# Fitted Finite Difference Method for Solving Singularly Perturbed Differential-Difference Equations

Thesis Submitted to the Department of Mathematics, Jimma University in Partial Fulfillment for the Requirements of the Degree of Masters of Science in Mathematics

**(NUMERICAL ANALYSIS)**


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**October, 2017**

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## DECLARATION

I undersigned declare that, this thesis entitled "**Fitted finite difference method for solving singularly perturbed differential-difference equations**" is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged.

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**ACKNOWLEDGMENT**

First of all, I am indebted to my almighty God who gave me long life and helped me to reach this precious time. Next, my special heartfelt thanks go to my advisor Gemechis File (PhD) and co-advisor Gashu Gadisa (MSc) for their constructive and critical comments throughout the preparation of this thesis work. Also, I would like to thanks Tesfaye Aga (MSc) for his constructive comments.

## ABSTRACT

*In this thesis, fitted fourth order finite difference method is presented for solving singularly perturbed differential-difference equations. First the given singularly perturbed differential-difference equations are transformed into an asymptotically equivalent singularly perturbed boundary value problem. Using fitted finite difference approximation, the given differential equation is transformed into a three-term recurrence relation, which can easily be solved by Thomas Algorithm. The stability and convergence of the method have been investigated. To validate the applicability of the proposed method three model examples have been considered and solved for different values of parameters and mesh size h. Both theoretical error bounds and numerical rate of convergence have been established for the method. The numerical results have been presented in tables and further to examine the effect of delay and advance parameters on the left and right boundary layer of the solution; graphs have been given for different values of parameters. Concisely, the present method gives better result than some existing numerical methods reported in the literature.*

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# CHAPTER ONE

# INTRODUCTION

## 1.1. Background of the Study

Numerical analysis is a branch of mathematics concerned with theoretical foundations of numerical algorithms for the solution of problems arising in scientific applications, (Wasow, 1942). Science and technology develop many practical problems, such as the mathematical boundary layer theory or approximation of solution of various problems described by differential equations and almost all physical phenomena in nature are modeled using differential equations, and singularly perturbed problems are vital class of these kinds of problems, (Cengizci, 2017). In general, a singular perturbation problem is a differential equation that is controlled by a small positive parameter $0 < \varepsilon \ll 1$ that exists as multiplier to the highest derivative term in the differential equation. As $\varepsilon$ tends to zero, the solution of problem exhibits interesting behaviors (rapid changes) since the order of the equation reduces. The region where these rapid changes occur is called inner region and the region in which the solution changes mildly is called outer region. A singularly perturbed differential-difference equation (SPDDE) is an ordinary differential equation in which the highest derivative is multiplied by a small parameter and involving at least one delay or advance term.

We often encounter many problems which are described by parameter dependent differential equations. Any system involving a feedback control will almost involve time delays. These arise because a finite time is required to sense information and then react to it. If we restrict the class of delay differential equations to a class in which the highest derivative is multiplied by a small positive parameter and involving at least one delay term, then it is said to be a singularly perturbed delay differential equation. In this problem typically there are thin transition layers where the solution varies rapidly or jumps abruptly, while away from the layers the solution behaves regularly and varies slowly. In recent years, there has been a growing interest in the numerical treatment of such differential equations. This is due to the versatility of such type of differential equations in the mathematical modeling of various physical and biological phenomena. For example, population ecology, control theory, viscous elasticity, and materials with thermal memory. Boundary value problems in differential-difference equations arise in a very natural way in studying variation problems in control theory where the problem is complicated by the effect of time delays in signal transmission (Elsgol'cs, 1964). According to Kadalbajoo and Gupta, (2010) these singularly perturbed problems arise in the modeling of various modern complicated processes, such as fluid flow at high Reynolds numbers, water quality problems in rivers networks, drift diffusion equation of semi-conductor device modeling, electro-magnetic field problem in moving media, financial modeling of option pricing, turbulence model, simulation of oil extraction from under-ground reservoirs, theory of plates and shells, atmospheric pollution, and groundwater transport.

Some researchers are tried to find the approximate solutions of SPDDEs.

$$\varepsilon y''(x)+a(x)y'(x-\delta)+b(x)y(x)+c(x)y(x-\tau)=f(x),\quad 0\le x\le 1 \tag{1.1}$$

subject to the interval and boundary conditions,

$$y(x)=\alpha(x)\ ,\ -\delta,-\tau\le x\le 0 \text{ and } y(1)=\beta(x). \tag{1.2}$$

where $\varepsilon$ is a perturbation parameter $(0<\varepsilon\ll 1)$, $\delta$ and $\tau$ are delay parameters with $0<\delta=o(\varepsilon),\ \ 0<\tau=o(\varepsilon)$ and $a(x),\ b(x),\ c(x),\ f(x),\ \alpha(x)$ and $\beta(x)$ are smooth functions.

According to Doolan et al, (1980) still there is a lack of accuracy and convergence because of the treatment of singular perturbation problems is not trivial and the solution depends on perturbation parameter and mesh size. Due to this, numerical treatment of singularly perturbed boundary value problems needs improvement.

For example; Awoke and Reddy (2013) presented parameter fitted scheme to solve singularly perturbed delay differential equations. Soujanya et al., (2013) presented an exponentially fitted non symmetric numerical method for singularly perturbed differential equations with layer behavior**.** Chakravarthy et al., (2015) presented fitted numerical scheme to solve singularly perturbed delay differential equation. Gemechis et al., (2015) presented a fitted-stable central difference method for solving singularly perturbed two point boundary value problems with the boundary layer at one end left or right of the interval. Erdogan, (2009) presented an exponentially fitted method to solve singular perturbed delay differential equation, using exponentially fitted difference schemes (Erdogan et al., 2012), and (Kadalbajoo and Ramesh, 2007; Kadalbajoo and Kumar, 2008; Sirisha and Reddy, 2015) are developed a numerical methods for solving singularly perturbed differential-difference equations and so on. Thus, in this study we develop a fitted finite difference method for solving singularly perturbed differential-difference equations.

## 1.2. Statement of the Problem

Singularly perturbed differential-difference equations occur very frequently in mathematical modeling and control theory. More specifically, the boundary value problems (BVPs) for differential-difference equations (DDEs) come up in the study of the effect of time delays in signal transmission of control theory and in the phenomenon where Markov process governs, such as persistence times of population with large number of random fluctuations and time between the impulses of the nerve cell, (Bellman and Cooke, 1963). Depending on the parameter values, the resulting solutions of the class of differential-difference equations exhibit oscillations, boundary and interior layers, or turning point behavior. Thus, existing numerical methods produce good results only when we take step size $h < \varepsilon$. This shows that there is a challenge for singularly perturbed boundary value problems to get more accurate solution due to perturbation parameter is sufficiently small and no good result when $\varepsilon \leq h$.

Recently, some researchers are tried to develop a numerical methods for solving singularly perturbed differential-difference equations. For examples; Sirisha and Reddy, (2014) presented solution of singularly perturbed delay differential equations with dual layer behaviour using numerical integration; Swamy et al., (2016) introduced a Galerkin Method for solving this problem; Kanth and Kumar, (2017) also introduced a fitted tension spline method for solving such problem; Cengizci, (2017) also used two-term Taylor series expansion for the delayed convection term. However, the issue of accuracy and convergence of the method still needs attention and improvement. Therefore, it is important to develop an alternative numerical method which may be more accurate, stable and convergent for solving singularly perturbed differential-difference equations.

Owning to this, the present study attempts to answer the following questions:

1. How does the fitted finite difference method be described for solving singularly perturbed differential-difference equations?
2. To what extent the proposed method approximate the solutions?
3. To what extent the proposed method is stable and convergent?

### 1.3. Objectives of the Study

#### 1.3.1. General Objective

The general objective of this study is to develop fitted finite difference method for solving singularly perturbed differential–difference equations.

#### 1.3.2. Specific Objectives

The specific objectives of the present study are:

1. To formulate the fitted finite difference method for solving singularly perturbed differential–difference equations.
2. To investigate the accuracy of the present method.
3. To establish the stability and the convergence of the present scheme.

### 1.4. Significance of the Study

The outcomes of this study may help to introduce the application of numerical methods in solving problems arising in different field of studies and serve as reference material for scholars' who works on this area.

### 1.5. Delimitation of the Study

The singularly perturbed problems are perhaps arises in variety of mathematical and physical problem. However, this study is delimited to fitted finite difference method for solving singularly perturbed differential–difference equations of the form:

$$\varepsilon y''(x)+a(x)y'(x)+b(x)y(x-\delta)+c(x)y(x)+d(x)y(x+\eta)=f(x),\quad 0\le x\le 1 \qquad (1.3)$$

subject to the interval and boundary conditions,

$$\begin{aligned} y(x)&=\alpha(x),\ -\delta\le x\le 0,\\ y(x)&=\beta(x),\ 1\le x\le 1+\eta. \end{aligned} \qquad (1.4)$$

where $\varepsilon$ is a perturbation parameter $(0<\varepsilon\ll 1)$, $\delta$ is delay parameter, $\eta$ is advance parameter with $0<\delta=o(\varepsilon),\ 0<\eta=o(\varepsilon)$ and $a(x),\ b(x),\ c(x),\ d(x),\ f(x),\ \alpha(x)$ and $\beta(x)$ are smooth bounded functions.

# CHAPTER TWO

# REVIEW OF RELATED LITERATURE

## 2.1. Singularly Perturbation Differential-Difference Equation

The study of many theoretical and applied problems in science, engineering and technology leads to boundary value problems for singularly perturbed differential equations that have a multi-scale character. However, most of the problems cannot be completely solved by analytic techniques. Consequently, numerical simulations are of fundamental importance in gaining some useful insights on the solutions of the singularly perturbed differential equations.

The differential-difference equation plays an important role in the mathematical modeling of various practical phenomena in the biosciences and control theory. Any system involving a feedback control will almost always involve time delays. These arise because a finite time is required to sense information and then react to it. Many phenomena in real life and science may be modeled mathematically by delay differential or differential difference equations (DDEs). Equations of this type arise widely in scientific fields such as biology, medicine, ecology and physics, in which the time evolution depends not only on present states but also on states at or near a given time in the past. If we restrict the class of DDEs to a class in which the highest derivative is multiplied by a small parameter, then we get a class of singularly perturbed differential difference equations (SPDDEs). These equations are used to model a large variety of practical phenomena, for instance, variational problems in control theory, (Glizer, 2003). Like ordinary differential equations, the order of a DDE is the order of the highest derivative term. A DDE is classified into two categories, retarded and neutral. A delay-differential equation is said to be of retarded DDE if the delay argument does not occur in the highest order derivative term, otherwise it is said to be neutral DDE. A differential-difference equation model incorporating stochastic effects due to neuronal variability and the solution to this model was approximated by Monte Carlo techniques. More generalization of this model, to deal with distribution of post synaptic potential amplitudes, was discussed by Stein, (1967). Asymptotic approach to study general boundary-value problems for singularly perturbed differential-difference equations was given in a series of papers by Lange and Miura, (1994). A variety of numerical approaches have been presented by Kadalbajoo and Sharma, (2004) for singularly perturbed differential-difference equations with only negative shift and as well as with both positive and negative shifts.

### 2.2. Fitted Finite Difference Method

As Phaneendra et al., (2014) presented a numerical finite difference approach to solve the boundary-value problem for singularly perturbed differential-difference equation, which contains only negative shift in the differentiated term. In this method, first approximate the shifted term by Taylor series and apply a fitted finite difference scheme. The existence and uniqueness of the discrete problem along with stability estimates are discussed. The effect of small shifts on the boundary layer solution of the problem has been given by considering several numerical experiments. Frequently, delay/advance differential equations or differential–difference equations have been reduced to differential equations with coefficients that depend on the delay/advance by means of first-order accurate Taylor series expansions of the terms that involve either delay or advance, and the resulting differential equations have been solved either analytically when the coefficients of these equations are constant or numerically, when they are not. The arguments for small delay problems are found throughout the literature on epidemics and population where these small shifts play an important role in the modeling of various real life phenomena, (Kuang, 1993). Many researchers have investigated the effect of small shift on the layer behavior of the solution and observed that it is very small and affects the solution significantly. In this direction, Lange and Miura, (1994) provided asymptotic approach to SPDDE Eq. (1.3) with Eq. (1.4) and showed that the effect of the small delay and shift terms on the solution cannot be neglected. The numerical study of singularly perturbed delay differential equations is initiated in (Kadalbajoo and Sharma, 2002).

It is well known that standard discretization methods for solving singular perturbation problems are not useful and fail to give accurate results when the perturbation parameter $\varepsilon$ tends to zero. This motivates the need for other methods that have $\varepsilon$-uniform convergence. In general there are two approaches for construction $\varepsilon$-uniform methods. The first one is the fitted operator method which contains specially designed finite difference operator which reflects the singularly perturbed nature of the solution. Extensive details of $\varepsilon$-uniform fitted operator methods can be found in (Miller et al., 1996). The second one is the fitted mesh method which contains finite difference operator on specially designed meshes such as non-uniform layer-adapted meshes. In this study the proposed finite difference method to solve singularly perturbed differential equation with small shifts. To overcome the defect and weakness of the standard methods, use a piecewise uniform mesh and approximate the terms containing small shifts (delay and advance) by Taylor series, then apply fitting factor on finite difference method of uniform mesh. Both cases, when boundary layer occurs in left and right side of interval will be study. The method is useful for obtaining numerical solution of considered problem in both cases. The advantages of this method are that it is simple to implement and it achieves high accuracy comparing with other methods. In this section, we consider a boundary-value problem for SPDDE with mixed type of small shifts of the form Eqs. (1.3) and (1.4). It is noted that if $\delta = \eta = 0$, then Eqs. (1.3) and (1.4) reduces to the singularly perturbed differential equation (SPDE), which is studied by numerous researchers, see for example, Miller, et al (1996) for sufficiently small $\delta$ and $\eta$, we follow the same technique in, Lange and Miura, (1994). To tackle the shift terms, we expand the shift terms through Taylor series expansions assuming smoothness condition on the solution of Eqs. (1.3) and (1.4).

As introduced in the literature, most researchers try to find approximate solution for singularly perturbed differential-difference equations, but mainly focuses on constant coefficients, and some others those who have done for variable coefficients did not get more accurate solutions. Owing this, we find a more accurate and convergent numerical method for solving singularly perturbed differential-difference equations, by using fitted finite difference method.

## CHAPTER THREE

## METHODOLOGY

### 3.1. Study Sites

This study is conducted at Jimma University under the department of Mathematics from September 2016 to September 2017. Conceptually, the study focus on the area of fitted finite difference method for solving singularly perturbed differential–difference equations.

### 3.2. Study Design

The study employed mixed-design (documentary review and numerical experimentation) on singularly perturbed problem with delay and advanced parameters.

### 3.3. Source of Information

The relevant sources of information for this study are books, published articles & related studies from internet and the experimental result obtained by using MATLAB ver. 2013a.

### 3.4. Mathematical Procedures

Important materials and data for the study are collected by means of documentary review. Hence, in order to achieve the stated objectives, the study follows the following steps:

1. Defining the problem,
2. Discretizing the solution domain/interval,
3. Replacing the derivatives in the differential equation by the finite difference approximation and fit the scheme
4. Finding the fitting factor and obtain the fitted schemes,
5. Rewriting the obtained schemes into tri-diagonal systems and solve by Thomas Algorithm,
6. Establishing the stability and convergence of the method,
7. Writing MATLAB code for the Algorithm,
8. Validating the schemes by using numerical examples,
9. Presenting the numerical results in different forms (tables and graphs),
10. Discussing the results against the previous findings.

# CHAPTER FOUR

# DESCRIPTION OF THE METHOD, ANALYSIS AND DISCUSSION

## 4.1. Description of the Method

Consider the singularly perturbed differential-difference equation of the form:

$$\varepsilon y''(x)+a(x)y'(x)+b(x)y(x-\delta)+c(x)y(x)+d(x)y(x+\eta)=f(x),\quad 0\le x\le 1 \tag{4.1}$$

subject to the interval and boundary conditions,

$$\begin{aligned} y(x)&=\alpha(x),\ -\delta\le x\le 0,\\ y(x)&=\beta(x),\ 1\le x\le 1+\eta \end{aligned} \tag{4.2}$$

where $\varepsilon$ is a perturbation parameter $(0<\varepsilon\ll 1)$, $\delta$ is delay parameter, $\eta$ is advance parameter with $0<\delta=o(\varepsilon),\ 0<\eta=o(\varepsilon)$ and $a(x),\ b(x),\ c(x),\ d(x),\ f(x),\ \alpha(x)$ and $\beta(x)$ are smooth bounded functions. The boundary layer and oscillatory behaviour of the problem under consideration is maintained for $\delta\ne 0$ but sufficiently small, depending on the sign of $b(x)+c(x)+d(x)$, for all $x\in(0,1)$. If $b(x)+c(x)+d(x)<0$, the solution of the problem in Eqs. (4.1) and (4.2) exhibits boundary layer behaviour at both end points $x=0$ and $x=1$, and if $b(x)+c(x)+d(x)>0$, it exhibits oscillatory behaviour. The solution $y(x)$ should be continuous on $[0,1]$, continuously differentiable on $(0,1)$ and also satisfies Eqs. (4.1) and (4.2).

By using Taylor series expansion in the neighborhood of the point $x$, we have:

$$y(x-\delta)\approx y(x)-\delta y'(x)+o(\delta^2) \tag{4.3}$$

$$y(x+\eta)\approx y(x)+\eta y'(x)+o(\eta^2) \tag{4.4}$$

Substituting Eq. (4.3) and Eq. (4.4) into Eq. (4.1), we obtain an asymptotically equivalent singularly perturbed two point boundary value problem of the form:

$$Ly(x)\equiv \varepsilon y''(x)+p(x)y'(x)+q(x)y(x)=f(x) \tag{4.5}$$

under the boundary conditions,

$$y(0)=\alpha_0 \text{ and } y(1)=\beta_0 \tag{4.6}$$

where, $p(x)=a(x)-\delta b(x)+\eta d(x)$ and $q(x)=b(x)+c(x)+d(x)$.

The transition from Eq. (4.1) to Eq.(4.5) is admitted, because of the condition that $0<\delta,\eta<<1$ are sufficiently small. Further details on the validity of this transition can be found in Elsgolt's and Norkin (1973).

Now dividing the solution domain or interval $[0,1]$ into $N$ equal parts with constant mesh length *h*, we have $x_i = x_0 + ih,\ i = 0,1,2,...,N$ and let $y(x_i) = y_i$.

By using Taylor series expansion, we obtain:

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!}y_i'' + \frac{h^3}{3!}y_i^{(3)} + \frac{h^4}{4!}y_i^{(4)} + o(h^5) \tag{4.7}$$

$$y_{i-1} = y_1 - hy_i' + \frac{h^2}{2!}y_i'' - \frac{h^3}{3!}y_i^{(3)} + \frac{h^4}{4!}y_i^{(4)} + o(h^5) \tag{4.8}$$

Subtracting Eq. (4.8) from Eq. (4.7), we obtain, the second order finite difference approximation $(\delta_c^1 y_i)$ for the first derivative of $y_i$ as:

$$\delta_c^{\,1} y_i = \frac{y_{i+1} - y_{i-1}}{2h} + T_1 \tag{4.9}$$

where $T_1 = -\frac{h^2}{6}y_i^{(3)}$.

Similarly by adding Eq. (4.7) and Eq. (4.8), we obtain, the second order finite difference approximation $(\delta_c^2 y_i)$ for the second derivative of $y_i$ is:

$$\delta_c^{\,2} y_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + T_2 \tag{4.10}$$

where $T_2 = -\frac{h^2}{12}y_i^{(4)}$.

Substituting Eqs. (4.7) and (4.8) into Eq. (4.9) yields:

$$\delta_c^1 y_i = y_i' + \frac{h^2}{6}y_i^{(3)} + T_3 \tag{4.11}$$

where $T_3 = \frac{h^4}{120}y_i^{(5)} + T_1 = \frac{h^4}{120}y_i^{(5)} - \frac{h^2}{6}y_i^{(3)}$.

Again substituting Eqs. (4.7) and (4.8) into Eq. (4.10) yields:

$$\delta_c^2 y_i = y_i'' + \frac{h^2}{12}y_i^{(4)} + T_4 \tag{4.12}$$

where $T_4 = \dfrac{h^4}{360} y_i^{(6)} - \dfrac{h^2}{12} y_i^{(4)}$

Applying $\delta_c^2$ to $y_i'$ in Eq. (4.9), we get:

$$y_i^{(3)} = \delta_c^2 y_i' - T_1^{(2)} \tag{4.13}$$

Substituting Eq. (4.13) into Eq. (4.11), we obtain:

$$\delta_c^1 y_i = y_i' + \frac{h^2}{6} \delta_c^2 y_i' + T_5 \tag{4.14}$$

where $T_5 = \dfrac{13h^4}{360} y_i^{(5)} - \dfrac{h^2}{6} y_i^{(3)}$

Applying $\delta_c^2$ to $y_i''$ in Eq. (4.10) we get a four order finite difference scheme for Eq. (4.5) as:

$$y_i^{(4)} = \delta_c^2 y_i'' - T_2^{(2)} \tag{4.15}$$

Substitution Eq. (4.15) into Eq. (4.12), we obtain:

$$\delta_c^2 y_i = y_i'' + \frac{h^2}{12} \delta_c^2 y_i'' + T_6 \tag{4.16}$$

where $T_6 = \dfrac{7h^4}{720} y_i^{(6)} - \dfrac{h^2}{12} y_i^{(4)}$

From Eqs. (4.14) and (4.16), we get:

$$y_i' = \frac{\delta_c^1 y_i - T_5}{1 + \frac{h^2}{6} \delta_c^2} \quad \text{and} \quad y_i'' = \frac{\delta_c^2 y_i - T_6}{1 + \frac{h^2}{12} \delta_c^2} \tag{4.17}$$

After evaluating Eq. (4.5) at nodal point $x_i$ and using Eq. (4.17), we obtain:

$$\varepsilon \left( \frac{\delta_c^2 y_i - T_6}{1 + \frac{h^2}{12} \delta_c^2} \right) + p_i \left( \frac{\delta_c^1 y_i - T_5}{1 + \frac{h^2}{6} \delta_c^2} \right) + q_i y_i = f_i \tag{4.18}$$

Simplifying Eq. (4.18), we have:

$$\begin{aligned} &\varepsilon \delta_c^2 y_i + \varepsilon \frac{h^2}{6} \delta_c^4 y_i - \varepsilon \left( 1 + \frac{h^2}{6} \delta_c^2 \right) T_6 + p_i \delta_c^1 y_i + \frac{h^2 p_i}{12} \delta_c^3 y_i + q_i y_i + \frac{h^2 q_i}{4} \delta_c^2 y_i + \frac{h^4 q_i}{72} \delta_c^4 y_i \\ &= f_i \left( 1 + \frac{h^2}{4} \delta_c^2 + \frac{h^4}{72} \delta_c^4 \right) + p_i T_5 \left( 1 + \frac{h^2}{12} \delta_c^2 \right) \end{aligned} \tag{4.19}$$

By successively differentiating both sides of Eq. (4.5) and evaluating at $x_i$, we have:

$$\delta_c^3 y_i = \frac{1}{\varepsilon}\left( f_i' - p_i\delta_c^2 y_i - (p_i' + q_i)\delta_c^1 y_i - q_i y_i \right) \tag{4.20}$$

$$\delta_c^4 y_i = \frac{1}{\varepsilon}\left\{ f_i'' - \frac{p_i}{\varepsilon} f_i' - \left( 2p_i' + q_i - \frac{p_i^2}{\varepsilon} \right)\delta_c^2 y_i + \left( \frac{p_i(p_i' + q_i)}{\varepsilon} - p_i'' - 2q_i' \right)\delta_c^1 y_i + \left( \frac{p_i q_i'}{\varepsilon} - q_i'' \right) y_i \right\} \tag{4.21}$$

Substituting Eqs. (4.20) and (4.21) into Eq. (4.19), we get:

$$\begin{aligned}
&\left( \varepsilon - \frac{h^2}{6}\left(2p_i'\right) + \frac{h^2 q_i}{12} + \frac{h^2 p_i^2}{12\varepsilon} - \frac{h^4 q_i}{72\varepsilon}\left( 2p_i' + q_i - \frac{p_i^2}{\varepsilon} \right) \right)\delta_c^2 y_i - \varepsilon\left( 1 + \frac{h^2}{6}\delta_c^2 \right) T_6 \\
&\quad + \left\{ -\frac{h^2 p_i}{12\varepsilon}(p_i' + q_i) + \frac{h^4 q_i}{72\varepsilon}\left( \frac{p_i(p_i' + q_i)}{\varepsilon} - p_i'' - 2q_i' \right) + \frac{h^2}{6}\left( \frac{p_i(p_i' + q_i)}{\varepsilon} - p_i'' - 2q_i' \right) + p_i \right\}\delta_c^1 y_i \\
&\quad + \left\{ \frac{h^4 q_i}{72\varepsilon}\left( \frac{p_i q_i'}{\varepsilon} - q_i'' \right) - \frac{h^2 p_i q_i'}{12\varepsilon} + q_i + \frac{h^2}{6}\left( \frac{p_i q_i'}{\varepsilon} - q_i'' \right) \right\} y_i \\
&\quad = f_i + \frac{h^2}{4}\delta_c^2 f_i + \frac{h^4}{72}\delta_c^2 f_i'' + p_i T_5\left( 1 + \frac{h^2}{12}\delta_c^2 \right) - \frac{h^2}{6} f_i'' + \frac{h^2 p_i}{12\varepsilon} f_i' - \frac{h^4 q_i}{72\varepsilon} f_i'' - \frac{h^4 p_i q_i}{72\varepsilon^2} f_i'
\end{aligned} \tag{4.22}$$

Introducing the fitting factor $(\sigma)$ into Eq. (4.22), we have:

$$\begin{aligned}
&\sigma\varepsilon\left\{ \left( 1 - \frac{h^2}{6\varepsilon}\left(2p_i'\right) + \frac{h^2 q_i}{12\varepsilon} + \frac{h^2 p_i^2}{12\varepsilon^2} - \frac{h^4 q_i}{72\varepsilon^2}\left( 2p_i' + q_i - \frac{p_i^2}{\varepsilon} \right) \right)\delta_c^2 y_i - \left( 1 + \frac{h^2}{6}\delta_c^2 \right) T_6 \right\} \\
&\quad + \left\{ -\frac{h^2 p_i}{12\varepsilon}(p_i' + q_i) + \frac{h^4 q_i}{72\varepsilon}\left( \frac{p_i(p_i' + q_i)}{\varepsilon} - p_i'' - 2q_i' \right) + \frac{h^2}{6}\left( \frac{p_i(p_i' + q_i)}{\varepsilon} - p_i'' - 2q_i' \right) + p_i \right\}\delta_c^1 y_i \\
&\quad + \left\{ \frac{h^4 q_i}{72\varepsilon}\left( \frac{p_i q_i'}{\varepsilon} - q_i'' \right) - \frac{h^2 p_i q_i'}{12\varepsilon} + q_i + \frac{h^2}{6}\left( \frac{p_i q_i'}{\varepsilon} - q_i'' \right) \right\} y_i = f_i + \frac{h^2}{4}\delta_c^2 f_i + \frac{h^4}{72}\delta_c^2 f_i'' \\
&\quad + p_i T_5\left( 1 + \frac{h^2}{12}\delta_c^2 \right) - \frac{h^2}{6} f_i'' + \frac{h^2 p_i}{12\varepsilon} f_i' - \frac{h^4 q_i}{72\varepsilon} f_i'' - \frac{h^4 p_i q_i}{72\varepsilon^2} f_i'
\end{aligned} \tag{4.23}$$

Using the central difference approximation for $\delta_c^2 y_i$ and $\delta_c^1 y_i$, we get:

$$\frac{\sigma\varepsilon}{h^2}\left\{\left(1-\frac{h^2}{6\varepsilon}(2p_i')+\frac{h^2q_i}{12\varepsilon}+\frac{h^2p_i^2}{12\varepsilon^2}-\frac{h^4q_i}{72\varepsilon^2}\left(2p_i'+q_i-\frac{p_i^2}{\varepsilon}\right)\right)(y_{i+1}-2y_i+y_{i-1})-h^2\left(1+\frac{h^2}{6}\delta_c^2\right)T_6\right\}$$
$$+\frac{1}{2h}\left\{\left(-\frac{h^2p_i}{12\varepsilon}(p_i'+q_i)+\frac{h^4q_i}{72\varepsilon}\left(\frac{p_i(p_i'+q_i)}{\varepsilon}-p_i''-2q_i'\right)+\frac{h^2}{6}\left(\frac{p_i(p_i'+q_i)}{\varepsilon}-p_i''-2q_i'\right)+p_i\right)(y_{i+1}-y_{i-1})\right\}$$
$$+\left\{\frac{h^4q_i}{72\varepsilon}\left(\frac{p_iq_i'}{\varepsilon}-q_i''\right)-\frac{h^2p_iq_i'}{12\varepsilon}+q_i+\frac{h^2}{6}\left(\frac{p_iq_i'}{\varepsilon}-q_i''\right)\right\}y_i=f_i+\frac{h^2}{4}\delta_c^2f_i+\frac{h^4}{72}\delta_c^2f_i''$$
$$+p_iT_5\left(1+\frac{h^2}{12}\delta_c^2\right)-\frac{h^2}{6}f_i''+\frac{h^2p_i}{12\varepsilon}f_i'-\frac{h^4q_i}{72\varepsilon}f_i''-\frac{h^4p_iq_i}{72\varepsilon^2}f_i'$$

Multiplying both sides of Eq. (4.23) by $h$ and taking the limit as $h\to 0$, we have:

$$\lim_{h\to 0}\frac{\sigma}{\rho}\left\{\left(1-\frac{h^2}{6\varepsilon}(2p_i')+\frac{h^2q_i}{12\varepsilon}+\frac{h^2p_i^2}{12\varepsilon^2}-\frac{h^4q_i}{72\varepsilon^2}\left(2p_i'+q_i-\frac{p_i^2}{\varepsilon}\right)\right)(y_{i+1}-2y_i+y_{i-1})-h^2\left(1+\frac{h^2}{6}\delta_c^2\right)T_6\right\}$$
$$+\frac{1}{2}\lim_{h\to 0}\left\{\left(-\frac{h^2p_i}{12\varepsilon}(p_i'+q_i)+\frac{h^4q_i}{72\varepsilon}\left(\frac{p_i(p_i'+q_i)}{\varepsilon}-p_i''-2q_i'\right)+\frac{h^2}{6}\left(\frac{p_i(p_i'+q_i)}{\varepsilon}-p_i''-2q_i'\right)+p_i\right)(y_{i+1}-y_{i-1})\right\}$$
$$+\lim_{h\to 0}\left\{\frac{h^5q_i}{72\varepsilon}\left(\frac{p_iq_i'}{\varepsilon}-q_i''\right)-\frac{h^3p_iq_i'}{12\varepsilon}+hq_i+\frac{h^3}{6}\left(\frac{p_iq_i'}{\varepsilon}-q_i''\right)\right\}y_i$$
$$=\lim_{h\to 0}h\left\{\left(1+\frac{h^2}{4}\delta_c^2+\frac{h^4}{72}\delta_c^4\right)f_i+p_iT_5\left(1+\frac{h^2}{12}\delta_c^2\right)-\frac{h^2}{6}f_i''-\frac{h^2p_i}{12\varepsilon}f_i'-\frac{h^4q_i}{72\varepsilon}f_i''\right\}$$

$$\Rightarrow\frac{\sigma}{\rho}\left\{\left(1+\frac{\rho^2p_i^2}{12}\right)\lim_{h\to 0}(y_{i+1}-2y_i+y_{i-1})\right\}+\frac{p_i}{2}\lim_{h\to 0}(y_{i+1}-y_{i-1})=0,\ \text{where}\ \rho=\frac{h}{\varepsilon}$$

$$\Rightarrow\frac{\sigma}{12\rho}(12+\rho^2p_i^2)\lim_{h\to 0}\{y_{i+1}-2y_i+y_{i-1}\}+\frac{p_i}{2}\lim_{h\to 0}(y_{i+1}-y_{i-1})=0 \tag{4.24}$$

From the theory of singular perturbations, we have two cases for $p(x)>0$ and $p(x)<0$, and is of the form in O'Malley, (1974).

**Case 1**: For $p(x)<0$ (right-end boundary layer), we have:

$$\lim_{h\to 0}(y_{i-1}-2y_i+y_{i+1})=\left\{(\alpha_0-y_0(0))e^{-p(0)\left(\frac{-1}{\varepsilon}+i\rho\right)}\left(e^{-p(0)\rho}+e^{p(0)\rho}-2\right)\right\}$$

$$\lim_{h\to 0}(y_{i+1}-y_{i-1})=\left\{(\alpha_0-y_0(0))e^{-p(0)\left(\frac{-1}{\varepsilon}+i\rho\right)}\left(e^{-p(0)\rho}-e^{p(0)\rho}\right)\right\}$$

Thus, from Eq. (4.24), we get:

$$\sigma(0)=\frac{-\frac{p(0)}{2}\left\{\left(\alpha_0-y_0(0)\right)e^{-p(0)\left(\frac{-1}{\varepsilon}+i\rho\right)}\left(e^{-p(0)\rho}-e^{p(0)\rho}\right)\right\}}{\frac{(12+\rho^2p^2(0)}{12\rho}\left\{\left(\alpha_0-y_0(0)\right)e^{-p(0)\left(\frac{-1}{\varepsilon}+i\rho\right)}\left(e^{-p(0)\rho}+e^{p(0)\rho}-2\right)\right\}}$$

$$\Rightarrow\sigma(0)=\frac{6\rho p_i\left(e^{p(0)\rho}-e^{-p(0)\rho}\right)}{(12+\rho^2p_i^2)\left(e^{-p(0)\rho}+e^{p(0)\rho}-2\right)}\ \text{, as}\ \begin{cases}e^x-e^{-x}=\left(e^{\frac{x}{2}}+e^{\frac{-x}{2}}\right)\left(e^{\frac{x}{2}}-e^{\frac{-x}{2}}\right)\\ e^x+e^x-2=\left(e^{\frac{x}{2}}-e^{\frac{-x}{2}}\right)^2\end{cases}$$

$$\Rightarrow\sigma(0)=\frac{6\rho p(0)\left(e^{\frac{p(0)\rho}{2}}+e^{\frac{-p(0)\rho}{2}}\right)}{(12+\rho^2p_i^2)\left(e^{\frac{p(0)\rho}{2}}-e^{\frac{-p(0)\rho}{2}}\right)}$$

$$\Rightarrow\sigma(0)=\frac{6\rho p(0)}{(12+\rho^2p^2(0)}\coth\left(\frac{p(0)\rho}{2}\right)$$

**Case 2**: For $p(x)>0$ (left-end boundary layer), we have:

$$\lim_{h\to0}\left(y_{i-1}-2y_i+y_{i+1}\right)=\left\{\left(\beta_0-y_0(1)\right)e^{-p(1)\left(\frac{-1}{\varepsilon}+i\rho\right)}\left(e^{-p(1)\rho}+e^{p(1)\rho}-2\right)\right\}$$

$$\lim_{h\to0}\left(y_{i+1}-y_{i-1}\right)=\left\{\left(\beta_0-y_0(1)\right)e^{-p(1)\left(\frac{-1}{\varepsilon}+i\rho\right)}\left(e^{-p(1)\rho}-e^{p(1)\rho}\right)\right\}$$

By following the same procedure in case 1 above, we obtain:

$$\sigma(1)=\frac{6\rho p(1)}{\left(12+\rho^2p^2(1)\right)}\coth\left(\frac{p(1)\rho}{2}\right)$$

In general, for discretization we take a variable fitting factor as:

$$\sigma_i=\frac{6\rho p_i}{\left(12+\rho^2p_i^2\right)}\coth\left(\frac{p_i\rho}{2}\right)\tag{4.25}$$

Now using Eqs. (4.9) and (4.10) into Eq. (4.23) for $\delta_c^1 y_i$ and $\delta_c^2 y_i$ and making use of

$\delta_c{}^2 f_i = \dfrac{f_{i+1} - 2f_i + f_{i-1}}{h^2}$ and $\delta_c{}^2 f_i'' = \dfrac{f_{i+1}'' - 2f_i'' + f_{i-1}''}{h^2}$, we obtain:

$$\begin{aligned}
&\left\{\sigma_i\left(\frac{\varepsilon}{h^2} - \frac{1}{6}(2p_i') + \frac{q_i}{12} + \frac{p_i^2}{12\varepsilon} - \frac{h^2 q_i}{72\varepsilon}\left(2p_i' + q_i - \frac{p_i^2}{\varepsilon}\right)\right)\right.\\
&\left.-\frac{1}{2}\left(-\frac{hp_i}{12\varepsilon}(p_i'+q_i) + \frac{h^3 q_i}{72\varepsilon}\left(\frac{p_i(p_i'+q_i)}{\varepsilon} - p_i'' - 2q_i'\right)_i + \frac{h}{6}\left(\frac{p_i(p_i'+q_i)}{\varepsilon} - p_i'' - 2q_i'\right) + \frac{p_i}{h}\right)\right\} y_{i-1}\\
&+\left\{-2\sigma_i\left(\frac{\varepsilon}{h^2} - \frac{1}{6}(2p_i') + \frac{q_i}{12} + \frac{p_i^2}{12\varepsilon} - \frac{h^2 q_i}{72\varepsilon}\left(2p_i' + q_i - \frac{p_i^2}{\varepsilon}\right)\right)\right.\\
&\left.+\frac{h^4 q_i}{72\varepsilon}\left(\frac{p_i q_i'}{\varepsilon} - q_i''\right) - \frac{h^2 p_i q_i'}{12\varepsilon} + q_i + \frac{h^2}{6}\left(\frac{p_i q_i'}{\varepsilon} - q_i''\right)\right\} y_i\\
&+\left\{\sigma_i\left(\frac{\varepsilon}{h^2} - \frac{1}{6}(2p_i') + \frac{q_i}{12} + \frac{p_i^2}{12\varepsilon} - \frac{h^2 q_i}{72\varepsilon}\left(2p_i' + q_i - \frac{p_i^2}{\varepsilon}\right)\right)\right.\\
&\left.+\frac{1}{2}\left(-\frac{hp_i}{12\varepsilon}(p_i'+q_i) + \frac{h^3 q_i}{72\varepsilon}\left(\frac{p_i(p_i'+q_i)}{\varepsilon} - p_i'' - 2q_i'\right) + \frac{h}{6}\left(\frac{p_i(p_i'+q_i)}{\varepsilon} - p_i'' - 2q_i'\right) + \frac{p_i}{h}\right)\right\} y_{i+1}
\end{aligned}$$

$$= f_i + \frac{1}{4}(f_{i+1} - 2f_i + f_{i-1}) + \frac{h^2}{72}(f_{i+1}'' - 2f_i'' + f_{i-1}'') - \frac{h^2}{6} f_i'' + \frac{h^2 p_i}{12\varepsilon} f_i' - \frac{h^4 q_i}{72\varepsilon} f_i'' - \frac{h^4 p_i q_i}{72\varepsilon^2} f_i' + T \quad (4.26)$$

where,

$T = p_i \dfrac{h^4}{45} y_i^{(5)} - \sigma_i \varepsilon \dfrac{h^4}{240} y_i^{(6)} + O(h^5)$ is the local truncation error.

From Eq. (4.26), we get the tri-diagonal system of equation of the form:

$$L^N \equiv E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \text{ for } i = 1, 2, \ldots, N-1 \quad (4.27)$$

where,

$$\begin{aligned}
E_i = \sigma_i &\left\{\frac{\varepsilon}{h^2} - \frac{1}{6}(2p_i') + \frac{q_i}{12} + \frac{p_i^2}{12\varepsilon} - \frac{h^2 q_i}{72\varepsilon}\left(2p_i' + q_i - \frac{p_i^2}{\varepsilon}\right)\right\}\\
&-\frac{1}{2}\left\{-\frac{hp_i}{12\varepsilon}(p_i'+q_i) + \frac{h^3 q_i}{72\varepsilon}\left(\frac{p_i(p_i'+q_i)}{\varepsilon} - p_i'' - 2q_i'\right)_i + \frac{h}{6}\left(\frac{p_i(p_i'+q_i)}{\varepsilon} - p_i'' - 2q_i'\right) + \frac{p_i}{h}\right\}
\end{aligned}$$

$$F_i = 2\sigma_i \left\{ \frac{\varepsilon}{h^2} - \frac{1}{6}(2p_i') + \frac{q_i}{12} + \frac{p_i^2}{12\varepsilon} - \frac{h^2 q_i}{72\varepsilon}\left(2p_i' + q_i - \frac{p_i^2}{\varepsilon}\right) \right\}$$

$$- \frac{h^4 q_i}{72\varepsilon}\left(\frac{p_i q_i'}{\varepsilon} - q_i''\right) + \frac{h^2 p_i q_i'}{12\varepsilon} - q_i - \frac{h^2}{6}\left(\frac{p_i q_i'}{\varepsilon} - q_i''\right)$$

$$G_i = \sigma_i \left\{ \frac{\varepsilon}{h^2} - \frac{1}{6}(2p_i') + \frac{q_i}{12} + \frac{p_i^2}{12\varepsilon} - \frac{h^2 q_i}{72\varepsilon}\left(2p_i' + q_i - \frac{p_i^2}{\varepsilon}\right) \right\}$$

$$+ \frac{1}{2}\left\{ -\frac{hp_i}{12\varepsilon}(p_i' + q_i) + \frac{h^3 q_i}{72\varepsilon}\left(\frac{p_i(p_i' + q_i)}{\varepsilon} - p_i'' - 2q_i'\right) + \frac{h}{6}\left(\frac{p_i(p_i' + q_i)}{\varepsilon} - p_i'' - 2q_i'\right) + \frac{p_i}{h} \right\}$$

$$H_i = f_i + \frac{1}{4}(f_{i+1} - 2f_i + f_{i-1}) + \frac{h^2}{72}(f_{i+1}'' - 2f_i'' + f_{i-1}'') - \frac{h^2}{6}f_i'' + \frac{h^2 p_i}{12\varepsilon}f_i' - \frac{h^4 q_i}{72\varepsilon}f_i'' - \frac{h^4 p_i q_i}{72\varepsilon^2}f_i'$$

The tri-diagonal system in Eq. (4.27) can be easily solved by the method of Discrete Invariant Imbedding Algorithm.

## 4.2. Stability and Convergence Analysis

### 4.2.1. Stability Analysis

The stability analysis is followed the approach of Gashu et al., (2017).

**Case 1:** When $q(x) < 0$, *i.e.* $b(x) + c(x) + d(x) < 0$, for $x \in (0,1)$.

**Lemma 4.1**: If $y(0) \geq 0$ and $Ly(x) \leq 0$, for all $x \in (0,1)$, then the solution $y(x) \geq 0$ for all $x \in (0,1)$ for Eqs. (4.5) and (4.6).

**Proof**

Suppose $t \in (0,1)$, such that $y(t) = \min\limits_{x \in (0,1)} y(x)$ and $y(t) < 0$. Since, $t \notin \{0,1\}$ and is a point of minima, then $y'(t) = 0$ and $y''(t) \geq 0$.

Therefore, we have:

$$Ly(t) \equiv \varepsilon y''(t) + p(t)y'(t) + q(t)y(t) > 0, \text{ since } y(t) < 0 \text{ (by assumption) and } q(t) < 0.$$

But, this is a contradiction. Then, it follows that $y(t) \geq 0$ and therefore, $y(x) \geq 0$ for all $x \in (0,1)$.

**Theorem 4.1:** If the solution of the problem in Eqs. (4.5) and (4.6) satisfies $\left|y(x)\right| \le C\max\left\{\left|y(0)\right|, \max_{x\in(0,1)}\left|Ly(x)\right|\right\}$ for some constant $C \ge 1$, then the solution is stable.

**Proof:**

We define two functions, $\psi^{\pm} = C\max\left\{\left|y(0)\right|, \max_{x\in(0,1)}\left|Ly(x)\right|\right\} \pm y(x)$, then we get:

$\psi^{\pm}(0) \ge 0$ and

$L\psi^{\pm}(x) \equiv Cq_i \max\left\{\left|y(0)\right|, \max_{x\in(0,1)}\left|Ly(x)\right|\right\} \pm Ly(x) \le 0,$ since $q(x) < 0$ and for suitable choice of C.

Therefore, by Lemma 4.1, we get, $\psi^{\pm}(x) \ge 0,$ for all $x \in (0,1)$. So,

$$\left|y(x)\right| \le C\max\left\{\left|y(0)\right|, \max_{x\in(0,1)}\left|Ly(x)\right|\right\}.$$

Hence, the stability of the solutions of the problem in Eqs. (4.5) and (4.6) is proved for the case of boundary layer behaviour.

**Lemma 4.2**: The finite difference operator $L^N$ in Eq. (4.27) satisfies the discrete minimum principle, *i.e.*, if $w_i$ is any mesh function such that $w_o \ge 0$ and $L^N w_i \le 0$, for all $x_i \in (0,1)$, then $w_i \ge 0$ for all $x \in (0,1)$.

**Proof**

Suppose there exists a positive integer k such that $w_k < 0$ and $w_k = \min_{0\le i\le N} w_i$.

Then, from Eq. (4.27), we have:

$$L^N w_k \equiv E_k w_{k-1} - F_k w_k + G_k w_{k+1}$$

$$= \left\{\left(\frac{\sigma_k \varepsilon}{h^2} + \frac{p_k^2}{12\varepsilon} + \frac{q_k}{12} - \frac{p_k'}{3}\right) + \frac{h^2 q_k}{72\varepsilon}\left(\frac{p_k^2}{\varepsilon} - 2p_k' - q_k\right)\right\}(w_{k-1} - w_k)$$
$$+ \left\{\left(\frac{\sigma_k \varepsilon}{h^2} + \frac{p_k^2}{12\varepsilon} + \frac{q_k}{12} - \frac{p_k'}{3}\right) + \frac{h^2 q_k}{72\varepsilon}\left(\frac{p_k^2}{\varepsilon} - 2p_k' - q_k\right)\right\}(w_{k+1} - w_k)$$
$$+ \left\{\frac{p_k}{h} + \frac{hp_k\left(p_k' + q_k\right)}{12\varepsilon} - \frac{hp_k''}{6} - \frac{hq_k'}{3} + \frac{h^3 q_k}{72\varepsilon}\left(\frac{p_k\left(p_k' + q_k\right)}{\varepsilon} - p_k'' - 2q_k'\right)\right\}(w_{k+1} - w_{k-1}) + A_k w_k$$

where,

$$A_k = \frac{h^2}{6}\left(\frac{p_k q_k'}{\varepsilon} - q_k''\right) - \frac{h^2 p_k q_k'}{12\varepsilon} + q_k + \frac{h^4 q_k}{72\varepsilon}\left(\frac{p_k q_k'}{\varepsilon} - q_k''\right)$$

For sufficiently small $h$ $(i.e.\ h \to 0)$ and for suitable value of $p_k$, we obtain:

$L^N w_k > 0$. Since, $w_k < 0$ (by assumption), $\varepsilon, \sigma_k > 0$ and $A_k \to q_k < 0$.

But, this is a contradiction. Hence, $w_i \geq 0$ for all $x_i \in (0,1)$.

**Theorem 4.2**: The finite difference operator $L^N$ in Eq. (4.27) is stable for $b(x)+c(x)+d(x)<0,$ if $w_i$ is any mesh function, then $|w_i| \leq C \max\left\{|w_0|, \max_{x_i \in (0,1)} |Lw_i|\right\}$, for some constant $C \geq 1$.

**Proof:**

We define two functions, $\psi_i^{\pm} \equiv C \max\left\{|w_0|, \max_{x_i \in (0,1)} |Lw_i|\right\} \pm w_i$, then similar to Theorem 4.1, we get:

$\psi_0^{\pm} \geq 0$ and

$L\psi_i^{\pm} \equiv Cq_i \max\left\{|w_0|, \max_{x_i \in (0,1)} |Lw_i|\right\} \pm Lw_i \leq 0,$ since $b_i + c_i + d_i < 0 \Rightarrow q_i < 0$ and $C \geq 1$.

Therefore by Lemma 4.2, we get:

$\psi_0^{\pm} \geq 0$ for all $x_i \in (0,1)$.

$\Rightarrow \psi_i^{\pm} \equiv C \max\left\{|w_0|, \max_{x_i \in (0,1)} |Lw_i|\right\} \pm w_i \geq 0.$

Thus, $|w_i| \leq C \max\left\{|w_0|, \max_{x_i \in (0,1)} |Lw_i|\right\}$.

This proves the stability of the scheme for the case of boundary layer behaviour.

**Case 2**: When $q(x) > 0$, *i.e.* $b(x)+c(x)+d(x)>0$, for $x \in (0,1)$.

The continuous maximum principle and stability of the solution of Eqs. (4.5) and (4.6) are presented as follows for the case of oscillatory behaviour.

**Lemma 4.3:** If $y(0) \geq 0$ and $Ly(x) \geq 0$, for all $x \in (0,1)$, then the solution $y(x) \geq 0$ for all $x \in (0,1)$ for Eqs. (4.5) and (4.6).

**Proof**

Suppose $t \in (0,1)$, such that $y(t) = \max\limits_{x\in(0,1)} y(x)$ and $y(t) < 0$. Since, $t \notin \{0,1\}$ and is a point of maxima, then $y'(t) = 0$ and $y''(t) \le 0$.

Therefore, we have:

$$Ly(t) \equiv \varepsilon y''(t) + p(t)y'(t) + q(t)y(t) < 0, \text{ since } y(t) < 0 \text{ (by assumption) and } q(t) > 0.$$

But, this is a contradiction. Then, it follows that $y(t) \ge 0$ and therefore, $y(x) \ge 0$ for all $x \in (0,1)$.

**Theorem 4.3:** If the solution of the problem in Eqs. (4.5) and (4.6) satisfies

$\left|y(x)\right| \le k \max\left\{\left|y(0)\right|, \max\limits_{x\in(0,1)}\left|Ly(x)\right|\right\}$, for some constant $k \ge 1$, then the solution is stable.

**Proof**: The proof is analogous to Theorem 4.1.

Hence, the stability of the solutions of the problem in Eqs. (4.5) and (4.6) is proved for the case of oscillatory behaviour.

Now, we present the stability of the discrete problem in Eq. (4.27) for the case of oscillatory behaviour.

**Lemma 4. 4:** The finite difference operator $L^N$ in Eq. (4.27) satisfies the discrete maximum principle, *i.e.*, if $w_i$ is any mesh function such that $w_o \ge 0$ and $L^N w_i \ge 0$, for all $x_i \in (0,1)$, then $w_i \ge 0$ for all $x \in (0,1)$.

**Proof**

Suppose there exists a positive integer *k* such that $w_k < 0$ and $w_k = \max\limits_{0\le i\le N} w_i$.

Then, from Eq. (4.27), we have:

$$L^N w_k \equiv E_k w_{k-1} - F_k w_k + G_k w_{k+1}$$

$$
\begin{aligned}
&=\left\{\left(\frac{\sigma_k\varepsilon}{h^2}+\frac{p_k^2}{12\varepsilon}+\frac{q_k}{12}-\frac{p_k'}{3}\right)+\frac{h^2q_k}{72\varepsilon}\left(\frac{p_k^2}{\varepsilon}-2p_k'-q_k\right)\right\}\left(w_{k-1}-w_k\right)\\
&+\left\{\left(\frac{\sigma_k\varepsilon}{h^2}+\frac{p_k^2}{12\varepsilon}+\frac{q_k}{12}-\frac{p_k'}{3}\right)+\frac{h^2q_k}{72\varepsilon}\left(\frac{p_k^2}{\varepsilon}-2p_k'-q_k\right)\right\}\left(w_{k+1}-w_k\right)\\
&+\left\{\frac{p_k}{h}+\frac{hp_k\left(p_k'+q_k\right)}{12\varepsilon}-\frac{hp_k''}{6}-\frac{hq_k'}{3}+\frac{h^3q_k}{72\varepsilon}\left(\frac{p_k\left(p_k'+q_k\right)}{\varepsilon}-p_k''-2q_k'\right)\right\}\left(w_{k+1}-w_{k-1}\right)+A_kw_k
\end{aligned}
$$

For sufficiently small $h$ and for suitable value of $p_k$, we obtain:

$L^N w_k<0$. Since, $w_k<0$ (by assumption), $\varepsilon, \sigma_k>0$ and $A_k\to q_k>0$.

But, this is a contradiction. Hence, $w_i\geq 0$ for all $x_i\in(0,1)$.

**Theorem 4.4**: The finite difference operator $L^N$ in Eq. (4.27) is stable for $b(x)+c(x)+d(x)>0,\quad (i.e.\ q(x)>0)$, if $w_i$ is any mesh function, then $|w_i|\leq k\max\left\{|w_0|, \max_{x_i\in(0,1)}|Lw_i|\right\}$, for some constant $k\geq 1$.

**Proof**: The proof is similar to Theorem 4. 2.

This proves the stability of the scheme for the case of oscillatory behaviour.

#### 4.2.2. Convergence Analysis

Writing the tri-diagonal system (4.27) in matrix vector form, we get:

$$AY = D \tag{4.28}$$

where $A=\left(m_{i\,j}\right),\ i, j=1,2,\ ...,\ N\text{-}1$ is a tri-diagonal matrix of order $N\text{-}1$ from Eq. (4.27).

Multiplying both sides of Eq. (4.27) by $\left(-h^2\right)$ we get:

$$m_{i\,i+1} = -\sigma_i\left(\varepsilon - \frac{h^2}{6}\left(2p_i'\right) + \frac{h^2p_i^2}{12\varepsilon} + \frac{h^2q_i}{12} - \frac{h^4q_i}{72\varepsilon}\left(2p_i' + q_i - \frac{p_i^2}{\varepsilon}\right)\right)$$
$$-\frac{h}{2}\left(p_i + \frac{h^2}{6}\left(\frac{p_i(p_i'+q_i)}{\varepsilon} - p_i'' - 2q_i'\right) - \frac{h^2p_i(p_i'+q_i)}{12\varepsilon} + \frac{h^4q_i}{72\varepsilon}\left(\frac{p_i(p_i'+q_i)}{\varepsilon} - p_i'' - 2q'\right)\right)$$

$$m_{i\,i} = 2\sigma_i\left(\varepsilon - \frac{h^2}{6}\left(2p_i'\right) + \frac{h^2p_i^2}{12\varepsilon} + \frac{h^2q_i}{12} - \frac{h^4q_i}{72\varepsilon}\left(2p_i' + q_i - \frac{p_i^2}{\varepsilon}\right)\right)$$
$$-h^2\left(\frac{h^2}{6}\left(\frac{p_iq_i'}{\varepsilon} - q_i''\right) - \frac{h^2p_iq_i}{12\varepsilon} + q_i + \frac{h^4q_i}{72\varepsilon}\left(\frac{p_iq_i'}{\varepsilon} - q_i''\right)\right)$$

$$m_{i\,i-1} = -\sigma_i\left(\varepsilon - \frac{h^2}{6}\left(2p'\right) + \frac{h^2p_i^2}{12\varepsilon} + \frac{h^2q_i}{12} - \frac{h^4q_i}{72\varepsilon}\left(2p_i' + q_i - \frac{p_i^2}{\varepsilon}\right)\right)$$
$$+\frac{h}{2}\left(p_i + \frac{h}{6}\left(\frac{p_i(p_i'+q_i)}{\varepsilon} - p_i'' - 2q_i'\right) - \frac{h^2p_i(p_i'+q)}{12\varepsilon} + \frac{h^4q_i}{72\varepsilon}\left(\frac{p_i(p_i'+q_i)}{\varepsilon} - p_i'' - 2q'\right)\right)$$

and $D=\left(d_i\right)$ is a column vector, where,

$$d_1 = -h^2\left(H_1 - E_1\alpha_0\right)$$

$$d_i = -h^2H_i \quad \text{for } i = 2,3,\dots,N-2$$

$$d_{N-1} = -h^2\left(H_{N-1} - G_{N-1}\beta_0\right)$$

with a local truncation error:

$$T_i\left(h\right) = \frac{h^6}{15}K + O\left(h^7\right) \tag{4.29}$$

where $K = \dfrac{p_i}{3}y_i^{(5)} - \dfrac{\sigma_i\varepsilon}{16}y_i^{(6)}$.

We also have

$$A\overline{Y}-T(h)=D \tag{4.30}$$

where $\overline{Y}=\left(\overline{y}_0,\overline{y}_1,\overline{y}_2,\ \dots,\ \overline{y}_N\right)^T$ denotes the actual solution and

$T(h)=\left(T_0(h),T_1(h),T_2(h),T_3(h),\dots,T_N(h)\right)^T$ is the local truncation error.

From Eqs. (4.28) and (4.30), we get:

$$A\left(\overline{Y}-Y\right)=T(h) \tag{4.31}$$

Thus we obtain the error equation

$$AE=T(h) \tag{4.32}$$

where $E=\overline{Y}-Y=\left(e_0,e_1,e_2,\dots,e_N\right)^T$.

Let $S_i$ be the sum of elements of the $i^{th}$ row of A, then we get:

$$S_i=\sigma_i\varepsilon+h\left(-\frac{p_i}{2}\right)+h^2\left(-\frac{\sigma}{6}\left(2p_i'\right)+\frac{\sigma p_i^2}{12\varepsilon}+\frac{\sigma q_i}{12}-q_i\right)+\frac{h^3}{6}\left(p_i''+2q_i'-\frac{h^3p_i(p_i'+q_i)}{2\varepsilon}\right)$$
$$+O\left(h^4\right)\quad \text{for, } i=1$$
$$S_i=h^2\left(-q_i\right)+O\left(h^4\right)\text{ for, } i=2,3,\dots,N-2$$

$$S_i=\sigma_i\varepsilon+h\left(\frac{p_i}{2}\right)+h^2\left\{-\frac{\sigma}{6}\left(2p_i'\right)+\frac{\sigma p_i^2}{12\varepsilon}+\frac{\sigma q_i}{12}+\frac{1}{12}\left(\frac{p_i(p_i'+q_i)}{\varepsilon}-p_i''-2q_i'\right)-q_i\right\}$$
$$+h^3\left(-\frac{p_i(p_i'+q_i)}{24\varepsilon}\right)+O\left(h^4\right)\quad \text{for, } i=N-1$$

For sufficiently small *h*, the matrix A is irreducible and monotone (Gemechis et al., 2017). Then it follows that $A^{-1}$ exists and its elements are non-negative.

Hence from Eq. (4.32), we get:

$$E=A^{-1}T(h) \tag{4.33}$$

and

$$\|E\|\le\left\|A^{-1}\right\|.\|T(h)\| \tag{4.34}$$

Let $\overline{m}_{k,i}$ be the $(k,i)^{th}$ element of $A^{-1}$. Since $\overline{m}_{k,i}\ge 0$, from the theory of matrices we have:

$$\sum_{i=1}^{N-1}\overline{m}_{k,i}\,S_i=1,\qquad k=1,2,\dots,N-1 \tag{4.35}$$

Therefore,

$$\sum_{i=1}^{N-1}\bar{m}_{k,i} \le \frac{1}{\min\limits_{1\le i\le N-1} S_i} = \frac{1}{h^2\left|B_{i_0}\right|} \tag{4.36}$$

where $B_{i_0} = -q_i$

We define $\left\|A^{-1}\right\| = \max\limits_{1\le k\le N-1}\sum\limits_{i=1}^{N-1}\left|\bar{m}_{k,i}\right|$ and $\left\|T(h)\right\| = \max\limits_{1\le i\le N-1}\left|T(h)\right|$

From Eqs.(4.28), (4.33), (4.34) and (4.36), we get:

$$e_i = \sum_{k=1}^{N-1}\bar{m}_{k,i}\, T_i(h)\,, \quad i = 1, 2, \ldots, N-1 \tag{4.37}$$

which implies

$$e_i \le \left(\sum_{k=1}^{N-1}\bar{m}_{k,i}\right)\max_{1\le i\le N-1}\left|T(h)\right| \le \frac{h^6 K}{15h^2\left|B_{i_0}\right|} = \frac{h^4 K}{15\left|B_{i_0}\right|} \tag{4.38}$$

where $i_0$ is some number between $i$ and $N$.

Therefore, $\left\|E\right\| = O\left(h^4\right)$.

Hence our method gives a fourth order convergence for uniform mesh.

### 4.3. Numerical Examples

To demonstrate the applicability of the method, three model examples having constant and variable coefficients with left-end and right-end boundary layers have been carried out. The exact solution of singularly perturbed differential-difference equation of Eqs. (4.1) and (4.2), with constant coefficients is given by:

$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \frac{f}{c_3} \tag{4.39}$$

where

$$m_1 = \frac{-(a - \delta b + \eta d) + \sqrt{\left(-(a - \delta b + \eta d)\right)^2 - 4\varepsilon c_3}}{2\varepsilon}$$

$$m_2 = \frac{-(a - \delta b + \eta d) - \sqrt{\left(-(a - \delta b + \eta d)\right)^2 - 4\varepsilon c_3}}{2\varepsilon}$$

$$c_3 = b + c + d\ ,$$

$$c_1 = \frac{\left(-f + \beta c_3 + e^{m_2}\left(f - \alpha c_3\right)\right)}{c_3\left(e^{m_1} - e^{m_2}\right)}, \qquad c_2 = \frac{\left(f - \beta c_3 + e^{m_1}\left(-f + \alpha c_3\right)\right)}{c_3\left(e^{m_1} - e^{m_2}\right)}$$

For the variable coefficients, the maximum absolute errors are computed using double mesh principle given by:

$$z_h = \max_i \left| y_i^h - y_i^{h/2} \right|, \quad i = 1, 2, ..., N-1 \tag{4.40}$$

where, $y_i^h$ is the numerical solution on the mesh $\{x_i\}_1^{N-1}$ at the nodal point $x_i$ and $x_i = x_0 + ih,\ i = 1, 2, ..., N-1$ and $y_i^{h/2}$ is the numerical solution on a mesh, obtained by bisecting the original mesh with N number of mesh intervals, (Doolan et al.,1980).

**Example 4.1:** Consider the singularly perturbed differential-difference equation with right end boundary layer (Swamy et al., 2016)

$$\varepsilon y''(x) - y'(x) - 2y(x-\delta) + y(x) - 2y(x+\eta) = 0$$

subject to the interval and boundary conditions,

$$y(x) = 1,\ -\delta \le x \le 0, \quad y(1) = -1,\ 1 \le x \le \eta.$$

The maximum absolute errors are presented in Tables 4.1 and 4.4 for different values of $\varepsilon$ and $\delta$. The graph of the computed solution for $\varepsilon = 0.1$ and different values of $\delta$ and $\eta$ is also given in Figure 4.1.

**Example 4.2:** Consider the singularly perturbed differential-difference equation with left end boundary layer (Swamy et al., 2016)

$$\varepsilon y''(x) + 0.5y'(x) - 3y(x-\delta) - 2y(x) + 2y(x+\eta) = 1$$

subject to the interval and boundary conditions,

$$y(x) = 1, \ -\delta \le x \le 0, \quad y(1) = 0, 1 \le x \le \eta.$$

The maximum absolute errors are presented in Tables 4.2 and 4.4 for different values of $\varepsilon$ and $\delta$. The graph of the computed solution for $\varepsilon = 0.1$ and different values of $\delta$ and $\eta$ is also given in Figure 4.2.

**Example 4.3:** Consider the singularly perturbed differential-difference equation with right end boundary layer (Gemechis and Reddy, 2012)

$$\varepsilon y''(x) - \left(1 + e^{x^2}\right) y'(x) - xy(x-\delta) + x^2 y(x) - \left(1 - e^{-x}\right) y(x+\eta) = 1$$

subject to the interval and boundary conditions,

$$y(x) = 1, \ -\delta \le x \le 0, \quad y(1) = -1, \ 1 \le x \le \eta.$$

The maximum absolute errors are presented in Table 4.3 and 4.4 for different values of $\varepsilon$ and $\delta$. The graph of the computed solution for $\varepsilon = 0.1$ and different values of $\delta$ and $\eta$ is also given in Figure 4.3.

## 4.4. Numerical Results

**Table 4.1:** Maximum Absolute errors of **Example 4.1** for different values of δ, η and ε = 0.1.

| $N$ | 8 | 32 | 128 | 512 |
|---|---|---|---|---|
| **Fitted FD (Present Method)** | | $\eta = 0.5\varepsilon$ | | |
| $\delta \downarrow$ | | | | |
| 0.00 | 4.3229e-03 | 1.5775e-05 | 6.1456e-08 | 2.4006e-10 |
| 0.05 | 3.8440e-03 | 1.3769e-05 | 5.4036e-08 | 2.1092e-10 |
| 0.09 | 3.4760e-03 | 1.2460e-05 | 4.8494e-08 | 1.8940e-10 |
| **Swamy et al., (2016)** | | | | |
| 0.00 | 0.031377538 | 0.001800241 | 0.000112071 | 7.0036e−06 |
| 0.05 | 0.029748010 | 0.001700026 | 0.000105418 | 6.5860e−06 |
| 0.09 | 0.028294285 | 0.001611053 | 9.9793e−05 | 6.2344e−06 |
| **Fitted FD (Present Method)** | | $\delta = 0.5\varepsilon$ | | |
| $\eta \downarrow$ | | | | |
| 0.00 | 3.3862e-03 | 1.2139e-05 | 4.7199e-08 | 1.8429e-10 |
| 0.05 | 3.8440e-03 | 1.3769e-05 | 5.4036e-08 | 2.1092e-10 |
| 0.09 | 4.2256e-03 | 1.5339e-05 | 5.9891e-08 | 2.3403e-10 |
| **Swamy et al., (2016)** | | | | |
| 0.00 | 0.027910529 | 0.001587651 | 9.8361e−05 | 6.1442e−06 |
| 0.05 | 0.029748010 | 0.001700026 | 0.000105418 | 6.5860e−06 |
| 0.09 | 0.031068500 | 0.001781207 | 0.000110800 | 6.9223e−06 |

**Table 4.2:** Maximum Absolute errors **Example 4.2** for different values of δ, η and ε = 0.1.

| $N$ | 8 | 32 | 128 | 512 |
|---|---|---|---|---|
| **Fitted FD (Present Method)** | | $\eta = 0.5\varepsilon$ | | |
| $\delta \downarrow$ | | | | |
| 0.00 | 2.9005e-03 | 1.0342e-05 | 4.0567e-08 | 1.5841e-10 |
| 0.05 | 3.5885e-03 | 1.2831e-05 | 4.9745e-08 | 1.9433e-10 |
| 0.09 | 4.1815e-03 | 1.4979e-05 | 5.8027e-08 | 2.2664e-10 |
| **Swamy et al., (2016)** | | | | |
| 0.00 | 0.025347510 | 0.001425327 | 8.9204$e$ −05 | 5.5742$e$ −06 |
| 0.05 | 0.027533826 | 0.001567710 | 9.7155$e$ −05 | 6.0690$e$ −06 |
| 0.09 | 0.028669770 | 0.001645550 | 0.000102186 | 6.3826$e$ −06 |
| **Fitted FD (Present Method)** | | $\delta = 0.5\varepsilon$ | | |
| $\eta \downarrow$ | | | | |
| 0.00 | 1.7013e-03 | 1.1139e-05 | 4.3477e-08 | 1.6984e-10 |
| 0.05 | 3.5885e-03 | 1.2831e-05 | 4.9745e-08 | 1.9433e-10 |
| 0.09 | 3.9801e-03 | 1.4251e-05 | 5.5183e-08 | 2.1551e-10 |
| **Swamy et al., (2016)** | | | | |
| 0.00 | 0.026174618 | 0.001478341 | 9.2083$e$ −05 | 5.7527$e$ −06 |
| 0.05 | 0.027533826 | 0.001567710 | 9.7155$e$ −05 | 6.0690$e$ −06 |
| 0.09 | 0.028348272 | 0.001623113 | 0.00010057 | 6.2854$e$ −06 |

**Table 4.3:** Maximum Absolute errors **Example 4.3** for different values of δ, η and ε = 0.1.

| $\underrightarrow{N}$ | 8 | 32 | 128 | 512 |
|---|---|---|---|---|
| | $\eta = 0.5\varepsilon$ | | | |
| $\delta\downarrow$ | | | | |
| 0.00 | 9.1099e-02 | 1.1121e-02 | 6.3825e-04 | 4.0044e-05 |
| 0.05 | 9.0471e-02 | 1.0955e-02 | 6.3063e-04 | 3.9502e-05 |
| 0.09 | 8.9962e-02 | 1.0822e-02 | 6.2443e-04 | 3.9063e-05 |
| | $\delta = 0.5\varepsilon$ | | | |
| $\eta\downarrow$ | | | | |
| 0.00 | 9.6047e-02 | 1.1165e-02 | 6.4582e-04 | 3.9245e-05 |
| 0.05 | 9.6212e-02 | 1.1248e-02 | 6.4941e-04 | 3.9502e-05 |
| 0.09 | 9.6342e-02 | 1.1313e-02 | 6.5223e-04 | 3.9705e-05 |

**Table 4.4:** Maximum Absolute errors for different values of ε and $h$ for δ, η = 0.5ε.

| $\varepsilon/h$ | $2^{-5}$ | $2^{-6}$ | $2^{-7}$ | $2^{-8}$ | $2^{-9}$ |
|---|---|---|---|---|---|
| **Example 4.1** | | | | | |
| $2^{-3}$ | 7.8680e-06 | 4.8845e-07 | 3.0476e-08 | 1.9040e-09 | 1.1899e-10 |
| $2^{-4}$ | 4.8096e-05 | 2.9675e-06 | 1.8546e-07 | 1.1578e-08 | 7.2378e-10 |
| $2^{-5}$ | 3.4554e-04 | 1.9965e-05 | 1.2232e-06 | 7.6501e-08 | 4.7753e-09 |
| $2^{-6}$ | 2.1829e-03 | 1.5228e-04 | 8.8556e-06 | 5.4340e-07 | 3.3824e-08 |
| $2^{-7}$ | 9.4645e-03 | 1.0205e-03 | 7.0761e-05 | 4.1300e-06 | 2.5365e-07 |
| $2^{-8}$ | 2.8510e-02 | 4.4918e-03 | 4.9169e-04 | 3.4000e-05 | 1.9883e-06 |
| **Example 4.2** | | | | | |
| $2^{-3}$ | 7.8654e-06 | 8.8695e-07 | 3.0790e-08 | 3.4972e-09 | 1.2029e-10 |
| $2^{-4}$ | 3.6536e-05 | 2.2720e-06 | 1.4204e-07 | 8.8711e-09 | 5.5450e-10 |
| $2^{-5}$ | 1.8558e-04 | 1.1830e-05 | 7.3360e-07 | 4.5832e-08 | 2.8636e-09 |
| $2^{-6}$ | 1.2043e-03 | 6.8432e-05 | 4.2579e-06 | 2.6466e-07 | 1.6561e-08 |
| $2^{-7}$ | 6.7569e-03 | 4.7969e-04 | 2.7698e-05 | 1.6967e-06 | 1.0622e-07 |
| $2^{-8}$ | 2.8776e-02 | 2.9911e-03 | 2.0743e-04 | 1.2083e-05 | 7.4178e-07 |
| **Example 4.3** | | | | | |
| $2^{-3}$ | 8.3545e-03 | 2.0138e-03 | 4.9861e-04 | 1.2495e-04 | 3.1217e-05 |
| $2^{-4}$ | 1.7199e-02 | 4.3785e-03 | 1.0417e-03 | 2.5713e-04 | 6.4294e-05 |
| $2^{-5}$ | 2.5179e-02 | 8.8894e-03 | 2.2383e-03 | 5.2902e-04 | 1.3037e-04 |
| $2^{-6}$ | 3.1540e-02 | 1.2943e-02 | 4.5167e-03 | 1.1313e-03 | 2.6648e-04 |
| $2^{-7}$ | 4.4783e-02 | 1.6224e-02 | 6.5594e-03 | 2.2763e-03 | 5.6865e-04 |
| $2^{-8}$ | 7.8783e-02 | 2.3176e-02 | 8.2240e-03 | 3.3015e-03 | 1.1426e-03 |

### The effect of delay and advance parameters on boundary layers

The following graphs (Figure 4.1-Figure 4.3) show the numerical solution obtained by the present method for different values of delay and advance $(\delta$ and $\eta)$ parameters.

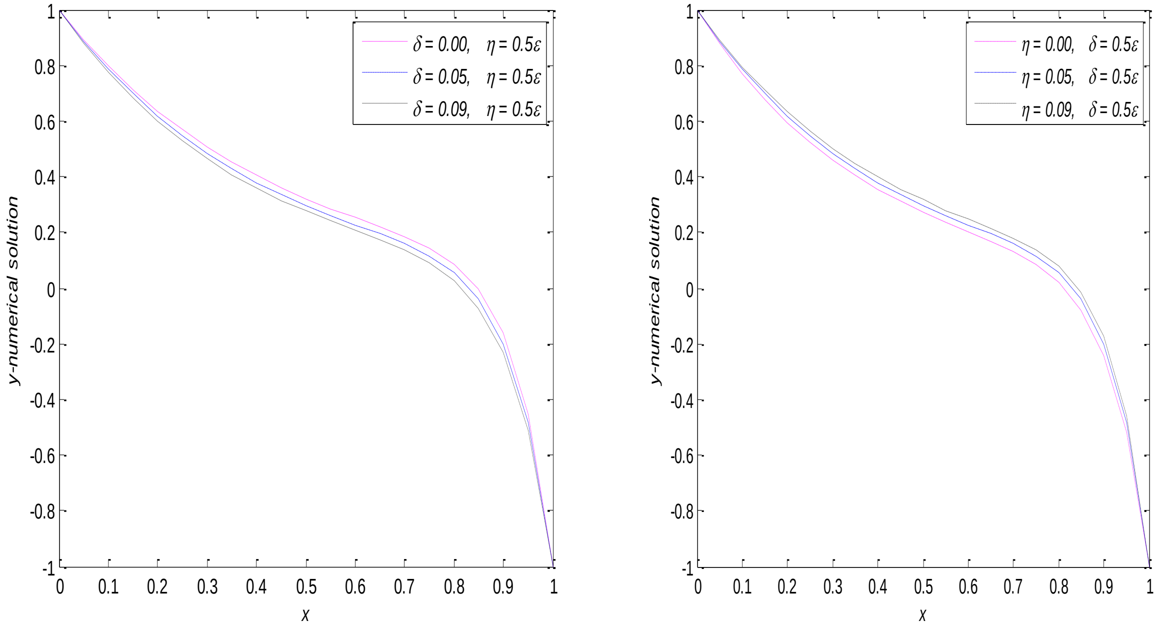


**Figure 4.1:** Numerical solution of **Example 4.1** for ε = 0.1 and N = 20.

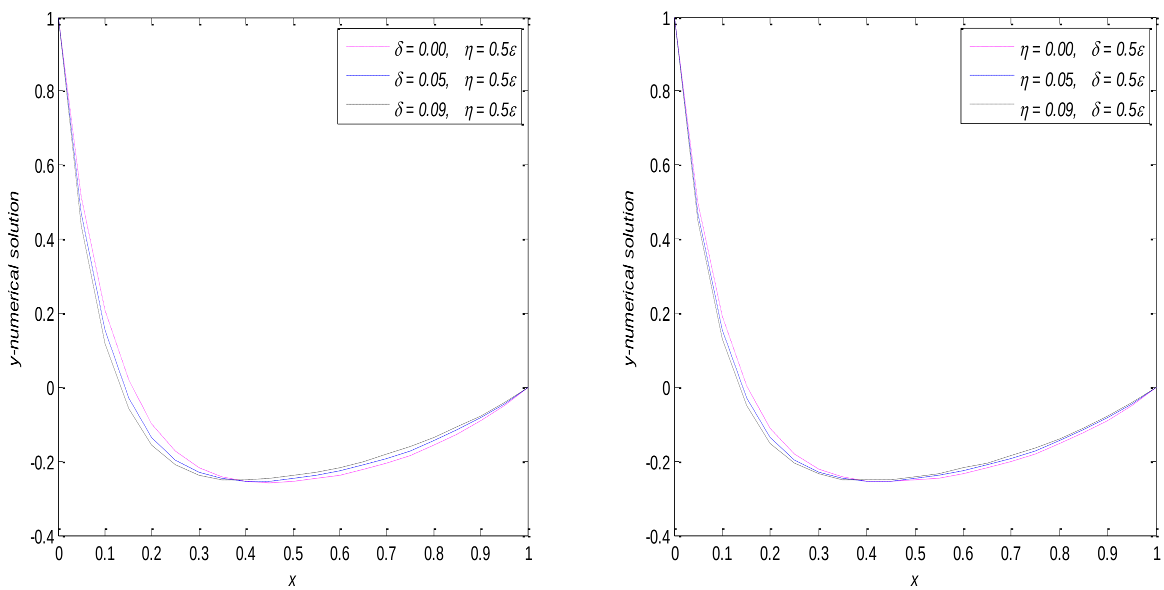


**Figure 4.2:** Numerical solution of **Example 4.2** for ε = 0.1 and N = 20.

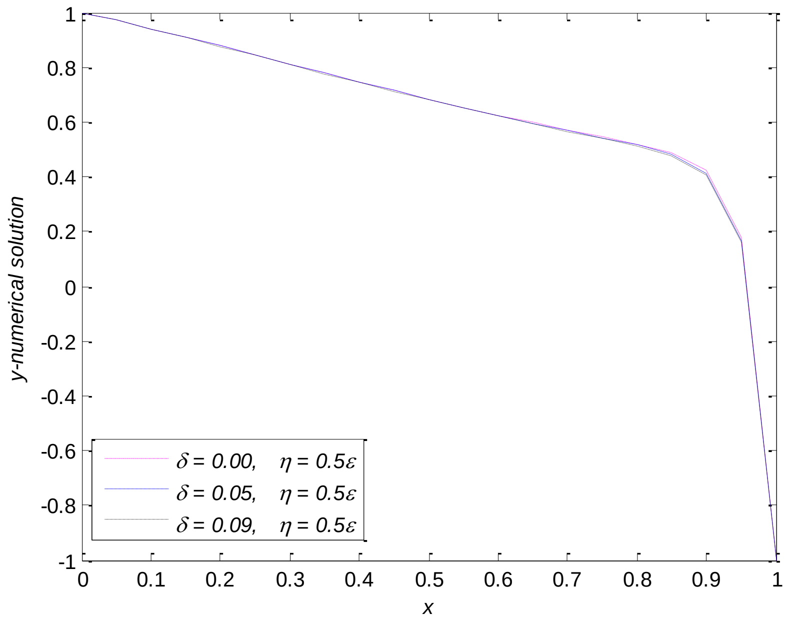


**Figure 4.3:** Numerical solution of **Example 4.3** for ε = 0.1 and N = 20.

The following graphs (Figure 4.4-Figure 4.6) show the pointwise absolute errors obtained by the present method for different values of mesh size $h$.

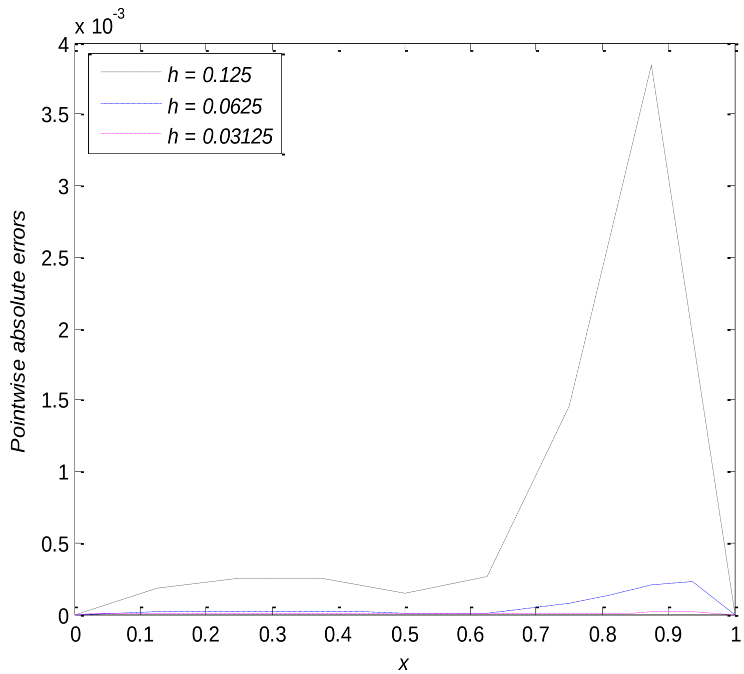


**Figure 4.4:** Pointwise absolute errors of **Example 4.1** for different value of $h$ with ε = 0.1 and δ, η = 0.5ε.

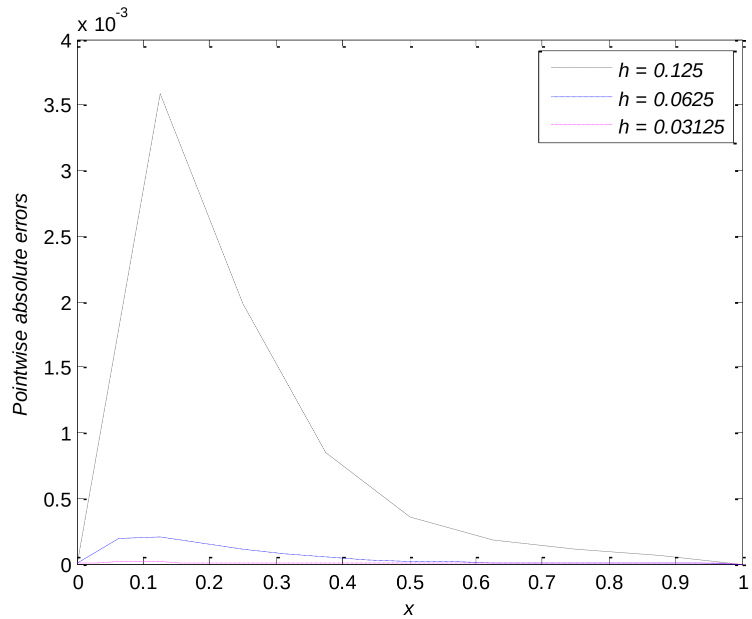


**Figure 4.5:** Pointwise absolute errors of **Example 4.2** for different value of $h$ with $\varepsilon = 0.1$ and $\delta, \eta = 0.5\varepsilon$.

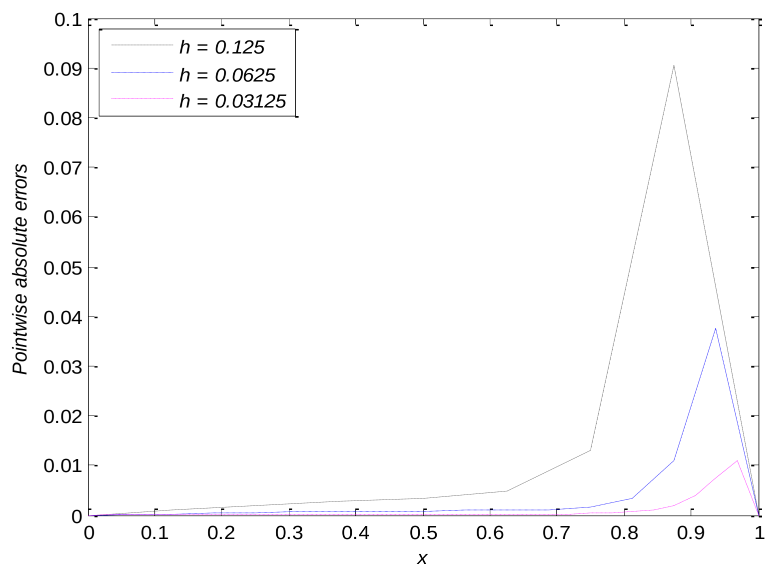


**Figure 4.6:** Pointwise absolute errors of **Example 4.3** for different value of $h$ with $\varepsilon = 0.1$ and $\delta, \eta = 0.5\varepsilon$.

**Rate of Convergence**

In some way in Eq. (4.39) one can define $z_{h/2}$ by replacing $h$ by $h/2$ and $N-1$ by $2N-1$, that is

$z_{h/2} = \max_i \left| y_i^{h/2} - y_i^{h/4} \right|$, for $\mathrm{i} = 1, 2, \ldots, 2N-1$. The computational rate of convergence $\rho$ is also obtained by using the double mesh principle defined as, (Doolan et at., 1980):

$$\rho = \left( \frac{\log(z_h) - \log(z_{h/2})}{\log 2} \right).$$

**Table 4.5:** Rate of convergence ρ for different values of $\delta$ with $\varepsilon = 0.1$, $\eta = 0.5\varepsilon$.

| $\delta/h$ | 16 | 32 | 64 | 128 |
|---|---|---|---|---|
| **Example 4.1** | | | | |
| 0.00 | 4.0655 | 4.0164 | 4.0041 | 4.0010 |
| 0.05 | 4.0566 | 4.0144 | 4.0036 | 4.0009 |
| 0.09 | 4.0500 | 4.0126 | 4.0031 | 4.0008 |
| **Example 4.2** | | | | |
| 0.00 | 4.0256 | 4.0064 | 4.0016 | 4.0004 |
| 0.05 | 4.0349 | 4.0087 | 4.0022 | 4.0005 |
| 0.09 | 4.0435 | 4.0108 | 4.0027 | 4.0007 |

### 4.5. Discussion

In this thesis, fitted finite difference method is presented for solving singularly perturbed differential-difference equations (SPDDEs) with delay as well as advance parameters. First, SPDDEs is converted into an asymptotically equivalent singularly perturbed boundary value problem by using the Taylor series expansion for the delay and advance terms. Then, the given interval is discretized, and using fitting finite difference approximation the given differential equation is transformed into a three-term recurrence relation, which can easily be solved using Thomas Algorithm. The stability and convergence of the method have been investigated. The numerical results have been presented in Tables (4.1) – (4.4) for different values of the perturbation parameter $\varepsilon$, delay parameter $\delta$, advance parameter $\eta$ and number of mesh points $N$. It can be observed from the Tables that the present method gives better results than some reported literatures.

Kadalbajoo and Ramesh, (2007) states that, the accuracy of the problem increases by increasing the number of the nodal points. Thus, it can be observed from the Tables that, the maximum absolute errors decrease rapidly as N increases, which in turns shows the convergence of computed solution. Doolan et al, (1980); Kadalbajoo and Sharma, (2004) states that no good result for singularly perturbed boundary value problem when $\varepsilon < h$ . But we get a good result for $\varepsilon < h$, Table (4.4).

To demonstrate the effect of delay and advance parameters on the left and right boundary layer of the solution, the graphs for different values of delay parameter $\delta$ and advance parameter $\eta$ are plotted in Figures (4.1) - (4.3); accordingly, depending on the sign of $p(x)$ one can see that, from Figures (4.1) and (4.3) as $\delta$ increases the width of the right boundary layer decreases for fixed value of $\eta$ but, as $\eta$ increases the width of the right boundary layer increases for fixed value of $\delta$ while the width of the left boundary layer decreases when $\delta$ or $\eta$ increases Figure (4.2). Figures (4.4) – (4.6) shows as a mesh size $h$ decreases the errors goes to zero.

# CHAPTER FIVE

# CONCLUSION AND SCOPE OF THE FUTURE WORK

## 5.1. Conclusion

This study is implemented on three model examples by taking different values of perturbation parameter, delay parameter and advance parameter, and the computational results are presented in the Tables and Figures. One can conclude that, the results observed from the Tables demonstrate that the present method approximate the solution very well. A numerical result presented in this thesis shows the betterment of the proposed method over some existing methods reported in the literature. Furthermore, the stability and convergence of the method is established well. The results presented (Table 4.5) confirmed that the computational rate of convergence as well as theoretical estimates indicates that the present method is of fourth order convergence. The effect of the delay and advance parameters on the solution of singularly perturbed differential-difference equation is showed by sketching graphs (Figures 4.1 – 4.3). Furthermore, as $h$ decreases the absolute error also deceases (see Tables (4.1) – (4.4) and Figures (4.4) – (4.6)).

In general, the present method is stable, convergent and more accurate for solving singularly perturbed differential-difference equations.

## 5.2. Scope of the Future Work

In this thesis, the numerical method based on fitted finite difference method is introduced for solving singularly perturbed differential-difference equations. Hence, the scheme proposed in this thesis can also be extended to sixth order and higher order finite difference method for solving singularly perturbed differential-difference equations.

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