JIMMA UNIVERSITY COLLEGE OF NATURAL SCIENCES

DEPARTMENT OF MATHEMATICS

ε - Uniform Numerical Method for Singularly Perturbed 1D Parabolic Convection-Diffusion Problems

A Research Submitted to the Department of Mathematics, Jimma University in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics.

(Numerical Analysis)

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Declaration

I here by declare that this work which is being presented in this thesis titled ε - Uniform Numerical Method for Singularly Perturbed 1D Parabolic Convection-Diffusion Problems is an authentic record of my own work. It has not been submitted elsewhere (Universities or Institutions) for the award of any other degree, and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

Name:

Derartu Ayensa: Sign:__________ Date:__________

This study research has been presented with approval of the advisor

Name:

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Contents

List of Tables

List of Figures

Acronyms

- CDE-convection diffusion equation.
- 1D -One Dimensional.
- MOL-method of lines.
- NSDFM- non-standard finite difference method.
- ODE -Ordinary Differential Equation.
- PDE -Partial Differential Equation.
- SPP - Singularly Perturbed Problem

Abstract

In this thesis , ε - Uniform Numerical Method for solving Singularly Perturbed 1D Parabolic Convection-Diffusion Problems is developed using non-standard finite difference method with Runge-Kutta method by applying the method of lines procedure. First, discretizing the spatial domain using uniform mesh and applying non-standard finite difference methods for the spatial direction of singularly perturbed 1D parabolic convection-diffusion problem. Then, the given differential equation transformed to system of initial value problems(IVP) which is solved by Runge-Kutta method of order two and three implicit. To validate the applicability of the proposed method two model examples were considered and solved for different values of perturbation parameter and mesh sizes. Numerical experiments are carried out extensively to support the theoretical results. The stability is analyzed and the present numerical scheme is of first-order convergence.

Chapter 1

Introduction

1.1 Background of the Study

Numerical analysis is a technique used to solve mathematical problems on a computer and also widely used by scientists and engineers to solve some problems. It does not strive for exactness. Instead, attempts to devise a method which yields an approximation differing from exactness by less than a specified tolerance, or by an amount which has less than a specified probability of exceeding that tolerance. The ultimate aim of the field of numerical analysis is to provide convenient methods for obtaining useful solutions to mathematical problems and for extracting useful information from available solutions which are not expressed in tractable forms. Such problems may each be formulated, for example, in terms of algebraic or transcendental equation, an ordinary or partial differential equation, or in terms of a set of such equations.

A partial differential equation (PDE) is a mathematical equation that involves two or more independent variables, as unknown function (dependent on those variables), and partial derivatives of the unknown function with respect to the independent variables. Linear second order PDEs with two independent variables (x,t) and one dependent variable u has general form of

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial t}+C\frac{\partial^2 u}{\partial t^2}+D=f(x,t) \tag{1.1}$$

where A,B and C are functions of x,t and D is a function of $x,t,u,\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial t}$. If the discriminant $B^2-4AC=0$, it is called parabolic PDE.

The wide use of computing techniques, combined with the demands of scientific and technical practices, has stimulated the development of numerical methods to a great extent, and in particular, methods for solving differential equations. The efficiency of such methods is governed by their accuracy, simplicity in computing the discrete solution and also their relative insensitivity to parameters in the problem. At present, numerical methods for solving partial differential equations, in particular, finite difference scheme, are well developed for wide classes of problems Shishkin(2009).

Differential equations whose highest-order derivative(s) are multiplied by a perturbation parameter $\varepsilon, 0 < \varepsilon << 1$ is called singularly perturbed differential equation O'Malley (1991), Vishik and Lyusternik (1960, 1961)). Solutions of singularly perturbed problems, unlike regular problems, have boundary and/or interior layers, that is, narrow sub domains specified by the parameter on which the solutions vary by a finite value. The derivatives of the solution in these sub domains grow without bound as ε tends to zero.

In the case of singularly perturbed problems, the use of numerical methods developed for solving regular problems leads to errors in the solution that depend on the value of the parameter ε. Errors of the numerical solution depend on the distribution of mesh points and become small only when the effective mesh-size in the layer is much less than the value of the parameter ε(Shishkin ,1992; Miller et al., 1996 and Farrell et al., 2000). Such numerical methods turn out to be in applicable for singularly perturbed problems.

Due to this, there is an interest in the development of special numerical methods where solution errors are independent of the parameter or that converge ε - uniformly. When the solutions of a PDE are ε-uniformly convergent, we call these methods and solutions robust, Farrell .et al, (2000). At present, only few methods are devoted to the development of numerical methods for solving singularly perturbed problems. Grid methods (fitted mesh methods) for partial differential equations are considered in the book of Miller et al., (1996).

Singularly perturbed 1D parabolic convection-diffusion problem has the form:

$$\frac{\partial u}{\partial t}-\varepsilon\frac{\partial^2 u}{\partial x^2}+a(x)\frac{\partial u}{\partial x}+b(x)u(x,t)=f(x,t),\ \ (x,t)\in\Omega_x\times Q=(0,1)\times(0,T] \tag{1.2}$$

with the boundary conditions and the initial condition

$$\begin{aligned} u(0,t)=&\mu_0(t),\quad t\in[0,T]\\ u(1,t)=&\mu_1(t),\quad t\in[0,T]\\ &\text{and}\\ u(x,0)=&\phi(x),\quad x\in[0,1] \end{aligned} \tag{1.3}$$

where ε is the perturbation parameter such that $0<\varepsilon\ll 1$ the coefficient functions $a(x),b(x)$ and $f(x,t)$ are sufficiently smooth. The convection-diffusion-reaction equation is classified into three processes Makungu et al. (2012). The first process is called convection and is due to movement of materials from one region to another. The second process is called diffusion and is due to movement of materials from region of high concentration to a region of low concentration. The last process is called reaction and is due to decay, adsorption and reaction of substances with other components. The convection-diffusion-reaction PDE provides a very useful and important mathematical model in wide range of applications in natural sciences and engineering Mickens et al., (1999). These applications includes the transport of air, adsorption of pollutants in soil, diffusion of neutrons, food processing, modeling of biological systems, modeling of semiconductors, oil reservoir flow transport and reaction of chemical species etc. In many of these applications, the unknown variables in the governing PDE represent physical quantities that cannot take negative values such as pollutants, population, and concentration of chemical compounds Chen-Charpentier and Kojouharov, (2013).

Singularly Perturbed Parabolic Equation Models

In the real life, there are many singular perturbation models which arise in parabolic partial differential equations. We leave out the techniques used to solve these models. Interested readers can see the references cited along with these models, for more details.

a. Think about the time-dependent Navier-Stokes problem in two space variables x and y (Roos et al., 2008) given by:

$$\frac{\partial u}{\partial t} - \frac{1}{Re}\triangle(U.\nabla)u = -\nabla_p \tag{1.4}$$

In the upper half-plane $y > 0$.

$$\nabla.U = 0, \tag{1.5}$$

in the same domain

$$u = 0 \tag{1.6}$$

on the boundary $y = 0$ at large Reynolds number Re. One can regard the boundary $y = 0$ as a fixed plate,and we assume that the velocity U at $y = \infty$ is parallel to the x-axis with magnitude U. We seek as flow, at constant pressure p whose velocity is parallel to the plate and independent of x. Then Eq.(1.4) requires to:

$$\frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} \tag{1.7}$$

where $\varepsilon = \frac{1}{Re}$ Set $\Upsilon = \frac{y}{2\sqrt{\varepsilon t}}$ and let $u(y,t) = Uf(\Upsilon)$.A complication leads to:

$$u = U\frac{2}{\sqrt{\pi}}\int_0^{\Upsilon} \rho^{s^{-2}} ds \tag{1.8}$$

Eqs.(1.8) show that there is a narrow region near $y = 0$ where u departs significantly from the constant flow U.We say that u has a boundary layer at $y = 0$. Linearizion of Eqs.(1.5-1.7) yields an equation of the form:

$$\frac{\partial U}{\partial t} - \varepsilon\triangle u + b\nabla u + cu = f \tag{1.9}$$

where b is independent of u. Such convection-diffusions model many fluid flows;the appear in the well know ocean equation and in related subjects like water pollution problems,simulation of oil extraction from underground preservers,flows in chemical reactors and convective heat transport problems with large p'eclet numbers(Roos et al.,2008).

b. Ground water flow and solute transport

The movement of water and solutes through the unsaturated zone has been of importance in traditional applications of ground water hydrology, soil physics and agronomy.

In one dimension, the theoretical basis for modeling the liquid phase water movement in unsaturated porous media can be described by a combination of the darcy's law and the equation of continuity (Kadalbajoo and Gupta, 2010)

$$\frac{\partial c(x,t)}{\partial t} = D\frac{\partial^2 c(x,t)}{\partial x^2} - V\frac{\partial c(x,t)}{\partial x} - \lambda c(x,t), \forall_{x,t} > 0 \tag{1.10}$$

where t is time,x is horizontal distance taken zero at the soil center and measured positive to the right of the soil center;c(x,t) is the solute concentration at time t;distance x; D is the soil water diffusivity; v is the average velocity and λ is the decay coefficient.The contamination in ground water can be calculated by means of Eqs.(1.10).

The solute transport Eqs.(1.10) represents the mathematical modeling for the unknown concentration $c(x,t)$. We now scale this mathematical problem by selecting the characteristic values for the dependent and independent variables.

Consequently,we define dimensionless variables by:

$$T = \frac{t}{\lambda^{-1}}, X = \frac{x}{v\lambda^{-1}}, C = \frac{c}{c_0} \tag{1.11}$$

Reformulating the problem in terms of these scaled variables easily gives the scaled problem.

$$\frac{\partial c(x,t)}{\partial t} = \varepsilon\frac{\partial^2 c(x,t)}{\partial x^2} - v\frac{\partial c(x,t)}{\partial x} - c(x,t), \forall_{x,t} > 0, \quad \text{where} \quad \varepsilon = \frac{\lambda D}{v^2} \ll 1 \tag{1.12}$$

Definition: ε - uniform convergence

Consider (P_ε) be a family of singularly perturbed parabolic PDEs parametrized by a singular perturbation parameter ε, where ε satisfies $0 < \varepsilon \ll 1$. Assume that each problem in (P_ε) has unique solution denoted by u_ε, and that each u_ε is approximated by sequence of numerical solutions

$\{(U_\varepsilon, \bar{D}^{M,\Delta t})\}$ obtained by using a monotone numerical method $(P_\varepsilon^{M,\Delta t})$ where U_ε is defined on the mesh $\bar{D}^{M,\Delta t}$ and; M and Δt are discretization parameters. Then U_ε is said to converge $\varepsilon-$ uniformly to the exact solution u_ε, if there exists positive integers M_0, K_0 and positive numbers C, p and q, such that for all $M \geq M_0$ and $K \geq K_0$, where $K = T/\Delta t$, we have

$$\sup_{0<\varepsilon\leq 1} \|U_\varepsilon - u_\varepsilon\|_\infty \leq C\left(M^{-q} + (\Delta t)^p\right) \tag{1.13}$$

where M_0, K_0, C, p and q are all independent of ε. Here p and q are called the $\varepsilon-$ uniform order of convergence of the temporal and spatial direction respectively, and C is called the $\varepsilon-$ uniform error constant.

The Method of Lines (MOL) is a technique that enables us to convert partial differential equations into sets of ordinary differential equations that, in some sense, are equivalent to the former PDEs. The basic idea behind the MOL methodology is straight forward. The method of lines is a general way of viewing a partial differential equation as a system of ordinary differential equations . The partial derivatives with respect to the space variables are discretized to obtain a system of ODE's in the variable t .

The theoretical basis of non-standard discrete modeling method is based on the concept of"exact" and "best" finite difference schemes. Mickens (2005) presented techniques for constructing non-standard finite difference methods. According to Mickens rules, to construct a discrete scheme, denominator function for the discrete derivatives must be expressed in terms of more complicated functions of step sizes than those used in the standard procedure. These complicated functions constitutes a general property of these schemes, which is useful while designing reliable schemes for such problems. On the spatial domain [0, 1], uniform meshes with mesh length is introduced, where N is the number of mesh points in spatial direction. And then apply non-standard finite difference method for the spatial derivatives.

In recent years, different authors developed different numerical methods for solving such differential equations. It is well known that classical numerical methods for solving singular perturbation problems are unstable and fail to give accurate results when the perturbation parameter ε is small. But, still the accuracy and convergence of the methods need attention, because of the treatment of singular perturbation problems is not trivial and the solution depends on perturbation parameter and mesh size h, Doolan et al., (1980). Due to this, numerical treatment of singularly perturbed 1D parabolic convection-diffusion problems needs improvement. The presence of the singular perturbation parameter ε, leads to occurrences of oscillations or divergence in the computed solutions while using classical numerical methods. In order to avoid these oscillations or divergence, an unacceptably large number of mesh points are required when ε is very small.

Therefore, in order to overcome this drawback associated with classical numerical methods, we need to develop a method based on method of lines (MOL) using non-standard finite difference method in spatial direction together with Runge-Kutta method of order two and three implicit for temporal direction, which treat the problem without creating an oscillation. Thus, this study present an accurate and ε-uniform convergent numerical method for solving singular perturbation 1D parabolic convection-diffusion problem using methods of lines with non-standard finite difference method.

1.2 Objectives of the study

1.2.1 General objective

General objective of this study is to solve singularly perturbed 1D parabolic convection-diffusion problem using methods of lines with non-standard finite difference method.

1.2.2 Specific Objectives

The specific objectives of this study are:

- To apply MOL with non-standard finite difference method on singularly perturbed 1D parabolic convection-diffusion problems.

- To establish the ε-uniform convergence of the present scheme .

1.3 Significance of the Study

The result obtained from this study may:

- be used as a reference material for scholars who works on this area.
- help graduate students to acquire research skills and scientific procedures.
- be used a numerical method for solving singularly perturbed parabolic convection-diffusion problems.

1.4 Delimitation of the Study

This study is delimited to solve the singularly perturbed 1D parabolic convection-diffusion problems of the form:

$$\frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} + a(x)\frac{\partial u}{\partial x} + b(x)u(x,t) = f(x,t), \ \ (x,t) \in \Omega_x \times Q = (0,1) \times (0,T] \tag{1.14}$$

with the boundary conditions and the initial condition

$$\begin{aligned} u(0,t) =& \mu_0(t), \quad t \in [0,T] \\ u(1,t) =& \mu_1(t), \quad t \in [0,T] \\ & \text{and} \\ u(x,0) =& \phi(x), \quad x \in [0,1] \end{aligned} \tag{1.15}$$

where ε is the perturbation parameter such that $0 < \varepsilon \ll 1$ the coefficient functions $a(x), b(x)$ and $f(x,t)$ are sufficiently smooth and satisfy the following. $a(x) \geq \alpha > 0, \ b(x) \geq \beta > 0, \ \forall x \in [0,1]$. In general, the problem in Eqn. (1.14) - (1.15) admits a unique solution $u(x,t)$ which exhibits a regular boundary layer of width $O(\varepsilon)$ at $x = 1$.

Chapter 2

Literature Review

Ng-Stynes et al. (1998) presented the numerical methods for time-dependent convection-diffusion equations. The authors consider the initial-boundary value problem: $\varepsilon y_{xx} + a(x,t)y_x - b(x,t)y - d(x,t)y_t = f(x,t), (x,t) \in \Omega = (0,1) \times (1,T]$ $y(0,1) = q_0(t)$ for $0 \leq t \leq T$, $y(1,t) = q_1(t)$ for $0 \leq t \leq T, y(x,0) = s(x)$ for $0 \leq t \leq 1$, and a, b, d and f are sufficiently smooth with $\alpha^* \geq a(x,t) \geq \alpha > 0, \beta^* \geq b(x,t) \geq \beta > 0, \delta^* \geq d(x,t) \geq \delta > 0, \text{on}[0,1] \times [0,T]$. In this article,they examined a singularly perturbed linear parabolic initial-boundary value problem in one space variable. Various finite difference schemes are derived for this problem using a semi-discrete Petrov-Galerkin and finite element methods. The schemes do not have a cell Reynolds number restriction and are shown to be first-order accurate, uniformly in the perturbation parameter.

Clavero et al. (2003), proposed “A uniformly convergent scheme on a nonuniform mesh for convection diffusion parabolic problems”. The authors consider the problem:
$\frac{\partial U}{\partial t} - \varepsilon \frac{\partial^2 U}{\partial x^2} + a(x)\frac{\partial U}{\partial x} + b(x)U = f(x,t), (x,t) \in D \equiv \Omega \times (0,1) \equiv (0,1) \times (0,T), u(x,0) = u_0(x), x \in \Omega, u(0,t) = u(1,t) = 0, t \in [0,T]$. In this paper they constructed a numerical method to solve one-dimensional time-dependent convection diffusion problem with dominating convection term. They use the classical Euler implicit method for the time discretization and the simple upwind scheme on a special nonuniform mesh for the spatial discretization. They show that the resulting method is uniformly convergent with respect to the diffusion parameter. The main lines for the analysis of the uniform convergence carried out here can be used for the study of more general singular perturbation problems and also for more complicated numerical schemes.

Ramos et al., (2005) studied “An exponentially-fitted method for singularly perturbed, one dimensional, parabolic problems”. They consider the following singularly perturbed, one dimensional (linear) parabolic problem of the advection diffusion reaction type: $\frac{\partial U}{\partial t} + a(x,t)\frac{\partial U}{\partial x} + b(x,t)U = \varepsilon\frac{\partial^2 u}{\partial x^2} + f(x,t), 0 < x < 1, t \geq 0$ subject to $u(0,t) = u(1,t) = 0, t \geq 0, U(x,0) = u_0, 0 \leq x \leq 1$ where $0 < \varepsilon \ll 1$ is the diffusion coefficient or perturbation parameter,x and t denote the spatial coordinate and time, respectively, u is the dependent variable, $a(x,t)$; is the speed, and $f(x,t) - b(x,t)u$; is the reaction term. Those authors had been proposed the method based on an exponentially-fitted method for singularly perturbed, one-dimensional, linear, convection-diffusion-reaction equations in equally-spaced grids. The method is based on the implicit discretization of the time derivative, freezing of the coefficients of the resulting ordinary differential equations at each time step, and the analytical solution of the resulting convection-diffusion differential operator. This solution is of exponential type and exact for steady, constant-coefficients convection- diffusion equations with constant sources.

Kadalbajoo and Awasthi (2006) presented a parameter uniform difference scheme for singularly perturbed parabolic problem in one space dimension. They consider the following singularly perturbed parabolic problem: $L_\varepsilon u(x,t) \equiv u_t - \varepsilon u_{xx} + a(x)u_x + b(x)u = f(x,t), (x,t) \in \Omega$, where $\Omega = (0,1) \times (0,T]$ and $\partial\Omega = \bar{\Omega}/\Omega$, with initial condition $u(x,0) = u_0(x), 0 \leq x \leq 1$ and boundary conditions $u(0,t) = 0 = u(1,t), 0 \leq t \leq T$. They made a numerical study to examine a singularly perturbed parabolic initial-boundary value problem in one space dimension on a rectangular domain. The solution of this problem exhibits the boundary layer on the right side of the domain. The Crank-Nicholson finite difference method consisting of an upwind finite difference operator on a fitted piecewise uniform mesh is constructed. The resulting method has been shown almost first order accurate in space and second order in time. The authors have shown that the resulting method is uniformly convergent with respect to the singular perturbation parameter. It is shown that a numerical method consisting of same finite difference operator on uniform mesh does not converge uniformly with respect to the singular perturbation parameter.

Kadalbajoo et al., (2008) proposed“A uniformly convergent B-spline collocation method on a nonuniform mesh for singularly perturbed one-dimensional time-dependent linear convection-diffusion problem. They consider the 1D parabolic convection -diffusion problem: $\frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} + a(x)\frac{\partial}{\partial x} + b(x)u = f(x,t), (x,t) \in D \equiv \Omega \times (0,T] \equiv (0,1) \times (0,t], u(x,0) = u_0(x), x \in \bar{\Omega}, u(0,t) = 0, u(1,t) = 0, t \in [0,t]$. They proposed a numerical method for solving this problems. The method comprises a standard implicit finite difference scheme to discretize in temporal direction on a uniform mesh by means of Rothes method and B-spline collocation method in spatial direction on a piecewise uniform mesh of Shishkin type. The method is shown to be unconditionally stable and accurate of order $O((\triangle x)^2 + \triangle t)$. An extensive amount of analysis has been carried out to prove the uniform convergence with respect to the singular perturbation parameter. Several numerical experiments have been carried out in support of the theoretical results.

Rashidinia et al., (2013), presented “Application of Sinc-Galerkin method to singularly perturbed parabolic convection-diffusion problems. They consider the singularly perturbed problem of the form: $\frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 U}{\partial x^2} + b(x,t)\frac{\partial U}{\partial x} + d(x,t)U = f(x,t)$ subject to the initial and boundary condition $u(x,0) = s(x), u(0,t) = q_0(t)$ and $u(1,t) = q_1(t)$. In this article, they apply the Sinc-Galerkin method to solve the stated problems first their method is based on the discretization of the time variable by means of the implicit Euler method and freezing the coefficient of the resulting ordinary differential equation at each time step. Second they use Sinc-Galerkin method on the yield linear ordinary differential equation at each time step resulting from the time semi-discretization. In the Sinc-Galerkin method the test functions are defined by the Sinc-function $S(x) = sin(\frac{\pi x}{\pi x})$. This method has many advantages over classical methods that use polynomials as bases. As they mentioned, in the presence of singularities, it gives a much better rate of convergence and accuracy than polynomials method. Those scholars shown and concluded that the convergence analysis and stability of the proposed method are presented with an exponential convergence was achieved as well. But, even if the convergence analysis of the proposed method shown as an exponential convergence was achieved as well, there is no confirmation of theoretical with experimental results.

Suayip and Niyazi (2013), presented “Numerical solutions of singularly perturbed one dimensional parabolic convection diffusion problems by the Bessel collocation method. They considered the problem: $\frac{\partial u}{\partial t}-\varepsilon\frac{\partial^2 U}{\partial x^2}+b(x,t)\frac{\partial U}{\partial x}+d(x,t)U=f(x,t)$ subject to the initial and boundary condition $u(x,0)=s(x),u(0,t)=q_0(t)\text{and}u(1,t)=q_1(t)$. The authors proposed the method based on the Bessel collocation method used for some problems of ordinary differential equations. The method was implemented within the following procedure: first the approximate solution of the problem in the truncated Bessel series form was obtained by this method. Secondly, substituting the truncated Bessel series solution into the problem and then by using the matrix operations and the collocation points; the suggested scheme reduces the problem to a linear algebraic equation system. Finally, by solving these algebraic equations, the unknown Bessel coefficients computed. An error estimation technique is given for the considered problem and the method. To show the accuracy and the efficiency of the method, two model numerical examples are implemented and the numerical result comparisons are given with the other methods that are developed by Clavero et al., (2003); Ramos (2005) and Kadalbajoo et. al., (2008). In this article, the comparison of numerical results for the particular model example given as: $\frac{\partial u}{\partial t}-\varepsilon\frac{\partial^2 U}{\partial x^2}+(2-x^2))\frac{\partial U}{\partial x}+xU=10t^2\exp(-t)x(1-x),(x,t)\in(0,1)\times(0,1]$subject to the condition $U(x,0)=0,0\leq x\leq 1\text{and}U(0,t)=0=U(1,t),0<t\leq 1$. The exact solution of this problem is not known. But from the comparison for the perturbation parameter 0^{-6},the maximum absolute error given with the method developed by Suayip and Niyazi (2013) is $4.2727*10^{-02}$, Kadalbajoo et al., (2008) is $3.7566*10^{-03}$ and the method developed by Ramos (2005) is $2.25*10^{-02}$.This implies that, though the authors try to improve the accuracy and the efficiency of the method, the proposed method is not more accurate than previously developed methods.

Gowrisankar and Natesan ,(2014) proposed “Robust numerical scheme for singularly perturbed convection diffusion parabolic initial boundary value problems on equidistributed grids which studies the numerical solution of singularly perturbed parabolic convection diffusion problems exhibiting regular boundary layers. To solve these problems, they use the classical upwind finite difference scheme on layer-adapted non-uniform meshes. The non-uniform meshes are obtained by equidistributing a positive monitor function, which depends on the second-order spatial derivative of the singular component of the solution. The truncation error and the stability analysis are obtained with the convergence of first order convergent. Parameter-uniform error estimates are derived for the numerical solution and this scheme is also appropriate to solve the linear and semi-linear initial boundary value problems. For numerical experiments, they consider four (two linear and the others are non-linear) model examples are carried out to support the theoretical results. From the considered linear model example, one is the same with the example considered by Suayip and Niyazi (2013), and the second example is given by: $\frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 U}{\partial x^2} + (1x(1-x))\frac{\partial U}{\partial x} = f(x,t), (x,t) \in (0,1) \times (0,1]$subject to the condition $U(x,0) = U_0(x), 0 \leq x \leq 1andU(0,t) = 0 = U(1,t), 0 < t \leq 1$. For this particular example the initial data $U_0(x)$ and the source function $f(x,t)$ to fit the exact solution $U(x,t) = \exp(t)(\exp(-1/\varepsilon) + x(1-\exp(-1/\varepsilon)) - exp(-1+x/\varepsilon))$. The authors confirmed the theoretical and experimental results of the proposed method for the two linear examples with obtaining $9.2101 * 10^{02}$ maximum absolute errors and 1.1019 maximum rate of convergence at different values of mesh sizes formed by number of intervals $32 \leq N \leq 1024$ and time step size $\frac{1}{10} \leq \triangle t \leq \frac{1}{320}$ for the perturbation parameter, $10^{-08} \leq \varepsilon \leq 10^{-02}$. Since, the numerical results obtained by those authors had been compared with Shishkin and Bakhvalov meshes; it verify that, the recent one can be used as an alternative mesh generating rather than more efficient than the two meshes.

Das and Natesan, (2015) presented uniformly convergent hybrid numerical scheme for singularly perturbed delay parabolic convection-diffusion problems on Shishkin mesh. A simplified mathematical description of the overall control system is given by: $\frac{\partial u(x,t)}{\partial t} = \varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} + v(g(u(x,t-\tau)))\frac{\partial u(x,t)}{\partial x} + c[f(u(x,t-\tau)) - u(x,t)]$ defined on a one dimensional spatial domain $0 < x < 1$, where v is the instantaneous material strip velocity depending on a prescribed spatial average of the time-delayed temperature distribution $u(x,t-\tau)$ and f represents a distributed temperature source function depending on $u(x,t-\tau)$. This article studies the numerical solution of singularly perturbed delay parabolic convection- diffusion initial-boundary-value problems. Since the solution of these problems exhibit regular boundary layers in the spatial variable, the authors use the piecewise-uniform Shishkin mesh for the discretization of the domain in the spatial direction, and uniform mesh in the temporal direction. The time derivative is discretized by the implicit-Euler scheme and the spatial derivatives are discretized by the hybrid scheme. For the proposed scheme, the stability analysis was carried out, and parameter-uniform error estimates are derived. Numerical examples are given to show the accuracy and efficiency of the scheme.

Munyakazi (2015) presented "A robust finite difference method for two parameter parabolic convection-diffusion problems. They author considered the singularly perturbed problem: $\frac{\partial u}{\partial t} - \varepsilon\frac{\partial^2 U}{\partial x^2} + \mu b(x,t)\frac{\partial U}{\partial x} + d(x,t)U = f(x,t)$ subject to the initial and boundary condition $u(x,0) = s(x), u(0,t) = q_0(t)$ and $u(1,t) = q_1(t)$ To implement this method, the basic procedures are; first discretize the time variable by means of the classical backward Euler method and at each time level a two-point boundary value problem is obtained. Second, these problems are, in turn, discretized in space on a uniform mesh following the nonstandard methodology of Mickens and then discrete operator satisfies a minimum principle. Third, the error analysis shows that the method is uniformly convergent with respect to the perturbation parameters. Finally, validate the developed numerical scheme compared with Shishkin (1988), and the experimental results to test the parameter-uniform convergence and the comparison with Kadalbajoo and Yadaw (2012) by considering only the particular example given as: $\varepsilon\frac{\partial^2 U}{\partial x^2} + \mu(1+x)\frac{\partial U}{\partial x} - U(x,t) - \frac{\partial U}{\partial t} = 16x^2(1-x)^2, (x,t) \in (0,1)\times(0,1]$ subject to the initial and boundary conditions: $u(x,0) = u_0(x), x \in [0,1] \text{and} u(0,t) = 0 = u(1,t), t \in [0,1]$ respectively. Since, the exact solution for this example is not known; they use a variant of the double mesh principle. The comparison of numerical results with respect to accuracy (with the maximum absolute error is $1.49 * 10^{-03}$ and order of convergence (the maximum and minimum order convergence are 1.06 and 1.02) are presented as via schemes at the perturbation parameters $\varepsilon = 2^{-5}$ and $\mu \leq 2^{-6}$ within the range of number of intervals $128 \leq N \leq 1024$ in the space direction and number of intervals $M = 2N$ in time direction. Using the numerical experiment on this model example, they had shown that the numerical scheme developed by Munyakazi (2015) approximate the exact solution very well than Kadalbajoo and Yadaw (2012). Yet, there is no more favored method identified and there is still the need to construct better methods than those which are available.

Yanping and Li-Bin(2016) presented "An Adaptive Grid Method for Singularly Perturbed

Time-Dependent Convection-Diffusion Problems".They consider the following singularly perturbed time-dependent convection-diffusion problem: $u_t(x,t)+L_{x,\varepsilon}(x,t) = f(x,t),(x,t) \in G = \Omega\times(0,T] \equiv (0,1)\times(0,T], u(x,0) = u_0(x), x\in\Omega, u(0,t) = u(1,t) = 0, t \in (0,T]$ where $L_{x,\varepsilon} \equiv -\varepsilon u_{xx} + a(x)u_x + b(x)u$ The authors study the numerical solution of singularly perturbed time dependent convection-diffusion problems. To solve these problems, the backward Euler method is first applied to discretize the time derivative on a uniform mesh, and the classical upwind finite difference scheme is used to approximate the spatial derivative on an arbitrary nonuniform grid. Then, in order to obtain an adaptive grid for all temporal levels, they construct a positive monitor function, which is similar to the arclength monitor function. Furthermore, the ε-uniform convergence of the fully discrete scheme is derived for the numerical solution. Finally, some numerical results are given to support our theoretical results.

Chandru et al., (2017) presented "Numerical treatment of two-parameter singularly perturbed parabolic convection diffusion problems with non-smooth data". They considered the following two-parameter parabolic initial-boundary value problem (IBVP) on the domain $\Gamma = \Omega_x \times \Omega_t$, which combines the reaction-diffusion and convection-diffusion forms: $L_\varepsilon, \mu y(x,t) \equiv (\varepsilon y_{xx} + \mu a y_x - b y_x - c y_t)(x,t) = f(x,t), (x,t) \in (\Gamma^- \cup \Gamma^+), y(x,t) = p(x,t), (x,t) \in \Gamma_c$,
$y(x,t) = q(x,t), (x,t) \in \Gamma_l, y(x,t) = r(x,t), (x,t) \in \Gamma_r$. Here, $0 < \varepsilon, 0 \le \mu \le 1$ are two singular perturbation parameters. The coefficient functions $b(x,t), c(x,t)$ are assumed to be sufficiently smooth functions on Γ such that $b(x,t) \ge \beta > 0, c(x,t) \ge \nu > 0$. In additional,they assume $a(x,t), f(x,t)$ are sufficiently smooth on $(\Gamma^- \times \Gamma^+)$ such that $a(x,t) \le -\alpha_1 < 0, (x,t) \in \Gamma^-$ and $a(x,t) \ge \alpha_2 > 0, (x,t) \in \Gamma^+$ Here, α_1, α_2 are positive constants. Let $\alpha = \min \alpha_1, \alpha_2$. In addition, They assume the jumps of $a(x,t)$ and $f(x,t)$ at $d(x,t)$ satisfying $|[\alpha](d,t)| \le c, |[f](d,t)| \le c$, where the jump of ω at (d,t) is defined as $[\omega](d,t) = \omega(d_+,t) - \omega(d_-,t)$. In this paper, they consider a parabolic convection-diffusion-reaction problem where the diffusion and convection terms are multiplied by two small parameters, respectively. In addition, the author assume that the convection coefficient and the source term of the partial differential equation have a jump discontinuity. The presence of perturbation parameters leads to the boundary and interior layers phenomena

whose appropriate numerical approximation is the main goal of this paper. they have developed a uniform numerical method, which converges almost linearly in space and time on a piecewise uniform space adaptive Shishkin-type mesh and uniform mesh in time. Error tables based on several examples show the convergence of the numerical solutions. In addition, several numerical simulations are presented to show the effectiveness of resolving layer behavior and their locations.

As we see in the above literature, most researchers try to find numerical solution for singularly perturbed 1D parabolic convection diffusion problems. In this thesis, we tried to develop more accurate and ε-uniformly convergent numerical method for this problem. We used methods of lines with non standard finite difference method for solving singular perturbed 1D parabolic convection-diffusion problems.

Chapter 3

Methodology

3.1 Study Site and Period

This study was conducted at Jimma University in the department of Mathematics from September 2018 to June 2019.

3.2 Study Design

The study employed both documentary review and numerical experimentation .

3.3 Source of Information

The relevant sources of information such as; books, published articles and related studies from internet are used.

3.4 Mathematical Procedures

In order to achieve the stated objectives, the study followed the following mathematical procedures.

1. Defining the problem.

2. Discretizing the spatial domain using uniform mesh.

3. Applying non-standard finite difference methods for the spatial direction of singularly perturbed 1D parabolic convection-diffusion problem.

4. Solving the obtained system of initial value problems by using implicit Runge-Kutta method of order two and three .

5. Establishing ε-uniform convergence of the obtained scheme.

6. Writing MATLAB code for the developed scheme.

7. Validate the schemes by using numerical examples and results.

8. Compare the obtained result with the finding of previous studies.

Chapter 4

Description of the Method, Result and Discussion

4.1 Description of the Method

4.1.1 Properties of Analytical solution

In order to show on the bounds of the solution $u(x,t)$, we assume,without loss of generality the initial to be zero (Bobisud et.al,1968). Since $u_0(x)$ is sufficiently smooth and using the property of the norm, we can prove the following lemma.

Lemma 1: (Continuous maximum principle)

let $\psi \in C^{2,1}(\bar{D})$ and be such that $\psi \geq 0, \forall (x,t) \in \partial D$. Then $L_\varepsilon \psi(x,t) > 0, \forall (x,t) \in D$ implies that $\psi(x,t) \geq 0, \forall (x,t) \in \bar{D}$.

Proof: Let (x^*,t^*) be such that $\psi(x^*,t^*) = \min_{(x,t)\in\bar{D}} \{\psi(x,t)\}$ and suppose that $\psi(x^*,t^*) < 0$. It is clear that $\psi(x^*,t^*) \notin \partial D$. So we have

$$L\psi(x^*,t^*) = \psi_t(x^*,t^*) - \varepsilon \psi_{xx}(x^*,t^*) + a(x)\psi_x(x^*,t^*) + b(x)\psi(x^*,t^*)$$

Since $\psi(x^*,t^*) = \min_{(x,t)\in\bar{D}} \{\psi(x,t)\}$ which implies $\psi_x(x^*,t^*) = 0$, $\psi_t(x^*,t^*) = 0$ and $\psi_{xx}(x^*,t^*) \geq 0$ and implies that $L\psi(x^*,t^*) < 0$ which is contradiction to the assumption that made above. So we have $L\psi(x^*,t^*) > 0, \quad \forall (x,t) \in D$. Hence $\psi(x,t) \geq 0, \quad \forall (x,t) \in D$

Lemma 2: The bound on the solution $u(x,t)$ of the continuous problem Eqs.(1.14-1.15) is given by

$$|u(x,t)| \leq C, \forall (x,t) \in \bar{D}$$

Proof: From inequality $|u(x,t) - u(x,0)| - |u(x,t) - u_0(x)| \leq Ct$, we have

$$|u(x,t)| - |u_0(x)| \leq |u(x,t) - u(x,0)| \leq Ct$$

$$\Rightarrow |u(x,t)| \leq Ct + |u_0(x)|, \forall_{x,t} \in \bar{D}$$

since $t \in [0,T]$ and $u_0(x)$ is bounded it implies $|u(x,t)| \leq C$

Lemma 3 (Stability estimate)

Let $u(x,t)$ be the solution of problem (1.14-1.15). Then we have

$$||u|| \leq \beta^{-1}||f|| + \max(|u_0(x)|, \max(\mu_0(x,t), \mu_1(x,t)))$$

Under the smoothness and compatibility conditions, proved that the exact solution and its derivatives satisfy

$$\left|\frac{\partial^{i+j}u(x,t)}{\partial x^i \partial t^j}\right| \leq C(1 + \varepsilon^{-i}\exp(-\alpha(1-x)/\varepsilon)), (x,t) \in \bar{D}, i = 0,1, 0 \leq i \leq 3, 0 \leq i+j \leq 3$$

Proof: Define barrier functions $\vartheta^{\pm}$ as

$$\vartheta^{\pm}(x,t) = \beta^{-1}||f|| + \max\{u_0(x), \max\{\mu_0(x,t), \mu_1(x,t)\}\} \pm u(x,t)$$

At the initial value:

$$\begin{aligned}\vartheta^{\pm}(x,0) &= \beta^{-1}||f|| + \max\{u_0(x), \max\{\mu_0(x,t), \mu_1(x,t)\}\} \pm u(x,0)\\ &= \beta^{-1}||f|| + \max\{u_0(x), \max\{\mu_0(x,t), \mu_1(x,t)\}\} \pm u_0(x)\\ &\geq 0.\end{aligned}$$

at the boundary points:

$$\begin{aligned}\vartheta^{\pm}(0,t) &= \beta^{-1}||f|| + \max\{u_0(0), \max\{\mu_0(0,t), \mu_1(0,t)\}\} \pm u(0,t)\\ &= \beta^{-1}||f|| + \max\{u_0(0), \max\{\mu_0(0,t), \mu_1(0,t)\}\} \pm u_0(0)\\ &\geq 0.\\ \vartheta^{\pm}(1,t) &= \beta^{-1}||f|| + \max\{u_0(1), \max\{\mu_0(1,t), \mu_1(1,t)\}\} \pm u(1,t)\\ &= \beta^{-1}||f|| + \max\{u_0(1), \max\{\mu_0(1,t), \mu_1(1,t)\}\} \pm u_0(1)\\ &\geq 0. \text{ and}\end{aligned}$$

$$\begin{aligned}L\vartheta^{\pm}(x,t) &= \vartheta_t^{\pm}(x,t) - \varepsilon\vartheta_{xx}^{\pm}(x,t) + a(x)\vartheta_x^{\pm}(x,t) + b(x)\vartheta^{\pm}(x,t)\\ &= (\max\{\mu_{0_t}(x,t), \mu_{1_t}(x,t)\} \pm u_t(x,t))\\ &- \varepsilon(\max\{\mu_{0_{xx}}(x,t), u_{0xx}(x), \mu_{1_{xx}}(x,t)\} \pm u_{xx}(x,t))\\ &+ a(x)\bigg(\max\{u_{0x}(x,t), \max\{\mu_{0_x}(x,t), \mu_{1_x}(x,t)\}\} \pm u_x(x,t)\bigg) +\\ &b(x)\bigg(\beta^{-1}||f|| + \max\{u_0(x), \max\{\mu_0(x,t), \mu_1(x,t)\}\} \pm u(x,t)\bigg)\\ &\geq 0\end{aligned}$$

since $\varepsilon \geq 0, a(x) \geq \alpha > 0$ and $b(x) \geq \beta > 0$.

which implies that

$$L\vartheta^{\pm}(x,t) \geq 0$$

Hence by maximum principle we have,

$$\vartheta^{\pm}(x,t) \geq 0, \quad \forall (x,t) \in \bar{D}$$

$$u(x,t) \leq \beta^{-1}||f|| + \max\{u_0(x), \max\{\mu_0(x,t), \mu_1(x,t)\}\}$$

Hence, the proof is completed.

Lemma 4. The bound on the derivative of the solution $u(x,t)$ with respect to x is given by:

$$\left|\frac{\partial^i u(x,t)}{\partial x^i}\right| \leq C\left(1+\varepsilon^{-i}\exp(-\alpha(1-x)/\varepsilon)\right),\ \ (x,t)\in\bar{D},\ \ i=0,1,2,3,4.$$

Proof: See Clavero et al.,(2003)

4.1.2 Discretization in Spatial direction

On the spatial domain [0, 1], we introduce the equidistant meshes with uniform mesh length $\triangle x = h$ such that $\Omega_x^M = x_i = x_0 + ih, i = 1,2,3,...M, x_0 = 0, x_M = 1, h = \frac{1}{M}$ where M is the number of mesh points in the spatial direction.

$$\frac{\partial u}{\partial t} - \varepsilon\frac{\partial^2 u}{\partial x^2} + a(x)\frac{\partial u}{\partial x} + b(x)u(x,t) = f(x,t),\ \ (x,t)\in\Omega_x\times Q = (0,1)\times(0,T] \tag{4.1}$$

with the boundary conditions and the initial condition

$$\begin{aligned} u(0,t) =& \mu_0(t),\quad t\in[0,T] \\ u(1,t) =& \mu_1(t),\quad t\in[0,T] \\ &\text{and} \\ u(x,0) =& \phi(x),\quad x\in[0,1] \end{aligned} \tag{4.2}$$

For the problem in the form Eqs.(4.1),we consider the sub- equation which is more influenced by the perturbation parameter.

$$-\varepsilon\frac{d^2u}{dx^2} + a(x)\frac{du}{dx} = 0 \tag{4.3}$$

Then using the finite difference scheme as

$$-\varepsilon\frac{U_{i+1} - 2U_i + U_{i-1}}{\rho^2} + a(x)\frac{U_i - U_{i-1}}{h} = 0 \tag{4.4}$$

we calculate for the denominator function ρ^2 using the following procedures. First we rewrite the Eqs.(4.3) equivalently as a system of two first order coupled differential equations as:

$$\frac{du}{dx} = y \tag{4.5}$$

$$\frac{dy}{dx} = \frac{a(x)}{\varepsilon}y \tag{4.6}$$

which implies that $y = \exp\left(\frac{a(x)}{\varepsilon}x\right)$ written in the form of : $y_i = \exp\left(\frac{a(x_i)}{\varepsilon}x_i\right)$. To get the discrete difference scheme for y, we apply first order difference scheme Eqs.(4.5) as

$$y_i = \frac{U_{i+1} - U_i}{h} \Rightarrow hy_i = U_{i+1} - U_i \text{ and } hy_{i-1} = U_i - U_{i-1} \tag{4.7}$$

and solving for ρ^2 from equation (4.4)

Now substituting y_i in (4.4) we obtain:

$$\varepsilon\frac{hy_i - hy_{i-1}}{\rho^2} = a(x_i)y_i \Rightarrow \rho^2 = h\frac{\varepsilon}{a(x_i)}\left(\frac{y_i - y_{i-1}}{y_{i-1}}\right)$$
$$\Rightarrow \rho^2 = \frac{h\varepsilon}{a(x_i)}\left(\exp\left(\frac{ha(x_i)}{\varepsilon}\right) - 1\right)$$

By using the denominator function ρ^2 into the main scheme we get

$$\frac{dU}{dt}(x_i,t) - \varepsilon\frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{\rho^2} + a(x_i)\frac{U_i(t) - U_{i-1}(t)}{h} + b(x_i)U_i(t) = f(x_i,t)$$

where ρ^2 is the denominator function. From Eqs.(4.1-4.2) reduces to semi- discrete form as

$$\begin{aligned} L^hU_i(t) = &\frac{dU_i(t)}{dt} - \varepsilon\frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{\rho^2} + a(x_i)\frac{U_i(t) - U_{i-1}(t)}{h} \\ &+ b(x_i)U_i(t) = f_i(t) \end{aligned} \tag{4.8}$$

with the semi-discrete boundary and initial condition

$$\begin{aligned} U_0(t) =& \mu_0(0,t), \quad t \in [0,T] \\ U_M(t) =& \mu_1(1,t), \;\; t \in [0,T] \\ &\text{and} \\ U_i(0) =& \phi(x_i) \quad , i = 1,2,...,M \end{aligned} \tag{4.9}$$

The above system of equations of IVP in Eqs.(4.8-4.9) can be written in the form:

$$\frac{dU_i(t)}{dt} + AU_i(t) = F_i(t) \tag{4.10}$$

where A is a tridiagonal matrix of $M-1 \times M-1$ and $U_i(t)$ and $F_i(t)$ are $M-1$ column vectors. The entries of A and F are given as:

$$\begin{aligned} A_{ii} &= \frac{2\varepsilon}{\rho_i^2} + \frac{a(x_i)}{h} + b(x_i), i = 1(1)M-1 \\ A_{ii+1} &= \frac{-\varepsilon}{\rho_i^2}, i = 1(1)M-2 \\ A_{ii-1} &= \frac{-\varepsilon}{\rho_i^2} - \frac{a(x_i)}{h}, i = 2(1)M-1 \\ F_1(t) &= f_1(t) + (\frac{\varepsilon}{\rho^2} + \frac{a(x_1)}{h})\mu_0(0,t), \\ F_i(t) &= f_i(t), i = 2(1)M-2 \\ F_{M-1}(t) &= f_{M-1}(t) + (\frac{\varepsilon}{\rho_{M-1}^2})\mu_1(1,t) \end{aligned} \tag{4.11}$$

respectively. Now we need to show the semi-discrete operator L^h also satisfies the maximum principle and the uniform stability estimate.

Theorem 1. (Semi-discrete maximum principle).

The operator defined by the discrete scheme in Eqs.(4.8) satisfies a semi-discrete maximum principle. That is, Suppose $U_0(t) \geq 0, U_M(t) \geq 0$. Then $L^h U_i(t) \geq 0, \forall i = 1(1)M-1$ implies that $U_i(t) \geq 0, \forall i = 1(1)M$.

Proof: Suppose there exists $s \in \{0,1,2,...M\}$ such that $U_s(t) = \min_{0\leq i\leq M}\{U_i(t)\}$. Suppose that $U_s(t) < 0$ which implies $s \neq 0, M$. Also we have $U_{s+1} - U_s > 0$ and $U_s - U_{s-1} < 0$. Now we have

$$L^h U_s(t) = \frac{dU_s(t)}{dt} - \varepsilon\frac{U_{s+1}(t) - U_s(t) - (U_s(t) - U_{s-1}(t))}{\rho_s^2} + a_s\frac{U_s(t) - U_{s-1}(t)}{h} + b_s U_s(t) < 0$$

Using the assumption, we get $L^h U_i(t) < 0$ for $i = 1(1)M-1$.

Thus the supposition $U_i(t) < 0, i = 1(1)M-1$ is wrong. Hence $U_i(t) \geq 0, \forall_i = 0(1)M$.

Lemma 5. The solution $U_i(t)$ of the semi-discrete problem in Eqs.(4.10) satisfy the following bound.

$$|U_i(t)| \leq \beta^{-1}\max|L^h U_i(t)| + \max\left(|u_0(x_i)|, \max(\mu_0(x_i,t), \mu_1(x_i,t))\right)$$

Proof: Let $s = \beta^{-1}\max|L^h U_i(t)| + \max(|u_0(x_i)|, \max(\mu_0(x_i,t), \mu_1(x_i,t)))$

and define the barrier function $\Psi_i^{\pm}(t)$ by: $\Psi_i^{\pm}(t) = s \pm U_i(t)$

At the boundary points we have

$$\Psi_0^{\pm}(t) = s \pm U_0(t) = s \pm \mu_0(0,t) \geq 0$$

$$\Psi_M^{\pm}(t) = s \pm U_M(t) = s \pm \mu_1(1,t) \geq 0$$

On the discretized domain $0 < i < M$, we have

$$\begin{aligned} L^h\Psi_i^{\pm}(t) =& \frac{d(s \pm U_i(t))}{dt} - \varepsilon\left(\frac{s \pm U_{i+1}(t) - 2(s \pm U_i(t)) + s \pm U_{i-1}(t)}{\rho^2}\right) \\ &+ a_i\left(\frac{s \pm U_i(t) - s \pm U_{i-1}(t)}{h}\right) + b_i(s \pm U_i(t)) \\ =& b_i s \pm L^h U_i(t) \\ =& b_i(\beta^{-1}\max|L^h U_i(t)| + max\left(|u_0(x_i)|, \max\{\mu_0(x_i,t), \mu_1(x_i,t)\}\right) \pm f_i(t)) \geq 0, \text{since } \ b_i \geq \beta \end{aligned}$$

from theorem(1), using the discrete maximum principle, we obtain $\Psi_i^{\pm}(t) \geq 0, \forall (x_i,t) \in \bar{\Omega}^M \times Q$

4.1.3 Error estimate for semi-discrete scheme

Now let us analyze these spatial discretization for convergence, we prove above the semi-discrete operator L^h satisfy the maximum principle and the uniform stability estimate.

Theorem 2. Let the coefficient functions $a(x)$, $b(x)$ and f in Eqs.(4.1) be sufficiently smooth functions so that $u(x,t) \in C^4[0,1] \times [0,T]$. Then the semi-discrete solution $U_i(t)$ of the Eqs.(4.1)-(4.2) satisfies.

$$|L^h(U(x_i,t) - U_i(t))| \le Ch\left(1 + \sup_{0\le i\le M} \frac{\exp(-\alpha(1-x_i)/\varepsilon)}{\varepsilon^3}\right) \tag{4.12}$$

Proof: Consider

$$\begin{aligned}
|L^h(U(x_i,t) - U_i(t))| =& |L^h U(x_i,t) - L^h U_i(t)| \\
\le& C| - \varepsilon(U_{xx}(x_i,t) - \frac{D_x^+ D_x^- h^2}{\rho^2} U(x_i,t)) + a_i(U_x(x_i,t) - D_x^- U(x_i,t))| \\
\le& C\varepsilon|(U_{xx}(x_i,t) - D_x^+ D_x^- U(x_i,t))| + C\varepsilon|(\frac{h^2}{\rho_i^2} - 1)D_x^+ D_x^- U(x_i,t)| + Ch|U_{xx}(x_i,t)| \\
\le& C\varepsilon h^2 |U_{xxxx}(x_i,t)| + Ch|U_{xx(x_i,t)}|
\end{aligned}$$

Above used estimate $\varepsilon\left|\frac{h^2}{\rho_i^2} - 1\right| \le Ch$ is based on the non-standard denominator function behavior. Let define $\gamma = a_i h/\varepsilon, \gamma \in (0,\infty)$. Then

$$\varepsilon\left|\frac{h^2}{\rho_i^2} - 1\right| = a_i h \left|\frac{1}{\exp(\gamma) - 1} - \frac{1}{\gamma}\right| =: a_i h R(\gamma)$$

where

$$R(\gamma) = \frac{\exp(\gamma) - 1 - \gamma}{\gamma(\exp(\gamma) - 1)}$$

and from this we have

$$\lim_{\gamma\to 0} R(\gamma) = \frac{1}{2}, \ \lim_{\gamma\to\infty} R(\gamma) = 0$$

Therefore

$$R(\gamma) \leq C, \ \ \gamma(0,\infty)$$

so,the error estimate becomes

$$|L^h(U(x_i,t)) - U_i(t))| \leq C\varepsilon h^2|U_{xxxx}(x_i,t)| + Ch|U_{xx}(x_i,t)|. \tag{4.13}$$

From (4.13) and boundedness of derivatives of solution in lemma (4),we obtain:

$$\begin{aligned}
|L^h(U(x_i,t) - U_i(t))| \leq & C\varepsilon h^2|1+\varepsilon^{-4}\exp\left(\frac{-\alpha(1-x_i)}{\varepsilon}\right)| + Ch|1+\varepsilon^{-2}\exp\left(\frac{-\alpha(1-x_i)}{\varepsilon}\right)| \\
& \leq Ch^2|\varepsilon+\varepsilon^{-3}\exp\left(\frac{-\alpha(1-x_i)}{\varepsilon}\right)| + Ch|1+\varepsilon^{-2}\exp\left(\frac{-\alpha(1-x_i)}{\varepsilon}\right)| \\
& \leq Ch^2|1+\varepsilon^{-3}\exp\left(\frac{-\alpha(1-x_i)}{\varepsilon}\right)| + Ch|1+\varepsilon^{-3}\exp\left(\frac{-\alpha(1-x_i)}{\varepsilon}\right)| \\
& \leq Ch\left(1+\max_{i\in 0,1,\ldots,M}\frac{\exp\big(-\alpha(1-x_i)/\varepsilon\big)}{\varepsilon^3}\right), \ \text{since} \ \varepsilon^{-2} \leq \varepsilon^{-3}.
\end{aligned}$$

Lemma 6. For a fixed mesh and for $\varepsilon \to 0$, it holds

$$\lim_{\varepsilon\to 0}\max_{1\leq i\leq M-1}\frac{\exp(-\alpha(1-x_i)/\varepsilon)}{\varepsilon^n} = 0, \quad n = 1,2,3,\ldots$$

where

$$x_i = ih, \ h = 1/M, \forall_i = 1(1)M-1$$

Proof: Consider the partition $[0,1] : 0 = x_0 < x_1 < \ldots < x_{M-1} < x_M = 1$ for the interior grid points, we have

$$\max_{1\leq i\leq M-1}\frac{(\exp(-\alpha x_i)/\varepsilon)}{\varepsilon^n} \leq \frac{(\exp(-\alpha x_1)/\varepsilon)}{\varepsilon^n} = \frac{(\exp(-\alpha h)/\varepsilon)}{\varepsilon^n} \ \text{ and}$$

$$\max_{1\leq i\leq M-1}\frac{(\exp(-\alpha(1-x_i)/\varepsilon)}{\varepsilon^n} \leq \frac{(\exp(-\alpha(1-x_{M-1})/\varepsilon)}{\varepsilon^n} = \frac{(\exp(-\alpha h)/\varepsilon)}{\varepsilon^n}, \text{since} \ \ x_1 = h, 1-x_{M-1} = h$$

then application of L'Hospital's rule gives

$$\lim_{\varepsilon \to 0} \frac{\exp(-\alpha h/\varepsilon)}{\varepsilon^n} = \lim_{s=1/\varepsilon \to \infty} \frac{s^n}{\exp(\alpha hs)} = \lim_{s=1/\varepsilon \to \infty} \frac{n!}{(\alpha h)^n \exp(\alpha hs)} = 0$$

this complete the proof

Theorem 3. Under the hypothesis of boundedness of semi-discrete solution, lemma (6) and theorem (2) above, the semi-discrete solution satisfy the following bound.

$$\sup_{0<\varepsilon\ll 1} ||U(x_i,t) - U_i(t)||_{\Omega^M \times [0,T]} \leq CM^{-1} \tag{4.14}$$

Proof: Immediate result from boundedness of solution, lemma (6) and theorem (2) will give the required estimates.

4.1.4 Discretization in temporal direction

On the time domain $[0,T]$, we introduce the discretization in time direction step $\triangle t_j = t_{j+1} - t_j, j = 0(1)K$ such that $Q^K = \Omega_t^K$ where K denote the number of mesh in the temporal direction. At this stage we use low order numerical method to discretize the system of IVPs in Eqs.(4.10) using special type of Runge-Kutta method developed by Bogacki and Shampine in 1989 with order two and three implicit given in (Shampine et.al,1997). First rewrite Eqs. (4.10) in the form:

$$\frac{dU_i(t)}{dt} = f(t, U_i(t)) \tag{4.15}$$

with the initial condition $U(x_i,0)=\phi(x_i), i=0(1)M$, here $f(t,U_i(t))=-AU_i(t)+F_i(t)$ so for each $j=1(1)K$ we write the scheme as:

$$
\begin{aligned}
K_1 =& f(t_j,U_{i,j}),\;\; j=1(1)K-1\\
K_2 =& f(t_j+\frac{1}{2}\triangle t_j, U_{i,j}+\frac{1}{2}\triangle t_j K_1),\;\; j=1(1)K-1\\
K_3 =& f(t_j+\frac{3}{4}\triangle t_j, U_{i,j}+\frac{3}{4}\triangle t_j K_2),\;\; j=1(1)K-1\\
U^*_{i,j+1} =& U_{i,j}+\frac{2}{9}\triangle t_j K_1+\frac{1}{9}\triangle t_j K_2+\frac{4}{9}\triangle t_j K_3,\;\; j=1(1)K-1\\
K_4 =& f(t_j+\triangle t_j, U^*_{i,j+1}),\;\; j=1(1)K-1\\
U_{i,j+1} =& U_{i,j}+\frac{7}{24}\triangle t_j K_1+\frac{1}{4}\triangle t_j K_2+\frac{1}{3}\triangle t_j K_3+\frac{1}{8}\triangle t_j K_4,\;\; j=1(1)K-1
\end{aligned}
$$

It is stated in (Lamba and Stuart,1998) that, for $j=1(1)K$ the local approximation $U_{i,j+1}$ to $U_i(t_{j+1})$ has third order accuracy (i.e.$(\triangle\, t)^3$).

Let $\triangle t=\max_{0\leq j\leq K}\triangle t_j$ then we have the following lemma.

Lemma 7. From the above approximation method in temporal direction, the global error estimates in this direction are given by

$$||E_{j+1}||_\infty = ||U_i(t_{j+1})-U_{i,j+1}||\Omega^M\times\Omega^K \leq C(\triangle t)^2$$

where E_{j+1} is the global error in the temporal direction at $(j+1)^{th}$ time level.

Proof: Using the local error estimate e_j up to j^{th} time step, we obtain the global error estimate at

$(j+1)^{th}$ time step.

$$
\begin{aligned}
||E_{j+1}||_\infty =& \sum_{i=1}^{j} ||e_i||_\infty, \;\; j \leq K \\
\leq& ||e_1||_\infty + ||e_2||_\infty + ... + ||e_j||_\infty, ||e_j||_\infty = jC(\triangle t_j)^3 \\
\leq& C_1(j\triangle t)(\triangle t)^2 \\
\leq& C_1 T(\triangle t)^2, \;\text{ since }\; j\triangle t \leq T \\
\leq& C(\triangle t)^2
\end{aligned}
$$

Then using the boundedness of the solution and lemma (7) implies

$$
\sup_{0<\varepsilon\ll 1} ||U_i(t_{j+1}) - U_{i,j+1}||_{\Omega^M\times\Omega^K} \leq C(\triangle t)^2) \tag{4.16}
$$

this show that the discretization in temporal direction is consistent and global error is bounded. Now we use(4.16) to prove the parameter uniform convergence of the fully discrete scheme as

$$
\begin{aligned}
\sup_{0<\varepsilon\ll 1} ||U(x_i,t_j) - U_{i,j}||_{\Omega^M\times\Omega^K} \leq& \sup_{0<\varepsilon\ll 1} ||U(x_i,t_j) - U_i(t_j)||_{\Omega^M\times\Omega^K} \\
&+ \sup_{0<\varepsilon\ll 1} ||U_i(t_j) - U_{i,j}||_{\Omega^M\times\Omega^K}
\end{aligned} \tag{4.17}
$$

Using boundedness of the solution,theorem(3), lemma(7) and Eqn.(4.17) we obtain:

$$
\sup_{0<\varepsilon\ll 1} ||U(x_i,t_j) - U_{i,j}||_{\Omega^M\times\Omega^K} \leq C\bigg(M^{-1} + (\triangle t)^2\bigg) \tag{4.18}
$$

Remark: The inequality in (4.18) shows the parameter uniform convergence of the proposed scheme with order $O\bigg(M^{-1} + (\triangle t)^2\bigg)$,for $h = M^{-1}$.

4.2 Numerical results

To verify the established theoretical results in this thesis, we perform some experiments using the proposed numerical scheme on the problem of the form given in equation (1.2) - (1.3).

Example 1: Consider the following parabolic initial boundary value problem:
$\frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} + (2-x^2)\frac{\partial x}{\partial x} + xu(x,t) = 10t^2 e^{-t} x(1-x), \quad (x,t) \in (0,1)\times(0,1]$ with initial condition $u(x,0) = 0,\ x \in (0,1)$, boundary conditions $u(0,t) = 0,\ t \in [0,1]$, $u(1,t) = 0,\ t \in [0,1]$

Example 2: Consider the following parabolic initial boundary value problem:
$\frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} + (1 + x(1-x))\frac{\partial u}{\partial x} = f(x,t), \quad (x,t) \in (0,1)\times(0,1]$ with initial condition $u(x,0) = u_0(x),\ x \in (0,1)$, boundary conditions $u(0,t) = 0,\ t \in [0,1]$, $u(1,t) = 0,\ t \in [0,1]$ where we choose the initial and the source functions $f(x,t)$ are from the exact solution
$u(x,t) = e^{-t}(c_1 + c_2 x - e^{-(1-x)\varepsilon})$ where $c_1 = e^{-\frac{1}{\varepsilon}}$ and $c_2 = 1 - e^{-\frac{1}{\varepsilon}}$.

Exact solution is not available for the first example, therefore the point-wise and maximum nodal errors are calculated by using the double mesh principle given as:

$$E_{\varepsilon}^{M,\triangle t} = \max_{1\le i\le M-1, 1\le j\le K-1} \left|U_{i,j}^{M,\triangle t} - U_{i,j}^{2M,\triangle t/2}\right|$$

where M the number of mesh points in x and $\triangle\ t$ is the mesh length in t direction. $U_{i,j}^{M,\triangle t}$ are the computed solution of the problem using $M, \triangle\ t$ mesh numbers and $U_{i,j}^{2M,\triangle t/2}$ are computed solution on double number of mesh points $2M, \triangle\ t/2$ by adding the mid points $x_{i+1/2} = \frac{x_{i+1}+x_i}{2}$ and $t_{j+1/2} = \frac{t_{j+1}+t_j}{2}$ into the mesh points . For any value of the mesh points M and $\triangle\ t$ the ε-uniform error estimate are calculated using the formula

$$E_{\varepsilon}^{M,\triangle t} = \max_{\varepsilon} |E^{M,\triangle t}|$$

The rate of convergence of the method is calculated using the formula

$$r^{M,\triangle t} = \log_2\left(E^{M,\triangle t}/E^{2M,\triangle t/2}\right) = \frac{\log\left(E^{M,\triangle t}\right) - \log\left(E^{2M,\triangle t/2}\right)}{\log 2}$$

Table 4.1: Maximum absolute error for Example 1 and result in Gowrisankar and Natesan(2014)

ε	M=32	M=64	M=128	M=256	M=512	M=1024
	$\triangle\, t = \frac{1}{10}$	$\triangle\, t = \frac{1}{20}$	$\triangle\, t = \frac{1}{40}$	$\triangle\, t = \frac{1}{80}$	$\triangle\, t = \frac{1}{160}$	$\triangle\, t = \frac{1}{320}$
present method						
10^{0}	4.5685e-03	2.3613e-03	1.1974e-03	6.0256e-04	3.0220e-04	1.5133e-04
10^{-4}	6.1336e-03	3.5748e-03	1.9245e-03	9.9750e-04	5.0763e-04	2.6114e-04
10^{-6}	6.1336e-03	3.5748e-03	1.9245e-03	9.9750e-04	5.0763e-04	2.5606e-04
10^{-8}	6.1336e-03	3.5748e-03	1.9245e-03	9.9750e-04	5.0763e-04	2.5606e-04
$E_{\varepsilon}^{M,\triangle t}$	6.1336e-03	3.5748e-03	1.9245e-03	9.9750e-04	5.0763e-04	2.5606e-04
Result in	Gowrisankar	and Natesan(2014				
10^{0}	9.2151e-04	4.6408e-04	2.3891e-04	1.2182e-04	6.2135e-05	3.1334e-05
10^{-4}	1.1342e-02	6.2851e-03	3.2988e-03	1.7175e-03	8.6996e-04	4.3954e-04
10^{-6}	1.3838e-02	6.6509e-03	3.4377e-03	1.7677e-03	8.9286e-04	4.4781e-04
10^{-8}	1.4524e-02	6.7667e-03	3.6247e-03	1.7939e-03	8.9428e-04	4.4947e-04
$E_{\varepsilon}^{M,\triangle t}$	1.4524e-02	6.7667e-03	3.6247e-03	1.7939e-03	8.9428e-04	4.4947e-04

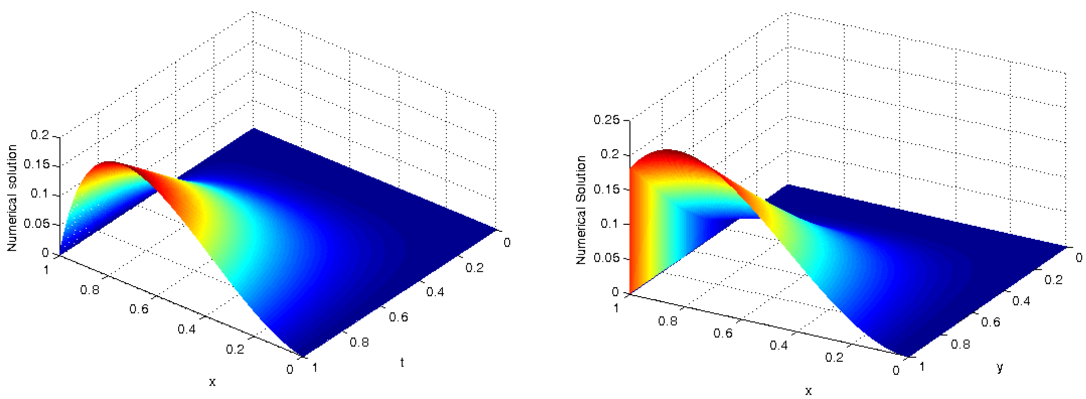

Figure 4.1: 3D plot of the numerical solution of Example 1 with $\varepsilon = 10^{-1}$ in (a),and $\varepsilon = 10^{-5}$ in (b)

Rate of convergence of the solution for Example 1

ε	M=32 $\triangle t = \frac{1}{10}$	M=64 $\triangle t = \frac{1}{20}$	M=128 $\triangle t = \frac{1}{40}$	M=256 $\triangle t = \frac{1}{80}$	M=512 $\triangle t = \frac{1}{160}$
present method					
10^0	0.9521	0.9797	0.9907	0.9956	0.9978
10^{-4}	0.7789	0.8934	0.9481	0.9745	0.9590
10^{-6}	0.7789	0.8934	0.9481	0.9745	0.9590
10^{-8}	0.7789	0.8934	0.9481	0.9745	0.9590
Result in	Gowrisankar	and Natesan(2014)			
10^0	0.9896	0.9579	0.9717	0.9712	0.9876
10^{-4}	0.8517	0.9299	0.9416	0.9812	0.9849
10^{-6}	1.0570	0.9521	0.9595	0.9853	0.9955
10^{-8}	1.1019	0.9005	1.0148	1.0043	0.9925

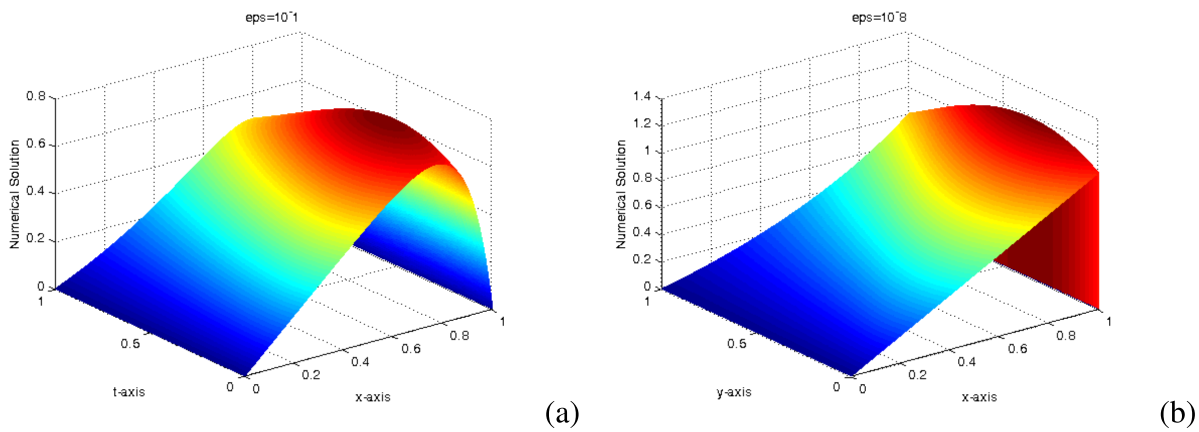

Figure 4.2: 3D plot of the numerical solution of Example 2 with $\varepsilon = 10^{-1}$ in (a) and $\varepsilon = 10^{-4}$ in (b)

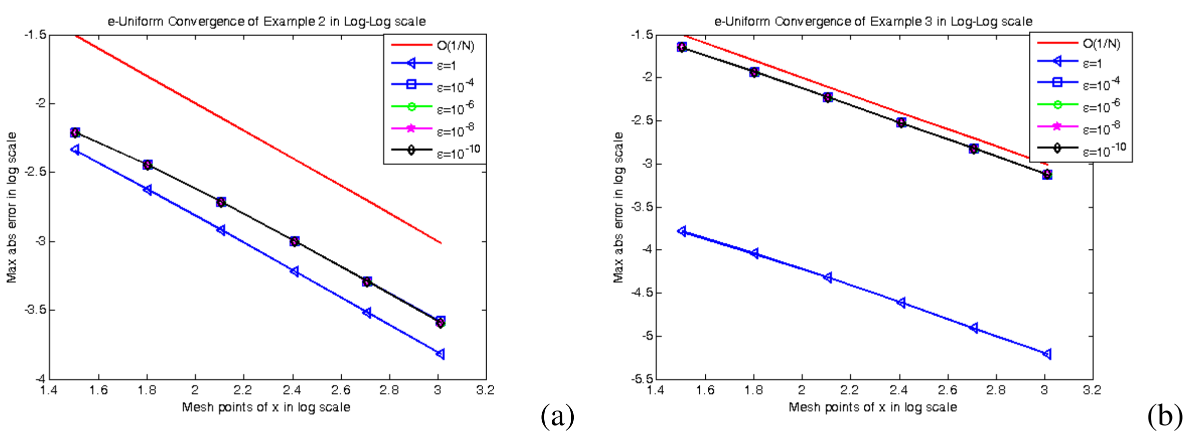

Figure 4.3: Loglog plot of maximum point-wise error of the solution for Example 1 in (a) and Example 2 in (b)

Table 4.2: Maximum absolute error for Example 2 and results in Gowrisankar and Natesan(2014) and Yanping and Li-Bin(2016)

ε	M=32 $\triangle t=\frac{1}{10}$	M=64 $\triangle t=\frac{1}{20}$	M=128 $\triangle t=\frac{1}{40}$	M=256 $\triangle t=\frac{1}{80}$	M=512 $\triangle t=\frac{1}{160}$	M=1024 $\triangle t=\frac{1}{320}$
present Method						
10^0	1.6239e-04	9.0171e-05	4.7384e-05	2.4264e-05	1.2281e-05	6.1797e-06
10^{-4}	2.2557e-02	1.1668e-02	5.9325e-03	2.9872e-03	1.4970e-03	7.4735e-04
10^{-6}	2.2557e-02	1.1668e-02	5.9325e-03	2.9872e-03	1.4970e-03	7.4899e-04
10^{-8}	2.2557e-02	1.1668e-02	5.9325e-03	2.9872e-03	1.4970e-03	7.4899e-04
$E_\varepsilon^{M,\triangle t}$	2.2557e-02	1.1668e-02	5.9325e-03	2.9872e-03	1.4970e-03	7.4899e-04
Result in	Gowrisankar	and Natesan(2014)				
10^0	7.5333e-04	4.1685e-04	2.1748e-04	1.0922e-04	5.4081e-05	2.6669e-05
10^{-4}	8.1473e-02	4.2740e-02	2.1362e-02	1.0776e-02	5.4258e-03	2.7360e-03
10^{-6}	8.7828e-02	4.3819e-02	2.2051e-02	1.1062e-02	5.5220e-03	2.7590e-03
10^{-8}	9.2101 e-02	4.5172 e-02	2.3305 e-02	1.1137 e-02	5.5484 e-03	2.7720 e-03
$E_\varepsilon^{M,\triangle t}$	9.2101e-02	4.5172e-02	2.3305e-02	1.1137e-02	5.5484e-03	2.7720e-03
Result in	Yanping	and Li-Bin(2016)				
10^0	6.8921e-04	3.7085e-04	1.9290e-04	9.8440e-05	4.9739e-05	
10^{-4}	9.3382e-02	5.5430e-02	3.9185e-02	2.1997e-02	1.1787e-02	
10^{-6}	9.7044e-02	6.0392e-02	3.8509e-02	1.8888e-02	1.1989e-02	
10^{-8}	9.7889e-02	5.9632e-02	3.9439e-02	2.0684e-02	1.2039e-02	
$E_\varepsilon^{M,\triangle t}$	9.7889e-02	5.9632e-02	3.9439e-02	2.0684e-02	1.2039e-02	

4.3 Discussion

The solution of examples given in 1- 2 has a boundary layer at the right side of the x-domain (see figure in 4.1 and 4.2). The computed solutions $U_{i,j}$ of example 1 and 2 for different values of perturbation parameters are also shown in figures (4.1a - 4.1b) and (4.2a -4.2b) respectively. The numerical results displayed in tables 4.1 and 4.2 clearly indicate that the proposed method based on MOL by using a NSFDM in spatial direction with Runge-Kutta method in temporal direction is parameter-uniform convergent. From the results in tables 4.1 and 4.2 , we observe that the maximum point-wise error decreases as M, K increases for each value of ε. We see that the maximum point-wise error is stable as $\varepsilon \to 0$ for each $M, \triangle t$. Using these two examples we confirm that the proposed numerical method is more accurate, stable and ε-uniform convergent with rate of convergence one. Numerical results shows the parameter-uniformness of the proposed scheme on equidistant mesh . The results in the proposed method is better than that obtained in Gowrisankar and Natesan(2014) and Yanping and Li-Bin(2016).

Chapter 5

Conclusion and Scope of Future Work

5.1 Conclusion

In this thesis, ε -uniform numerical method is presented for solving singularly perturbed 1D parabolic Convection-Diffusion Problems that has a boundary layer on the right side of the domain. The developed method is based on method of lines that constitute the non-standard finite difference for the spatial discretization and the implicit Runge-Kutta method of order 2 and 3 in the temporal direction for the system of initial value problem resulting from the spatial discretization. Stability and convergence analysis of the proposed scheme is shown. This study is implemented on two model examples by taking different values for the perturbation parameter ε and the computational results are presented in the tables and graphs. Also stability is analyzed and proposed numerical scheme is first-order convergence. The performance of the proposed scheme is investigated by comparing the results with prior studies result in Gowrisankar and Natesan (2014) and Yanping and Li-Bin (2016) . It has been found that the proposed method gives more accurate and stable numerical results.

5.2 Scope of Future Work

In this thesis, ε - Uniform Numerical Method for solving singularly perturbed 1D Parabolic Convection-Diffusion Problems. Hence, the schemes proposed in this thesis can also be extended to singularly perturbed delay DEs. And also, it is possible to extend for non linear problems. Additionally, this method can be extended to higher dimension PDEs.

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