EXTRAPOLATED NEWTON RAPHSON METHOD FOR SOLVING FUNCTIONS OF TWO VARIABLES

A THESIS SUBMITED TO THE DEPARTMENT OF MATHEMATICS, JIMMA UNIVESITY IN PARTIAL FULLFILMENT FOR THE REQUIRMENTS OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS (NUMERICAL ANALYSIS)

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Declaration

I, the undersigned declare that, the research entitled "**Extrapolated Newton Raphson Method For SolvingFunctions Of Two Variables"** is original and it has not been submitted to any institution elsewhere for the award of any academic degree or like, where other sources of information that have been used, they have been acknowledged .

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Acronyms

NM- Newton’s Method

ENRM- Extrapolated Newton Raphson method

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Abstract

This study mainly focuses on extrapolated Newton's method for functions of two variables for systems of nonlinear equations. In this study, extrapolated Newton's method of two variables is developed, prove for the convergence and the order of the new method is shown. The criteria adopted are number of iterations and accuracy of each method.

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CHAPTER NONE

1. Introduction

1.1 Background of the Study

The major goal of numerical analysis is to find the approximation solution to difficult problem (or that cannot be solved analytically) by using different numerical techniques, particularly when analytical solutions are not available or if it is very difficult to obtain its solution analytically the subject addresses variety of questions ranging from the approximation of functions and approximate solutions of algebraic equations with particularemphasis on the stability, accuracy,efficiency and reliability of numerical algorithm.

One of the important methods to find approximation solution of numerical analysis is Newton-Raphson method. Newton-Raphson method is a powerful technique for solving equations numerically. In numerical analysis Newton's Method is known as Newton –Raphson method, named after Isaac Newton and Joseph–Raphson is a methods for finding successively better approximations to the roots (or Zeroes) of real valued function [9].

Newton's method was used by 17^{th} century Japanese mathematician [sekikowa] to solve single variables, through the connection with calculus was missing. Newton's method was first published in 1685 in "A treatise of algebra both historical and practical'' by [John Wallis] in 1690,[Joseph Raphson] published a simplified description in "Analysis acquationum universalis". Raphson again viewed Newton's method purely as algebraic method.

Solutions of equation by numerical iteration are of interest to many science and engineering problems. For this purpose, several methods which require evaluation of one function and its derivatives at each steps of the iteration have an order of convergence two in many cases [4].

Several methods developed later on over the years are largely based on modification of Newton's method and have been shown to give increasing higher order convergence [8].

In the problems of finding root of an equation (or a solution of a system of equations) an iterative method uses an initial guess to generate successive approximation to solution. In contrast direct methods attempt to solve the problem at a finite sequence of operations. In the absence of rounding off errors, direct methods would deliver an exact solution (like solving linear system of equations $Ax = b$ by Gaussian elimination). Furthermore, extrapolation is a powerful tool available to numerical analysis for improvingthe performance of a wide variety of mathematical methods. Since many interesting problems cannot be solved analytically, we often have to use computers to derive approximate solutions. In order to make this process efficient we have to think carefully about our implementations, this is where extrapolation comes in [4].

However, iterative methods are often useful even for linear problems involving large number of variables, where direct method would be prohibitively expensive.

Our main concern would be 2x2 nonlinear systems of equations convergence and analysis of methods for approximating this solution of systems of equations; direct methods for the solution of non-linear equations are usually feasible only for small systems of every special form. Consequently, our attention, and certainly the most central to our consideration, is Newton's method.

$$z^{k+1} = z^k - \left(F'\left(x^k\right)\right)^{-1} F\left(x^k\right) \qquad (1.1)$$

Where, k=0, 1, 2. . .

Here**,** $F'(x^k)$ denotes the derivative or Jacobian matrix of F evaluated at x^k and $(F'(x^k))^{-1}$ is its inverse.

The reason that we are studying the extension of Newton-Raphson method in this research is that it can also solve 2×2 non- linear system of equations using Jacobainmatrix and its inverse.

Recently, in [10] extrapolate Newton's method in function of one variable was developed. The study is aimed to extend the result of [10] in functions of two variables and in addition we develop the order of convergence of extrapolated Newton's method.

1.2 Statement of the Problem.

Numerical methods can be suitable for problems that are very difficult or impossible to solve analytically [13].Perhaps, a more interesting and more useful application of root finding is to solve systems of nonlinear equations. In many areas of science, we want to model the whole systems, which can in many cases prove difficult to solve analytically, so we can use the method described here to approximate some solutions [14].

Over the years, Newton-Raphson [12] iterative method has proved to be a very efficient method for solving non-linear equations.

Therefore, this study is intended to answer the following three basic questions:

- ✓ What are the procedures and techniques that can be followed to extrapolate Newton-Raphson methods involving functions of two variables?
- ✓ To what extent the proposed method would converge?
- ✓ What are the advantages of the proposed method over the other methods?

1.3. Objective of the study

1.3.1. General objective

The general objective of this study is to develop extrapolated Newton-Raphson method for solving functions of two variables.

1.3.2. Specific objectives

The specific objectives of the study are:

- To describe the procedures and the techniques that can be followed in extrapolated Newton Raphson method for solving functions of two variables.
- To analyze the convergence of the proposed method.
- To explain the advantages of the proposed method over the existing methods.

1.4. Significance of the study

The outcomes of this study may have the following importance:

- It contributes to research activities in this area.
- It provides some background information for other researchers who want to work on similar topics.
- To enhances the research skill and scientific communication of the researcher.

1.5. Delimitation of the study

This study is delimited to the extrapolated Newton-Raphson method for solving functions of two variables.

CHAPTER TWO

2.1. Review of related Literature

Newton’s method is probably the simplest, most flexible, best known, and most used numerical method. However, as it’s well known difficulty in the application of Newton’s method is the selection of initial guess, which must be chosen sufficiently close to the true solution in order to guarantee the convergence [16].

In numerical analysis, Newton’s method (also known as the Newton- Raphson method named after, Isaac Newton and Joseph- Raphson is method for finding successively better approximation to the Zeroes or roots of real valued functions. The algorithm is first in the class of householder’s methods succeeded by Halley’s method.

In [9] the conditions for convergence of Newton's method for a system of nonlinear equations have been discussed. The convergence is quadratic if the first derivatives are sufficiently smooth and the initial point is not too far from one of the roots of the equations.

Obviously, the solution of a system of nonlinear equations is much more complex than for a single equation. Similar convergence problems may result, but now no simple geometric visualization is possible. Furthermore, pointing out that problems involving 50 or more equations are difficult to solve unless a good estimate is available before iteration. However, this is only true for systems in which the Jacobian matrix is filled, or nearly so, with nonzero elements. In one-dimensional, unsteady-flow analysis, the Jacobian has a special structure and contains many zeros. For large stream systems, less than 1 percent of the elements in the Jacobian will be nonzero; all other elements are known in advance to be zero.

In 1994, Newton’s method worked best on functions which are as nearly linear as possible in the neighborhood of the root being sought to rederive Halley’s method wonder fully elegant fashion, by recasting the geometric property in the form of the equation [11].

However, the literature on the solution of non-linear systems of equations refers to two basic methods, the successive substitution methods and Newton-Raphson method. The successive substitution method is generally simple to program and demands low computer memory, but may lead to divergence unless the equations are appropriately ordered, whilst the Newton-Raphsonmethod is more sufficiently close to the solution [3]. When convergent, the Newton-

Raphson possesses quadratic order so accelerating the convergence rate as it approaches the solution [8]. More than three hundred years passed since a procedure for solving an algebraic equation was proposed by Newton in 1669 and later by Raphson in 1990 [12] the method is now called Newton's method or the Newton-Raphson method and is still central technique for solving nonlinear equations[16].

Many topics related to Newton's method still affront attention from researchers. For example, the construction of globally convergent effective iterative methods for solving non-iterative methods in R^n or C^n is an important Research area in the fields of numerical analysis and optimization [14].

One topic which has always been of paramount importance in numerical analysis is that of approximating roots of non-linear equation in one variable, be they algebraic or transcendental.

In the literature of numerical analysis extrapolated Newton-Raphson method for functions of two variables, many authors have attempted to obtain higher accuracy rapidly by using a numerous methods. Among the methods available in the literature; [10] developed extrapolated Newton's method in functions of one variable, [1] developed an implicit function theorem and modified Newton-Raphson method for roots of functions between finite dimensional space without assuming non singularity of Jacobian at the initial approximation and [16] generalized inverse of jacobian can be used in a Newton's method whose limit points are stationary pints of $\|f\|^2$. As clearly explained above, most of authors have attempted to obtain simple and accuracy method by using different mechanisms. This study mainly deepened on the results of [1] and [10] in which extrapolated Newton-Raphson method for function of one variable is extended to functions of two variables.

CHAPTER THREE

3. Methodology

This chapter consists of methods and materials that are used to carry out the study .These are study area and period, source of information, study design, study and ethical procedure considerations.

3.1. Study Area and period

The study was conducted in JimmaUniversity College of Natural science Department of Mathematics in 2014/2015 Academic year. This study focuses on Extrapolated Newton's Raphson method for solving functionsof two variables.

3.2. Source of information

The data was collected from the relevant source of information to achieve the objective of the study such that; journals, books review and different related studies

3.3 study design

The study has been used documentary review article design to conduct the study.

3.4. Study procedures

Important materials and data for the study was collected using documentary analysis as an instrument. In order to achieve the intended objectives the study followed the following mathematical steps:

1. Extrapolated Newton-Raphson method for solving function of two variables is extended.
2. The free parameters are estimated.
3. Order of convergence for the new method is established.
4. Validation example is developed.
5. Conclusions and recommendations are drawn.

3.5 Ethical Considerations

The researcher wastaking care of ethical considerations through official letter support from the college or department. The consent of the personalities involved in data source confirmed and acknowledged.

CHAPTER FOUR

4. RESULT AND DISCUSSION

4.1. Preliminary

Definition: 4.1.1

Let $D \subset \Re^2$.A function f of two variables is a rule that assigns to each ordered pair (x, y) in D ,a unique real number denoted by $f(x, y)$.The set D is the domain of f and its range is the set of values that f takes on, that is, $\{f(x, y)\, such\, that\ (x, y) \in D\}$

Definition: 4.1.2

A continuous function $F : \Re^2 \to \Re^2$ is continuously differentiable at $z \in \Re^2$ if each component functions are continuously differentiable at z.The derivatives of $F\ at\ z$ is called the Jacobian matrix of $F\ at\ z$ and its transpose is called the gradient of $F\ at\ z$.The common notations are:

$$F'(z) \in \Re^2 X \Re^2, F'(z) = J(z) = \nabla F^T(z)$$

$$J(z) = \begin{bmatrix} f_x(z) & f_y(z) \\ g_x(z) & g_y(z) \end{bmatrix}$$

In these cases z is the function of $x\, and\ y$.

Consider the system of two equations and two unknowns:

$$f(x,y)=0$$

$$g(x,y)=0 \qquad (4.1)$$

Let (x_k, y_k) be a suitable approximation to the root (ξ, η) of the system (4.1).let Δx be an increment in x_k and Δy be an increment in y_k such that $(x_k+\Delta x, y_k+\Delta y)$ is the exact solution, that is:

$$f(x_k+\Delta x, y_k+\Delta y)\equiv 0$$

$$g(x_k+\Delta x, y_k+\Delta y)\equiv 0 \qquad (4.2)$$

Expanding in Taylor series about the point(x_k , y_k), we get

$$f(x_k,y_k)+\left[\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right]f\left(x_k,y_k\right)+\frac{1}{2!}\left[\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right]^2 f(x_k,y_k)+...=0$$

$$g(x_k,y_k)+\left[\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right]g(x_k,y_k)+\frac{1}{2!}\left[\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right]^2 g(x_k,y_k)+...=0 \qquad (4.3)$$

Neglecting second and higher powers of $\Delta x and \Delta y$,we obtain

$$f(x_k,y_k)+\Delta x f_x(x_k,y_k)+\Delta y f(x_k,y_k)=0$$

$$g(x_k,y_k+\Delta x g_x(x\ ,y_k)+\Delta y g_y(x_k,y_k)=0 \qquad (4.4)$$

Where suffixes with respect to x and y represent partial differentiation solving equation (4.4) for $\Delta x\, and\, \Delta y$ we get;

$$\Delta x=-\frac{1}{D_k}\left[f(x_k,y_k)g_y(x_k,y_k)-g(x_k,y_k)f_y(x_k,y_k)\right]$$

$$\Delta y = \frac{1}{D_k}\left[g(x_{k,} y_k) f_x(x_k, y_k) - f(x_k, y_k) g_x(x_k, y_k)\right] \qquad (4.5)$$

Where: $D_k = f_x(x_k, y_k) g_y(x_k, y_k) - g_x(x_k y_k) f_y(x_k, y_k) \neq 0$

We obtain,

$x_{k+1} = x_k + \Delta x \; and \; y_{k+1} = y_k + \Delta y$

Writing the equation (4.4) in matrix form, we get

$$\begin{bmatrix} f_x(x_k, y_k) & f_y(x_k, y_k) \\ g_x(x_k, y_k) & g_y(x_k, y_k) \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = -\begin{bmatrix} f(x_k, y_k \\ g(x_k, y_k \end{bmatrix} \qquad (4.6)$$

$or;$
$J_k \Delta x = -F(x_k, y_k$

Where; $J_k = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}_{(x_k, y_k)}$, $F = \begin{bmatrix} f \\ g \end{bmatrix}_{(x_k, y_k)}$

$$let \;\; \Delta X = \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

The solution of the system (4.6) is given by

$$\Delta X = J_k^{-1} F(x_k, y_k)$$

Where;

$$J_k^{-1} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}^{-1}_{(x_k, y_k)} = \frac{1}{D_k}\begin{bmatrix} g_y & -f_y \\ -g_x & f_x \end{bmatrix}_{(x_k, y_k)}$$

Therefore, we can write,

$$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = J_k^{-1} \begin{bmatrix} f(x_k, y_k) \\ g(x_k, y_k) \end{bmatrix}$$

And;

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} + \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - J_k^{-1} \begin{bmatrix} f(x_k, y_k) \\ g(x_k, y_k) \end{bmatrix} \qquad (4.7)$$

Where;

K=0, 1, 2,…

Or;

$$X^{(k+1)} = X^{(k)} - J_k^{-1} F(x^{(k)}) \qquad (4.8)$$

Where,

$$X^{(k)} = \begin{bmatrix} x^{(k)} & y^{(k)} \end{bmatrix}^T$$

And,

$$F(x^{(k)}) = [f(x_k, y_k) \;\; g(x_k, y_k)]^T$$

The method given by (4.8) is an extension of the Newton- Raphson method of equation $f(x) = 0$ to system of equations.

4.2. Extrapolated Newton's Method

Newton-Raphson method for solving systems of two nonlinear equations:

$$\begin{matrix} f(x,y)=0 \\ g(x,y)=0 \end{matrix}$$, in two variables $x\, and\, y$

In order to do this we must combine these two equations into a single equations form:

$$F(z)=0 \tag{4.9}$$

Where; $z=(x,y)$

Where F must give us a two component column vector and '0' is also a two component zero column vectors. That is;

$$F(z)=\begin{bmatrix} f(z) \\ g(z) \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which is equivalent to the system of two equations $f(z)=0$ and $g(z)=0$

Newton-Raphson method for solving (4.9) is given by:

$$z^{k+1}=z^k-J^{-1}(z^k)F(z^k) \tag{4.10}$$

Where; $J^{-1}(z^k)=F'(z^k)$

And k=0, 1, 2,…

And also we have from [6] the method (4.10) is converge if $\left|J^{-1}(z^k)\right|<1$

We extrapolateNewton's method,:

$$z^{k+1}=z^k-J^{-1}(z^k)F(z^k)$$

$$z^{k+1}=\alpha_n z^{*k+1}+(1-\alpha_n)z^k$$

Where;

$$z*^{k+1}=z^k-J^{-1}(z^k)F(z^k)$$

$$\Rightarrow z^{k+1} = z^k - \alpha_n J^{-1}(z^k)F(z^k) \tag{4.11}$$

By Fernando.T.G, in [3] we have the following relations:

Where; $F'(z^k) \in R^2 X R^2$

$$F'(z^k) = J(z^k) = \nabla F^T(z^k) \tag{4.12}$$

We can write equation (4.9) in the form $z = \phi(z)$ and generating sequence of approximation defined by:

$$z^{k+1} = \phi(z^k) \tag{4.13}$$

From equation (4.13) and equation (4.11), we have;

$$\phi(z^k) = z^k - \alpha_n J^{-1}(z^k)F(x^k)$$

The method of (4.11) is convergent if $|\phi'(z^k)| < 1$

$$\Rightarrow |\phi'(z^k)| = \left|1 - \alpha_n \left(\frac{F(z^k)}{J(z^k)}\right)'\right| < 1$$

$$= \left|1 - \alpha_n \left(\frac{F'(z^k)J(z^k) - J'(z^k)F(z^k)}{J^2(z^k)}\right)\right| < 1$$

$$= \left|1 - \alpha_n \left(J^{-2}(z^k)[F'(z^k)J(z^k) - J'(z^k)F(z^k)]\right)\right| < 1$$

$$= \left|1 - \alpha_n \left(J^{-1}(z^k)F'(z^k) - \frac{J'(z^k)F(z^k)}{J^2(z^k)}\right)\right| < 1$$

$$= \left|1 - \alpha_n \left(\frac{J(z^k)F(z_k)}{J(z^k)F(z_k)} - \frac{J'(z^k)F^2(z^k)}{J^2(z^k)F(z_k)}\right)\right| < 1$$

By multiplying $\frac{F(z^k)}{F(z^k)}$ in bracket

$$= \left|1 - \alpha_n \left(1 - \frac{J'(z^k)F(z^k)}{J(z^k)F(z^K)}\left(\frac{F(z^k)}{J(z^k)}\right)\right)\right| < 1$$

$$=\left|1-\alpha_n\left(1-\frac{J'(z^k)F^2(z^k)}{J^2(z^k)F(z^k)}\right)\right|<1$$ Because from (4.12) we have;

$$F'(z^k)=J(z^k)$$

$$=\left|1-\alpha_n+\alpha n\left(\frac{J'(z^k)F^2(z^k)}{J^2(z^k)F(z^k)}\right)\right|<1$$

Let $\rho_n=\dfrac{J'(z^k)F^2(z^k)}{J^2(z^k)F(z^k)}$ (4.14)

Where, (n=0, 1, 2,,..)

Or, by using equation (4.12), we obtain

$$\rho_n=\frac{F''(z^k)F^2(z^k)}{F'^2(z^k)F(z^k)}$$

Where; $F'(z^k)=J(z^k)=\nabla F^T(z^k)$

$$\Rightarrow F''(z^k)=J'(z^k)=\nabla(\nabla F^T(z^k)) \tag{4.15}$$

Then, $\left|\phi'(z^k)\right|=\left|1-\alpha_n+\alpha_n\rho_n\right|<1$

Since the iteration method (4.13) is convergent under the condition of extrapolated Newton's method for functions of two variables will converge if,

$$\left|\phi'(z^k)\right|=\left|1-\alpha_n+\alpha_n\rho_n\right|<1$$

Let $\mu=\left|\phi'(z^k)\right|$

$$\Rightarrow \mu=\left|1-\alpha_n+\alpha_n\rho_n\right|<1 \quad ; \quad \forall z\subset D\in\Re^2 \tag{4.16}$$

4.2.1. Pre analysis convergence of the method

From (4.16) we have;

$$\Rightarrow |1-\alpha_n+\alpha_n\rho_n|<1$$

$$\Rightarrow -1<(1-\alpha_n+\alpha_n\rho_n)<1$$

$$\Rightarrow -1<(1-\alpha_n(1-\rho_n))<1$$

$$\Rightarrow -2<(-\alpha_n(1-\rho_n))<0$$

$$\Rightarrow -2<-\alpha_n(1-\rho_n)\text{ And }-\alpha_n(1-\rho_n)<0$$

$\Rightarrow 2>\alpha_n(1-\rho_n)\text{ And }\alpha_n(1-\rho_n)>0$; by multiply '-ve' on both sides

$$\alpha_n<\frac{2}{1-\rho_n}\ \&\ \alpha_n>0$$

$$\Rightarrow \alpha_n\in\left(0,\frac{2}{1-\rho_n}\right)$$

Similarly;

$$1-\rho_n<\frac{2}{\alpha_n}\ \&\ \rho_n<1$$

$$\Rightarrow \rho_n>1-\frac{2}{\alpha_n}\ \&\ \rho_n<1$$

$$\Rightarrow \rho_n\in\left(\left(1-\frac{2}{\alpha_n}\right),1\right),\quad \forall\alpha_n\in\left(0,\frac{2}{1-\rho_n}\right)$$

$$\Rightarrow \rho_n\in(-\infty,1)$$

Without loss of generality let consider the positive value of ρ_n that is $\rho_n\in(0,1)$.this is possible when we assume $J'(z^k)$ and $F(z^k)$ have the same sign.

4.2.2. Estimation of the parameter α_n

We need to find a positive real value of α_n for an each iteration, which minimizes μ of (4.16).

Since ρ_n of (4.11) is positive and real for all, we have in general;

$$a = 0 \le \rho_n \le \frac{J'(z^k)F^2(z^k)}{J^2(z^k)F(z^k)} = b \tag{4.17}$$

Where; k=0, 1, 2, . . .

The process of minimizing μ of (4.16) keeping in view of (4.17) with respect to α_n using the procedure give in young [15] gives the optimal choices for α_n as:

$$\partial_n(opt) = \frac{2}{2-(a+b)}$$

$$= \frac{2}{2-\rho_n} \tag{4.18}$$

With the choice of computational parameter α_n, we have

$$\phi'(z^k) = 1 - \frac{2}{2-\rho_n} + \frac{2\rho_n}{2-\rho_n}$$

$$= \frac{(2-\rho_n) - 2 + 2\rho_n}{2-\rho_n}$$

$$= \frac{\rho_n}{2-\rho_n} \tag{4.19}$$

Theorem: 4.2.2.1: Extrapolated Newton-Raphson method for functions of two variables (4.11) is converges if the method (4.10) is convergent.

Proof:

We have here (4.10) is convergent.so that from [6] we have $\left|J^{-1}(z^k)\right| < 1$

$and\ \rho_n \in [0,1)$

$$.\Rightarrow |\rho_n| = \left|\frac{J'(z^k)F^2(z^k)}{J^2(z^k)F(z^k)}\right| < 1$$

$$\Rightarrow \left|\frac{J'(z^k)F(z^k)}{J^2(z^k)}\right| < 1$$; This one is the condition convergence of (4.10)

Thus; (4.10) is convergent.

But from (4.16) by letting we have;

$$\mu = \left|\phi'(z^k)\right|$$

$$\mu < 1$$

$$\Rightarrow \left|\phi'(z^k)\right| < 1$$

$$\Rightarrow \frac{\rho_n}{2-\rho_n} < 1\ ;\ \forall z \in \Re^2 \qquad (n=0, 1, 2,\ldots)$$

$$\Rightarrow \left|1-\alpha_n+\alpha_n\rho_n\right| < 1\ ;\ \forall z \in \Re^2$$

Therefore, the extrapolated (4.11) is convergent.

4.2.3. Analysis of convergence of the method

Theorem: 4.2.3.1

Let $F : D \to \Re^2$ for an open disc D. Assume that F has first and second derivatives in the region D. If it has a simple root at $\delta \in D$ and z_0 is sufficiently close to δ, the extrapolated Newton Raphson Method defined by (4.11) is $3rd$ order convergent.

Proof:

Here, we have

$$F = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}, \begin{cases} z = (x,y) \\ z_0 = (x_0, y_0), e_k = z_k - \delta \\ \delta = (\beta_1, \beta_2) \end{cases} \text{ and } c_j = \frac{1}{j!} \frac{F^{(j)}(\delta)}{F^{(1)}(\delta)}$$

In this case β_1 *and* β_2 are roots of $f(x,y)$ *and* $g(x,y)$, respectively. Then the Extrapolated Newton-Raphson Method is given by:

$$z^{k+1} = z^k - \alpha_n J^{-1}(z^k) F(z^k)$$

Where;

$$\alpha_n = \frac{2}{2-\rho_n} \quad \text{And} \quad \rho_n = \frac{J'(z^k)F^2(z^k)}{J^2(z^k)F(z^k)}$$

This gives us:

$$z^{k+1} = z^k - \frac{2F'(z^k)F^2(z^k)}{2F'(z^k)^2 F(z^k) - F''(z^k)F^2(z^k)} \text{ For k=1, 2. . .}$$

Or;

$$z^{k+1} = z^k - \frac{2J(z^k)F^2(z^k)}{2J^2(z^k)F(z^k) - J'(z^k)F^2(z^k)}$$

From (4.15) we have;

$$F'(z^k) = J(z^k)$$

$$F''(z^k) = J'(z^k)$$

Let δ be simple root of $F(z)$ $(i.e\ F(\delta) = 0 \text{ and } F'(\delta) \neq 0)$

And $e_k = z^k - \delta$ (4.20)

Be the error at k^{th} iteration. Then by Taylor's series we have;

$$F(z^k) = F(\delta + e_k) = F(\delta) + F'(\delta)e_k + \frac{1}{2!}F''(\delta){e_k}^2 + \frac{1}{3!}F'''(\delta){e_k}^3 + \frac{1}{4!}F^{(4)}(\delta){e_k}^4 + O({e_k}^5)$$

$$= F'(\delta)[e_k + \frac{{e_k}^2}{2!}\frac{F''(\delta)}{F'(\delta)} + \frac{{e_k}^3}{3!}\frac{F'''(\delta)}{F'(\delta)} + \frac{{e_k}^4}{4!}\frac{F^{(4)}(\delta)}{F'(\delta)} + O({e_k}^5)]$$

$$= F'(\delta)[e_k + c_2{e_k}^2 + c_3{e_k}^3 + c_4{e_k}^4 + O({e_k}^5)] \quad (4.21)$$

And similarly;

$$F^2(z^k) = F'^2(\delta)[{e_k}^2 + 2c_2{e_k}^3 + (2c_3 + {c_2}^2){e_k}^4 + O({e_k}^5)] \quad (4.22)$$

Again by the same procedure we have;

$$F'(z^k) = F'(\delta + e_k) = F'(\delta) + F''(\delta)e_k + \frac{1}{2!}F'''(\delta){e_k}^2 + \frac{1}{3!}F^{(4)}(\delta){e_k}^3 + \frac{1}{4!}F^{(5)}(\delta){e_k}^4 + O({e_k}^5)$$

$$= F'(\delta)[1 + \frac{F''(\delta)}{F'(\delta)}e_k + \frac{{e_k}^2}{2!}\frac{F'''(\delta)}{F'(\delta)} + \frac{{e_k}^3}{3!}\frac{F^{(4)}(\delta)}{F'(\delta)} + \frac{{e_k}^4}{4!}\frac{F^{(5)}(\delta)}{F'(\delta)} + O({e_k}^5)]$$

$$= F'(\delta)[1 + 2c_2e_k + 3c_3{e_k}^2 + 4c_4{e_k}^3 + 5c_5{e_k}^4 + O({e_k}^5)] \quad (4.23)$$

Multiplying (4.23) by itself, we obtain:

$$F'(z^k).F'(z^k) = F'(\delta + e_k).F'(\delta + e_k) = F'^2(\delta + e_k)$$
$$= F'^2(\delta)[1 + 2c_2e_k + 3c_3{e_k}^2 + 4c_4{e_k}^3 + 5c_5{e_k}^4 + O({e_k}^5)]^2$$

$$= F'^2(\delta)[1 + 4c_2e_k + (4{c_2}^2 + 6c_3){e_k}^2 + (8c_4 + 12c_2c_3){e_k}^3 + (8c_2c_4 + 9{c_3}^2 10c_5){e_k}^4 + O({e_k}^5)] \quad (4.24)$$

And

$$F''(z^k) = F''(\delta + e_k) = F''(\delta) + F'''(\delta)e_k + \frac{1}{2!}F^{(4)}(\delta){e_k}^2 + \frac{1}{3!}F^{(5)}(\delta){e_k}^3 + \frac{1}{4!}F^{(6)}(\delta){e_k}^4 + O({e_k}^5)$$

Then by multiplying right hand side by $\frac{F'(\delta)}{F'(\delta)}$, we obtain:

$$= F'(\delta)[\frac{F''(\delta)}{F'(\delta)} + \frac{F'''(\delta)}{F'(\delta)}e_k + \frac{{e_k}^2}{2!}\frac{F^{(4)}(\delta)}{F'(\delta)} + \frac{{e_k}^3}{3!}\frac{F^{(5)}(\delta)}{F'(\delta)} + \frac{{e_k}^4}{4!}\frac{F^{(6)}(\delta)}{F'(\delta)} + O({e_k}^5)]$$

$$= F'(\delta)[2c_2 + 6c_3e_k + 12C_4e_k{}^2 + 20c_5e_k{}^3 + 30c_6e_k{}^4 + O(e_k{}^5)] \quad (4.25)$$

From (4.22) and (4.23) we obtain:

$$F'(z^k).F^2(z^k) = F'(\delta + e_k).F^2(\delta + e_k)$$

$$= F'(\delta)[1 + 2c_2e_k + 3c_3e_k{}^2 + 4c_4e_k{}^3 + 5c_5e_k{}^4 + O(e_k{}^5)] \text{ X } [$$
$$F'^2(\delta)[e_k{}^2 + 2c_2e_k{}^3 + (2c_3 + c_2{}^2)e_k{}^4 + O(e_k{}^5)]]$$

$$= F'^3(\delta)[e_k{}^2 + 4c_2e_k{}^3 + (5c_3 + 5c_2{}^2)e_k{}^2 + O(e_k{}^5)]$$

Thus;

$$2F'(z^k)F^2(z^k) = F'^3(\delta)[e_k{}^2 + 4c_2e_k{}^3 + (5c_3 + 5c_2{}^2)e_k{}^2 + O(e_k{}^5)] \quad (4.26)$$

From (4.25) and (4.22) we obtain;

$$F''(z^k)F^2(z^k) = F''(\delta + e_k)F^2(\delta + e_k)$$

$$=[F'(\delta)[2c_2 + 6c_3e_k + 12C_4e_k{}^2 + 20c_5e_k{}^3 + 30c_6e_k{}^4 + O(e_k{}^5)] \text{ X } [$$
$$F'^2(\delta)[e_k{}^2 + 2c_2e_k{}^3 + (2c_3 + c_2{}^2)e_k{}^4 + O(e_k{}^5)]]$$

$$= F'^3(\delta)[2c_2e_k{}^2 + (4c_2{}^2 + 6c_3)e_k{}^3 + (16c_2c_3 + 2c_2{}^3 + 12c_4)e_k{}^4 + O(e_k{}^5)]$$

Thus;

$$F''(z^k)F^2(z^k) = 2F'^3(\delta)[c_2e_k{}^2 + (2c_2{}^2 + 3c_3)e_k{}^3 + (8c_2c_3 + c_2{}^3 + 6c_4)e_k{}^4 + O(e_k{}^5)] \quad (4.27)$$

From (4.24) and (4.21) we have;

$$F'^2(z^k)F(z^k) = F'^2(\delta)[1 + 4c_2e_k + (4c_2{}^2 + 6c_3)e_k{}^2 + (8c_4 + 12c_2c_3)e_k{}^3 + (8c_2c_4 + 9c_3{}^3 + 10c_5)e_k{}^4 + O(e_k{}^5)]$$
$$\text{X } [F'(\delta)[e_k + c_2e_k{}^2 + c_3e_k{}^3 + c_4e_k{}^4 + o(e_k{}^5)]]$$

$$2F'^2(z^k)F(z^k) = 2F'^3(\delta)[e_k + 5c_2e_k{}^2 + (8c_2{}^2 + 7c_3)e_k{}^3 + (9c_4 + 22c_2c_3 + 4c_2{}^3)e_k{}^4 + O(e_k{}^5)] \quad (4.28)$$

Then from equations (4.27) and (4.28) we have;

$$2F'^2(z^k)F(z^k) - F''(z^k)F^2(z^k) =$$
$$2F'^3(\delta)[e_k + 4c_2e_k{}^2 + (6c_2{}^2 + 4c_3)e_k{}^3 + (14c_2c_3 + 3c_2{}^3 + 3c_4)e_k{}^4 + O(e_k{}^5)] \quad (4.29)$$

From equations (4.25) and (4.29) we obtain;

$$\frac{2F'(z^k)F^2(z^k)}{2F'^2(z^k)F(z^k)-F''(z^k)F^2(z^k)}=\frac{e_k{}^2+4c_2e_k{}^3+(5c_3+5c_2{}^2)e_k{}^4+o(e_k{}^5)}{e_k+4c_2e_k{}^2+(4c_3+6c_2{}^2)e_k{}^3+(14c_2c_3+3c_2{}^3+3c_4)e_k{}^4+o(e_k{}^5)}$$

$$=\frac{e_k}{e_k}\left(\frac{e_k+4c_2e_k{}^2+(5c_3+5c_2{}^2)e_k{}^3+o(e_k{}^4)}{1+4c_2e_k+(4c_3+6c_2{}^2)e_k{}^2+(14c_2c_3+3c_2{}^3+3c_4)e_k{}^3+o(e_k{}^4)}\right)$$

$$=\left(\frac{e_k+4c_2e_k{}^2+(5c_3+5c_2{}^2)e_k{}^3+o(e_k{}^4)}{1+4c_2e_k+(4c_3+6c_2{}^2)e_k{}^2+(14c_2c_3+3c_2{}^3+3c_4)e_k{}^3+o(e_k{}^4)}\right)$$

$$=[e_k+4c_2e_k+(5c_3+5c_2{}^2)e_k{}^2+o(e_k{}^4)]\text{X}$$

$$[e_k+4c_2e_k+(5c_3+5c_2{}^2)e_k{}^2+o(e_k{}^4)]$$

$$[1-\{(4c_2e_k+(6c_2{}^2+4c_3)e_k{}^2+(14c_2c_3+3c_2{}^3+3c_4)e_k{}^3+o(e_k{}^4)]+$$

$$[4c_2e_k+(6c_2{}^2+4c_3)e_k{}^2+(14c_2c_3+3c_2{}^3+3c_4)e_k{}^3+o(e_k{}^4)]^2-\ldots\}$$

$$=[e_k+(c_3-33c_2{}^2)e_k{}^3+o(e_k{}^4)]\;[1+4c_2e_k+(6c_2{}^2+4c_3)e_k{}^2+(14c_2c_3+3c_2{}^3+3c_4)e_k{}^3+o(e_k{}^4)]^{-1}$$

Therefore;

$$z^{k+1}=z^k-\frac{2J(z^k)F^2(z^k)}{2J(z^k)F(z^k)-J'(z^k)F^2(z^k)}\text{ is ENRM}$$

$$z^{k+1}-z^k = =[e_k+(c_3-33c_2{}^2)e_k{}^3+o(e_k{}^4)]$$

$$e_{k+1}-e_k=-e_k-(c_3-33c_2{}^2)e_k{}^3+O(e_k{}^4)$$

$$e_{k+1}=(33c_2{}^2-c_3)e_k{}^3+O(e_k{}^4)$$

This shows that the method (4.11) or ENRM has a third order of convergence.

4.3. Numerical Example

In order to show the importance and applicability of the method we need to take problems and solve using these methods. After obtaining the approximate solution we compare the result with the numerical solution and each methodagainst the other method.

Example: Consider the system of nonlinear equations.

$$f(x, y) = x^2 - y - 1 = 0$$

$$g(x, y) = y^2 - x = 0$$

The solution of the above system is obtained by using the Newton's method and Extrapolated Newton Raphson method taking initial approximation for x's and y's, error tolerance with 5x 10^{-4} for both methods and the results are presented in Table4.1andTable4.2 respectively.

Table 4.1. Newton's method example

K	$Z^{(K)}$	$Z^{(K+1)}$	$F(Z^K)$
0	1	1.6667	-1
	1	1.3333	0
1	1.6667	1.5023	0.4444
	1.3333	1.23	0.1111
2	1.5023	1.4887	0.027
	1.23	1.2474	0.0107
3	1.4887	1.4903	-0.03117
	1.2474	1.2208	0.0073
4	1.4903	1.4902	0.0001
	1.2208	1.2207	0.00005

Table 4.2. Extrapolated Newton's method example

K	$Z^{(K)}$	$Z^{(K+1)}$	$F(Z^K)$
0	1	1.3333	-1
	1	1.25	0
1	1.3333	1.48760	-0.4723
	1.25	1.2257	0.2292
2	1.48760	1.4907	-0.0127
	1.2257	1.2204	0.0147
3	1.4907	1.4901	0.0017
	1.2204	1.2206	-0.0013

4.4. Discussion

In this study the researcher developed the extrapolated Newton Raphson method which is completely consistent with Newton Raphson method. The other thing is the interval of extrapolated parameter which controls the converges of the method and optimal parameter is obtained byusing the procedure of Young [15].It is observed that the newly developed method is better in accuracy than the Newton-Raphson method as its order of convergence is third order while that of the later one is second order convergent. One practical example of systems of nonlinear equations is considered. The analysis of the results shows that the Newton's method takes 4 iterations to converge and Extrapolated Newton's method converges after three iteration as per the error of tolerance considered. As it shown in the analysis part and the example we considered the newly method is better than Newton-Raphson method for solving system of non-linear equations.

Table4.3.Comparison of Numerical Test for system of nonlinear equation

Methods	Number of iterations	Numerical solutions Z^{k+1}	Numerical solutions Z^{K}	Absolute Errors ($e=z^{k+1}-z^{k}$)
NM	0	1.6667	1	0.6667
		1.3333	1	0.3333
	1	1.5023	1.6667	0.1644
		1.23	1.3333	0.1033
	2	1.4887	1.5023	0.0136
		1.2474	1.23	0.0174
	3	1.4903	1.4887	0.0084
		1.2208	1.2474	0.0066
	4	1.4902	1.4903	0.0001
		1.2207	1.2208	0.0001
ENRM	0	1.3333	1	0.3333
		1.25	1	0.25
	1	1.48760	1.3333	0.1543
		1.2257	1.25	0.0243
	2	1.4907	1.48760	0.0031
		1.2204	1.2257	0.0053
	3	1.4907	1.4901	0.0006
		1.2204	1.2206	0.0002

CHAPTER FIVE

5. CONCLUSIONAND FUTURE WORK

5.1. CONCLUSION

The intention of this study is to introduced a new method and compare its result with that of Newton-Raphson method. Thus, from discussion of the results, it is easily observed that Extrapolated Newton's method is the efficient method to solve systems of nonlinear equations with two variables. One practical example of nonlinear equations is considered. The analysis of the results shows that the Newton's method takes 4 iterations to converge and Extrapolated Newton's method converges after three iteration as per the error of tolerance considerd.As it shown in the analysis part and the example we considered the newly method is better than Newton-Raphson method for solving system of nonlinear equations.

5.2. Future Work

This study is concentrated on Extrapolated Newton's Raphson method for solving functions of two variables .In future work, it would be interesting to investigate some other methods for solving nonlinear systems of equations and to extend in N by N matrices.

In general, ExtrapolatedNewton's method is suitable for solving nonlinear systems of equations of two variables with two equations. Particularly, it fits in solving 2x2 matrices with two variables that arise in the application of engineering and approximation solution of Numerical analysis.

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