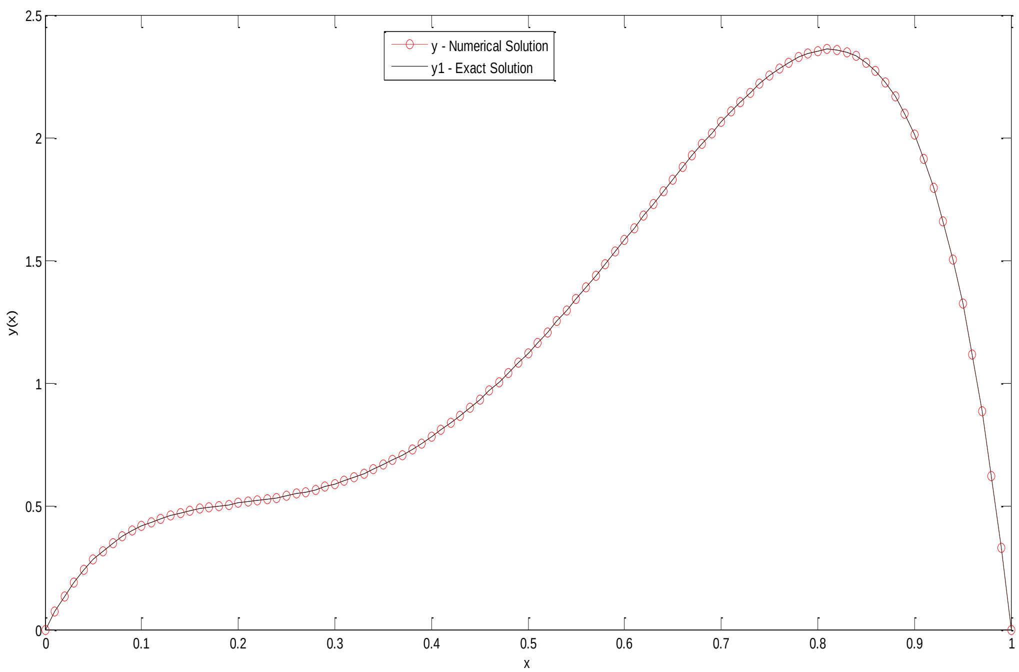


Figure 4.8: Numerical Solution of Example 4.3 for $\varepsilon = 0.01$ and $h = 0.01$

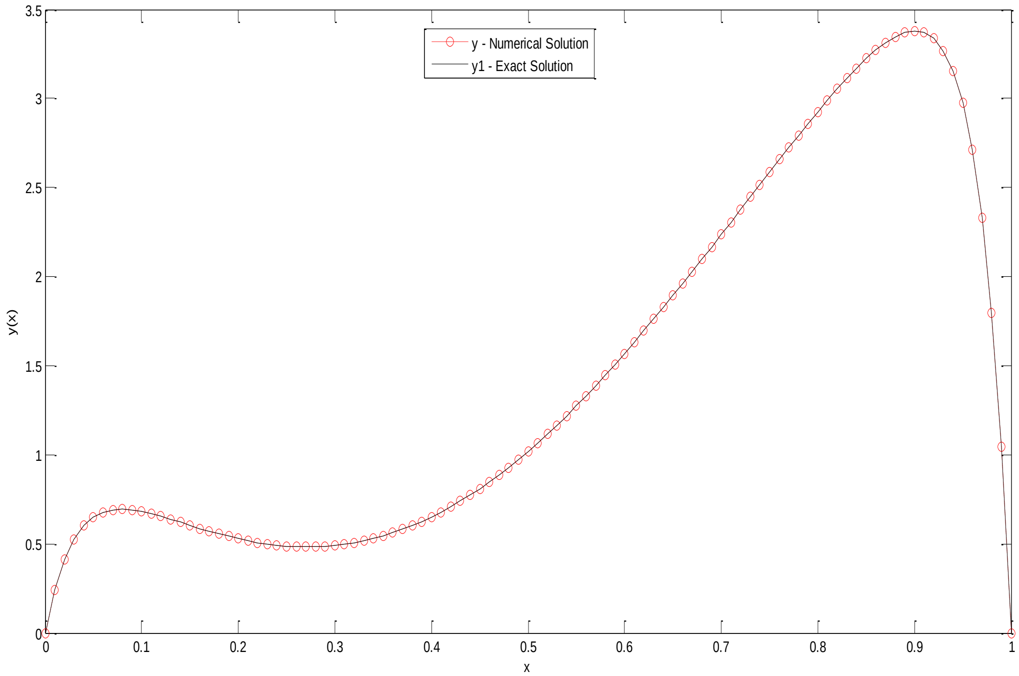


Figure 4.9: Numerical Solution of Example 4.3 for $\varepsilon = 0.001$ and $h = 0.01$

4.6. Discussion

In this thesis, eighth order compact finite difference method has been presented for solving singularly perturbed one dimensional reaction diffusion problems. First, the given interval is discretized and the given differential equation is replaced by finite difference approximations. Then, the given differential equation is transformed to linear systems of algebraic equations and then using Taylor's series and central finite difference approximation, it is reduced to a three term recurrence relation which can be easily solved by using Thomas Algorithm. The results of the present method has been compared with numerical results obtained by Rashidinia et al. [28] and Surla et al. [31-32] which are reported in the literature (See Tables 4.1- 4.3).

As it can be observed from the tables (4.1-4.3), the present method approximates the exact solution better than the methods proposed by Rashidinia et al. [28] and Surla et al. [31-32]. Further, as it can be observed from the tables and graphs the present method approximates the exact solution very well for $h \geq \varepsilon$ for which most of the existing methods fails to give good results. Moreover, all the maximum absolute errors decrease rapidly as N increases.

To validate the applicability of the proposed method, the graphs have been plotted in Figures 4.1-4.6 for exact solutions versus the numerical solutions obtained for different values of provides a good agreement of results presenting exact as well as numerical solutions, which proves the reliability of the compact finite difference method. Figures 4.7-4.9 provides the numerical problem without exact value were evaluated by double mesh principles.

Both the theoretical and numerical error bounds have been established for the method. Table 4.4 shows the present method have the rate of convergence which is in agreement with the theoretical proofs.

CHAPTER FIVE

CONCLUSION AND SCOPE OF FUTURE WORK

5.1. Conclusion

In this thesis, eighth order compact finite difference method has been presented for solving singularly perturbed one dimensional reaction diffusion problems. This study has been implemented on three model examples by taking different values for the perturbation parameter ε and the computational results are presented in the tables and graphs. The results obtained shows the present method approximate the exact solution very well. Further, numerical results presented in this thesis show the improvement of the proposed method over some existing methods reported in the literature.

The results presented in the new method confirmed that the computational rate of convergence as well as theoretical estimates indicate as it is an eighth order convergent. In brief manner, the present method is conceptually simple, easy to use and readily adaptable for computer implementation for solving singularly perturbed one dimensional reaction-diffusion equation.

5.2. Scope of the Future Work

In the present thesis, the numerical method based on eighth order compact finite difference schemes were constructed for solving singularly perturbed one dimensional reaction-diffusion problems. Hence, the schemes proposed in this thesis can also be extended to higher compact finite difference methods for singularly perturbed one dimensional reaction-diffusion problems. And also, this thesis considered the uniform mesh length. So, one can be extended this to non-uniform mesh length. Additionally, this method can also be extended to partial differential equation.

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