Existence and Uniqueness of Solutions for Initial Value Problems of Caputo Fractional Order Ordinary Differential Equation

A Thesis Submitted to the Department of Mathematics, Jimma University for Partial Fulfillment of the Requirements of the Degree of Masters of Sciences in Mathematics.

(Differential Equation)

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September, 2015
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Declaration

This study is my original work and has not been presented for a degree or like in any other University, where other sources of information have been used.

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Acknowledgment

First of all, my special heartfelt thanks go to my adviser Dr. Yesuf Obsie and co-advisor Dr. Cherinet Tuge for their unreserved support, advice and guidance throughout the preparation of this research. Next, I would like to express my sincere thanks to my wife Ababa Unatu and my daughter Fenet Reta for their financial help and encouraging me to prepare this study. Lastly, I would like to thank Dr. Alemayew Geremu for his comment on the analytic part of my study and all my friends for their constructive comments and provision of some reference materials and any other materials while I was preparing this research.

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Abstract

This research is intended to study the relationship between the fractional order integral of a function h*, the Riemann-Liouville fractional order derivative of the function* h *and the Caputo fractional order derivative of the function* h *and also to prove the existence and uniqueness of solutions for initial value problems (IVP) of Caputo fractional order ordinary differential equations using Schaefer fixed point theorem.*

CHAPTER ONE

INTRODUCTION

1.1 Background of the study

Fractional calculus is a name for the theory of derivatives and integrals of arbitrary order. It generalizes the concepts of integer-order differentiation and n-fold integration, [17]

Fractional calculus is a branch of mathematics that deals with operators having a non-integer order, that is ,fractional derivatives and fractional integrals .Fractional derivatives and integrals are mathematical operators involving differentiation and integration of an arbitrary (non-integer) order such as $D^{\gamma} f(x) = \frac{d^{\gamma} f(x)}{dx^{\gamma}}$,where γ can be taken to be non-integer, [25] .

Differential equations of fractional order is a generalization of ordinary differential equations and integration to arbitrary non integer orders, [5].

There are well-known approaches to fractional derivatives, by Riemann-Liouville, Grunwald -Letnikov and Caputo, [1,11,21] .The most commonly used definitions are by Riemann-Liouville and Caputo. Caputo introduced an alternative definition which has the advantage of defining integer order initial conditions for fractional order differential equations.

Applied problems require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions, which contain y (0), y' (0), etc. . . . The Caputo fractional derivative is important because it allows traditional initial conditions to be included in the formulation of the problem. Caputo'*s* fractional derivative satisfies these demands. Thus, Caputo fractional derivative is the base for fractional differential equations with integer order initial conditions.

For more details on the geometric and physical interpretation for fractional derivatives of both Riemann-Liouville and Caputo types, [11,23].

Fractional differential equations have proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. So, there are numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic and other areas of applications, [9, 10, 13, 17, 29]. There has been a significant development in fractional and partial differential equations in recent years; see the monographs of [1, 6,14,16,17] , and [26].

For a good bibliography on this topic we refer to [25]. As a consequence there is an intensive development of the theory of differential equations of fractional order. We refer to the monographs of [1] .

So solving fractional order differential equation of Caputo type is completely important in the circumstance of applied mathematics, theoretical physics and engineering sciences,[22].To understand Caputo fractional order differential equations as well as further applying it in practical scientific research, it is important to find their exact solutions, [22].

However, to solve Caputo fractional order differential equations using different methods, first it is important to know the existence and uniqueness of solutions for Caputo fractional order differential equation.

Earlier the existence and uniqueness of solutions for fractional order differential equation is studied by using Sadovskii's fixed point theorem, [18] and other method, [21,27,30]. In this research the researcher studies the existence and uniqueness of solutions for initial value problem (IVP) of Caputo fractional order ordinary differential equation using Schaefer fixed point theorem which belongs in fixed point theory. For the fixed point theory and many related results, interested reader may consult [28]. Caputo fractional order derivative has a lot of applications in many areas of mathematics, physics and engineering. This motivates the present study for showing the existence and uniqueness of solution for initial value problems of Caputo fractional order ordinary differential equation.

Therefore, the main purpose of this study is to proof the existence and uniqueness of solutions for initial value problem of Caputo fractional order ordinary differential equation in Banach space using Schaefer fixed point theorem.

1.2. Statements of the problem

Finding the solutions of fractional order differential equation is possible using different methods, [3].

This study is targeted answering the following questions.

- ❖ What is the relationship between the Riemann-Liouville fractional order integral, the Riemann-Liouville fractional order derivative, and the Caputo fractional order derivative?

- ❖ What is the proof of the existence and uniqueness of solution for initial value problems of Caputo fractional order ordinary differential equations using methods of Schaefer$'s$ fixed point theorem in Banach space?

1.3 Objectives of the study

1.3.1. General Objective

The general objective of the study is to show the existence and uniqueness of solutions for Caputo fractional order differential equations by using fixed point theorem.

1.3.2. Specific Objectives

The specific objectives of the studies are

- ❖ To identify the relationship between the Riemann-Liouville fractional integral order, the Riemann-Liouville fractional order derivative, and the Caputo fractional order derivative.
- ❖ To prove the existence and uniqueness of solutions for initial value problems of Caputo fractional order ordinary differential equations in Banach space using methods of Schaefer$'s$ fixed point theorem.

1.4. Significance of the study

The study is essential for the following reasons.

- It develops the researcher's skill to know the relationship between the Riemann-Liouville fractional integral order, the Riemann-Liouville fractional order derivative, and the Caputo fractional order derivative.
- It provides techniques of proving existence and uniqueness of solutions for initial value problems of Caputo fractional order ordinary differential equations by using Schaefer's fixed point theorem.
- It used as reference material for anyone who works on the similar study.

1.5. Delimitation of the study

The study was restricted to prove existence and uniqueness of solutions for initial value problems of Caputo fractional order ordinary differential equations using methods of Schaefer's fixed point theorem.

CHAPTER TWO

2. RELATED LITERATURE REVIEW

Fractional differential equations have become important in recent years as mathematical models of phenomena in engineering, chemistry, physics, and other sciences by using mathematical tools from the theory of derivatives and integrals of fractional non-integer order, [33].

Different Authors have proved the existence and uniqueness of solutions for fractional order differential equations by using different methods and conditions.

In [4] sufficient conditions are established for the existence and uniqueness of solutions for various classes of initial and boundary value problem for fractional differential equations and inclusions involving the Caputo fractional derivative.

In 20012, Sadovskii's fixed point method is used to investigate the existence and uniqueness of solutions of Caputo fractional order differential equations[18].

Krasnoselskii's fixed point method has been successfully employed to study the existence and uniqueness of solutions for nonlocal impulsive fractional integro differential equations in a Banach space, [15].

In[2], Rothe's method is used to establish the existence and uniqueness of a strong solution for a semi-linear fractional differential equation.

In [34], the existence and uniqueness of solutions for fractional differential equations by using Schauder's fixed point method is shown.

In 2009, different conditions are used to prove the existence and uniqueness of solutions of impulsive fractional differential equations [21].

In [19], Burton-kirk fixed point method is used to prove the existence of solutions to partial functional differential equations with impulses and infinite delay, involving the Caputo fractional derivative.

In [30], Banach contraction principle and Krasnoselskii's fixed point theorem are used as a main tool to establish the existence and uniqueness results of fractional order ordinary and delay differential equations and applications.

In 2013, by using Schaefer fixed point theorem, sufficient conditions are established for the existence and uniqueness of solutions for a class of impulsive integro-differential equations with nonlocal conditions involving Caputo fractional derivative[35].

In [32], theorems on existence and uniqueness of the solution are established under some sufficient conditions on nonlinear terms.

In [4], Banach contraction principle and Leray-Schaudre's Alternative fixed point theorem are used to prove the existence and uniqueness solutions for impulsive fractional integro-differential equations in Banach spaces.

In addition, Schaefer fixed point method has been successfully employed to discuss the existence and uniqueness of solutions of initial value problem for impulsive fractional mixed integro differential equations, [27].

Therefore, the main purpose of this study is to investigate the existence and uniqueness of solutions for initial value problem of Caputo fractional order ordinary differential equations in Banach space by using Schaefer fixed point theorem.

CHAPTER THREE

3. METHODOLOGY

3.1 Study area and period

The study was conducted under Mathematics Department at Jimma University which is Ethiopia's first innovative community oriented educational institution of higher learning from September, 2014 to September, 2015.

3.2 Study Design

The study was made analytically.

3.3 Sources of data

The relevant data has been collected from secondary sources such as, published research articles and reference books.

3.4 Procedure of the study

In order to achieve the objectives of the study, i.e. to prove the existence and uniqueness of solutions for initial value problems of Caputo fractional order ordinary differential equation using Schaefer fixed point theorem the study has been followed the following steps.

Consider initial value problem of Caputo fractional order ordinary differential equation:

$$\frac{d^{\alpha}x(t)}{dt^{\alpha}} = f(t, x(t)) \qquad , t \in I = [0, T], \tag{1}$$

$$x(0) = x_0 \ , \ 0 < \alpha < 1 \ and \ 0 < T \leq 1 \tag{2}$$

Let the function F is defined as: $F(x(t)) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)ds$ .Then *by* considering the following assumptions:

H_1 .The function $f : I \times R \rightarrow R$ is continuous.

H_2. There exists a constant $N_1 > 0$ such that $\|f(t,x)\| < N_1$ for each $t \in I$ and each $x \in R$

H_3 .There exists a constant $N_2 > 0$ such that

$$\left\| f(t,x\ ) - f(t,y) \right\| \le N_2 \left\| x\ - y\ \right\|$$

H_4. There exist N>0 such that $\|x_0\| \le N$. Then the study has shown the following steps, [7].

$Step$ 1: Let { x_n } be a sequence such that $x_n \to x$ as n$\to\infty$, then show that

$$\left\| F(x_n(t)) - F(x(t)) \right\| \to 0 \qquad (F \text{ is continuous})$$

$Step$ 2: Show that for any $\mu > 0$, there exist a positive ε such that for each

$x \in B_\eta = \{ x \in PC(I,R) : \|x\|_\infty \le \mu \}$, we have $\|F(x)\|_\infty \le \varepsilon$

(F *maps bounded sets in to bounded sets in* $PC(I,R)$).

$Step$ 3: F maps bounded sets in to eqicontinuous sets of $PC(I,R)$.

$Step$ 4: Show that € = { $x \in PC(I,R) : x = \lambda F(x)$ } for some 0< λ <1 is bounded, where

$PC(I,R)$ denotes the Banach space defined as $PC(I,R) = \{x : I \to R | x \in C(I,X)\}$

with the norm: $\|x\|_{PC}$ = $\sup \|x(t) : t \in I\|$ and $I = [0,T]$

3.5 Ethical Issues

Cooperation letter was received from Education Office of researcher work place to facilitate the collection of data from the sources mentioned.

CHAPTER FOUR

4. RESULT AND DISCUSSION

This work devoted to the study of the existence and uniqueness solutions for initial value problem($IVP, for\ short$) of Caputo fractional ordinary differential equation of order $\alpha \in (0,1)$ which is defined as:

$$\frac{d^{\alpha} x(t)}{dt^{\alpha}} = \mathrm{f}(t, x(t)),\ t \in I = [0, T] \tag{1}$$

$$x(0) = x_0 \tag{2}$$

in Banach space under suitable condition on f by using Schaefer fixed point theorem where :

$f : I \times R \to R$ is a given function, $\frac{d^{\alpha} x(t)}{dt^{\alpha}} = {}^{C}D^{\alpha} x(t)$ is the Caputo fractional order derivative of the function x(t), $x_0 \in R$,0= $t_0 < t_1 < \ldots < t_m < t_{m+1}$ = T and $0 < T \le 1$

Equation (1) and (2) are equations of initial value problems of Caputo fractional order ordinary differential equation.

In this work, the study is discussed the existence and uniqueness of solutions for equation (1) and (2).

4.1 Preliminaries'

In this section, the researcher introduces notations, definitions and preliminary facts which are used throughout this research. The study presents some definitions from [1,7,8,24] as follows.

***Difinition*4.1.1 a.** A normed vector space is a vector space X over a filed F together with a function (the norm), $\|.\| : X \to R^{+}$ satisfying

$i.$ $\|x\|$ =0 if and only if x =0

$ii.$ $\|\lambda x\| = |\lambda|\ \|x\|$, $\forall x \in X$ and λ an element of the filed.

$iii.$ $\|x + y\| \le \|x\| + \|y\|$.

b. A sequence { x_n } in a normed space X is said to be converge to an element x of X if

$$\lim_{n\to\infty}\|x_n - x\| = 0$$

c. A sequence x_n of elements of a normed vector space is called Cauchy sequence if given $\varepsilon > 0$, there exist $N > 0$ such that for every $m, n \geq N$ we have $\|x_m - x_n\| < \varepsilon$

d. A mapping T from A to itself is said to be continuous if $T(x_n) \to T(x)$ for every convergent sequence $(x_n) \subseteq A$ with $x_n \to x \in A$.

e. A Banach space over a field of real or complex is a complete normed vector space.

***Definition*4.1.2:** A mapping T of a normed space E in to itself is said to satisfy a lipschitzim condition with Lipschitz constant k if $\|T(x) - T(y)\| \leq k\|x - y\|$, $x, y \in E$,

***Theorem*4.1.1**: **(Picard's Existence and Uniqueness theorem):** Let f be continuous and bounded on the domain D. Suppose that f satisfies a lipshitz condition on D with respect to its second argument that is constant k such that for, $(t,x),(t,v) \in D$, $|f(t,x) - f(t,v)| \leq k|x - v|$ then the IVP $x' = f(t,x)$, $x(t_0) = x_0$ *where* $t_0, x_0 \in D$ has a unique solution.

***Definition*4.1.3:** A family $F \subseteq C([a,b])$ is eqicontinuous if every $\varepsilon > 0$, there is a $\delta > 0$ such that if $|x - y| < \delta$, *then* $|f(x) - f(y)| < \varepsilon$ for any $f \in F$.

***Theorem*4.1.2(Arzela-Ascoli theorem)**: $F \subseteq C([a,b])$ is compact if and only if F is eqicontinuous and there is M >0 such that $\|f\|_\infty \leq M$ *for all* $f \in F$.

Let $C(I, R)$ be the Banach space of all continuous function from I in to R with the norm:

$$\|x\|_\infty = \sup\{\|x(t)\| : t \in I = [0,T]\} \text{ for } x \in C(I,R).$$

Let us recall the following known definitions.

***Definition* 4.1.4.*a*.** The fractional order integral of a function h (t) of order $\alpha > 0$ is defined

$$by: \quad I_0^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds \qquad (3)$$

b. The Riemann-Liouville fractional derivative of order 0< α <1 of a function $h\ (t)$ is defined as:

$$D^{\alpha}\ h(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{0}^{t}(t-s)^{-\alpha}\ h(s)ds \tag{4}$$

c. The Caputo fractional order derivative of a function h (t) is defined as:

$$^{C}D^{\alpha}h(t) = \frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}(t-s)^{-\alpha}h'(s)ds \tag{5}$$

where Γ denotes the gamma function which is defined as Γ (t) = $\int_{0}^{\infty}e^{-t}t^{x-1}dt$.

4.2 RESULTS

4.2.1 The relationship between the Riemann-Liouville fractional orders integral of a function $h(t)$, the Riemann-Liouville fractional order derivative of a function $h(t)$ and the Caputo fractional order derivative of a function $h(t)$

The study adopts the following from definitions of (3), (4) and (5).

$$I_{0}^{\alpha}h(t) = \frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-s)^{\alpha-1}h(s)ds\,.$$

$$D^{\alpha}h(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{0}^{t}(t-s)^{-\alpha}h(s)ds$$

$$^{C}D^{\alpha}h(t) = \frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}(t-s)^{-\alpha}h'(s)ds$$

Theorem 4.2.1.1 Let $I_0^{\alpha} h(t)$ be the Riemann-Liouville fractional order integral of a function $h(t)$, $D^{\alpha} h(t)$ be the Riemann-Liouville fractional order derivative of a function $h(t)$ and ${}^{C}D^{\alpha} h(t)$ be the Caputo fractional order derivative of a function h(t) , then

1. $D_0{}^{\alpha} h(t) = \frac{d}{dt} I_0^{1-\alpha} h(t)$
2. ${}^{C}D^{\alpha} h(t) = I_0{}^{1-\alpha} h'(t)$
3. $D^{\alpha} h'(t) = \frac{d}{dt}\, {}^{C}D^{\alpha} h(t)$

Proof of 1: From the definition:

$$I_0{}^{\alpha} h(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \text{ ,we have}$$

$$I_0{}^{1-\alpha} h(t) = \left(\int_0^t \frac{(t-s)^{(1-\alpha)-1}}{\Gamma(1-\alpha)} h(s)ds \right).$$

Then $\frac{d}{dt} I_0{}^{1-\alpha} h(t) = \frac{d}{dt} \left(\int_0^t \frac{(t-s)^{(1-\alpha)-1}}{\Gamma(1-\alpha)} h(s)ds \right)$

$$= \frac{d}{dt}\left(\frac{1}{\Gamma(1-\alpha)} \right) \int_0^t (t-s)^{-\alpha} \text{ h(s) ds} + \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{\alpha-1} h(s)ds$$

$$= \frac{d}{dt} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \text{ h(s) ds} \quad \text{because} \quad \frac{d}{dt}\left(\frac{1}{\Gamma(1-\alpha)} \right) = 0$$

$$= D^{\alpha} \text{ h (t)} \quad \text{(Definition of 4.1.1.b)}$$

Therefore D^{α} h (t) $= \frac{d}{dt} I_0{}^{1-\alpha} h(t)$.

Proof of 2: Again from $I_0{}^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds$, we have

$$I_0{}^{1-\alpha} h(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{((1-\alpha)-1)} \text{ h(s) ds}$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \text{ h(s) ds.}$$

So that $I_0^{1-\alpha} \frac{d}{dt} h(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{d}{ds} \text{h(s) ds}$

$$= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} h'(s) ds$$

$$= {}^C D^{\alpha} \text{ h (t)}$$

Therefore ${}^C D^{\alpha}$ h (t) $= I_0^{1-\alpha} h'(t)$.

Proof of 3: From the definition:

$$D^{\alpha} h(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} h(s) ds \text{ ,we get}$$

$$D^{\alpha} \frac{d}{dt} h(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} \frac{d}{ds} h(s) ds$$

$$= \frac{d}{dt} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} h'(s) ds$$

$$= \frac{d}{dt} {}^C D^{\alpha} \text{ h (t)}$$

Therefore $D^{\alpha} \frac{d}{dt} h(t)$ is equal to $\frac{d}{dt} {}^C D^{\alpha}$ h (t).

In general the above proof and its theorem shows the equivalence of $I_0^{\alpha} h(t)$, $D^{\alpha} h(t)$ and ${}^C D^{\alpha} h(t)$.

4.2.2 EXISTENCE AND UNIQUENESS OF SOLUTIONS

The second result is the main result by using Schaefer's fixed point theorem.

***Lemma* 4.2.2.1 (Schaefer's fixed point theorem).** Let X be a Banach space and $F: X \to X$ be a completely continuous operator. If the set $E = \{x \in X: x = \lambda F(x) \ ,0 < \lambda < 1\}$ is bounded, then F has at least a fixed point in X, [12]

***Difintion* 4.2.2.1** A function $x \in PC(I, R)$ is said to be a solution of equation (1) and (2) in Banach space X with norm $\|.\|_X$ if x satisfies the equation:

$$\frac{d^\alpha x(t)}{dt^\alpha} = {}^C D^\alpha x(t) = f(t, x(t)) \text{ on } I, \text{ and the initial condition: } x(0) = x_0$$

***Lemma* 4.2.2.2** Let $\alpha > 0$, then the differential equation ${}^C D^\alpha$ h (t) =0 has solutions h(t) = $c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}$, $c_i \in R$, i=1,2,…,n-1 ,n= $[\alpha]+1$,($[\alpha]$ is the integer part of α, [30].

***Lemma* 4.2.2.3** Let $\alpha > 0$, then $I^\alpha D^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}$, $c_i \in R$, i=1,2,…,n-1,n=$[\alpha]+1$,[30].

***Lemma* 4.2.2.4** Let a function: $f(t, x(t)): I \times R \to R$ be continuous. Then equations (1) and (2) are equivalent to the integral equation:

$$x(t) = \begin{cases} x_0 + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} h(s) ds & \text{if } t \in [0, t_1] \\ x_0 + \dfrac{1}{\Gamma(\alpha)} \displaystyle\sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} h(s) ds + \dfrac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} h(s) ds & \text{if } t \in [t_k, t_{k+1}] \end{cases} \qquad (6)$$

if and only if ${}^C D^\alpha x(t) = f(t, x(t))$

$$x(0) = x_0$$

where $h \in C(I, R)$ satisfies the functional equation $h(t) = f(t, h(t))$. In other words, any solution of (1) and (2) is also solution of (6).

Proof: Suppose $^{C}D^{\alpha}x(t) = f(t, x(t))$.

If $t \in [0, t_1]$, then by applying the Riemann-Liouville integral and *lemma* 4.2.2.3 we get

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds$$

Therefore for each $t \in [0, t_1]$, $x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds$.

If $t \in (t_1, t_2]$, by applying the Riemann-Liouville integral and *lemma* 4.2.2.3 we get

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} h(s) ds$$

Therefore for each $t \in [0, t_2]$, $x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} h(s) ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} h(s) ds$

If $t \in (t_2, t_3]$, by applying the Riemann-Liouville integral and *lemma* 4.2.2. 3 we get

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_{t_2}^t (t-s)^{\alpha-1} h(s) ds$$

Thus for each $t \in [0, t_3]$, $x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} h(s) ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} h(s) ds$ +

$\frac{1}{\Gamma(\alpha)} \int_{t_2}^t (t-s)^{\alpha-1} h(s) ds$

If $t \in (t_k, t_{k+1}]$, by the same procedures with the above, we get

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} h(s) ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} h(s) ds \text{ for each } t \in [0, t_{k+1}]$$

Conversely assume that x *satisfies* $x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds$. If $t \in [0, t_1]$, then $x(0) = x_0$ and using the fact that $^{C}D^{\alpha}$ *is the inverse of* I^{α}, we get $^{C}D^{\alpha}x(t) = f(t, x(t))$, *for each* $t \in [0, t_1]$. If $t \in [t_k, t_{k+1}], k = 1,2, \dots, m$ and using the fact that $^{C}D^{\alpha}C = 0$ where C is a constant, we get $^{C}D^{\alpha}x(t) = f(t, x(t))$.

Thus every solution of (1) and (2) is also a solution of the integral equation (6).

To prove the existence of solutions for equation (1) and (2) the study stated the following theorems.

Theorem 4.2.2.1 Assume that:

H_1 .The function $f: I \times R \to R$ is continuous.

H_2.There exists a constant $N_1 > 0$ such that $\| f(t,x) \| < N_1$ for each $t \in I$ and each $x \in R$.

H_3 .There exists a constant $N_2 > 0$ such that

$$\| f(t,x) - f(t,y) \| \le N_2 \| x - y \|, x, y \in R, t \in [0,T]$$

H_4. There exists a constant $N_3 > 0$ such that

$$|x_0| \le N_3 \ .$$

Then equations (1) and (2) have at least one solution under Schaefer fixed point theorem on $C([0,T], R)$.

Before proving the theorem let transform problem (1) and (2) in to a fixed point problem. Applying Riemann-Liouville integral on both sides and using theorem 4.2.1.1 and fractional calculus, equations (1) and (2) can be represented by:

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s, x(s)) ds$$

Define the operator $F: PC(I,R) \to PC(I,R)$ by

$$F(x(t)) = x_0 + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f(s, x(s)) ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(s, x(s)) ds ,$$

$k = 1, 2, \ldots, m+1$

Using Schaefer fixed point theorem the proof is divided in to four main steps in which the study has shown operator F has a fixed point under the assumptions of the theorem.

***Step* 1**: Under the assumptions of the theorem show that F is continuous.

Let $\{x_n(t)\}$ be a sequence such that $x_n \to x$ in $PC(I,R)$ as $n \to \infty$.

Then for each $t \in [0, t_1]$, we have:

$$F(x_n(t)) - F(x(t)) = [x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, x_n(s))ds] - [x_0 + \int_0^t (t-s)^{\alpha-1} f(s, x(s))ds]$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, x_n(s))ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, x(s))ds$$

$$= \frac{1}{\Gamma(\alpha)} [\int_0^t (t-s)^{\alpha-1} f(s, x_n(s))ds - \int_0^t (t-s)^{\alpha-1} f(s, x(s))ds]$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^t [((t-s)^{\alpha-1} f(s, x_n(s)) - ((t-s)^{\alpha-1} f(s, (x(s))]ds$$

$$\Rightarrow \|F(x_n(t) - F(x(s))\| = \left\|\frac{1}{\Gamma(\alpha)}\int_0^t [((t-s)^{\alpha-1} f(s, x_n(s))) - ((t-s)^{\alpha-1} f(s, x(s))]ds\right\|$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^t \|[(t-s)^{\alpha-1} f(s, x_n(s) - (t-s)^{\alpha-1} f(s, x(s))]\|ds$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s, x_n(s) - f(s, x(s)\|\, ds$$

$$\to \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s, x(s) - f(s, x(s)\|\, ds$$

$$=0$$

Thus $\|F(x_n(t) - F(x(s))\| \to 0$

Therefore, for $t \in [o, t_1]$, $\|F(x_n(t) - F(x(s))\| \to 0$

For each $t \in [t_k, t_{k+1}]$, *we have*

$$F(x_n(t)) - F(x(t)) = \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f(s, x_n(s))ds - \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k} (t-s)^{\alpha-1} f(s, x(s))ds\ s$$

$$= \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k} [(t_k - s)^{\alpha-1} f(s, x_n(s)) - (t_k - s)^{\alpha-1} f(s, x(s))]\, ds$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \ \left[f(s, x_n(s) - f(s, x(s)) \right] \text{ ds}$$

Thus $\|F(x_n(t) - F(x(s))\| = \frac{1}{\tau(\alpha)} \sum_{0<t_k<t} \left\| \int_{t_{k-1}}^{t_k} (t_i - s)^{\alpha-1} [f(s, x_n) - f(s, x(s)] \right\| \text{ds}$

$\leq \frac{1}{\tau(\alpha)} \sum_{0<t_k<t}^{k} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \| f(s, x_n(s)) - f(s, x(s)) \| ds \to 0$, since f is continuous and as n$\to \infty$, $x_n \to x$.

Hence as $n \to \infty, \|F(x_n(t) - F(x(s))\| \to 0$. This proves the continuity of F for t$\in [t_k, t_{k+1}]$.

***Step* 2**: F maps bounded sets in to bounded sets in $PC(I, R)$. To show this, it is enough to show that for any $\mu > 0$, there exist a positive ε such that for each;

$$x \in B_\mu = \{x \in PC\ (I, R) : \|x\|_\infty \leq \mu\}, \text{we have } \|F(x)\|_\infty \leq \ \varepsilon .$$

By H_1, H_2 *and* H_4 , for each $t \in [0, t_1]$, we have:

$$F(x(t)) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s, x(s)) \text{ ds}$$

$$\|F(x(t))\| = \left\| x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s, x(s)) ds \right\|$$

$$\leq \|x_0\| + \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s, x(s)) \right\| \text{ds}$$

$$\leq \|x_0\| + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} \| f(s, x(s)) \| ds$$

$$\leq \|x_0\| + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} N_1 \ ds$$

$$= \|x_0\| + \frac{N_1}{\Gamma(\alpha)} \int_0^t \ (t - s)^{\alpha-1} ds$$

$$= \|x_0\| + \frac{N_1}{\alpha\Gamma(\alpha)} t^\alpha$$

$$=\|x_0\|+\frac{N_1}{\Gamma(\alpha+1)}\mathrm{t}^{\alpha}$$

$$\leq N_3+\frac{N_1}{\Gamma(\alpha+1)}\mathrm{T}^{\alpha}\ \text{ where T = t}_1$$

$$\textit{Thus for } t\in[0,t_1], \|F(x(t))\|\leq N_3+\frac{N_1}{\Gamma(\alpha+1)}\mathrm{T}^{\alpha}$$

For t $\in [t_k, t_{k+1}]$, (k= 1,2,… , m), we have :

$$F(x(t)) = x_0 + \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}f(s,x(s))\,\mathrm{ds} + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}f(s,x(s))\,\mathrm{ds}$$

$$\|F(x(t))\| = \left\|x_0+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}f(s,x(s))ds+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}f(s,x(s))ds\right\|$$

$$\leq\|x_0\|+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\|f(s,x(s))\|ds+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\|f(s,x(s))\|ds$$

$$=\|x_0\|+\sum_{0<t_k<t}\frac{N_1}{\alpha\Gamma(\alpha)}\left(t_k-t_{k-1}\right)^{\alpha}+\frac{N_1}{\alpha\Gamma(\alpha)}\left(t-t_k\right)^{\alpha}$$

$$\leq\|x_0\|+\frac{mN_1}{\Gamma(\alpha+1)}T^{\alpha}+\frac{N_1}{\Gamma(\alpha+1)}\mathrm{T}^{\alpha}$$

$$\leq N_3+\frac{(m+1)N_1T^{\alpha}}{\Gamma(\alpha+1)}\qquad\qquad \textit{where } T=t_{k+1}$$

Let ε = max {$N_3+\frac{mN_1T^{\alpha}}{\Gamma(\alpha+1)}$, $N_3+\frac{(m+1)N_1T^{\alpha}}{\Gamma(\alpha+1)}$ } , k = 1,2, ... , m

This implies $\|F(x(t))\|_{\infty}\leq\varepsilon$

Therefore F maps bounded sets in to bounded sets.

***Step* 3**: F maps bounded sets in to eqicontinuous sets of $PC\ (I, R)$.

Let $a_1, a_2 \in I$, $a_1 < a_2$, Bη be bounded set of $PC(I, R)$ as in step 2 , and let x $\in$ Bη . Then for $a_1, a_2 \in [0, t_1]$, we have:

$$\|F(x(a_2))-F(x(a_1))\| = \left\|\frac{1}{\Gamma(\alpha)}\int_0^{a_2}(a_2-s)^{\alpha-1}f(s,x(s))ds-\frac{1}{\Gamma(\alpha)}\int_0^{a_1}(a_1-s)^{\alpha-1}f(s,x(s))ds\right\|$$

$$=\frac{1}{\Gamma(\alpha)}\left[\| \int_0^{a_1}(a_2-s)^{\alpha-1}f(s,x(s))ds+\int_{a_1}^{a_2}(a_2-s)^{\alpha-1}f(s,x(s))ds-\int_0^{a_1}(a_1-s)^{\alpha-1}f(s,x(s))ds \|\right]$$

$$=\frac{1}{\Gamma(\alpha)}\left[\| \int_0^{a_1}\left((a_2-s)^{\alpha-1}-(a_1-s)^{\alpha-1} \right)f(s,x(s))ds+\int_{a_1}^{a_2}(a_2-s)^{\alpha-1}f(s,x(s))ds \|\right]$$

$$\leq\frac{1}{\Gamma(\alpha)}\left[\int_0^{a_1}\|\left((a_2-s)^{\alpha-1}-(a_1-s)^{\alpha-1} \right)f(s,x(s))ds \|+\| \int_{a_1}^{a_2}(a_2-s)^{\alpha-1}f(s,x(s))ds \|\right]$$

$$=\frac{1}{\Gamma(\alpha)}\left[\int_0^{a_1}\| (a_2-s)^{\alpha-1}-(a_1-s)^{\alpha-1} \|\times\| f(s,x(s)) \|ds+\int_{a_1}^{a_2}\| (a_2-s)^{\alpha-1} \|\times\|f(s,x(s))\|ds\right]$$

$$\leq\frac{N_1}{\Gamma(\alpha)}\left[\int_0^{a_1}\| (a_2-s)^{\alpha-1}-(a_1-s)^{\alpha-1} \|ds+\int_{a_1}^{a_2}\| (a_2-s)^{\alpha-1} \|ds\right]$$

$$\leq\frac{N_1}{\Gamma(\alpha)}\left[\int_0^{a_1}\left(\|(a_2-s)^{\alpha-1}\|ds\right)+\int_0^{a_1}\| (a_1-s)^{\alpha-1} \|ds+\int_{a_1}^{a_2}\|(a_2-s)^{\alpha-1}\|ds\right]$$

$$=\frac{N_1}{\Gamma(\alpha)}\left[\left\|\frac{-1}{\alpha}(a_2-a_1)^{\alpha}\right\|+\frac{1}{\alpha}\|(a_2)^{\alpha}\|+\frac{1}{\alpha}\|(a_1)^{\alpha}\|+\frac{1}{\alpha}\|(a_2-a_1)^{\alpha}\|\right]$$

$$=\frac{N_1}{\alpha\Gamma(\alpha)}\left[\|2(a_2-a_1)^{\alpha}\|+\|(a_2)^{\alpha}\|+\|(a_1)^{\alpha}\|\right]$$

$$=\frac{N_1}{\Gamma(\alpha+1)}\left[\|2(a_2-a_1)^{\alpha}\|+\|(a_2)^{\alpha}\|+\|(a_1)^{\alpha}\|\right]$$

Therefore $\|F(x(a_2))-F(x(a_1))\| \leq\frac{N_1}{\Gamma(\alpha+1)}\left[2(a_2-a_1)^{\alpha}+(a_2)^{\alpha}+(a_1)^{\alpha}\right]$

As $a_1 \to a_2$ the right hand side tends to 0.

For a_1, $a_2 \in [t_k, t_{k+1}]$, $k=1,2,\dots,m$, we have

$$\|F(x(a_2)-F(x(a_1))\| \leq \left\|\frac{1}{\Gamma(\alpha)}\int_0^{a_2}(a_2-s)^{\alpha-1}f(s,x(s))-\int_0^{a_1}(a_1-s)^{\alpha-1}f(s,x(s))ds\right\|$$

$$\leq \frac{1}{\Gamma(\alpha)}\left[\int_0^{a_1}\|\left((a_2-s)^{\alpha-1}-(a_1-s)^{\alpha-1} \right)f(s,x(s))ds \|+\| \int_{a_1}^{a_2}(a_2-s)^{\alpha-1}f(s,x(s))ds \|\right]$$

$$\leq \frac{1}{\Gamma(\alpha)}\left[\int_0^{a_1}\left\|\left((a_2-s)^{\alpha-1}-(a_1-s)^{\alpha-1}\right)\right\|\times\| f(s,x(s))\|ds+\int_{a_1}^{a_2}\|(a_2-s)^{\alpha-1}\|\times\|f(s,x(s))\|ds\right]$$

$$\leq \frac{N_1}{\Gamma(\alpha+1)}\left(\left\|2(a_2-a_1)^{\alpha}\right\|+\|a_2^{\alpha}\|+\|a_1^{\alpha}\|\right)$$

As $a_1 \to a_2$, the right-hand side of the above inequality tends to zero. By steps 1 to 3 together with the Arzela-Ascoli theorem, we conclude that $F: PC(I,R) \to PC(I,R)$ is completely continuous.

***Step* 4**: Under Schaefer fixed point theorem show that given set is bounded. To assure this, we need to show that the set:

$€ = \{x \in PC(I,C) : x = \lambda F(x) \text{ for some } 0 < \lambda < 1\}$ is bounded.

Let $x \in €$, then $x = \lambda F(x)$ for some $0 < \lambda < 1$ .Thus, for each $t \in I$, we have:

$$x(t) = \lambda F(x(t))$$

$$= \lambda x_0 + \frac{\lambda}{\Gamma(x)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}f(s,x(x))ds+\frac{\lambda}{\Gamma(x)}\int_{t_k}^{t}(t-s)^{\alpha-1}f(s,x(s))ds$$

By H_2 and H_4 as in step 2 that for each $t \in [t_k,t_{k+1}]$ we have

$$\|x(t)\| \leq \lambda\|x_0\| + \frac{\lambda m N_1 T^{\alpha}}{\Gamma(\alpha+1)}+\frac{\lambda N_1 T^{\alpha}}{\Gamma(\alpha+1)}$$

Thus for every $t \in I$, we have

$$\|x\|_{\infty} \leq \lambda N_3 + \frac{\lambda m N_1 T^{\alpha}}{\Gamma(\alpha+1)}+\frac{\lambda N_1 T^{\alpha}}{\Gamma(\alpha+1)} := r.$$

This shows that the set € is bounded. By steps 1,2 and 4, we conclude that:

$F: PC(I,R) \to PC(I,R)$ is completely continuous and the set € is bounded. Thus by Schaefer fixed point theorem, we conclude that F has a fixed point which is a solution of the problem (1) and (2).

Therefore theorem 4.2.2.1 proves the existence of solutions for initial value problems of Caputo fractional order ordinary differential equation defined in equation (1) and (2) by using Schaefer's fixed point theorem.

Now we are in a position to prove the uniqueness of a solution for equation (1) and (2).

Theorem4.2.2.2 Assume H_3 holds with lipschitz constant k. Then equation (1) and (2) has unique solution on $C([0,T],R)$.

Proof: Define the map F by:

$$F(x(t)) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x(s))ds$$

Then for $x(t), y(t) \in C([0,T],R)$ we have

$$F(x(t)) - F(y(t)) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x(s))ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,y(s))ds$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(f(s,x(s)) - f(s,y(s)) \right) ds$$

$$\Rightarrow \|F(x(s)) - F(y(t))\| = \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(f(s,x(s) - f(s,y(s) \right) ds \right\|$$

$$= \frac{1}{\Gamma(\alpha)} \left\| \int_0^t (t-s)^{\alpha-1} \left(f(s,x(s) - f(s,y(s) \right) ds \right\|$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t \| (t-s)^{\alpha-1} \| \times \| \left(f(s,x(s)) - f(s,y(s)) \right) \| ds$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t \| (t-s)^{\alpha-1} \| \times N_2 \| x-y \| ds$$

$$= \frac{N_2}{\Gamma(\alpha)} \int_0^t \| (t-s)^{\alpha-1} \| \times \| x-y \| ds$$

$$= \frac{N_2}{\Gamma(\alpha)} \|x-y\| \int_0^t \| (t-s)^{\alpha-1} \| ds$$

$$= \frac{N_2 t^\alpha}{\Gamma(\alpha+1)} \|x-y\|$$

$$\leq \frac{N_2 T^\alpha}{\Gamma(\alpha+1)} \|x-y\|$$

Thus F satisfies the lipschitz condition with k $= \frac{N_2 T^\alpha}{\Gamma(\alpha+1)}$.

In theorem 4.2.2.1 the study has shown that F is continuous and bounded also in theorem 4.2.2.2 F satisfies the lipschitz condition. Thus F satisfies the Picard$'s$ theorem. Therefore by theorem 4.2.2.2 and step1, 2, and 3 of theorem 4.2.2.1 together with the picard$'s$ existence and uniqueness theorem, we conclude that the solution of equation (1) and (2) is unique.

Finally, theorem 4.2.2.1 and theorem 4.2.2.2 shows the overall procedures to show the existence and uniqueness of solutions for initial value problems of Caputo fractional order ordinary differential equation expressed in equation (1) and (2) using Schaefer fixed point theorem.

EXAMPLES

1. Consider the initial value problem of Caputo fractional order differential equation:

$$^{C}D^{\alpha}x(t) \quad = \frac{1}{(t+2)^2}\frac{|x|}{1+|x|} \quad , t \in I = [0,1] \text{ and } \; x \in \Re$$

$$x(0) = x_0 .$$

Set $f(t,x) =. \frac{1}{(t+2)^2}\frac{|x|}{1+|x|}$ .Then

i. Clearly the function f is continuous. So that the IVP given above satisfy (H_1).

ii. For any $x, y \in [0, \infty)$ and $t \in [0,1]$

$$|f(t,x) - f(t,y)| \quad = \left| \frac{1}{(t+2)^2}\frac{x}{1+|x|} - \frac{1}{(t+2)^2}\frac{y}{1+|y|} \right|$$

$$= \left| \frac{1}{(t+2)^2}\left(\frac{x}{1+|x|} - \frac{y}{1+|y|} \right) \right|$$

$$= \left| \frac{1}{(t+2)^2}\left(\frac{x(1+|y|) - y(1+|x|)}{(1+|x|)(1+|y|)} \right) \right|$$

$$= \left| \frac{1}{(t+2)^2}\left(\frac{x + x|y| - y - y|x|}{(1+|x|)(1+|y|)} \right) \right|$$

$$= \left| \frac{1}{(t+2)^2} \right| \left| \frac{x + x|y| - y - y|x|}{(1+|x|)(1+|y|)} \right|$$

$$\leq \frac{1}{(t+2)^2}\|x - y\|$$

$$= \frac{1}{4}\|x - y\| \quad if \;\; t = 0 \;\; .$$

$Thus \; \|f(t,x) - f(t,y)\| \;\; \leq \; k\|x - y\| \quad where \, k = \frac{1}{4}$.

Hence, the condition H_2 and H_3 of Schaefer fixed point theorem is satisfied. Therefore by $theorem$4.2.2.1 $and\ theorem$ 4.2.2.2 , the problem given above has a unique solution.

2. Consider the initial value problem of fractional order differential equation:

$$ {}^{C}D^{\alpha}x(t) = \frac{e^{-t} + |x(t)|}{(9+e^{-t})(1+|x(t)|)} , \; t \in I = [0,1], 0 < \alpha \le 1 \; and \;\; x, y \in [0,\infty) $$

$$ x(0) = 0 . $$

Set $f(t,x) = \dfrac{e^{-t}x}{(9+e^{-t})(1+x)}$. Then

i. Clearly the function f is continuous. So that the IVP given above satisfy (H_1)

ii. For any $x, y \in [0,\infty)$ and $t \in [0,1]$, we have

$$ |f(t,x) - f(t,y)| = \left| \frac{e^{-t}x}{(9+e^{t})(1+x)} - \frac{e^{-t}y}{(9+e^{-t})(1+y)} \right| $$

$$ = \frac{e^{-t}}{e^{t}+9} \frac{|x(1+y) - y(x+1)|}{|(x+1)(y+1)|} $$

$$ = \frac{e^{-t}}{e^{t}+9} \frac{|x + xy - yx - y|}{|(x+1)(y+1)|} $$

$$ = \frac{e^{-t}}{e^{t}+9} \frac{|x-y|}{(x+1)(y+1)} $$

$$ \le \frac{e^{-t}}{e^{-t}+9} |x-y| $$

$$ = \frac{1}{10} |x-y| \qquad if \;\; t = 0 $$

Hence, the condition H_2 and H_3 of Schaefer fixed point theorem is satisfied with $k = \dfrac{1}{10}$

It follows from $theorem$ 4.2.2 and $theorem$ 4.2.3 the problem given has a unique solution.

CHAPTER FIVE

5. CONCLUSION AND FUTURE SCOPE

In this study, the researcher study initial value problems for Caputo fractional order ordinary differential equations in Banach space and establish the existence and uniqueness results of solutions by fixed point theorem via a given estimate conditions. The study's result is based on the well-known Schaefer's fixed point theorem to show the existence and uniqueness of solutions under certain conditions. The existence and uniqueness solutions of boundary value problems for Caputo fractional order ordinary differential equation is the open problems.

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