Existence of Positive Solutions for Fourth Order Two Point Sturm Liouville Boundary Value Problems

A Thesis Submitted to the Department of Mathematics in Partial Fulfillment for the Requirements of the Degree of Masters of Science in Mathematics

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Declaration

I, the undersigned declared that, the thesis entitled existence of positive solutions for fourth-order two-point Sturm-Liouville boundary value problems is original and it has not been submitted to any institution elsewhere for the award of any degree or like, where other sources of information that have been used, they have been acknowledge.

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Abstract

This thesis is concerned with fourth-order two-point non-linear Sturm-Liouville boundary value problems. It also focused on constructing Green's function for corresponding non-trivial homogeneous equation by using its properties. Under the suitable conditions, we established the existence of at least one positive solution by applying Guo-Krasnoselskii's fixed point theorem. We provided examples to demonstrate for the applicability of our main result.

Contents

1 Introduction

1.1 Background of the study

Boundary value problems (BVPs for short) of fourth-order ordinary differential equations have received much attention due to their striking applications in engineering, physics, material mechanics, fluid mechanics and so on.

Boundary value problems associated with linear as well as non-linear ordinary differential equations or finite difference equations have created a great deal of interest and play an important role in many fields of applied mathematics such as engineering design and manufacturing and, major industries like automobile, aerospace, chemical, pharmaceutical, petroleum, electronics and communications as well as emerging technologies like biotechnology and nanotechnology rely on the boundary value problems to simulate complex phenomena at different scales for designing and manufacturing of high-technological products.

In the field of differential equations, a boundary value problem is a differential equation together with a set of additional constraints, called the boundary conditions. A solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary conditions.

Boundary value problems for ordinary differential equations play a very important role in both theory and applications.

They are used to describe a large number of physical, biological and chemical phenomena. Fourth-order differential equations boundary value problem occurs in beam theory (Bernis, 1982, Zill and Cullen, 2001), such as a beam with small deformation: a beam of a material which satisfies a nonlinear power-like stress and strain law; a beam with two-sided links which satisfies a nonlinear power-like elasticity law.

All these can be described by some fourth-order differential equations along with their boundary conditions. For example, the work of Timoshenko, (Timoshenko, 1961) on elasticity, the monograph by Soedel (Soedel, 1993) on deformation of structure, and the work of Dulacska (Dulacska, 1992) on the effects of soil settlement are rich sources of such applications.

In addition, the other works for beam equation (Chen, Ni and Wang (2006), Graef, Qian and Yang (2003), Ma, Zhang and Fu, (1997) and Lian, (2004)) in details.

Many authors have studied the existence of positive solutions to some fourth-order BVPs by using Guo-Krasnoselskii's fixed point theorem in cones. However, it is necessary to point out that, in most of the existing literature, the Green's functions involved are nonnegative, which is an important condition in the study of positive solutions of BVPs.

In analyzing nonlinear phenomena, many mathematical models give rise to problems for which only positive solutions make sense.

Therefore, since the publication of the monograph positive solutions of Operator Equations in the year 1964 by academician M.A. Krasnoselskii's, hundred of research articles on the theory of positive solutions of nonlinear problems have appeared.

In this vast field of research, we focused on the existence of positive solutions for fourth-order two-point nonlinear Sturm-Lioville boundary value problems. Recently the existence of positive solutions of boundary value problems was studied by many researchers.

We list down few of them which are related to our particular problem.

Erbe, L.H and Haiyan Wang, (1994), Lian, Wong, and Yeh, (1996), Henderson and Wang, (1997), Xin Dong and Zhanbing Bai, (2008), Moustafa El-Shahed and Tahani Al-Dajani, (2008), R.Vrabel, (2015), Yongxiang Li, (2016), Yun Zhang Jian-Ping Sun and Juan Zhao, (2018), Dang Quang A and Ngo Thi Kim Quy, (2018) and Yongfang Wei, Qilin Song, Zhanbing Bai, (2019).

Motivated by the above mentioned results, in this thesis, we established the existence of positive solutions for fourth-order two-point Sturm-Liouville boundary value problems,

$$y^{(4)}(t) + k^2 y''(t) = f(t, y(t)), \qquad 0 \le t \le 1, \tag{1.1}$$

$$\alpha y(0) - \beta y'(0) = 0, \gamma y(1) + \delta y'(1) = 0, \tag{1.2}$$

$$y''(0) = y''(1) = 0, \tag{1.3}$$

where $k \in (0, \frac{\pi}{2})$ is a constant, α, β, γ and δ are positive constants such that $f : [0,1]\times[0,\infty) \longrightarrow [0,\infty)$ is continuous function by applying Guo-Krasnoselskii's fixed point theorem in a cone Banach space. We also provided examples to demonstrate the applicability of our main result. By a positive solution of (1.1)-(1.3) we understand a function $y(t)$ which is positive on $0 \leq t \leq 1$ and satisfies the differential equation (1.1) for $0 \leq t \leq 1$ and the Sturm-Liouville boundary conditions (1.2), (1.3).

The rest of this thesis organized as follows: We first present some definitions which are needed throughout this work and construct Green's function by using its properties for corresponding homogeneous boundary value problems and state fixed point result by applying the Guo-Krasnoselskii's fixed point theorem in a cone Banach space. Finally, we investigate the existence of at least one positive solution for fourth-order two-point Sturm-Liouville boundary value problems (1.1)-(1.3) and as an application, examples were included to verify the applicability of our result.

1.2 Statement of the problem

In this study we focused on establishing the existence of positive solutions for fourth-order two-point Sturm-Liouville boundary value problems (1.1)-(1.3).

1.3 Objectives

1.3.1 General objective:

The main objective of this thesis was establishing the existence of positive solutions for fourth-order two-point Sturm-Liouville boundary value problems by applying Geo-Krasnoselskii's fixed point theorem.

1.3.2 Specific Objectives:

This study has the following specific objectives:
i) To construct Green's function by following its properties for corresponding

homogeneous boundary value problem.

ii) To formulate the equivalent integral equation for the boundary value problems (1.1)-(1.3).

iii) To prove the existence of at least one positive solution by applying Guo-Krasnoselskii's fixed point theorem.

iv) To verify the main result by providing illustrative examples.

1.4 Significance of the study:

The result of this thesis may have the following importance:

1) It may build the research skill and scientific communication skill of the researcher.

2) It may develop the researcher knowledge on applied mathematics research.

3) It may provide some background information for other researchers who want to conduct a research on related topics.

1.5 Delimitation of the study

The study was delimited to finding the existence of at least one positive solution for fourth-order two-point Sturm-Liouville boundary value problems.

2 Literature Review

2.1 Overview of positive solution

A positive solution is very important in diverse disciplines of mathematics since it can be applied for solving various problems and it is one of the most dynamic research subjects in nonlinear analysis. In this area the first important and significant result the existence of positive solution was proved by Erbe and Wang in 1994. Due to the importance of existence of positive solutions have been investigated heavily by many researchers:
Erbe, L.H and Haiyan Wang, (1994), established positive solutions for the two-point boundary value problem,

$$u''(t) + a(t)f(u(t)) = 0, \qquad 0 < t < 1,$$

$$\alpha u(0) - \beta u'(0) = 0,$$

$$\gamma u(1) + \delta u'(1) = 0.$$

Lian, Wong, and Yeh, (1996), studied the existence of at least one positive solution and multiple positive solutions for the two-point boundary value problem.

$$u''(t) + f(t, u(t)) = 0, \qquad t \in (0, 1),$$

$$\alpha u(0) - \beta u'(0) = 0,$$

$$\gamma u(1) + \delta u'(1) = 0.$$

Henderson and Wang, (1997), determined eigenvalue intervals, for which there exist positive solutions of the boundary value problem,

$$u''(t) + \lambda a(t)f(u(t)) = 0, \qquad t \in (0, 1),$$

$$u(0) = u(1) = 0.$$

Xin Dong and Zhanbing Bai, (2008), considered the existence of one or two positive solutions for the fourth-order boundary value problem with variable parameters,

$$u^{(4)}(t) + B(t)u''(t) - A(t)u(t) = f(t, u(t), u''(t)), \quad 0 < t < 1,$$

$$u(0) = u(1) = u''(0) = u''(1) = 0,$$

where $A(t), B(t) \in C[0,1]$ and $f(t,u,v) : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous. Moustafa El-Shahed and Tahani Al-Dajani, (2008), established the existence of positive solutions to nonlinear fourth order boundary value problem,

$$u^{(4)}(t) + \lambda a(t) f(u(t)) = 0, \quad 0 \le t \le 1,$$

$$u(0) = u''(0) = u'''(0) = u(1) = 0,$$

where $\lambda > 0$, positive parameter, $a : (0,1) \to [0,\infty]$ is continuous and $\int_0^1 a(t)dt > 0$.
R.Vrabel, (2015), established the existence of solution of the fourth-order differential equation by using lower and upper solution, namely, the ordinary differential equation,

$$u^{(4)}(t) + \lambda u''(t) = h(t, u(t)), \quad \lambda < 0,$$

subject to the Lidestone boundary conditions

$$u(0) = u(1) = u''(0) = u''(1) = 0.$$

Yongxiang Li, (2016), discussed the existence of positive solutions of the fully fourth-order nonlinear boundary value problems,

$$u^{(4)}(t) = f(t, u', u'', u'''), \qquad 0 \le t \le 1,$$

$$u(0) = u'(0) = u''(1) = u'''(1) = 0,$$

where $f:[0,1]\times R_+^3\times R_-\to R_+$ is continuous.
Yun Zhang, Jian-Ping Sun and Juan Zhao, (2018), concerned with the following fourth-order three-point boundary value problem with sign-changing Green's function,

$$u^{(4)}(t)=f(t,u(t)),\qquad 0\le t\le 1,$$

$$u'(0)=u''(0)=u'''(\eta)=u(1)=0,$$

where $\eta\in[1/3,1)$.
Dang Quang A and Ngo Thi Kim Quy, (2018), investigated the solvability and iterative solution of a nonlinear fully fourth order boundary value problem,

$$u^{(4)}(t)=f(t,u(t),u'(t),u''(t),u'''(t)),\qquad 0<t<1,$$

$$u(0)=u(1)=u''(0)=u''(1)=0,$$

where $f:[0,1]\times R^4$ is continuous.
Yongfang Wei, Qilin Song, Zhanbing Bai, (2019), proved the existence of the iterative solution to a fourth order boundary value problems,

$$u^{(4)}(t)=f(t,u(t),u'(t)),\qquad 0<t<1,$$

$$u(0)=u'(0)=u'(1)=u''(1)=0.$$

2.2 Preliminaries

First we recall some known definitions and basic concepts on Green's function that we used in the proof our main results.

Definition 2.1 *Let X be a non-empty set. A map $T:X\to X$ is said to be a self-map with domain of $T=D(T)=X$ and range of $T=R(T)=T(X)\subset X$.*

Definition 2.2 *Let $T:X\to X$ be self-map. A point $x\in X$ is called a fixed point of T if $Tx=x$.*

Definition 2.3 *We consider the second-order linear differential equation*

$$p_0(t)y'' + p_1(t)y' + p_2(t)y = r(t), \qquad t \in J = [0,1]. \tag{2.1}$$

Where the functions $p_0(t), p_1(t), p_2(t)$ and $r(t)$ are continuous in J and boundary conditions of the form

$$\begin{aligned} l_1[y] &= a_0y(0) + a_1y'(0) + b_0y(1) + b_1y'(1) = A, \\ l_2[y] &= c_0y(0) + c_1y'(0) + d_0y(1) + d_1y'(1) = B, \end{aligned} \tag{2.2}$$

where $a_i, b_i, c_i, d_i, i = 0,1$ and A, B are given constants.

The boundary value problems $(2.1), (2.2)$ is called a nonhomogeneous two-point linear boundary value problems, whereas the homogeneous differential equation

$$p_0(t)y'' + p_1(t)y' + p_2(t)y = 0,\ t \in J = [0,1]. \tag{2.3}$$

together with the homogeneous boundary conditions

$$\begin{aligned} l_1[y] &= 0, \\ l_2[y] &= 0, \end{aligned} \tag{2.4}$$

be called a homogeneous two-point linear boundary value problems.

The function called a Green's function $G(t,s)$ for the homogeneous boundary value problems (2.3), (2.4) and the solution of the nonhomogeneous boundary value problems (2.1), (2.2) can be explicitly expressed in terms of $G(t,s)$.

Obviously, for the homogeneous problems (2.3), (2.4) the trivial solution always exists. Green's function $G(t,s)$ for the boundary value problems (2.3), (2.4) is defined in the square $[0,1] \times [0,1]$ and possesses the following fundamental properties:

i) $G(t,s)$ is continuous in $[0,1] \times [0,1]$,

ii) $\frac{\partial G(t,s)}{\partial t}$ is continuous in each of the triangles $0 \le t \le s \le 1$ and $0 \le s \le t \le 1$

moreover,

$$\frac{\partial G(s^+, s)}{\partial t} - \frac{\partial G(s^-, s)}{\partial t} = \frac{1}{P_0(s)},$$

where $\frac{\partial G(s^+,s)}{\partial t} = \lim_{t\to s, t>s} \frac{\partial G(t,s)}{\partial t}$ and $\frac{\partial G(s^-,s)}{\partial t} = \lim_{t\to s, t<s} \frac{\partial G(t,s)}{\partial t}$.

iii) For every fixed $s \in [0,1]$, $z(t) = G(t,s)$ is a solution of the differential equation

(2.3) in each of the intervals [0, s) and (s, 1],

iv) For every fixed $s \in [0,1]$, $z(t) = G(t,s)$ satisfies the boundary conditions (2.4).

These properties completely characterize Green's function $G(t,s)$.

Definition 2.4 *Let $-\infty < a < b < \infty$. A collection of real valued functions $A = \{f_i | f_i : [a,b] \to R\}$ is said to be*

Uniformly bounded, if there exists a constant $M > 0$ with $|f_i(t)| \leq M$, for all $t \in [a,b]$ and for all $f_i \in A$.

Definition 2.5. A normed linear space is a linear space X in which for each vector x there corresponds a real number, denoted by $\|x\|$ called the norm of x and has the following properties:

i) $\|x\| \geq 0$, for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$,

ii) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in X$,

iii) $\|\alpha x\| = |\alpha|\|x\|$, for all $x \in X$ and α being a scalar.

Definition 2.5 *Let X be a normed linear space with norm denoted by $\|.\|$. A sequence of elements x_n of X is a Cauchy sequence, if for every $\epsilon > 0$ there exists an integer N such that $\|x_n - x_m\| < \epsilon$, for all $m, n \geq N$.*

Definition 2.6 *A normed linear space X is said to be complete, if every Cauchy sequence in X converges to a point in X.*

Definition 2.7 *A complete normed space X is called a Banach space.*

Definition 2.8 *Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone, if it satisfies the following two conditions:*

i *) $y \in P$, $\alpha \geq 0$ implies $\alpha y \in P$, and*

***ii*)** *$y \in P$ and $-y \in P$ implies $y = 0$.*

Definition 2.9 *Let X and Y be two metric spaces. A map $T : X \to Y$ is said to be completely continuous, if it is continuous and maps bounded sets into precompact sets.*

Definition 2.10 *Let X and Y be Banach spaces and $T : X \to Y$ an operator T is said to be completely continuous, if T is continuous and for each bounded sequence $x_n \subset (X), (Tx_n)$ has a convergent subsequences.*

Definition 2.11 *Let E be a real Banach space with cone P.*
A map $f : P \to [0, \infty)$ is said to be a nonnegative continuous convex functional on P, if f is continuous and

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y),$$

for all $x, y \in P$ and $\lambda \in [0, 1]$.

Definition 2.12 *Let E be a real Banach space with cone P.*
A map $f : P \to [0, \infty)$ is said to be a nonnegative continuous concave functional on P, if f is continuous and

$$f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y),$$

for all $x, y \in P$ and $\lambda \in [0, 1]$.

Definition 2.13 *The function $y(t) \in C[0, 1] \cap C^4[0, 1]$ is a positive solution of the boundary value problems,*

$$y^{(4)}(t) + k^2 y''(t) = f(t, y(t)), \qquad 0 \leq t \leq 1,$$

$$\alpha y(0) - \beta y'(0) = 0,$$

$$\gamma y(1) + \delta y'(1) = 0,$$

$$y''(0) = y''(1) = 0.$$

If y(t) is positive on the given interval and satisfies both the differential equation and the boundary conditions.

3 Research Design and Methodology

This chapter contains study period and site, study design, source of information and mathematical procedures.

3.1 Study period and site

The study was conducted from September 2018 to June 2019 in Jimma University under the department of mathematics.

3.2 Study design

In order to achieve the objective of the study we employed analytical method of design.

3.3 Source of information

The relevant sources of information for this study were different mathematics books, published articles, journals and related studies from internet.

3.4 Mathematical procedure

In this study we followed the following procedures:

i) Defining fourth-order two-point Sturm-Liouville boundary value problems.

ii) Constructing Green's function by following its properties for the corresponding homogeneous equation.

iii) Formulating the equivalent integral equation for the boundary value problems (1.1)-(1.3).

iv) Determining the existence of positive fixed point of the integral equation by applying Guo-Krasnoselskii's fixed point theorem.

v) Verifying the main result by providing illustrative examples.

4 Main Result and Discussion

4.1 Construct Green's Function

In this section, we construct Green's function for the homogeneous problem corresponding to (1.1)- (1.3).
Let $G(t,s)$ be Green's function for the homogeneous problem,

$$y^{(4)}(t) + k^2 y''(t) = 0, \qquad 0 \le t \le 1,$$

with the same boundary conditions (1.2), (1.3).
Let $-y''(t) = u(t), \qquad y''(t) \le 0.$
Thus the differential equation (1.1) considering the boundary condition

$$-(u''(t) + k^2 u(t)) = 0, \qquad 0 \le t \le 1, \qquad k \in (0, \frac{\pi}{2}), \tag{4.1}$$

$$u(0) = u(1) = 0. \tag{4.2}$$

For the de (4.1) two linearly independent solutions are $u_1(t) = \cos kt$ and $u_2(t) = \sin kt$. Hence, the problem (4.1), (4.2) have only the trivial solution if and only if

$$\Delta = \begin{bmatrix} u_1(0) & u_2(0) \\ u_1(1) & u_2(1) \end{bmatrix} = \begin{bmatrix} \cos k(0) & \sin k(0) \\ \cos k(1) & \sin k(1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \cos k & \sin k \end{bmatrix} = \sin k \ne 0.$$

To show this $u_1(t)$ and $u_2(t)$ be two linearly independent solutions of the differential equation (4.1).
From the property(**iii**) there exist four functions, say, $\lambda_1(s), \lambda_2(s), \mu_1(s)$ and $\mu_2(s)$ such that

$$G(t,s) = \begin{cases} \cos kt\lambda_1(s) + \sin kt\lambda_2(s), & 0 \le t \le s \le 1, \\ \cos kt\mu_1(s) + \sin kt\mu_2(s). & 0 \le s \le t \le 1. \end{cases} \tag{4.3}$$

Now using properties (**i**) and (**ii**), we obtain the following two equations:

$$\cos kt\lambda_1(s) + \sin kt\lambda_2(s) = \cos kt\mu_1(s) + \sin kt\mu_2(s), \tag{4.4}$$

$$-k\sin kt\mu_1(s) + k\cos kt\mu_2(s) + k\sin kt\lambda_1(s) - k\cos kt\lambda_2(s) = -1, \tag{4.5}$$

$$\sin kt(\lambda_1(s) - \mu_1(s)) - \cos kt(\lambda_2(s) - \mu_2(s)) = \frac{-1}{k}.$$

Let $v_1(s) = \lambda_1(s) - \mu_1(s)$ and

$$v_2(s) = \lambda_2(s) - \mu_2(s),$$

so that (4.4), (4.5) can be written as

$$\cos ktv_1(s) + \sin ktv_2(s) = 0, \tag{4.6}$$

$$\sin ktv_1(s) - \cos ktv_2(s) = \frac{-1}{k}. \tag{4.7}$$

Since $\cos kt$ and $\sin kt$ are linearly independent the Wronskian
$W(\cos kt, \sin kt) \neq 0$ for all $t \in [0, 1]$.
Thus, the relations (4.6), (4.7) uniquely determine
$v_1(s) = -\frac{\sin ks}{k}$ and $v_2(s) = \frac{\cos ks}{k}$.
Now using the relations $\mu_1(s) = \lambda_1(s) + \frac{\sin ks}{k}$ and $\mu_2(s) = \lambda_2(s) - \frac{\cos ks}{k}$,
Green's function can be written as

$$G(t,s) = \begin{cases} \cos kt\lambda_1(s) + \sin kt\lambda_2(s), & 0 \le t \le s \le 1, \\ \cos kt\lambda_1(s) + \frac{1}{k}\cos kt\sin ks + \sin kt\lambda_2(s) - \frac{1}{k}\sin kt\cos ks, & 0 \le s \le t \le 1. \end{cases} \tag{4.8}$$

Finally, using the property (**iv**) on the boundary condition (4.2) of Green's function with the given interval, we find
$G(0, s) = \cos k(0)\lambda_1(s) + \sin k(0)\lambda_2(s)$=0,
implies $\lambda_1(s) = 0$,
$G(1, s) = \cos k(1)\lambda_1(s) + \frac{1}{k}\cos k(1)\sin ks + \sin k(1)\lambda_2(s) - \frac{1}{k}\sin k(1)\cos ks = 0,$

We get

$$\begin{cases} \lambda_1(s) = 0, \\ k\cos k\lambda_1(s) + k\sin k\lambda_2(s) = \sin k\cos ks - \cos k\sin ks. \end{cases} \tag{4.9}$$

$$\Delta = \begin{bmatrix} 1 & 0 \\ k\cos k & k\sin k \end{bmatrix} = k\sin k,$$

Hence,

$$\Delta = k\sin k \quad by(4.9).$$

From (4.9) which easily determine $\lambda_1(s)$ and $\lambda_1(s)$ as

$$\lambda_1(s) = \frac{\begin{bmatrix} 0 & 0 \\ \sin k\cos ks - \cos k\sin ks & k\sin k \end{bmatrix}}{k\sin k} = 0.$$

$$\lambda_2(s) = \frac{\begin{bmatrix} 1 & 0 \\ k\cos k & \sin k\cos ks - \cos k\sin ks \end{bmatrix}}{k\sin k} = \frac{\sin k\cos ks - \cos k\sin ks}{k\sin k}.$$

Substituting the value of $\lambda_1(s)$ and $\lambda_2(s)$ in (4.8) and letting $\Delta = k\sin k$.

$$\mu_1(s) = \frac{1}{k}\sin ks$$

$$\mu_2(s) = \lambda_2(s) - v_2(s) = \frac{\sin k\cos ks - \cos k\sin ks}{k\sin k} - \frac{1}{k}\cos ks = -\frac{\cos k\sin ks}{k\sin k}.$$

Which gives

$$G(t,s) = \begin{cases} \cos kt(0) + \sin kt(\frac{\sin k\cos ks - \cos k\sin ks}{k\sin k}), & 0 \le t \le s \le 1, \\ \cos kt(0) + \cos kt(\frac{1}{k}\sin ks) + \sin kt(\frac{\sin k\cos ks - \cos k\sin ks}{k\sin k} - \frac{\cos ks}{k}), & 0 \le s \le t \le 1. \end{cases}$$

$$G(t,s) = \begin{cases} \frac{\sin kt\sin k\cos ks}{k\sin k} - \frac{\sin kt\cos k\sin ks}{k\sin k}, & 0 \le t \le s \le 1 \\ \frac{1}{k}\cos kt\sin ks - \frac{\sin kt\cos k\sin ks}{k\sin k}. & 0 \le s \le t \le 1 \end{cases}$$

$$G(t,s)=\begin{cases}\frac{\sin kt\sin k(1-s)}{k\sin k}, & 0\le t\le s\le 1,\\ \frac{\sin ks\sin k(1-t)}{k\sin k}, & 0\le s\le t\le 1.\end{cases} \tag{4.10}$$

$$u(t)=\int_0^1 G(t,s)f(s,y(s))ds. \tag{4.11}$$

We consider $-y''(t)=u(t)=\int_0^1 G(t,s)f(s,y(s))ds$ with boundary condition (1.2).

$$-y''(t)=0, \qquad 0\le t\le l, \tag{4.12}$$

$$\alpha y(0)-\beta y'(0)=0,$$

$$\gamma y(1)+\delta y'(1)=0.$$

For the homogeneous differential equation (4.12) two linearly independent solutions are $y_1(t)=1$ and $y_2(t)=t$. Hence, the problem (4.12), (1.2) has only the trivial solution if and only if

$$\rho=\begin{bmatrix}\alpha u_1(0)-\beta u_1'(0) & \alpha u_2(0)-\beta u_2'(0)\\ \gamma u_1(1)+\delta u_1'(1) & \gamma u_2(1)+\delta u_2'(1)\end{bmatrix}=\begin{bmatrix}\alpha & -\beta\\ \gamma & \gamma+\delta\end{bmatrix}=\gamma\beta+\alpha\gamma+\alpha\delta\neq 0.$$

From the property (**iii**) there exist four functions, say, $\lambda_1(s),\lambda_2(s),\mu_1(s)$ and $\mu_2(s)$ such that

$$H(t,s)=\begin{cases}\lambda_1(s)+t\lambda_2(s), & 0\le s\le t\le 1,\\ \mu_1(s)+t\mu_2(s), & 0\le t\le s\le 1.\end{cases} \tag{4.13}$$

Now using properties (**i**) and (**ii**), we obtain the following two equations:

$$\lambda_1(s)+t\lambda_2(s)=\mu_1(s)+t\mu_2(s), \tag{4.14}$$

$$\mu_2(s)-\lambda_2(s)=-1. \tag{4.15}$$

Let $v_1(s)=\mu_1(s)-\lambda_1(s)$ and $v_2(s)=\mu_2(s)-\lambda_2(s)$.
Thus $v_1(s)=s$ and $v_2(s)=-1$, so that (4.14), (4.15) can be written as

$$v_1(s)+tv_2(s)=0, \tag{4.16}$$

$$v_2(s) = -1. \tag{4.17}$$

Since 1 and t both are linearly independent the Wronskian $W(1, t) \neq 0$ for all $t \in [0, 1]$.

Thus, the relations (4.16), (4.17) uniquely determine $v_1(s)$ and $v_2(s)$.

Now using the relations $\mu_1(s) = \lambda_1(s) + v_1(s)$

$$\mu_1(s) = \lambda_1(s) + s,$$

and

$$\mu_2(s) = \lambda_2(s) + v_2(s),$$

$$\mu_2(s) = \lambda_2(s) - 1,$$

Green's function can be written as

$$H(t, s) = \begin{cases} \lambda_1(s) + t\lambda_2(s), & 0 \leq s \leq t \leq 1, \\ \lambda_1(s) + s + t\lambda_2(s) - t, & 0 \leq t \leq s \leq 1. \end{cases} \tag{4.18}$$

Finally, using the property (iv) on the boundary condition (1.2) of Green's function with the given interval, we find

$H(0, s) = \alpha\lambda_1(s) - \beta\lambda_2(s) = 0,$

$$\alpha\lambda_1(s) - \beta\lambda_2(s) = 0.$$

$H(1, s) = \gamma(\lambda_1(s) - s + \lambda_2(s) - 1) + \delta(\lambda_2(s) - 1) = 0,$

$$\gamma\lambda_1(s) + \gamma s + \gamma\lambda_2(s) - \gamma + \delta\lambda_2(s) - \delta = 0.$$

We have

$$\begin{cases} \alpha\lambda_1(s) - \beta\lambda_2(s) = 0, \\ \gamma\lambda_1(s) + (\gamma + \delta)\lambda_2(s) = \gamma + \delta - \gamma s. \end{cases} \tag{4.19}$$

Let $\rho = \gamma\beta + \alpha\gamma + \alpha\delta \neq 0,$

$$\lambda_1(s) = \frac{\begin{bmatrix} 0 & -\beta \\ \gamma+\delta-\gamma s & \gamma+\delta \end{bmatrix}}{\rho} = \frac{\beta(\gamma+\delta-\gamma s)}{\rho}.$$

$$\lambda_2 = \frac{\begin{bmatrix} \alpha & 0 \\ \gamma & \gamma+\delta-\gamma s \end{bmatrix}}{\rho} = \frac{\alpha(\gamma+\delta-\gamma s)}{\rho}.$$

Substituting the value of $\lambda_1(s)$ and $\lambda_2(s)$ in eq (4.18), we have

$$H(t,s) = \begin{cases} \frac{1}{\rho}[\beta(\gamma+\delta-\gamma s)+t(\alpha(\gamma+\delta-\gamma s))], & 0 \le s \le t \le 1, \\ \frac{1}{\rho}[\beta(\gamma+\delta-\gamma s)-s+t(1+\alpha(\gamma+\delta-\gamma s))], & 0 \le t \le s \le 1. \end{cases} \tag{4.20}$$

Hence

$$H(t,s) = \begin{cases} \frac{1}{\rho}(\gamma+\delta-\gamma t)(\beta+\alpha s), & 0 \le s \le t \le 1, \\ \frac{1}{\rho}(\beta+\alpha t)(\gamma+\delta-\gamma s), & 0 \le t \le s \le 1. \end{cases} \tag{4.21}$$

$$y(t) = \int_0^1 H(t,s)u(s)ds.$$

$$y(t) = \int_0^1 H(t,s)(\int_0^1 G(s,\tau)f(\tau,y(\tau))d\tau)ds. \tag{4.22}$$

Remark. $H(t,s)$ and $G(t,s)$ are Green's function for the corresponding homogeneous fourth-order boundary value problems $(1.1)-(1.3)$.
Therefore, y(t) is a solution of the fourth-order two-point boundary value problems of (1.1)-(1.3).

4.2 Bounds

Lemma 4.1 *The Green's function G(t, s) satisfies the following inequalities.*
i) $G(t,s) > 0$, *for all* $t,s \in (0,1)$,
ii) $G(t,s) \le G(s,s)$, *for* $0 \le t,s \le 1$,
iii) $G(t,s) \ge NG(s,s)$, *for* $\frac{1}{4} \le t,s \le \frac{3}{4}$,

$$where\ N = \frac{\sin\frac{k}{4}}{\sin\frac{3k}{4}}.$$

Proof: i) The Green's function G(t, s) is positive for all $t, s \in (0, 1)$.
ii) Let $s \leq t$, then

$$\frac{G(t,s)}{G(s,s)} = \frac{\sin ks \sin k(1-t)}{\sin ks \sin k(1-s)} = \frac{\sin k(1-t)}{\sin k(1-s)} \leq 1$$

$G(t,s) \leq G(s,s)$ is bounded.
Let $t \leq s$, then

$$\frac{G(t,s)}{G(s,s)} = \frac{\sin kt \sin k(1-s)}{\sin ks \sin k(1-s)} = \frac{\sin kt}{\sin ks} \leq 1.$$

We have $G(t,s) \leq G(s,s)$ is bounded.
Furthermore,for $\frac{1}{4} \leq t \leq \frac{3}{4}$.
iii) Let $s \leq t$, then

$$\frac{G(t,s)}{G(s,s)} = \frac{\sin ks \sin k(1-t)}{\sin ks \sin k(1-s)} = \frac{\sin k(1-t)}{\sin k(1-s)} \geq \frac{\sin(\frac{k}{4})}{\sin(\frac{3k}{4})}.$$

$$\frac{G(t,s)}{G(s,s)} \geq \frac{\sin(\frac{k}{4})}{\sin(\frac{3k}{4})}.$$

Let $t \leq s$, then

$$\frac{G(t,s)}{G(s,s)} = \frac{\sin kt \sin k(1-s)}{\sin ks \sin k(1-s)} = \frac{\sin kt}{\sin ks} \geq \frac{\sin\frac{k}{4}}{\sin\frac{3k}{4}}.$$

$$\frac{G(t,s)}{G(s,s)} \geq \frac{\sin\frac{k}{4}}{\sin\frac{3k}{4}}.$$

where $N = \frac{\sin\frac{k}{4}}{\sin\frac{3k}{4}}$.
So that $G(t,s) \geq NG(s,s)$. The proof is complete.

Lemma 4.2 *The Green's function H(t, s) satisfies the following inequalities.*
i) $H(t,s) > 0$, *for all* $t, s \in (0, 1)$,

ii) $H(t,s) \le H(s,s)$, *for all* $0 \le t, s \le 1$,
iii) $H(t,s) \ge MH(s,s)$, *for all* $\frac{1}{4} \le t, s \le \frac{3}{4}$,

$$where\ M = min\{\frac{\gamma + 4\delta}{4(\gamma+\delta)}, \frac{\alpha+4\beta}{4(\alpha+\beta)}\} \le 1.$$

Proof: i) The Green's function H(t, s) is positive for all $t, s \in (0,1)$.
ii) Let $s \le t$, then

$$\frac{H(t,s)}{H(s,s)} = \frac{(\gamma+\delta-\gamma t)(\beta+\alpha s)}{(\gamma+\delta-\gamma s)(\beta+\alpha s)} = \frac{(\gamma+\delta-\gamma t)}{(\gamma+\delta-\gamma s)} \le 1.$$

So that $H(t,s) \le H(s,s)$ is bounded.
Let $t \le s$, then

$$\frac{H(t,s)}{H(s,s)} = \frac{(\gamma+\delta-\gamma s)(\beta+\alpha t)}{(\gamma+\delta-\gamma s)(\beta+\alpha s)} = \frac{(\beta+\alpha t)}{(\beta+\alpha s)} \le 1.$$

So that $H(t,s) \le H(s,s)$ is bounded.

Furthermore, for $\frac{1}{4} \le t \le \frac{3}{4}$.
iii) Let $s \le t$, then

$$\frac{H(t,s)}{H(s,s)} = \frac{(\gamma+\delta-\gamma t)(\beta+\alpha s)}{(\gamma+\delta-\gamma s)(\beta+\alpha s)} = \frac{\gamma+\delta-\gamma t}{\gamma+\delta-\gamma s} \ge \frac{\gamma+\delta-\gamma(\frac{3}{4})}{\gamma+\delta-\gamma(\frac{1}{4})} = \frac{\gamma+4\delta}{3\gamma+4\delta}.$$

$$\frac{H(t,s)}{H(s,s)} \ge \frac{\gamma+4\delta}{3\gamma+4\delta}.$$

Let $t \le s$, then

$$\frac{H(t,s)}{H(s,s)} = \frac{(\gamma+\delta-\gamma s)(\beta+\alpha t)}{(\gamma+\delta-\gamma s)(\beta+\alpha s)} = \frac{(\beta+\alpha t)}{(\beta+\alpha s)} \ge \frac{\beta+\alpha(\frac{1}{4})}{\beta+\alpha(\frac{3}{4})} = \frac{\alpha+4\beta}{3\alpha+4\beta}.$$

$$\frac{H(t,s)}{H(s,s)} \geq \frac{\alpha+4\beta}{3\alpha+4\beta},$$

$$\frac{H(t,s)}{H(s,s)} \geq M, \frac{1}{4} \leq t \leq \frac{3}{4},\ M = min\{\frac{\gamma+4\delta}{4(\gamma+\delta)}, \frac{\alpha+4\beta}{4(\alpha+\beta)}\} \leq 1.$$

Hence $H(t,s) \geq MH(s,s)$ for all $\frac{1}{4} \leq t,s \leq \frac{3}{4}$.
The proof is complete.

Lemma 4.3 *(Krasnoselskii's, M.A, 1964), Let E be a Banach space, and let $P \subset E$ be a cone in E. Assume Ω_1, Ω_2 are bounded open subsets of E with $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$, and let*
$T\colon P \cap (\bar{\Omega}_2 \setminus \Omega_1) \to P$
be completely continuous operator such that either
i) $\|Ty\| \leq \|y\|,\ y \in P \cap \partial\Omega_1$ and $\|Ty\| \geq \|y\|,\ y \in P \cap \partial\Omega_2$,
ii) $\|Ty\| \geq \|y\|,\ y \in P \cap \partial\Omega_1$, and $\|Ty\| \leq \|y\|,\ y \in P \cap \partial\Omega_2$,
Then T has a fixed point in $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

4.3 Results

In this thesis we consider the fourth-order boundary value problems

$$y^{(4)}(t) + k^2 y''(t) = f(t, y(t)), \qquad 0 \leq t \leq 1,$$

with boundary conditions (1.1)-(1.3).
The following conditions will be assumed throughout:
$C_1)$ $0 \leq \int_0^1 H(t,s)(\int_0^1 G(s,\tau) f(\tau, y(\tau))d\tau)ds < \infty$,
$C_2)$ $f : [0,1] \times [0,\infty) \longrightarrow [0,\infty)$ is continuous function,
$C_3)$ $\rho = \gamma\beta + \alpha\gamma + \alpha\delta > 0,\ \ \alpha,\beta,\gamma,\delta \geq 0,\ \ \Delta = k\sin k > 0$ and $k \in (0, \frac{\pi}{2})$.
By using a Guo-Krasnoselskii's fixed point theorem the existence of positive solutions of (1.1)-(1.3) is obtained in the case when, f is either superlinear or sublinear. To be precise, we define the nonnegative extended real numbers

f_0, f^0, f_∞ and f^∞

$$f^0 = \lim_{y\to 0^+} \sup_{t\in[0,1]} \frac{f(t,y(t))}{y(t)} = 0 \; and \; f_\infty = \lim_{y\to\infty} \inf_{t\in[0,1]} \frac{f(t,y(t))}{y(t)} = \infty$$

superlinear case.

$$f_0 = \lim_{y\to 0^+} \inf_{t\in[0,1]} \frac{f(t,y(t))}{y(t)} = \infty \quad and \; f^\infty = \lim_{y\to\infty} \sup_{t\in[0,1]} \frac{f(t,y(t))}{y(t)} = 0$$

sublinear case.

Assume that they will exist. When $f^0 = 0$ and $f_\infty = \infty$ correspond to the superlinear case, and $f_0 = \infty$ and $f^\infty = 0$ correspond to the sublinear case. By a positive solution of $(1.1) - (1.3)$ we understand a solution y(t) which is positive on $0 \leq t \leq 1$ and satisfies the differential equation (1.1) for $0 \leq t \leq 1$ and the boundary conditions (1.2),(1.3).

Let $E = C[0,1]$. For $y \in E$, define $\|y\| = \max_{t\in[0,1]} |y(t)|$. Then $(E, \|.\|)$ is a Banach space. Denote

$$P = \{y \in E : y(t) \geq 0,\; y''(t) \leq 0,\; \min_{t\in[\frac{1}{4},\frac{3}{4}]} |y(t)| \geq \omega\|y\|\}, \tag{4.23}$$

where $\omega = MN$.

It is obvious that P is a positive cone in E.

Let us define an operator $T : P \to E$ by

$$Ty(t) = \int_0^1 H(t,s)(\int_0^1 G(s,\tau)f(\tau,y(\tau))d\tau)ds, \quad y \in P. \tag{4.24}$$

We observe that a fixed point of T in E is positive solution of the boundary value problems.

We use the well-known cone expression and compression Guo-Krasnoselskii's fixed point theorem to show at least one fixed point for T.

Lemma 4.4 *Let C_1, C_2 and C_3 are hold, then the operator $T : P \to E$ defined as follows:*

Proof. Since $H(t,s)$ and $G(t,s)$ are positive, then $Ty(t) \geq 0$ for all $y(t) \in P$. If $y \in P$, then $Ty(t) \in P$.
Fix $R \geq 0$, and $\omega = \{y \in E : \|y\| \leq R\}$

$$Ty(t) = \int_0^1 H(t,s)(\int_0^1 G(s,\tau)f(\tau, y(\tau))d\tau)ds,$$
$$\leq \int_0^1 H(s,s)(\int_0^1 G(s,\tau)f(\tau, y(\tau))d\tau)ds,$$
$$\|Ty\| \leq \int_0^1 H(s,s)(\int_0^1 G(s,\tau)f(\tau, y(\tau))d\tau)ds,$$
$$\|Ty\| \leq \|y\|.$$

Therefore Ty(t) is uniformly bounded.

Lemma 4.5 *Let C_1, C_2 and C_3 are hold, then the operator $T : P \to P$ is completely continuous.*

Proof. From the continuity of f, we know $Ty \in E$ for each $y \in P$. It follows from the Lemma 4.1 and Lemma 4.2 that for $y \in P$,

$$Ty(t) = \int_0^1 H(t,s)(\int_0^1 G(s,\tau)f(\tau, y(\tau))d\tau)ds,$$

$$\leq \int_0^1 H(s,s)(\int_0^1 G(s,\tau)f(\tau, y(\tau))d\tau)ds,$$

Note that by the non-negativity of H, G and f, one has

$$\|Ty\| \leq \int_0^1 H(s,s)(\int_0^1 G(s,\tau)f(\tau, y(\tau))d\tau)ds,$$

from which we have

$$min_{\frac{1}{4} \leq t \leq \frac{3}{4}} Ty(t) = min_{\frac{1}{4} \leq t \leq \frac{3}{4}} \int_0^1 H(t,s)(\int_0^1 G(s,\tau)f(\tau, y(\tau))d\tau)ds,$$

$$\geq \int_0^1 MH(s,s)(\int_0^1 NG(s,\tau)f(\tau,y(\tau))d\tau)ds,$$

$$= MN\int_0^1 H(s,s)(\int_0^1 G(s,\tau)f(\tau,y(\tau))d\tau)ds,$$

$$= \omega\int_0^1 H(s,s)(\int_0^1 G(s,\tau)f(\tau,y(\tau))d\tau)ds,$$

$$\|Ty\| \geq \omega\|Ty\|,\ Ty \in P.$$

Therefore $T : P \to P$ is a self-map. Since $H(t,s), G(t,s)$ and $f(t,y)$ are continuous, it is easily known that $T : P \to P$ is completely continuous.

The proof is complete.

From above arguments, we know that the existence of positive solutions of (1.1)-(1.3) can be equivalent to the existence of positive fixed points of the operator T.

Theorem 4.3.1 Assume that the conditions $C_1 - C_3$ are satisfied. If $f^0 = 0$ and $f_\infty = \infty$, then the boundary value problem has at least one positive solution $y \in C[0,1] \cap C^4[0,1]$.

Proof. Now since $f^0 = 0$, there exists an $A_1 \geq 0$.

$\lim_{y\to 0^+} \sup_{t\in[0,1]} \frac{f(t,y(t))}{y(t)} = 0$,

$|\sup_{t\in[0,1]} \frac{f(t,y(t))}{y(t)} - 0| < \eta$ where $\eta \geq 0$,

$\sup \frac{f(t,y(t))}{y(t)} < \eta$,

so that $f(t,y(t)) < \eta y$, $\eta > 0$ satisfy for every $y \in P$, and $\|y\| = A_1$ for $0 < y < A_1$ where η satisfies

$$\eta\int_0^1 H(t,s)(\int_0^1 G(s,\tau)d\tau)ds \leq 1. \tag{4.25}$$

$$Ty(t) = \int_0^1 H(t,s)(\int_0^1 G(s,\tau)f(\tau,y(\tau))d\tau)ds,$$

$$\leq \int_0^1 H(s,s)(\int_0^1 G(s,\tau)f(\tau,y(\tau))d\tau)ds,$$

$$\leq \int_0^1 H(s,s)(\int_0^1 G(s,\tau)d\tau)ds\eta y,$$

$$\leq \eta \int_0^1 H(s,s)(\int_0^1 G(s,\tau)d\tau)ds\ \|y\|,$$
$$\leq \|y\| \ \ by \ \ (4.25).$$

Consequently, $\|Ty\| \leq \|y\|$. So, if we set

$$\Omega_1 = \{y \in E : \|y\| < A_1\},$$

then

$$\|Ty\| \leq \|y\|, for\ y \in P \cap \partial\Omega_1. \tag{4.26}$$

Next, considering $f_\infty = \infty$, $\lim_{y\to\infty} \inf_{t\in[0,1]} \frac{f(t,y(t))}{y(t)} = \infty$. There exists $\eta_2 > 0$ and $\bar{A}_2 > 0$. Let $A_2 = \min\{2A_1, \omega\bar{A}_2\}$ and let

$$\Omega_2 = \{y \in E : \|y\| < A_2\}.$$

If $y \in P$ with $\|y\| = A_2$, then $\inf \frac{f(t,y(t))}{y(t)} > \eta_2$,

$$f(t, y(t)) > \eta_2 y(t)\ for\ y \geq \bar{A}_2,$$

where η_2 satisfy

$$\eta_2\omega^2 \int_{\frac{1}{4}}^{\frac{3}{4}} H(s,s)ds(\int_{\frac{1}{4}}^{\frac{3}{4}} G(s,\tau)d\tau)ds \geq 1. \tag{4.27}$$

$$Ty(t) = \int_0^1 H(t,s)(\int_0^1 G(s,\tau)f(\tau,y(\tau))d\tau)ds,$$
$$\geq min_{\frac{1}{4}\leq t\leq\frac{3}{4}} \int_0^1 MH(s,s)(\int_0^1 NG(s,\tau)f(\tau,y(\tau))d\tau)ds,$$
$$\geq \eta_2 MN \int_0^1 H(s,s)(\int_0^1 G(s,\tau)d\tau)dsy,$$
$$\geq \omega\eta_2 \int_{\frac{1}{4}}^{\frac{3}{4}} H(s,s)(\int_{\frac{1}{4}}^{\frac{3}{4}} G(s,\tau)d\tau)ds\omega\|y\| \ \ by\ (4.23),$$

$$= \eta_2 \omega^2 \int_{\frac{1}{4}}^{\frac{3}{4}} H(s,s)(\int_{\frac{1}{4}}^{\frac{3}{4}} G(s,\tau)d\tau)ds\|y\|,$$
$$\geq \|y\| \quad by \ (4.27).$$

Thus, $\|Ty\| \geq \|y\|$. For this case, if we let

$$\Omega_2 = \{y \in E : \|y\| < A_2\},$$

then

$$\|Ty\| \geq \|y\|, \quad for \ y \in P \cap \partial\Omega_2. \tag{4.28}$$

Hence by combining of (4.26) and (4.28) we have, $\|Ty\| = \|y\|$.
Therefore, by the first part of the fixed point theorem, it follows that T has a fixed point in $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$ such that $A_1 \leq \|y\| \leq A_2$ by Guo-Krasnoselskii's fixed point theorem. Further since Green's function positive, it follows that $y(t) \geq 0$ for $0 \leq t \leq 1$ and $y(t)$ is a desired solution for $(1.1) - (1.3)$. The proof is complete.

Theorem 4.3.2 Assume that the conditions C_1, C_2 and C_3 are satisfied. If $f_0 = \infty$ and $f^\infty = 0$, then the boundary value problem has at least one positive solution that lies in P.

Proof. Let T be the cone preserving, completely continuous operator defined by Lemma 4.5. Beginning with $f_0 = \infty$, there exists an $A_1 > 0$, $\xi_1 > 0$ and satisfy

$$\eta_2 \omega \int_{\frac{1}{4}}^{\frac{3}{4}} H(s,s)(\int_{\frac{1}{4}}^{\frac{3}{4}} G(s,\tau)d\tau)ds \geq 1. \tag{4.29}$$

$$f_0 = \lim_{y \to 0^+} \inf_{t \in [0,1]} \frac{f(t,y(t))}{y(t)} = \infty.$$

$\frac{f(t,y(t))}{y(t)} \geq \xi_1$ for $0 < y \leq A_1$,

$$f(t,y(t)) \geq \xi_1 y.$$

Where $\xi_1 \geq \eta_2$ and η_2 is given above.Then for $y \in P$ and $\|y\| = A_1$, we have

$$Ty(t) = \int_0^1 H(t,s)(\int_0^1 G(s,\tau)f(\tau, y(\tau))d\tau)ds,$$

$$\geq MN \int_{\frac{1}{4}}^{\frac{3}{4}} H(s,s)(\int_{\frac{1}{4}}^{\frac{3}{4}} G(s,\tau)f(\tau, y(\tau))d\tau)ds,$$

$$\geq \omega\xi_1 \int_{\frac{1}{4}}^{\frac{3}{4}} H(s,s)(\int_{\frac{1}{4}}^{\frac{3}{4}} G(s,\tau)d\tau)dsy,$$

$$\geq \omega\xi_1 \int_{\frac{1}{4}}^{\frac{3}{4}} H(s,s)(\int_{\frac{1}{4}}^{\frac{3}{4}} G(s,\tau)d\tau)ds\|y\|,$$

$$\geq \|y\|\omega\eta_2 \int_{\frac{1}{4}}^{\frac{3}{4}} H(s,s)(\int_{\frac{1}{4}}^{\frac{3}{4}} G(s,\tau)d\tau)ds,$$

$$\geq \|y\| \quad by \text{ (4.29)}.$$

Thus, $\|Ty\| \geq \|y\|$. So, if we let

$$\Omega_1 = \{y \in E : \|y\| < A_1\},$$

then

$$\|Ty\| \geq \|y\|, for\ y \in P \cap \partial\Omega_1. \tag{4.30}$$

It remains to consider $f^\infty = 0$. There exists an $\bar{A}_2 > 0$ such that $f(t, y(t)) \leq \xi_2 y$, for all $y \geq \bar{A}_2$. Where $\xi_2 > 0$ satisfies

$$\xi_2 \int_0^1 H(s,s)(\int_0^1 G(s,\tau)f(\tau, y(\tau))d\tau)ds \leq 1. \tag{4.31}$$

There are two cases, case(i) f is bounded, and case(ii) f is unbounded.
For case(i). Suppose $N > 0$ is such that $f(t, y(t)) \leq N$, for $0 < y < \infty$,

$$f^\infty = \lim_{y\to\infty} sup_{t\in[0,1]} \frac{f(t, y(t))}{y(t)} = 0,$$

$f(t,y(t)) \le \xi_2 y(t) = N$ for $y(t) > A_2 > 0$.
Let $A_2 = \max\{2A_1, N\int_0^1 H(s,s)(\int_0^1 G(s,\tau)d\tau)ds\}$.
Then, for $y(t) \in P$ with $\|Ty\| = A_2$, we have

$$Ty(t) = \int_0^1 H(t,s)(\int_0^1 G(s,\tau)f(\tau,y(\tau))d\tau)ds,$$

$$\le \int_0^1 H(s,s)(\int_0^1 G(s,\tau)f(\tau,y(\tau))d\tau)ds,$$

$$\le \int_0^1 H(s,s)(\int_0^1 G(s,\tau)d\tau)ds\xi_2 y,$$

$$= N\int_0^1 H(s,s)(\int_0^1 G(s,\tau)d\tau)ds,$$

$$\le A_2 = \|y\|.$$

So that $\|Ty\| \le \|y\|$. So, if

$$\Omega_2 = \{y \in E : \|y\| < A_2\},$$

then

$$\|Ty\| \le \|y\|, \ for\ y \in P \cap \partial\Omega_2. \tag{4.32}$$

For case(ii). If f is unbounded, then let $A_2 > \max\{2A_1, \bar{A}_2\}$ be such that $f(t,y(t)) \le f(t,A_2)$, for $0 < y \le A_2$. Choosing $y \in P$ with $\|Ty\| = A_2$, we have

$$Ty(t) = \int_0^1 H(t,s)(\int_0^1 G(s,\tau)f(\tau,y(\tau))d\tau)ds,$$

$$Ty(t) \le \int_0^1 H(s,s)(\int_0^1 G(s,\tau)f(\tau,y(\tau))d\tau)ds,$$

$$\le \int_0^1 H(s,s)(\int_0^1 G(s,\tau)f(\tau,A_2)d\tau ds,$$

$$\le \int_0^1 H(s,s)(\int_0^1 G(s,\tau)d\tau)dsy\xi_2,$$

$$\leq A_2\xi_2\int_0^1 H(s,s)(\int_0^1 G(s,\tau)d\tau)ds,$$
$$\leq A_2 = \|y\| \;\; by\ (4.31).$$

and so $\|Ty\| \leq \|y\|$. For this case, if we let

$$\Omega_2 = \{y \in E : \|y\| < A_2\},$$

then

$$\|Ty\| \leq \|y\|, for y \in P \cap \partial\Omega_2, \tag{4.33}$$

Hence by combining of (4.30), (4.32) and (4.33) we have, $\|Ty\| = \|y\|$.
Thus, in either of the cases, an application of the second part of the Guo-Krasnoselskii's fixed point theorem yields a solution of boundary value problems $(1.1) - (1.3)$ has a positive solution which belongs to $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$ and completes the proof of the theorem.

4.4 Example

Let us consider examples to see validity of our main result for fourth-order two-point Sturm-Liouville boundary value problem.

Example 1

Now consider the following fourth-order differential equation,

$$y^{(4)}(t) + \frac{1}{4}y''(t) = f_1(t, y(t)), \qquad t \in (0,1),$$

subject to the boundary conditions,

$$y(0) - y'(0) = 0,$$

$$y(1) + 2y'(1) = 0,$$

$$y''(0) = y''(1) = 0,$$

where $f_1(t, y(t)) = ty^{\frac{3}{2}},\ t \in (0, 1)$ for superlinear case.
The Green's function G(t, s) for the homogeneous problem,

$$-(u''(t) + \frac{1}{4}u(t)) = 0, \qquad 0 \le t \le 1,$$

$$G(t,s) = \begin{cases} \frac{\sin\frac{1}{2}t \sin\frac{1}{2}(1-s)}{\frac{1}{2}\sin\frac{1}{2}}, & 0 \le t \le s \le 1, \\ \frac{\sin\frac{1}{2}s \sin\frac{1}{2}(1-t)}{\frac{1}{2}\sin\frac{1}{2}}, & 0 \le s \le t \le 1. \end{cases}$$

$$u(t) = \int_0^1 G(t,s) s y^{\frac{3}{2}} ds.$$

We consider $-y''(t) = u(t) = \int_0^1 G(t,s)sy^{\frac{3}{2}}ds$ with boundary condition (1.2).

$$-y''(t) = 0, \qquad 0 \le t \le l,$$

satisfying the boundary conditions (1.3) is given by

Hence $H(t,s) = \begin{cases} \frac{1}{4}(-ts - t + 3s + 3), & 0 \le s \le t \le 1, \\ \frac{1}{4}(-ts - s + 3t + 3), & 0 \le t \le s \le 1. \end{cases}$

$$y(t) = \int_0^1 H(t,s)u(s)ds.$$

$$y(t) = \int_0^1 H(t,s)(\int_0^1 G(s,\tau)\tau y^{\frac{3}{2}} d\tau)ds.$$

Example 2

Now consider the following fourth-order differential equation,

$$y^{(4)}(t) + \frac{1}{4}y''(t) = f_2(t, y(t)), \qquad t \in (0, 1),$$

subject to the boundary conditions,

$$y(0) - y'(0) = 0,$$

$$y(1) + 2y'(1) = 0,$$

$$y''(0) = y''(1) = 0,$$

where $f_2(t, y(t)) = ty^{\frac{2}{3}},\ t \in (0, 1)$ for sublinear case.
It satisfies the above Green's function G(t, s) and H(t, s).

$$y(t) = \int_0^1 H(t, s)u(s)ds,$$

where $u(t) = \int_0^1 G(t, s)sy^{\frac{2}{3}}ds$.

$$y(t) = \int_0^1 H(t, s)(\int_0^1 G(s, \tau)\tau y^{\frac{2}{3}}d\tau)ds.$$

Therefore these examples satisfy all the given conditions and the final main results.

5 Conclusion and Future scope

5.1 Conclusion

Based on the obtained result the following conclusion can be derived:
In this study, we defined fourth-order two-point Sturm-Liouville boundary value problems and used the properties of Green's function to construct it for corresponding homogeneous equation.
After these we formulated equivalent integral equation for the boundary value problem (1.1)-(1.3) in the given interval and determined the existence of positive fixed point of the integral equation by applying Guo-Krasnoselskii's fixed point theorem.
We established the existence of positive solutions for fourth-order two-point Sturm-Liouville bvp by applying Guo-Krasnoselskii's fixed point theorem.
Finally, it was established that, there exists at least one positive solution for fourth-order two-point Sturm-Liouville boundary value problems.

5.2 Future scope

This study focused on existence of positive solutions for fourth-order two-point Sturm-Liouville bvp. Any interested researchers may conduct the research on:

i) Existence of positive solutions for fourth-order multi-point (three, four, ...) Sturm-Liouville boundary value problem.

ii) Existence of positive solutions for n^{th}-order multi-point Sturm-Liouville boundary value problem.

iii) Uniqueness of these positive solutions for fourth-order two-point Sturm-Liouville boundary value problem.

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