Existence and Uniqueness of Positive Solutions for Fourth Order Two Point Sturm-Liouville BVP

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS

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June, 2019
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Declaration

I, the undersigned declare that, the thesis entitled ”existence and uniqueness of positive solutions for fourth order two point Sturm-Liouville boundary value problem”, is original and it has not been submitted to any institution elsewhere for the award of any degree or like, where other sources of information that have been used, they have been acknowledged.

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Acknowledgment

First of all I would like to thank the Almighty God who gave me long life and helped me to pass through different ups and downs to reach this time. Next my special heartfelt thanks go to my advisors Dr. Wesen Legese and Mr. Kefyalew Hailu, for their unreserved support, advice and guidance through the preparation of this thesis. Lastly, I would like to thank both Differential Equations and Functional Analysis Stream Post Graduate Students for their constructive comments and provision of some references while I was preparing this thesis.

Abstract

In this thesis, we consider fourth order two point Sturm-Liouville BVP. The main objective of this study was finding the non trivial solution for the homogeneous part of the differential equation by using Greens function properties, we determine the unique solution for the BVP, then by formulating equivalent operator equation prove existence of positive solution by the application of Guo Krasnoselskii fixed point theorem and finally we verify the uniqueness for the equivalent operator equation.

Contents

Chapter 1

Introduction

1.1 Background of the study

Boundary value problems associated with linear as well as nonlinear ordinary differential equation or finite difference equation have created a great deal of interest and play an important role in many fields of applied mathematics such as engineering design and manufacturing. Major industries like automobile, aerospace, chemical, petroleum, electronics and communication as well as emerging technologies like biotechnology and nanotechnology rely on the boundary value problems to simulate complex phenomena at different scales for designing and manufacturing of high-technological products. In these applied settings, positive solutions are meaningful.

Some theories such as the Krasnoselskii fixed point theorem, the Leggett-Williams fixed point theorem, Avery's generalization of the Leggett-Williams fixed point theorem and Avery-Henderson fixed point theorem have given a decisive impetus for the development of the modern theory of differential equations. The advantage of these techniques lies in that they do not demand the knowledge of solution, but have great power in application, in finding positive solutions, multiple solutions, and eigen value intervals for which there exists one or more positive solutions.

In analyzing nonlinear phenomena many mathematical models give rise to problems for which only positive solutions make sense. Therefore, since the publication of the monograph positive solution of operator equation in the year 1964 by the academician Krasnoselskii, hundreds of research articles on the theory of positive solutions of nonlinear problems have appeared.

Boundary value problems for ordinary differential equations arise in different areas of applied mathematics and physics and so on. Fourth-order differential equations boundary value problems have their origin in beam theory Bernis in(1982) and Zill and M. R (2001) , ice formation Myers and Charpin in (2002) and Chapman in (2004), fluids on lungs Halpern, Jensen, and Grotberg in (1998), brain warping Memoli, Sapiro, and Thompson in (2004) , designing special curves on surfaces

Hofer and Pottmann in (2004), and so forth. In beam theory, more specifically, a beam with a small deformation, a beam of a material that satisfies a nonlinear power-like stress and strain law, and a beam with two-sided links that satisfies a nonlinear power like elasticity law can be described by fourth order differential equations along with their boundary value conditions. For more background and applications, we refer the reader to the work by Timoshenko (1961), on elasticity, the monograph by Soedel (1993) on deformation of structure, and the work by Dulcska (1992), on the effects of soil settlement.

Fixed point methods are often used for proving the existence of such solutions. An overview of such results can be found in Guo, Lakshmikantham, Liuin (1996). Guo and Lakshmikantham (1998). And Demling (1985).

A few papers along these lines are Erbe and Wang in (1994) Erbe, Hu and Wang in (1994), Lian Wanga and Yeh in (1996), Henderson and Wang in (1997),Karakosta, Tsamatos in (2002), Henderson, in (2004), Yongxiang, (2016), Yun Zhang, Jian-Ping Sun and Juan Zhao, (2018) etc.

Motivated by the above works, in this thesis, we established the existence of at least one positive solution for fourth order two point Sturm-Liouville boundary value problem of the form

$$(-y''(t)+k^2y(t))'' = f(t,y(t)), \tag{1.1.1}$$

where $0 \leq t \leq 1 \quad k > 0$

$$\begin{aligned} \alpha y(0) - \beta y'(0) &= 0 \\ \gamma y(1) + \delta y'(1) &= 0 \\ y''(1) = y''(0) &= 0 \end{aligned} \tag{1.1.2}$$

where β, δ are positive constants α, γ are non-negative constants and
$f : [0,1] \times R^+ \longrightarrow R^+$ is continuous. By using an application of Guo-Krasnoselskii fixed point theorem.

1.2 Statements of the problem

This study focused on proving the existence and uniqueness of positive solutions for the fourth two point Sturm-Liouville boundary value problem.

1.3 Objectives of the study

1.3.1 General objective

General objective of this thesis is to prove the existence of positive solution for the fourth order two point Sturm-Liouville boundary value problem by applying Guo-Krasnoselskii fixed point theorem and verifying the unique positive solution.

1.3.2 Specific objectives

The specific objectives of the study are:-

- To construct the Green's function for homogeneous fourth order two point Sturm-Liouville boundary value problem.
- To determine a unique solution for fourth order two-point Sturm-Liouville boundary value problem.
- To formulating equivalent operator equation to determine existence of positive solution for the fourth order two point Sturm-Liouville boundary value problem.
- To verify the uniqueness of the fixed point for the equivalent operator equation.

1.4 Significance of the study

- It may provide basic research skills to the researcher.
- It may also serve as reference for researchers who have interest to be engaged in research activities in this line of research.

1.5 Delimitation of the Study

This study focused only in determining existence and uniqueness of a positive solution for the fourth order two point Sturm-Liouville boundary value problem by applying Guo-Krasnoselskii fixed point theorem.

Chapter 2

Review of Related Literatures

2.1 Over view of The Study

In mathematics in the field of differential equations, a boundary value problem is a differential equation together with a set of additional constraints, called the boundary conditions. A solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary conditions. Boundary value problems arise in several branches of physics as any physical differential equation will have them. Problems involving the wave equation, such as the determination of normal modes, are often stated as boundary value problems. Large classes of important boundary value problems are the Sturm-Liouville problems. The analysis of these problems involves the Eigen function of differential operator. Now a Positive solution is very important in diverse disciplines of mathematics since it can be applied for solving various problems and it is one of the most dynamic research subjects in nonlinear analysis.
The first important and significant of positive solution was proved by Erbe and Wanga in 1994, established positive solutions for the two-point boundary value problem,

$$y''(t) + a(t)f(y(t)) = 0,\ 0 < t < 1,$$

$$\alpha y(0) - \beta y'(0) = 0,$$

$$\gamma y(1) + \delta y\prime(1) = 0.$$

Erbe, Hu and Wang in (1994), studied the existence of at least two positive solutions for the two point boundary value problem,

$$y''(t) + f(t, y(t)) = 0, \quad 0 < t < 1,$$

$$\alpha y(0) - \beta y'(0) = 0,$$

$$\gamma y(1) + \delta y\prime(1) = 0.$$

Henderson and Wang in (1997), determined eigenvalue intervals for which there exist positive solutions of boundary value problem,

$$y''(t) + \lambda a(t) f(y(t)) = 0, \quad 0 < t < 1,$$

$$y(0) = y(1) = 0.$$

Henderson, in (2004), studied the existence of double positive solutions of three point boundary value problem,

$$y''(t) + f(y(t)) = 0,\ 0 \leq\ t \leq\ 1,$$

$$y(0) =\ 0, y(p) - y(1) = 0,\ 0 < p < 1.$$

Dong and Bai in 2008, studied existence of one and two positive solutions for

$$u^{(4)}(t) + B(t)u''(t) - A(t)u(t) = f(t, u(t), u''(t)),$$

$$u(0) = u(1) = u''(0) = u''(1) = 0,$$

where $A(t), B(t) \in C[0,1]$ *and* $f(t,u,v) : [0,1] \times [0,\infty) \times R \quad \to [0,\infty)$ is continuous. By using fixed point index theorem in cone.

Moustafa and Tahari, in 2008 investigated existence and nonexistence of positive solutions for

$$u^{(4)}(t) + \lambda a(t) f(u(t)) = 0, \quad 0 \leq t \leq 1,$$

$$u(0) = u(1) = u''(0) = u'''(0) \ = 0,$$

$$\alpha u'(1) + \beta u''(1) \ = 0,$$

Where $\lambda > 0$ is a positive parameter, $a : (0,1) \to [0,\infty)$ is continuous and $\int_0^1 a(t)dt > 0$. $f : [0,1] \times [0,\infty) \ \to \ [0,\infty)$. By using Krasnoselskiis fixed point theorem in cones.

Vrabel in (2015) established the existence theorem for solutions of

$$y^{(4)}(x) + \lambda y''(x) = h(x, y(x)), \quad \lambda < 0,\ x \in\ [0,1],$$

$$y(0) = y(1) = y''(0) = y''(1) = 0.$$

By using upper and lower solution method.
Yongxiang, (2016),Obtained existence results of positive solution to the fully fourth-order nonlinear boundary value problem,

$$u^{(4)}(t) = f(t,u,u^{'},u^{''}), \quad t \in [0,1],$$

$$u(0) = u^{'}(0) = u^{''}(1) = u^{'''}(1),$$

$f : [0,1] \times R^{3+} \times R^{-} \to R^{+}$ is continuous.
Zhang, Sun and Zhao, in (2018), concerned with the following fourth-order three-point BVP with sign-changing Greens function,

$$u^{(4)}(t) = f(t,u(t)) \;\; t \in [0,1],$$
$$u^{'}(0) = u^{''}(0) = u^{'''}(\eta) = u(1) = 0,$$

where $\eta \in [\frac{1}{3},1]$.
Quang and Thi Kim in (2018), proposed a method for investigating the solvability and iterative solution of a nonlinear fully fourth order boundary value problem,

$$u^{(4)}(x) = f(x,u(x),u^{'}(x),u^{''}(x),u^{'''}(x)), \; 0 < x < 1,$$
$$u(0) = u^{''}(0) = u^{''}(1) = 0,$$

where $f : [0,1] \times R^{4+}$ is continuous.

2.2 Preliminaries

In this section we provide some definitions, some basic concepts on Greens function, existence of positive solutions and statements of few standard definitions, which are frequently used in thesis.
Definition 2.2.1: We consider the second-order linear differential equation:

$$p_0(x)y^{''} + p_1(x)y^{'} + p_2(x)y = r(x), \;\; x \in J = [\alpha,\beta], \qquad (2.2.1)$$

where the function $p_0(x), p_1(x), p_2(x)$ and $r(x)$ are continuous in $[\alpha,\beta]$,$\alpha,\beta \in R$ and boundary conditions of the form

$$\begin{aligned} l_1[y] &= a_0 y(\alpha) + a_1 y'(\alpha) + b_0 y(\beta) + b_1 y'(\beta) = A \\ l_2[y] &= c_0 y(\alpha) + c_1 y'(\alpha) + d_0 y(\beta) + d_1 y'(\beta) = B, \end{aligned} \tag{2.2.2}$$

where a_i b_i, c_i d_i,, $i = 0,1$ A,B are given constants. The boundary value problem (2.2.1), (2.2.2) is called a non homogeneous two-point linear boundary value problem, whereas the homogeneous differential equation

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = 0, x \in J = [\alpha,\beta] \tag{2.2.3}$$

together with the homogeneous boundary conditions

$$l_1[y] = 0, l_2[y] = 0 \tag{2.2.4}$$

is called a homogeneous two-point linear boundary value problem. The function G(x, t) called a Green's function for the corresponding homogeneous boundary value problems (1.1.1)-(1.1.2) and the solution of the non homogeneous boundary value problem (2.2.1), (2.2.2) can be explicitly expressed in terms of G(x, t). Obviously, for the homogeneous problem (2.2.3), (2.2.4) the trivial solution always exists. Green's function G(x, t) for the boundary value problem(2.2.3), (2.2.4) is defined in the square $[\alpha,\beta,] \times [\alpha,\beta]$ and possesses the following fundamental properties

i. G(x, t) is continuous in $[\alpha,\beta,] \times [\alpha,\beta]$,

ii. $\frac{\partial}{\partial x}G(x,t)$is continuous in each of the triangles $\alpha \leq x \leq t \leq \beta$ and $\alpha \leq t \leq x \leq \beta$; moreover, $\frac{\partial}{\partial x}G(t^+,t) - \frac{\partial}{\partial x}G(t^-,t) = \frac{1}{p_0(t)}$,

iii. for every $t \in [\alpha,\beta]$, $z(x) = G(x,t)$ is a solution of the differential equation (2.2.3) in each of the intervals $[\alpha,t)$ *and* $(t,\alpha]$,

iv. for every $t \in [\alpha,\beta]$, $z(x) = G(x,t)$ satisfies the boundary conditions (2.2.4).

These properties completely characterize Green's function G(x,t).

Theorem 2.2.1 [Ravi.P and Donal.O (2000)]. Let the homogeneous problem (2.2.3), (2.2.4) have only the trivial solution. The following holds.

i. There exist a unique Green's function G(x,t) for the problem (2.2.3), (2.2.4).

ii. The unique solution y(x) of the non homogeneous problem (2.2.1), (2.2.4) can be represented by $y(x) = \int_{\alpha}^{\beta} G(x,t)r(t)dt$.

Definition 2.2.2: Let X be a non-empty set. A map $T : X \to X$ is said to be a self-map of X if $Tx = x$. An element $x \; in \; X$ is called a fixed point of X.

Definition 2.2.3: A normed linear space is a linear space X in which each vector x there corresponds a real number, denoted by $||x||$ called norm of x and has the following properties:

a. $||x|| \geq 0, \; for \; all \; x \in X \; and \; ||x|| = 0$, if and only if $x = 0$,

b. $||x+y|| \leq ||x|| + ||y||$, for all $x, y \in \; X$,

c. $||\alpha x|| = |\alpha| ||x||$, for all x$\in X$ and for any scalar α.

Definition 2.2.4: We say that T:$P \to P$ is a contraction mapping on a normed linear space P, provided there is a constant $\alpha \in (0,1)$ such that

$$||Tx - Ty|| \leq \alpha ||x-y||,$$

for all x,y$\in P$ we say $y_1 \in P$ is a fixed point of T provided $Ty_1 = y_1$.

Definition 2.2.5: A normed linear space is said to be Complete, if every Cauchy sequence in X converges to a point in X.

Definition 2.2.6: Let E be a real Banach space with cone P. A map $f : P \to [0, \infty)$ is said to be a nonnegative continuous convex functional on P, if f is continuous and $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \; for \; all \; x, y \in P \; and \; \lambda \in [0,1]$.

Definition 2.2.7: Let E be a real Banach space with cone P. A map $f : P \to [0,\infty)$ is said to be a nonnegative continuous concave functional on P, if f is continuous and $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$ *for all* $x, y \in P$ *and* $\lambda \in [0,1]$.

Definition 2.2.8: Let E be a real Banach space. A nonempty closed convex set P is the subset of E is called a cone, if it satisfies the following two conditions:

i $y \in P, \lambda \geq 0$ *implies* $\lambda y \in P$, *and*

ii $y \in P$ *and* $-y \in P$ *implies* $y = 0$.

Definition 2.2.9: The function $y(t) \in C[0,1] \cap C^4[0,1]$ is positive solution of the boundary value problems of (1.1.1),(1.1.2). y(t) is positive on the given interval and satisfies both the differential equation and the boundary conditions.

Chapter 3

Methodology

This chapter consists; study area and period, study design, source of information, and mathematical procedures.

3.1 Study area and period

The study was conducted at Jimma University department of mathematics from September 2018 to June 2019 G.C.

3.2 Study Design

This study employed analytical design.

3.3 Source of Information

The relevant sources of information for this study were books, published articles and related studies from internet.

3.4 Mathematical Procedure of the Study

This study was conducted based on the following procedures:-

- Defining fourth order two point Sturm-Liouville boundary value problem.
- Constructing the Greens function for the homogeneous fourth order two point Sturm-Liouville boundary value problem.
- Determining a unique solution for fourth order two-point Sturm-Liouville boundary value problem.

- Formulating equivalent operator equation to determine existence of positive solution for the fourth order two point Sturm-Liouville boundary value problem.

- Verifying the uniqueness for the equivalent operator equation.

Chapter 4

Main Results

4.1 Construction of Green's Function

In this section by using the properties of Green's functions we construct the Greens function for BVP (1.1.1),(1.1.2).
We consider

$$(-y''(t)+k^2y(t))''=0,\ 0\leq t\leq 1 \quad k>0, \tag{4.1.1}$$

with the same boundary condition (1.1.2).

$$let \quad y'' = -u(t) \quad 0\leq t\leq 1, \tag{4.1.2}$$

$$y''(0)=y''(1)=0. \tag{4.1.3}$$

Thus, the differential equation $(1.1.1)$, with its boundary condition be comes

$$u''(t)-k^2u(t)=f(t,y(t)), \tag{4.1.4}$$

with boundary condition

$$u(0)=u(1)=0. \tag{4.1.5}$$

We find the Green's function, from the homogeneous differential equation (4.1.4), thus the two linearly independent solutions are:-

$$u_1(t)=-sinhkt+coshkt \quad u_2(t)=sinhkt+coshkt.$$

The problem (4.1.4) has trivial solution if and only if $\rho\neq 0$.
Where, $\rho=\begin{vmatrix} u_1(0), & u_2(0) \\ u_1(1), & u_2(1) \end{vmatrix}$
= $\begin{vmatrix} 1, & 1 \\ (-sinhk+coshk), & sinhk+coshk \end{vmatrix} = sinhk+coshk-(-sinhk+coshk)=2sinhk$
$\neq 0$. for all k $\in (0,\infty)$.
By property (iii) there exist four functions say $\lambda_1(s),\lambda_2(s),\mu_1(s)$ *and* $\mu_2(s)$ such

that

$$G(t,s) = \begin{cases} u_1(t)\lambda_1(s) + u_2(t)\lambda_2(s) & 0 \le t \le s \le 1, \\ u_1(t)\mu 1(s) + u_1(t)\mu 1(s) & 0 \le s \le t \le 1. \end{cases} \tag{4.1.6}$$

Now using properties (i) and (ii) we have the following two equations

$$u_1(s)\lambda_1(s) + u_2(s)\lambda_2(s) = u_1(s)\mu_1(s) + u_2(s)\mu_2(s) = 0 \tag{4.1.7}$$

$$u_1^{'}(s)\mu 1(s) + u_2^{'}()\mu 2(s) - u_1^{'}(s)\lambda 1(s) - u_2^{'}()\lambda 2(s) = \frac{1}{p_0(t)} \tag{4.1.8}$$

these become

$(-sinhks + coshks)v_1(s) + (sinhks + coshks)v_2(s) = 0$

$k(sinhks - coshks)v_1(s) + k(sinhks + coshks)v_2(s) = -1$

where $v_1(s) = \lambda_1(s) - \mu_1(s), and v_2(s) = \lambda_2(s) - \mu_2(s).$

Thus $v_1(s) = \dfrac{1}{2k(sinhks - coshks)}$ and $v_2(s) = \dfrac{1}{2k(sinhks + coshks})$.

Now using the relation $\mu_1(s) = \lambda_1(s) + v_1(s)$ and $\mu_2(s) = \lambda_2(s) + v_2(s)$, equation (4.1.6) can be written

$$G(t,s) = \begin{cases} (-sinhkt + coshkt)\lambda_1(s) + (sinhkt + coshkt)\lambda_2(s) & 0 \le t \le s \le 1 \\ (-sinhkt + coshkt)(\lambda_1(s) + \dfrac{1}{2k(sinhks - coshks)}) + \\ (sinhkt + coshkt)(\lambda_2(s) - \dfrac{1}{2k(sinhks + coshks)}) & 0 \le s \le t \le 1. \end{cases}$$

Finally using property (iv) we find $\lambda_1(s)$ and $\lambda_2(s)$, thus

$$\begin{cases} u_1(0)\lambda_1(s) + u_2(0)\lambda_2(s) & = 0 \\ u_1(\lambda_1(s) + v_1(s)) + u_2(1)(\lambda 2(s) + v_2(s)) = 0 \end{cases} \tag{4.1.9}$$

$$\Rightarrow \begin{cases} \lambda_1(s) + \lambda_2(s) & = & 0 \\ (-sinhk + coshk)\lambda_1(s) + (sinhk + coshk)\lambda_2(s) = \\ \frac{1}{k}(sinhkcoshks - coshksinhks) \end{cases} \tag{4.1.10}$$

Hence $\lambda_1 = \frac{1}{\rho} \begin{vmatrix} 0, & 1 \\ \frac{1}{k}(sinhk\ coshks - coshk\ sinhks), & (sinhk + coshk) \end{vmatrix}$

$$\Rightarrow \lambda_1 = \frac{-1}{ksinhk}(sinhk\ coshks - coshksinhks).$$

and

$$\lambda_2 = \frac{1}{\rho}\begin{vmatrix} 1, & 0 \\ (-sinhk + coshk), & \frac{1}{k}(sinhk\ coshks - coshk\ sinhks) \end{vmatrix}$$

$$\Rightarrow \lambda_2 = \frac{1}{ksinhk}(sinhk\ coshks - coshk\ sinhks).$$

Thus by substituting in the equation (4.0.6) this becomes

$$G(t,s) = \begin{cases} \dfrac{sinhkt\ sinhk(1-s)}{ksinhk} & 0 \le t \le s \le 1, \\ \dfrac{sinhks\ sinhk(1-t)}{ksinhk} & 0 \le s \le t \le 1. \end{cases}$$

$$u(t) = \int_0^1 G(t,s) f(s,y(s))ds.$$

Lemma 4.1.1 The Greens function $G(t,s)$ satisfies the following inequalities:-

i. $G(t,s) \le G(s,s)$ for all $t,s \in (0,1)$.

ii. $G(t,s) \ge MG(s,s)$ for all $t,s\ [\frac{1}{4},\frac{3}{4}]$, where M=$\dfrac{sinh(\frac{1}{4})}{sinhk}$.

Proof (i). $G(t,s)$ is positive for all t,s ∈ (0,1).

For $0 \le s \le t \le 1$ we have

$$\frac{G(t,s)}{G(s,s)} = \frac{sinhks\ sinhk(1-t)}{sinhks\ sinhk(1-s)} = \frac{sinhk(1-t)}{sinhk(1-s)} \le 1$$

$\Rightarrow G(t,s) \le G(s,s)\ for\ \ t,\ s \in (0,1).$

For $0 \le t \le s \le 1$ we have

$$\frac{G(t,s)}{G(s,s)} = \frac{sinhkt\ sinhk(1-s)}{sinhks\ sinhk(1-s)} = \frac{sinkt}{sinhks} \le 1$$

$\Rightarrow G(t,s) \le G(s,s)\ for\ \ t,\ s \in [0,1].$

Therefore,

$$G(t,s) \le G(s,s)\ for\ all\ t,s \in (0,1).$$

(ii). If $t \leq s$ for all $t,s \in [\frac{1}{4},\frac{3}{4}]$, we have

$$\frac{G(t,s)}{G(s,s)} = \frac{sinhkt}{sinhks} \geq \frac{sinh(\frac{k}{4})}{sinh(\frac{3k}{4})} \geq \frac{sinh(\frac{k}{4})}{sinhk}$$

$$\Rightarrow \frac{G(t,s)}{G(s,s)} \geq \frac{sinh(\frac{k}{4})}{sinhk}.$$

If $s \leq t$ for all $t,s \in [\frac{1}{4},\frac{3}{4}]$ we have

$$\frac{G(t,s)}{G(s,s)} = \frac{sinhk(1-t)}{sinhk(1-s)} \geq \frac{sinhk(1-\frac{3}{4})}{sinhk(1-\frac{1}{4})} \geq \frac{sinh(\frac{k}{4})}{sinhk}$$

$$\Rightarrow \frac{G(t,s)}{G(s,s)} \geq \frac{sinh(\frac{k}{4})}{sinhk}$$

$$\Rightarrow G(t,s) \geq MG(s,s), where\ M = \frac{sinh(\frac{k}{4})}{sinh(k)}.$$

Therefore, $G(t,s) \geq MG(s,s)$ for all $(t,s) \in [\frac{1}{4},\frac{3}{4}]$.

Now consider $-y^{''}(t) = u(t)$ with the boundary condition

$$\alpha y(0) - \beta y^{'}(0) = 0$$

$$\gamma y(1) + \delta y^{'}(1) = 0.$$

For the above homogeneous differential equation, the two linearly independent solutions are $y_1(t) = 1$ and $y_2(t) = t$. The problem has the trivial solution if and only if $\Delta \neq 0$. Where, $\Delta = \begin{vmatrix} \alpha & -\beta \\ \gamma & \gamma+\beta \end{vmatrix}$

$$\Rightarrow \Delta = \alpha(\gamma+\delta) + \beta\gamma \neq 0.$$

By the property (iii) of the Green's functions there exist four function say $\lambda_1(s), \lambda_2(s), \mu_1(s)\ and\ \mu_2(s)$ such that

H(t,s)= $\begin{cases} y_1(t)\lambda_1(s) + y_2(t)\lambda_2(s) & 0 \leq s \leq t \leq 1 \\ y_1(t)\mu_1(s) + y_2(t)\mu_1(s) & 0 \leq t \leq s \leq 1. \end{cases}$

Using properties (i) and (ii) we obtain

$$y_1(s)\lambda_1(s)+y_2(s)\lambda_2(s)=y_1(s)\mu_1(s)+y_2(s)\mu_2(s)=0. \tag{4.1.11}$$

$$y_1'(s)\mu_1(s+y_2'(s)\mu_1(s)-y_1'(s)\lambda_1(s)-y_2'(s)\lambda_2(s)=1. \tag{4.1.12}$$

$$\Rightarrow \begin{cases} y_1(s)v_1(s)+y_2(s)v_2(s)=0 \\ y_1'(s)v_1(s)+y_2'(s)v_2(s)=-1 \end{cases} \Rightarrow \begin{cases} v_1(s)+sv_2(s)=0 \\ v_2(s)=-1. \end{cases}$$

Therefore $v_1(s)$ =s, and $v_2(s)=-1$.
Now using the relation $\mu_1(s)=\lambda_1(s)+v_1(s),\ \mu_2(s)=\lambda_2(s)+v_2(s)$ and substituting in the above Green's function formula we obtain

$$H(t,s)=\begin{cases} \lambda_1(s)+t\lambda_2(s) & 0\le t\le s\le 1 \\ \lambda_1(s)+s+t(\lambda_2(s)-1) & 0\le s\le t\le 1. \end{cases}$$

Finally using property (iv) we find $\lambda_1(s),\ \lambda_2(s)$

$$\begin{cases} \alpha(y_1(0)\lambda_1(s)+y_2(0)\lambda_2(s))-\beta(y_1'(0)\lambda_1(s)+y_2'(0)\lambda_2(s))=0 \\ \gamma(y_1(\lambda_1(s)+s))+y_2(1)(-1+\lambda_2(s))+\delta(y_1'(\lambda_1(s)+s)+y_2'(1)(-1+\lambda_2)) \end{cases}$$

$$\Rightarrow \begin{cases} \alpha\lambda_1(s)+\beta\lambda_2(s)=0 \\ \gamma\lambda_1(s)+(\gamma+\delta)\lambda_2(s)=-\gamma s+\gamma+\delta. \end{cases}$$

Now from this we obtain $\lambda_1(s)\ and \lambda_2(s)$

$$\lambda_1(s)=\frac{1}{\Delta}\begin{vmatrix} 0, & -\beta \\ -\gamma s+\gamma+\delta, & (\gamma+\delta) \end{vmatrix}=\frac{1}{\Delta}\beta(-\gamma s+\gamma+\delta).$$

$$\lambda_2(s)=\frac{1}{\Delta}\begin{vmatrix} \alpha, & 0 \\ \gamma, & -\gamma s+\gamma+\delta, \end{vmatrix}=\frac{1}{\Delta}\alpha(-\gamma s+\gamma+\delta).$$

Hence by substitution we obtain

$$\text{H(t,s)}=\frac{1}{\Delta}\begin{cases} (\beta+\alpha s)(-\gamma t+\gamma+\delta) & 0\le s\le t\le 1 \\ (\beta+\alpha t)(-\gamma s+\gamma+\delta) & 0\le t\le s\le 1 \end{cases}$$

where $\Delta=\alpha(\gamma+\delta)+\beta\gamma.$

Thus $y(t)=\int_0^1 H(t,s)u(s)ds=\int_0^1\int_0^1 H(t,s)G(s,r)f(r,y(r))drds$, is the solution of the fourth order boundary value problems of (1.1.1), (1.1.2).

Lemma 4.1.2 The Green's function $H(t,s)$ satisfies the following conditions:-

i. $H(t,s) \leq H(s,s)$ for all $t,s \in (0,1)$.

ii. $H(t,s) \geq NH(s,s)$ for all $\frac{1}{4} \leq t \leq \frac{3}{4}$, where $N = \min\left\{\frac{\gamma+4\delta}{4(\gamma+\delta)}, \frac{\alpha+4\beta}{4(\alpha+\beta)}\right\}$

Proof (i), The Green's function $H(t,s)$ is positive for all $(t,s) \in [0,1]$.
Now for $0 \leq t \leq s \leq 1$,

$$\frac{H(t,s)}{H(s,s)} = \frac{(\gamma+\delta-\gamma s)(\beta+\alpha t)}{(\gamma+\delta-\gamma s)(\beta+\alpha s)} = \frac{\beta+\alpha t}{\beta+\alpha s} \leq 1.$$

Thus, $H(t,s) \leq H(s,s)$.

Again for $0 \leq s \leq t \leq 1$,

$$\frac{H(t,s)}{H(s,s)} = \frac{(\gamma+\delta-\gamma t)(\beta+\alpha s)}{(\gamma+\delta-\gamma s)(\beta+\alpha s)} = \frac{(\gamma+\delta-\gamma t)}{(\gamma+\delta-\gamma s)} \leq 1.$$

So that, $H(t,s) \leq H(s,s)$.
Therefore, $H(t,s) \leq H(s,s)$. for all $t,s \in (0,1)$.

(ii). If $s \leq t$ for all $t,s \in [\frac{1}{4}, \frac{3}{4}]$ then we have

$$\frac{H(t,s)}{H(s,s)} = \frac{\gamma+\delta-\gamma t}{\gamma+\delta-\gamma s} \geq \frac{\gamma+\delta-\gamma(\frac{3}{4})}{\gamma+\delta-\gamma(\frac{1}{4})} = \frac{\gamma+4\delta}{3\gamma+4\delta} \geq \frac{\gamma+4\delta}{4(\gamma+\delta)}$$

$$\Rightarrow \frac{H(t,s)}{H(s,s)} \geq \frac{\gamma+4\delta}{4(\gamma+\delta)}.$$

If $t \leq s$, for all $t,s \in [\frac{1}{4}, \frac{3}{4}]$ then we have

$$\frac{H(t,s)}{H(s,s)} = \frac{(\gamma+\delta-\gamma s)(\beta+\alpha t)}{(\gamma+\delta-\gamma s)(\beta+\alpha s)} = \frac{\beta+\alpha(\frac{1}{4})}{\beta+\alpha(\frac{3}{4})} \geq \frac{\alpha+4\beta}{4(\alpha+\beta)}$$

$$\Rightarrow \frac{H(t,s)}{H(s,s)} \geq \frac{\alpha+4\beta}{4(\alpha+\beta)}.$$

Thus,

$$H(t,s) \geq NH(s,s), \ for \ all \ t,s \ \in [\frac{1}{4}, \frac{3}{4}],$$

where $N = \min\left\{\frac{\gamma+4\delta}{4(\gamma+\delta)}, \frac{\alpha+4\beta}{4(\alpha+\beta)}\right\}$.

In generally y(t)= $\int_0^1 H(t,s)u(s)ds \; = \int_0^1 \int_0^1$ H(t,s)G(s,r)f(r,y(r)) dr ds from this

$$\int_0^1 H(t,s)G(s,r)dr$$

is the Green's function for the homogeneous fourth order two point boundary value problem(1.1.2),(1.1.2).

4.2 Method of Superlinear and sublinear condition.

Define the non negative extended real numbers $f_0,\ f^0,\ f_\infty,\ and\ f^\infty,$
$f_0 = \lim_{y\to 0^+} min\ \frac{f(t,y(t))}{y(t)} = \infty, \quad f^0 = \lim_{y\to 0^+}\ max\ \frac{f(t,y(t))}{y(t)} = 0$
$f_\infty = \lim_{y\to\infty} min\ \frac{f(t,y(t))}{y(t)} = \infty \quad f^\infty = \lim_{y\to\infty}\ max\ \frac{f(t,y(t))}{y(t)} = 0,$ where $t \in [0,1]$.
In the following we formulate equivalent operator equation in order to determine the required solution, by assuming that f is either sublinear or superlinear. When $f^0 = 0$ and $f_\infty = \infty$ is called superlinear case and $f_0 = \infty,\ and f^\infty = 0$ is called the sublinear case.
Let $E = \{y|y \in (\ C[0,1] \cap C^4[0,1])\}$ be a Banach space with the norm
$\|y\| = max_{t\in[0,1]}|y(t)|$. Let P be a cone in E given by
$P = \{y \in E|y(t) \geq 0, \quad and\ min_{\frac{1}{4}\leq t\leq\frac{3}{4}}\ y(t) \geq \mu\|y\|\ \}$ where μ = MN.

Let an operator T:$P \to E$ be defined by

$$Ty(t) = \int_0^1 \int_0^1 H(t,s)G(s,r)f(r,y(r))dr\ ds. \qquad (4.2.1)$$

From the fourth order boundary value problem (1.1.1)- (1.1.2), in this thesis the following condition are assumed throughout

- V_1. $0 \leq \int_0^1 \int_0^1 H(t,s)G(s,r)f(r,y(r))dr\ ds < \infty$.
- V_2. $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous.
- V_3. β,δ are positive constants α,γ are nonnegative constants, $\rho = 2sinhk > 0$ and $\gamma\beta + \alpha\gamma + \alpha\delta > 0$.

By applying fixed point theorem on T and establishing suitable conditions on f we determine the existence of a unique fixed point in a cone.

Lemma 4.2.1 [Krasnoselskii,M.A. (1964).] Let E be a Banach space, and Let P $\subset$E be a cone in E. Assume Ω_1, Ω_2 are open subset of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let

$$T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$$

be completely continuous operator such that either

$$(i) \quad \|Ty\| \leq \|y\|, \quad y \in P \cap \partial\Omega_1, and\ \|Ty\| \geq \|y\|, y \in P \cap \partial\Omega_2;\ or$$

$$(ii) \quad \|Ty\| \geq \|y\|, \quad y \in P \cap \partial\Omega_1, and\ \|Ty\| \leq \|y\|, y \in P \cap \partial\Omega_2.$$

Then T has a fixed point in P$\cap(\overline{\Omega_2} \setminus \Omega_1)$.

Lemma 4.2.2 Let V_1, V_2 *and* V_3 hold. Then the operator $T : P \to P$ is completely continuous.

Proof: From the continuity of f, we know Ty $\in E$, for each y = y(t) $\in P$ by above lemma 4.1.1 and 4.1.2 we have

$$Ty(t) = \int_0^1 \int_0^1 H(t,s)G(s,r)f(r,y(r))drds \leq \int_0^1 \int_0^1 H(s,s)G(r,r)f(r,y(r))drds \tag{4.2.2}$$

and hence

$$Ty(t) \leq \int_0^1 \int_0^1 H(s,s)G(r,r)f(r,y(r))drds. \tag{4.2.3}$$

From non negativity of H, G *and* f we have

$$\|Ty\| \leq \int_0^1 \int_0^1 H(s,s)G(r,r)f(r,y(r))drds.$$

Now for $\frac{1}{4} \leq t \leq \frac{3}{4}$ $y \in P$,

$$\begin{aligned} Ty(t)min_{\frac{1}{4}\leq t\leq\frac{3}{4}} &= min_{\frac{1}{4}\leq t\leq\frac{3}{4}} \int_0^1 H(t,s) \int_0^1 G(s,r)f(r,y(r))drds, \\ &\geq \int_0^1 MH(s,s) \int_0^1 NG(r,r)f(r,y(r))drds \\ &\geq \mu \int_0^1 H(s,s) \int_0^1 G(r,r)f(r,y(r))drds \\ &\geq \mu\|Ty\|. \end{aligned}$$

Therefore, $T: P \to P$.
Since H(t,s), G(s,r) and f(r,y(r)) are continuous, it is easily known that
$T: P \to P$ is completely continuous.
Theorem 4.2.1 Assume that the condition $V_1,\ V_2\ and\ V_3$ satisfied, if $f^0 = 0\ and$
$f_\infty = \infty$, then the boundary value problem has at least one positive solution that lies in P.
Proof: Now since $f^0 = 0$, we may choose $Z_1 > 0,\ 0 < y \le Z_1$ so that
$\lim_{y \to 0^+} max\ \frac{f(t,y(t))}{y(t)} = 0$
$\Rightarrow |max \dfrac{f(t,y(t))}{y(t)} - 0| \le \eta \quad t \in [0,1]$
$\Rightarrow max \dfrac{f(t,y(t))}{y(t)} \le \eta$
$\Rightarrow f(t,y(t)) \le \eta y, \quad where\ \eta$ satisfies

$$\eta \int_0^1 H(s,s) \int_0^1 G(r,r) dr\, ds, \le 1. \tag{4.2.4}$$

Thus, if $y \in P\ \ and\ \ \|y\| = Z_1$ then from (4.2.3) and (4.2.4)

$$Ty(t) \le \int_0^1 H(s,s) \int_0^1 G(r,r) f(r,y(r)) dr\, ds \le \|y\|, \quad 0 \le t \le 1. \tag{4.2.5}$$

Now if we let

$$\Omega_1 = \{y \in E : \|y\| < Z_1\} \tag{4.2.6}$$

then (4.2.5) shows that

$$\|Ty\| \le \|y\| \quad P \cap \partial\Omega_1. \tag{4.2.7}$$

Further since $f_\infty = \infty$, that is $f_\infty = \lim_{y(t) \to \infty} min_{t \in [0,1]} \frac{f(t,y(t))}{y(t)}$, there exist $Z_2 > 0$ such that f(t,y(t)) $\ge \xi y,\ y \ge Z_2\ where\ \xi > 0$ is choose so that

$$\mu^2 \xi \int_{\frac{1}{4}}^{\frac{3}{4}} H(s,s) \int_{\frac{1}{4}}^{\frac{3}{4}} G(r,r) dr\, ds \ge 1. \tag{4.2.8}$$

Let $Z_1' = max\{2Z_1, \frac{Z_2}{\mu}\}$ and $\Omega_2 = \{y \in E : \|y\| < Z_1'\}$.
Then y$\in P\ and\ \|y\| = Z_1'$ implies

$$min_{\frac{1}{4} \le t \le \frac{3}{4}} y(t) \ge \mu \|y\| \ge Z_2$$

so that

$$
\begin{aligned}
Ty(t) &= \int_0^1 H(t,s)\int_0^1 G(s,r)f(r,y(r))dr\,ds \\
&\geq min_{\frac{1}{4}\leq t\leq\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}} H(s,s)\int_{\frac{1}{4}}^{\frac{3}{4}} G(r,r)f(r,y(r))\,dr\,ds \\
&\geq M\,N\int_{\frac{1}{4}}^{\frac{3}{4}} H(s,s)\int_{\frac{1}{4}}^{\frac{3}{4}} G(r,r)f(r,y(r))\,dr\,ds \\
&\geq \xi\mu\int_{\frac{1}{4}}^{\frac{3}{4}} H(s,s)\int_{\frac{1}{4}}^{\frac{3}{4}} G(r,r)\,y(r)\,dr\,ds \\
&\geq \|y\|\mu^2\xi\int_{\frac{1}{4}}^{\frac{3}{4}} H(s,s)\int_{\frac{1}{4}}^{\frac{3}{4}} G(r,r)\,dr\,ds \\
&\geq \|y\|.
\end{aligned}
$$

Hence$\|Ty\|\geq\|y\|\ for\ y\in P\cap\partial\Omega_2$.
Therefore by Guo-Krasnoselskii fixed point theorem, it follows that T has a positive fixed point in $P\cap\overline{\Omega_2}\setminus\Omega_1$ such that

$$Z_1\leq\|y\|\leq Z_1^{'}.$$

Hence the boundary value problem (1.1.1),(1.1.2) has a positive solution and this completes the proof of the theorem.
Theorem 4.2.2 Assume that the condition V_1,V_2 and V_3 satisfied, if $f_0=\infty\ and$ $f^{\infty}=0$, then the boundary value problem has at least one positive solution that lies in P.
Proof: suppose that $f_0=\infty=\ \lim_{y(t)\to\infty}\ min_{t\in[0,1]}\ \frac{f(t,y(t))}{y(t)}$,we first choose $Z_1>0$ such that $f(t,y(t))\geq\xi^{'}y(t)\ for\ 0<y\leq Z_1$, where

$$\xi^{'}\mu\int_{\frac{1}{4}}^{\frac{3}{4}} H(s,s)\int_{\frac{1}{4}}^{\frac{3}{4}} G(r,r)\,dr\,ds\geq 1. \tag{4.2.9}$$

Then for y $\in P$ and $\|y\|=Z_1$ we have

$$Ty(t)=\int_0^1 H(t,s)\int_0^1 G(s,r)f(r,y(r))dr\,ds$$

$$\geq\mu\int_{\frac{1}{4}}^{\frac{3}{4}} H(s,s)\int_{\frac{1}{4}}^{\frac{3}{4}} G(r,r)f(r,y(r))dr\,ds$$

$$\geq \xi' \mu \int_{\frac{1}{4}}^{\frac{3}{4}} H(s,s) \int_{\frac{1}{4}}^{\frac{3}{4}} G(r,r)\ y(r) dr\ ds$$

$$\geq Z_1 \xi' \mu \int_{\frac{1}{4}}^{\frac{3}{4}} H(s,s) \int_{\frac{1}{4}}^{\frac{3}{4}} G(r,r) dr\ ds$$

$$\geq \|y\| \xi' \mu \int_{\frac{1}{4}}^{\frac{3}{4}} H(s,s) \int_{\frac{1}{4}}^{\frac{3}{4}} G(r,r) dr\ ds$$

$$\geq \|y\| \qquad\qquad by\ (4.2.9).$$

Thus we may let $\Omega_1 = \{y \in E : \|y\| < Z_1\}$ so that

$$\|Ty\| \geq \|y\|\ for\ P \cap \partial\Omega_1.$$

Now since $f_\infty = 0$ *there exist* $Z_2 > 0$, so that f(t,y(t))$\leq \lambda y,\ for\ \ y \geq Z_2\ where\ \lambda > 0$ satisfies

$$\lambda \int_0^1 H(s,s) \int_0^1 G(r,r) dr\ ds \leq 1.$$

We consider two cases:

case (i). suppose f is bounded, say f(t,y(t)) $\leq S$ for all $y \in (0,\infty)$. In this case choose $Z_1' = max\{2Z_1, S\int_0^1 H(s,s)\int_0^1 G(r,r)dr\ ds\}$ So that for $y \in P$ with $\|y\| = Z_1'$ we have

$$\begin{aligned} Ty(t) &= \int_0^1 H(t,s) \int_0^1 G(s,r) f(r,y(r)) dr\ ds \\ &\leq \int_0^1 H(s,s) \int_0^1 G(r,r) f(r,y(r)) dr\ ds \\ &\leq S\int_0^1 H(s,s) \int_0^1 G(r,r) dr\ ds \\ &\leq Z_1' = \|y\|. \end{aligned}$$

Therefore,

$$\|Ty\| \leq \|y\|.$$

Case (ii). If f is unbounded, then let $Z_1' > max\{2Z_1, Z_2\}$ and such that f(t,y(t)) $\leq f(Z_1', y(Z_1'))\ for\ 0 < y \leq Z_1'$. (We are able to do this f is unbounded .) Then for y $\in P$ and $\|y\| = Z_1'$,we have

$$\begin{aligned}
Ty(t) &= \int_0^1 H(t,s)\int_0^1 G(s,r)f(r,y(r))dr\,ds\\
&\leq \int_0^1 H(s,s)\int_0^1 G(r,r)f(r,y(r))dr\,ds\\
&\leq \int_0^1 H(s,s)\int_0^1 G(r,r)\ \lambda y(s)\ drds\\
&\leq \lambda Z_1^{'}\int_0^1 H(s,s)\int_0^1 G(r,r)dr\,ds\\
&\leq Z_1^{'} = \|y\|.
\end{aligned}$$

Therefore, in either case we may put $\Omega_2 = \{y \in E : \|y\| < Z_1^{'}\},\ and$
$for\ y \in P \cap \partial\Omega_2$ we have $\|Ty\| \leq \|y\|$.
Thus, by the Fixed Point Theorem it follows that BVP (1.1.1), (1.1.2) has a positive solution, and this completes the proof of the theorem.
Theorem 4.2.3 Assume that $f(t,y(t))$ satisfies $|f(t,u(t)) - f(t,v(t))| < \alpha|u-v|$ for $t \in [0,1], u,v \in [0,\infty)\ and\ \alpha \in (0,1)$. Then the BVP (1.1.1) and (1.1.2) has a unique positive solution if $\int_0^1 H(t,s)\int_0^1 G(s,r)dr\,ds < 1$.

Proof For the Operator $T : P \to P$ we have ,

$$|(Tu - Tv)| = |\int_0^1\int_0^1 H(t,s)G(s,r)f(r,u(r))drds - \int_0^1\int_0^1 H(t,s)G(s,r)f(r,v(r))drds,| \tag{4.2.10}$$

$$\leq \int_0^1\int_0^1 H(t,s)G(s,r)|(f(r,u(r)) - f(r,v(r)))\ |\ drds, \tag{4.2.11}$$

$$\leq \|u-v\|\alpha\int_0^1 H(t,s)\int_0^1 G(s,r)drds$$

$$\leq \alpha\|u-v\|.$$

Thus, by Contraction Mapping Theorem T has a unique fixed point.

Chapter 5

Conclusion and Future scope

5.1 Conclusion

In this thesis the existence and uniqueness of positive solution for the fourth order two-point Strum-Liouville boundary value problem is verified. We changed fourth order differential equation to second order, in order to use Green's function to determine non trivial solution for homogeneous boundary value problem. Different lemmas are stated and proved for the Green's function, then the unique solution has been obtained for the BVP (1),(2). We formulate the operator equation for the fourth order two-point Strum-Liouville boundary value problem, we determined the existence of positive solution for the operator equation using sublinear and superlinear condition and by applying Guo-Krasnoleskii fixed point theorem. Finally we verified the uniqueness for the equivalent operator equation.

5.2 Future scope

Differential equation arise in our day to day life. Specially, BVP have many applications in different disciplines. Therefore finding the solution of such differential equations is important to solve the problems. Thus the researcher suggests that more research is necessary on finding the solutions of fourth order Strum-Liouville boundary value problems and one can conduct researches on existence and uniqueness of positive solutions by taking different coefficient and considering orders greater than four.

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