JIMMA UNIVERSITY COLLEGE OF NATURAL SCIENCES

DEPARTMENT OF MATHEMATICS

Fitted Mesh Finite Element Method for Reaction-Diffusion Problems in One Dimension

A Thesis Submitted to the Department of Mathematics, Jimma University in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics.

(Numerical Analysis)

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Declaration

I, undersigned, declare that " Fitted Mesh Finite Element Method for Reaction-Diffusion Problems in One Dimension" is original and it has not been submitted to any institution elsewhere for the award of any academic degree or like and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

Name: Worku Tilahun

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Date: ....

Dedication

To my beloved brother Kibir Tilahun. I never forget you ever.

Acknowledgment

First of all, I would like to thank my omnipotent GOD for his good will to do this thesis. Next to this, it is of great pleasure and proud privilege to express my deepest and sincere gratitude to my supervisor Dr. Gemechis File and my co-supervisor Mr.Mesfin Mekuria. I am grateful for their indispensable encouragement, persistent help, constructive suggestions and meticulous guidance throughout the period of my thesis work. My heartfelt gratitude goes to my younger brother Mr. Tafere Tilahun for his moral and financial support and guiding me to write my thesis by the latex software. I am also indebted to extend my thanks to my mother Ayew, My father Tilahun, my wife Yetinayet, my daughter Meklit, my brothers Dr. Habtamu and Amanuel and my sisters Emebet and Eyerus, you all behind me for my success and I have a big thank for you. Last but not least, I would like to thank Department of Mathematics, Jimma University for giving me this delightful chance to do thesis on mathematics.

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Acronyms

* BVP - Boundary Value Problem.

* SP - Singularly Perturbed.

* SPBVP - Singularly Perturbed Boundary Value Problems.

* SPP - Singularly Perturbed Problem.

* 1D -One Dimensional.

* FEM -Finite Element Method.

* FDM -Finite Difference Method.

* SPA -Spline Approximation.

* ODE -Ordinary Differential Equation.

* PDE -Partial Differential Equation.

Abstract

In this thesis, fitted mesh finite element method has been presented for solving singularly perturbed one dimensional reaction-diffusion equations. First, the given differential equation is transformed to it's weak form and using shishkin mesh discretization technique the given domain is discretized in to a finite number of mesh elements so that piecewise linear base functions are defined depending on this discretization. Then the approximate solution on each element is represented by taking the linear combination of the base functions. Substitution of the approximate solution to the weak form and applying Galerkin's method resulted a system of algebraic equations over each element. The obtained system of equations are then assembled to obtain the global system of equation and reduced to a nonsingular tridiagonal matrix which can be easily solved by inverse matrix method. To validate the applicability of the proposed method a model example is considered and solved for different values of the perturbation parameter and mesh elements. Numerical experiment is carried out to support the theoretical result using MATLAB software. The results have been presented in tables interms of maximum absolute error and graphs. The present method is ε*-uniform and approximates the exact solution very well.*

Chapter 1

Introduction

1.1 Background of the study

Due to the difficulties in finding the exact solution or analytical solution of a mathematical problems such as, the exact solution of differential equation, the root of non-linear equation, the evaluation of integration involving complex expression and etc, leads to the development of numerical analysis. Numerical analysis is a branch of mathematics that provides tools and methods for solving mathematical problems in a numerical form (Gautschi, 2011).

Many real life problems are modeled by a parameter dependent differential equations whose solution behavior depend on the magnitude of the parameter. A differential equation in which the highest order derivative is multiplied by a small positive parameter ε $(0 < \varepsilon \ll 1)$ is called singularly perturbed differential equation and the parameter is called the perturbation parameter. A second order singularly perturbed differential equation is said to be of reaction-diffusion type, if the order of the differential equation is reduced by two when the perturbation parameter ε is set to zero. That is, a differential equation of the form: $-\varepsilon u''(x) + b(x)u(x) = f(x) \quad 0 < x < 1$, u(0)=$\alpha$, u(1)=$\gamma$, where α, γ are constants, ε such that $0 < \varepsilon \ll 1$ is the perturbation parameter, $f(x)$ and $b(x)$ are sufficiently smooth functions on $0 \leq x \leq 1$, is called a second order singularly perturbed differential equation of reaction-diffusion type.

Singular perturbation problem (SPP) arises in the modeling of fluid dynamics, elasticity, quantum mechanics, reaction-diffusion process, chemical-reactor theory, plasma dynamics, meteorology, diffraction theory, aerodynamics, oceanography, modeling of semi-conductor, hydrodynamics and many others allied areas (Feyisa and Gemechis, 2017).

The numerical solution of a singularly perturbed differential equation exhibit a multi-scale character. That is there is(are) a thin layer(s) of the domain where the solution changes rapidly or jumps suddenly forming a boundary layer(s), while away from the layer(s) the solution behaves regularly or changes slowly in the outer region. As a result such problems are called boundary layer problems (Gemechis and Reddy, 2013).

Due to this multi-scale character of the solution, classical numerical methods which are effective in solving most mathematical problems perform badly when applied to singular perturbation problems. Since the error estimate for these methods depend explicitly on the derivatives of the solution and these derivatives are not bounded as $\varepsilon \to 0$ implies that such approximations are meaningless for singularly perturbed differential equation (Russell, 2016). Moreover, numerical methods that work well for non-singular perturbation problems generally breakdown and fail to give significant solutions for small values of ε. Finite Element Method (FEM), Finite Difference Method (FDM), Spline Approximation (SPA) and Finite Volume Methods (FVM) are some of the classical numerical methods used to solve mathematical problems numerically (Russell, 2016).

Earlier numerical solutions of singularly perturbed differential equations were obtained by using a standard finite difference operator on a uniform mesh. In this approach, as the perturbation parameter decreases in magnitude the mesh is refined sufficiently to capture the boundary layer(s) or interior of the layer(s). Hence, even for a problems in one dimension, such methods are inefficient and inaccurate. A natural question then arises: *Is it possible to construct a numerical methods that behave uniformly well for all values of the singular perturbation parameter ε, no matter how small the parameter is*? (Miller, 2012)

A parameter-uniform or $\varepsilon - uniform$ numerical method is defined as follows.

Definition

Consider a family of mathematical problems parameterized by a singular perturbation parameter ε, where ε lies in the semi-open interval $0 < \varepsilon \leq 1$. Assume that each problem in the family has a unique solution denoted by u_ε, and that each u_ε is approximated by a sequence of numerical solutions $\{(U_\varepsilon, \overline{\Omega}^N)\}$, where U_ε is defined on the mesh $\overline{\Omega}^N$ and N is a discretization parameter. Then, the numerical solutions U_ε are said to be converge ε-uniformly to the exact solution u_ε, if there exist a positive integer N_0 and positive numbers C and p, where N_0, C and p are all independent of N and ε, such that, for all $N > N_0$,

$$\sup_{0<\varepsilon\leq 1} ||U_\varepsilon - u_\varepsilon||_{\overline{\Omega}_\tau^N} \leq CN^{-p}$$

Here p is called the ε-uniform rate of convergence and C is called the ε-uniform error constant. (Miller, 2012)

In the construction of ε-uniform numerical methods two approaches generally can be taken. The first of these involves replacing the standard finite difference operator by a finite difference operator which reflects the singularly perturbed nature of the differential operator. Such finite difference operators are referred to, in general, as fitted operator finite difference method. In fitted operator method, exponentially fitting factor or artificial viscosity will be used to control the rapid growth or decay of the numerical solution in the boundary layer.

The second successful approach, to the construction of ε-uniform numerical methods, involves the use of a mesh that is adapted to the singular perturbation. That is a fitted mesh method uses a non uniform meshes, which is fine or dense in the boundary layer regions and coarse outside the boundary layer (Miller, 2012) and it is the main interest of this thesis.

A fitted mesh can be incorporated into both a finite difference and a finite element method. The simplest form of fitted mesh is a piecewise uniform mesh with a specially chosen transition points separating the coarse and fine meshes. These piecewise uniform fitted meshes were first introduced by Shishkin and the corresponding numerical methods were further developed and shown to be ε-uniform in a series of papers culminating in the book "Grid approximation of singularly perturbed elliptic and parabolic equations" (Miller, 2012). The first numerical results using a fitted mesh method were presented by Shishkin (Miller, 2012). What distinguishes a Shishkin mesh from any other piecewise uniform mesh is the choice of the so-called transition parameter(s), which are the point(s) at which the mesh size changes abruptly (Kopteva and O'Riordan, 2010).

Hence for a singularly perturbed 1D reaction-diffusion equation having a boundary layers at the end points of its interval, the mesh is constructed in such a way that the boundary layer regions have more mesh points relative to the region outside the boundary layers. To do this, two transition points are required and the mesh comprises three uniform pieces or subintervals. Thus the simplest construction is, to choose a τ satisfying $0 < \tau \leq \frac{1}{4}$; and to locate the transition points at τ and $1-\tau$. Assuming that $N = 2^r$, with $r \geq 3$, the interval $(0, \tau)$ and $(1-\tau, 1)$ are each divided into $\frac{N}{4}$ equal mesh elements while the interval $(\tau, 1-\tau)$ is divided in to $\frac{N}{2}$ equal mesh elements. If

$\tau = \frac{1}{4}$ the mesh becomes uniform. Therefore, the transition parameter τ for this mesh is defined as:

$$\tau = \min\{\tfrac{1}{4}, 2\sqrt{\tfrac{\varepsilon}{\beta}} \ln N\}$$

The mesh length is also given by

$$\mathrm{h}_i = x_i - x_{i-1} = \begin{cases} \frac{4\tau}{N}, & 1, ..., \frac{N}{4} \\ \frac{2(1-\tau)}{N}, & \frac{N}{4} + 1, ..., \frac{3N}{4} \\ \frac{4\tau}{N}, & \frac{3N}{4} + 1, ..., N \end{cases}$$

The resulting piecewise uniform mesh, illustrated in figure below which is taken from Miller et al.(2012), depends on just one parameter τ and is denoted by Ω_τ^N

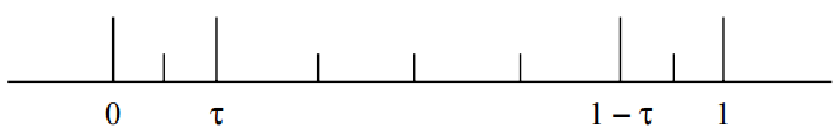


Figure 1.1: Dual layer Shishkin mesh on 8 mesh element

A variety of numerical methods are available in the literature of the numerical solution of perturbation and singular perturbation problems of second order ordinary differential equation (Nayfeh, 2011 and O'Malley, 1991) respectively. Many Scholars outlined in the literature (Rao and Kumer, 2010; Fasika et al. 2016a, 2016b, 2017; Kumar and Rao, 2010; Natesan et al., 2007 and Natesan and Bawa, 2007) have developed a numerical method of different order for the numerical solution of second order singularly perturbed differential equation of reaction-diffusion type.

In this study, we have constructed fitted mesh finite element method for solving second order singularly perturbed two point boundary value problem of reaction-diffusion type. The method involves dividing the domain of the solution in to a finite number of simple sub-domains, called elements and uses variational concepts to construct an approximate solution over the collection of elements. Because of the generality and reachness of the ideas underlying the method, it has been used with remarkable success in solving a wide range of problems in virtually all areas of engineering and mathematical physics. The main reason behind seeking approximate solution on a collection of sub-domain is, the fact that it is easier to represent a complicated functions as a collection of simple polynomials (Becker, 1981 and Reddy, 1993).

Analytical solutions are desirable because they are exact and numerical methods are developed to solve mathematical problems numerically. We take numerical solution or approximate solution when it is not possible to obtain analytical solution of the mathematical problem. However, as a mathematicians, we continue obtaining and developing numerical solutions to mathematical problems despite the analytical solution, in order to compare the deviation of the numerical solution from the analytical solution and after, to extend the numerical methods to more complex and larger mathematical problems in which the analytical solution is not known . Thus the increasing desire for the numerical solution to mathematical problems has become the present day scientific research area.

For a singularly perturbed reaction-diffusion problems, most numerical methods are not parameter-uniform (Russel, 2016). So developing a numerical method whose convergence doesn't depend on the perturbation parameter has a great importance to the scientific research area. Owning to this, this study dealt with formulating a parameter-uniform numerical method to find a numerical solution of a singularly perturbed 1D reaction-diffusion problems.

1.2 Objectives

1.2.1 General objective

The general objective of this study is to formulate Fitted Mesh Finite Element Method for the numerical solution of singularly perturbed reaction-diffusion problems.

1.2.2 Specific objectives

The specific objectives of this study are:

* To construct Finite Element Method on shishkin mesh for the numerical solution of singularly perturbed reaction-diffusion problems.
* To establish the parameter-uniform convergence of the present method.

1.3 Significance of the study

The result obtained from this study can:

* be used as a reference material for scholars who work on this area.
* help the graduate students to acquire research skills and scientific procedures.
* provide a numerical method for solving singularly perturbed reaction-diffusion problems.

1.4 Delimitation of the study

The study is delimited to solve a 1D reaction-diffusion equation of the form

$$-\varepsilon u''(x) + b(x)u(x) = f(x) \quad 0 < x < 1 \tag{1.1}$$

subjected to the boundary conditions

$$y(0) = \alpha, \quad y(1) = \gamma \tag{1.2}$$

where α and γ are constants, ε such that $0 < \varepsilon \ll 1$ is a perturbation parameter and $f(x)$ and $b(x)$ are sufficiently smooth functions such that $b(x) \geq \beta > 0 \quad$ for $0 \leq x \leq 1$, where β is a constants.

Chapter 2

Review of Related Literature

2.1 Singular Perturbation Theory

Lundwing Prandtl was the first to introduce the concept of boundary layer in 1904 at the Third International Congruence of Mathematics in Heidelberg Germany. His hypothesis was in the setting of fluid dynamics, fluid adjacent to the boundary sticks to the edge in a thin boundary layer due to friction but this friction has no effect to the flow of the fluid on the interior (Russell, 2016).

The term singular perturbation appears to have been first coined by Frendricks and Wasow in 1946 in their paper "Singular Perturbation of Nonlinear Oscillation" (Frendricks and Wasow, 1946). Wasow continued to contribute to the area of asymptotic methods over many years and his book "Asymptotic expansion for ordinary differential equation" (Wasow, 1965), attracted much interest in the area of singular perturbed boundary value problems.

A brief survey for the historical development of perturbation and singular perturbation problems is covered in a recent books by Nayfeh(2011) and O'Malley (1991) respectively. More precisely, a perturbation problem is a problem that contain a small parameter ε, called perturbation parameter. If the solution of the problem can be approximated by setting the value of the perturbation parameter equals to zero, then the problem is called regular perturbation problem, otherwise it is called singular perturbation problem. That is, if it is impossible to approximate the solution by an asymptotic expansion as the perturbation parameter tends to zero, then the problem is called singular.

Singular perturbed differential equation is a differential equation in which the highest order derivative term is multiplied by a small positive parameter ε called perturbation parameter. Such problems arises in the modeling of fluid dynamics, chemical reactor theory, nuclear reactor theory, reaction-diffusion process, meteorology, diffraction theory, semi-conductor devices and etc (Feyisa and Gemechis, 2017).

Whenever such problem arises in the modeling of materials, the material quantity usually changes rapidly over a very narrow region of the independent variable(s) called boundary layer(s) (Miller et al., 2012) implying that the problem depends on the perturbation parameter in such way that the solution varies rapidly in some part of the domain and varies slowly in some other part of the domain. So if we apply the classical numerical methods for solving such types of problems large oscillation may arises and disrupt the solution in the entire interval due to the rapid change of the solution in a very narrow region called boundary layer (Feyisa and Gemechis, 2017).

2.2 Singularly Perturbed Reaction-Diffusion Problems

Macmullen et al.(2001), constructed a parameter-uniform numerical method for singularly perturbed reaction-diffusion problems. They shown that a suitably designed discrete Schwartz method, based on standard finite difference operator with a uniform mesh on each sub-domain gives a numerical approximation which converges in a maximum norm to the exact solution uniformly with respect to the singular perturbation parameter ε.

By revising the existing spline collocation technique, Stojanovic (2002), introduced a piecewise interpolating polynomials for the driving terms in the numerical solution of the singularly perturbed reaction-diffusion 1D problems. They obtained an optimal difference scheme which is second order accurate.

In order to solve a singularly perturbed reaction-diffusion Robin boundary-value problems, Natesan and Bawa (2007), constructed a numerical method which involves both the cubic spline and classical finite difference scheme that is a hybrid numerical scheme. They applied on a piecewise uniform shishkin mesh and obtained almost a second order convergent scheme.

Rao et al. (2008), presented an exponential B-spline collocation method, which is a satisfactory ways of solving a self-adjoint singularly perturbed problems of reaction-diffusion type with Dirichlet boundary conditions. They have shown that the method is a second order uniform convergence.

Natesan et al. (2007), developed a numerical scheme which is a combination of the cubic-spline and the classical central difference scheme by applying on an appropriate piecewise uniform shishkin mesh. They have shown that the developed method is a second order uniformly convergent at the mesh points. Finally they had constructed the global solution using cubic splines which is uniformly convergent in the boundary layer regions.

Kadalbajoo and Arora (2008) developed a B-spline collocation method using artificial viscosity for a class of singularly perturbed reaction-diffusion equations. They used the artificial viscosity to capture the exponential features of the exact solution on a uniform mesh and used the B-spline collection method,which leads to a tridiagonal linear system. The method is shown to be uniformly convergent of second order.

Clavero et al. (2009), considered the finite difference hybrid scheme constructed by Natesan et al (2007), in order to obtain a uniformly convergent global solution and uniformly convergent normalized flux for self-adjoint singularly perturbed boundary value problems. The global solution is obtained from the numerical solution at the mesh point of this scheme, having almost a second order uniform convergence at the nodal point when it is constructed on a piecewise uniform shishkin mesh. They defined the solution and the normalized flux on the entire domain, using a classical cubic spline and proved that the uniform order of convergence of the global solution is the same as that of the hybrid scheme at the mesh pints

Kumer and Rao (2010), proposed a high order parameter robust finite difference method by discretizing the problem using a suitable combination of fourth order compact finite difference scheme and central difference scheme on generalized shishkin mesh. They have obtained almost fourth order uniform convergent method in a maximum norm.

In order to solve a singularly perturbed two point boundary value problems of reaction-diffusion type with Dirichlet boundary conditions, Feyisa and Gemechis (2017) developed an eight order numerical method based on finite difference scheme with uniform mesh. Again to solve a singularly perturbed differential equation with dual layer behavior, Phaneendra et al. (2015), proposed an exponentially fitted arithmetic average difference scheme by introducing a fitting factor in a three point arithmetic average discretization for the given problems and the fitting factor is obtained from the asymptotic approximation solution of singular perturbations.

To solve a second order singularly perturbed 1D reaction-diffusion problems with Dirichlet boundary conditions, were proposed a fourth order (Fasika et al., 2016a), six order (Fasika et al., 2017) and tenth order (Fasika et al., 2016b), compact finite difference methods. The methods are based on a finite difference scheme with uniform mesh. They have developed the methods by replacing the derivatives of the given differential equation by a finite difference approximations.

Rao et al. (2010) presented an exponential spline difference scheme based on spline in tension on a piecewise uniform shishkin mesh for singularly perturbed Dirichlet boundary value problem of reaction-diffusion type by using exponential spline identity relation based on second derivative formulation and obtained a second order ε-uniform convergence method. The method produces an exponential spline function which is useful to obtain the solution at any point of the interval.

2.3 Finite Element Method

Finite element method and its generalization are the most powerful computer oriented method ever devised to analyze practical application problems(Becker, 1981). The method is a numerical method like a finite difference method, but it is more general and powerful than any numerical methods (Reddy, 1993). In a finite element method a given domain is viewed as a collection of sub-domains and over each sub-domain the governing equation is approximated by any of the traditional variational methods, since it is easy to represent a complicated functions as a collection of simple polynomials (Becker, 1981 and Reddy, 1993).

2.4 Fitted Mesh Finite Element Method

The construction of ε-uniform numerical methods involves the use of a mesh that is adapted to the singular perturbation. Such methods are called fitted mesh methods. That is, a fitted mesh method uses non uniform meshes which is fine or dense in the boundary layer regions and coarse outside the boundary regions. A fitted mesh can be incorporated in to both a FDM and FEM. The simplest form of this mesh is a piecewise uniform mesh with specially chosen transition points separating the coarse and fine meshes. These piecewise uniform fitted meshes was first introduced by Grigorii Ivanovich Shishkin and the corresponding Numerical Methods were further developed in a series of papers culminating in the book "Grid approximation of singularly perturbed elliptic and parabolic equations" by Shishkin (Miller, 2012).

Hence for a singularly perturbed 1D reaction-diffusion equation having boundary layers at the end points of its interval, the mesh is constructed in such a way that the boundary layer regions have more mesh points relative to outside these regions. To do this, two transition points are required and the mesh comprises three uniform pieces or subintervals. Thus the simplest construction is, to choose a τ satisfying $0 < \tau \leq \frac{1}{4}$; and to locate the transition points at τ and $1-\tau$. Assuming that $N = 2^r$, with $r \geq 3$, the interval $(0, \tau)$ and $(1-\tau, 1)$ are each divided into $\frac{N}{4}$ equal mesh elements while the interval $(\tau, 1-\tau)$ is divided in to $\frac{N}{2}$ equal mesh elements. If $\tau = \frac{1}{4}$ the mesh becomes uniform. Therefore, the transition parameter τ for this mesh is defined as

$$\tau = \min\{\frac{1}{4}, 2\sqrt{\frac{\varepsilon}{\beta}}\ln N\}$$

The mesh length is also given by

$$h_i = x_i - x_{i-1} = \begin{cases} \frac{4\tau}{N}, & 1, ..., \frac{N}{4} \\ \frac{2(1-\tau)}{N}, & \frac{N}{4}+1, ..., \frac{3N}{4} \\ \frac{4\tau}{N}, & \frac{3N}{4}+1, ..., N \end{cases}$$

Chapter 3

Methodology

3.1 Study Site and Period

This study is conducted at Jimma University, College of Natural Science, Department of Mathematics from September 2018 to June 2019.

3.2 Study Design

This study is applied both the documentation review and numerical experimentation or mixed design

3.3 Source of Information

The relevant source of information for this study were books, published articles on reputable journals and related study from Internet.

3.4 Mathematical Procedure

In order to achieve the stated objectives, the study has followed the following mathematical procedures.

1. Defining the problem.
2. Transferring the given differential equation to it's weak form by using Galerkin Variational Method.
3. Discretizing the solution domain/generating elements.
4. Constructing a set of linear base functions based on the elements.
5. Representing the approximate solution by a linear combination of the base functions
6. Obtaining a system of equation on each elements.
7. Assembling the obtained system of equations in a tridiagonal form.
8. Writing a code for the obtained method by using MATLAB language.
9. Validating the scheme using numerical examples.

Chapter 4

Description of the Method, Result and Discussion

4.1 Description of the Method

Consider the following singularly perturbed reaction-diffusion equation of the form:

$$-\varepsilon u''(x) + b(x)u(x) = f(x), \qquad 0 < x < 1 \tag{4.1}$$

subjected to the boundary conditions

$$u(0) = \alpha \quad \text{and} \quad u(1) = \gamma \tag{4.2}$$

where $0 < \varepsilon \ll 1$, α and γ are a given constants, $f(x)$ and $b(x)$ are sufficiently smooth functions such that $b(x) \geq \beta > 0$ for $0 \leq x \leq 1$, where β is a constants.

Let H_0^1=The set of all functions whose order 1 or less is square integrable over $\Omega = [0, 1]$ and vanishes at the end point of the domain.

That is, if $v(x) \in H_0^1(0, 1)$, $\forall x \in (0, 1)$.

Then

i . $\int_0^1 (v'^2 + v^2)\,\mathrm{d}x < \infty$

ii . $v(0) = v(1) = 0.$

Now multiplying both sides of eqn(4.1) by $v(x) \in H_0^1$, gives

$$-\varepsilon u''(x)v(x) + b(x)u(x)v(x) = f(x)v(x) \tag{4.3}$$

Taking the integral of both sides of eqn(4.3), results

$$\int_0^1 \left[-\varepsilon u''(x)v(x) + b(x)u(x)v(x)\right]\,\mathrm{d}x = \int_0^1 f(x)v(x)\,\mathrm{d}x \tag{4.4}$$

Applying integration by part on the first term of eqn(4.4), gives

$$-\varepsilon u'(x)v(x)|_0^1 + \int_0^1 \varepsilon u'(x)v'(x)\,\mathrm{d}x + \int_0^1 b(x)u(x)v(x)\,\mathrm{d}x = \int_0^1 f(x)v(x)\,\mathrm{d}x \tag{4.5}$$

But $-\varepsilon u'(x)v(x)|_0^1 = 0$ since $v(0) = v(1) = 0$

Hence, it follows that

$$\int_0^1 \varepsilon u'(x)v'(x)\,\mathrm{d}x + \int_0^1 b(x)u(x)v(x)\,\mathrm{d}x = \int_0^1 f(x)v(x)\,\mathrm{d}x \tag{4.6}$$

$$\Rightarrow \int_0^1 \left[\varepsilon u'(x)v'(x) + b(x)u(x)v(x)\right]\mathrm{d}x = \int_0^1 f(x)v(x)\,\mathrm{d}x \tag{4.7}$$

Now we can rewrite eqn(4.7) as:

$$B(u, v) = L(v) \tag{4.8}$$

so that $B(u,v)$ is a bilinear function in u and v and $L(v)$ is a linear function in v. Now eqn(4.8) is called weak form of of eqn(4.1).

Next, we discretized the domain as follows. Applying shishkin mesh discretization technique we can choose a transition point τ such that, $\tau = \min\{\frac{1}{4}, 2\sqrt{\frac{\varepsilon}{\beta}}\ln N\}$. Then the whole domain is discretized in to three piecewise uniform sub-intervals of the form $(0,\tau)$, $(\tau, 1-\tau)$ and $(1-\tau, 1)$. The sub-intervals $(0,\tau)$ and $(1-\tau,1)$ are subdivided into $\frac{N}{4}$ elements and the sub-interval $(\tau, 1-\tau)$ is again subdivide into $\frac{N}{2}$ elements.

Now, let N be the number of mesh elements. Then we can define the mesh elements as:

$$\Omega_N = \{0 = x_1 < x_2 < x_3 < ... < x_{N+1} = 1\}$$

with $h_i = x_{i+1} - x_i$, where $i = 1, 2, 3, ..., N$

Depending on the above discretization, it is possible to construct a set of piecewise linear basis function of the form:

$$\phi_1 = \begin{cases} \frac{x_2 - x}{h_1} & \text{if} \quad x_1 < x < x_2 \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h_{i-1}} & \text{if} \quad x_{i-1} < x < x_i \\ \frac{x_{i+1} - x}{h_i} & \text{if} \quad x_i < x < x_{i+1} \\ 0 & \text{otherwise} \end{cases} \qquad \text{for} \quad i = 2, 3, 4, ..., N$$

$$\phi_{N+1} = \begin{cases} \frac{x - x_N}{h_N} & \text{if} \quad x_N < x < x_{N+1} \\ 0 & \text{otherwise} \end{cases}$$

which are often called the hat functions.

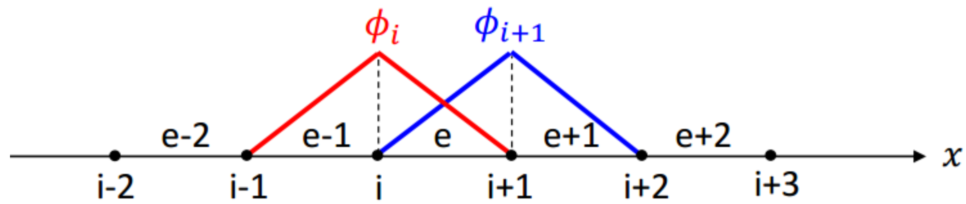


Figure 4.1: Diagram of a mesh and hat basis functions. (Fig. from Zhilin et al. (2018))

Now let us consider a typical element Ω_e in the finite element mesh with end points s_1^e and s_2^e. Thus our variational form over each element is of the form:

$$\int_{s_1^e}^{s_2^e} [\varepsilon u^{e\prime}(x)v'(x) + b(x)u^e(x)v(x)] \, \mathrm{d}x = \int_{s_1^e}^{s_2^e} f(x)v(x) \, \mathrm{d}x \tag{4.9}$$

Representing the approximate solution or the numerical solution by the linear combination of the basis function on each element,

$$u_N^e(x) = \sum_{j=1}^{N_e} c_j^e \phi_j^e(x) \tag{4.10}$$

where the coefficients c_j^e are unknowns to be determined, N_e is the number of nodes in Ω_e and $\phi_j^e(x)'$s are the basis function for this element.
Differentiating eqn(4.10) gives

$$u^{e\prime}{}_N(x) = \sum_{j=1}^{N_e} c_j^e \phi^{e\prime}{}_j(x) \tag{4.11}$$

Plugging the approximate solution eqn(4.10) and its derivative eqn(4.11) for the exact solution in eqn(4.9), results

$$\int_{s_1^e}^{s_2^e} [\varepsilon \sum_{j=1}^{N_e} c_j^e \phi^{e\prime}{}_j(x)v'(x) + b(x) \sum_{j=1}^{N_e} c_j^e \phi_j^e(x)v(x)] \, \mathrm{d}x = \int_{s_1^e}^{s_2^e} f(x)v(x) \, \mathrm{d}x \tag{4.12}$$

Rearranging eqn(4.12) gives

$$\int_{s_1^e}^{s_2^e}[\sum_{j=1}^{N_e}(\varepsilon {\phi^{e}}'_j(x)v'(x)+b(x)\phi_j^e(x)v(x)c_j^e)]\,\mathrm{d}x=\int_{s_1^e}^{s_2^e} f(x)v(x)\,\mathrm{d}x \tag{4.13}$$

Applying Galerkin method, we can choose $v(x)=\phi_i(x)$, for $i=1,2$, where ϕ_i is a piecewise linear base functions.

Thus

$$\int_{s_1^e}^{s_2^e}\sum_{j=1}^{N_e}(\varepsilon {\phi^{e}}'_j(x){\phi^{e}}'_i(x)+b(x)\phi_j^e(x)\phi_i^e(x)c_j^e)\,\mathrm{d}x=\int_{s_1^e}^{s_2^e} f(x)\phi_i^e(x)\,\mathrm{d}x \tag{4.14}$$

$$\Rightarrow \sum_{j=1}^{N_e}[\int_{s_1^e}^{s_2^e}\varepsilon {\phi^{e}}'_j(x){\phi^{e}}'_i(x)+b(x)\phi_j^e(x)\phi_i^e(x)]\,\mathrm{d}x c_j^e=\int_{s_1^e}^{s_2^e} f(x)\phi_i^e(x)\,\mathrm{d}x \tag{4.15}$$

We can rewrite eqn(4.15) as:

$$\sum_{j=1}^{N_e} k_{ij}^e c_j^e = f_i^e \qquad \text{for} \quad i=1,2 \tag{4.16}$$

where

$$k_{ij}=\int_{s_1^e}^{s_2^e}\varepsilon {\phi^{e}}'_j(x){\phi^{e}}'_i(x)+b(x)\phi_j^e(x)\phi_i^e(x)\,\mathrm{d}x \quad \text{called the stiffness matrix and}$$

$$f_i=\int_{s_1^e}^{s_2^e} f(x)\phi_i^e(x)\,\mathrm{d}x \qquad \text{called the load vector.}$$

Since our base functions are linear, each element has two nodes and therefore, we have two equations per element of the following form.

$$\begin{aligned} k_{11}^e c_1^e + k_{12}^e c_2^e &= f_1^e \\ k_{21}^e c_1^e + k_{22}^e c_2^e &= f_2^e \end{aligned} \tag{4.17}$$

Here the subscripts 1 and 2 are labels of the endpoint nodes on a typical element Ω_e. These subscripts are to be relabeled upon assembling the elements so as to coincide with appropriate nodes $1, 2, 3, ..., N+1$ in the final mesh.

We now assemble the equations describing the entire collection of elements comprising our mesh by sweeping through all elements, one at a time and using equations eqn(4.17) to calculate the contribution of each of them. Since our mesh contain N elements and $N+1$ nodes, we have $N+1$ equations in $N+1$ degree of freedom describing the assembled system of elements. Thus we anticipate calculating an $(N+1) \times (N+1)$ stiffness matrix $\mathbf{K} = [K_{ij}]$ and an $(N+1) \times 1$ load vector $\mathbf{F} = \{F_i\}$, for $i, j = 1, 2, 3, ..., N+1$

Thus for Ω_1, between nodes 1 and 2, we have

$$\begin{aligned} k_{11}^1 c_1 + k_{12}^1 c_2 &= f_1^1 \\ k_{21}^1 c_1 + k_{22}^1 c_2 &= f_2^1 \end{aligned} \tag{4.18}$$

Next, we proceed to the second element Ω_2. It lies between nodes 2 and 3. Thus using eqn(4.17), we have

$$\begin{aligned} k_{11}^2 c_2 + k_{12}^2 c_3 &= f_1^2 \\ k_{21}^2 c_2 + k_{22}^2 c_3 &= f_2^2 \end{aligned} \tag{4.19}$$

Adding eqn(4.18) and eqn(4.19), gives

$$\begin{gathered} k_{11}^1 c_1 + k_{12}^1 c_2 = f_1^1 \\ k_{21}^1 c_1 + (k_{22}^1 + k_{11}^2) c_2 + k_{12}^2 c_3 = f_2^1 + f_1^2 \\ k_{21}^2 c_2 + k_{22}^2 c_3 = f_2^2 \end{gathered} \tag{4.20}$$

Similarly for the element Ω_3, we have

$$
\begin{aligned}
k_{11}^3 c_3 + k_{12}^3 c_4 &= f_1^3 \\
k_{21}^3 c_3 + k_{22}^3 c_4 &= f_2^3
\end{aligned}
\tag{4.21}
$$

Thus, the system becomes:

$$
\begin{aligned}
k_{11}^1 c_1 + k_{12}^1 c_2 &= f_1^1 \\
k_{21}^1 c_1 + (k_{22}^1 + k_{11}^2) c_2 + k_{12}^2 c_3 &= f_2^1 + f_1^2 \\
k_{21}^2 c_2 + (k_{22}^2 + k_{11}^3) c_3 + k_{12}^3 c_4 &= f_2^2 + f_1^3 \\
k_{21}^3 c_3 + k_{22}^3 c_4 &= f_2^3
\end{aligned}
\tag{4.22}
$$

Continuing this process through the entire system of N elements, we arrive at:

$$
\begin{aligned}
k_{11}^1 c_1 + k_{12}^1 c_2 &= f_1^1 \\
k_{21}^1 c_1 + (k_{22}^1 + k_{11}^2) c_2 + k_{12}^2 c_3 &= f_2^1 + f_1^2 \\
k_{21}^2 c_2 + (k_{22}^2 + k_{11}^3) c_3 + k_{12}^3 c_4 &= f_2^2 + f_1^3 \\
k_{21}^3 c_3 + (k_{22}^3 + k_{11}^4) c_4 + k_{12}^4 c_5 &= f_2^3 + f_1^4 \\
&\vdots \\
k_{21}^{N-1} c_{N-1} + (k_{22}^{N-1} + k_{11}^N) c_N + k_{12}^N c_{N+1} &= f_2^{N-1} + f_1^N \\
k_{21}^N c_N + k_{22}^N c_{N+1} &= f_2^N
\end{aligned}
\tag{4.23}
$$

Then the linear system of equations for the entire mesh is of the form:

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ K_{21} & K_{22} & K_{23} & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & K_{32} & K_{33} & K_{34} & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & K_{43} & K_{44} & K_{45} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & K_{N,N-1} & K_{N,N} & K_{N,N+1} \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & K_{N+1,N} & K_{N+1,N+1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ \vdots \\ c_N \\ c_{N+1} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ \vdots \\ F_N \\ F_{N+1} \end{bmatrix} \tag{4.24}$$

where

$$\begin{aligned} K_{11} =& k_{11}^1, K_{12} = k_{12}^1 \\ K_{21} =& k_{21}^1, K_{22} = k_{22}^1 + k_{11}^2, K_{23} = k_{12}^2 \\ & \vdots \\ K_{N,N-1} =& k_{21}^{N-1}, K_{N,N} = k_{22}^{N-1} + k_{11}^N, K_{N,N+1} = k_{12}^N \\ K_{N+1,N} =& k_{21}^N, K_{N+1,N+1} = k_{22}^N \quad \text{and} \end{aligned}$$

$$\begin{aligned} F_1 =& f_1^1, \\ F_2 =& f_2^1 + f_1^2, \\ F_3 =& f_2^2 + f_1^3, \\ F_4 =& f_2^3 + f_1^4, \\ & \vdots, \\ F_N =& f_2^{N-1} + f_1^N, \\ F_{N+1} =& f_2^N \end{aligned}$$

Now applying the given Dirichlet boundary conditions gives, $u_N(0) = \alpha$ and $u_N(1) = \gamma$ so that $N - 1$ unknown nodal values c_2, c_3, c_4,..., c_N remain. Then eqn(4.24) reduces to the $N - 1 \times N - 1$ system of equation of the following type.

$$\begin{bmatrix} K_{22} & K_{23} & 0 & 0 & \dots & 0 & 0 \\ K_{32} & K_{33} & K_{34} & 0 & \dots & 0 & 0 \\ 0 & K_{43} & K_{44} & K_{45} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \dots & K_{N,N-1} & K_{N,N} \end{bmatrix} \begin{bmatrix} c_2 \\ c_3 \\ c_4 \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} F_2 - K_{21}\alpha \\ F_3 \\ F_4 \\ \vdots \\ F_N - K_{N,N+1}\gamma \end{bmatrix} \tag{4.25}$$

and the two auxiliary equation corresponding to nodes 1 and $N + 1$ are:

$$\begin{aligned} K_{11}\alpha + K_{12}c_2 &= F_1 \\ K_{N+1,N}c_N + K_{N+1,N+1}\gamma &= F_{N+1} \end{aligned} \tag{4.26}$$

The reduced stiffness matrix in eqn(4.25) is nonsingular, so that it can be solved for the unknown nodal values c_2, c_3, c_4,..., c_N.

4.2 Convergence Analysis of the Scheme

Here, we consider the reaction-diffusion problem in eqn(4.1) with the homogeneous form of the boundary condition in eqn(4.2). The differential operator for this problem is given by: $L_\varepsilon = -\varepsilon \frac{d^2}{dx^2} + b$ and it satisfies the following maximum principle.

Lemma 1: Maximum Principle. Assume that $\psi(0) \geq 0$ and $\psi(1) \geq 0$.

Then, $L_\varepsilon \psi(x) \geq 0$, for all $x \in \Omega$, implies that $\psi(x) \geq 0$,for all $x \in \overline{\Omega}$.

Proof.

Let $\psi \in C^2(\overline{\Omega})$ and x^* be such that $\psi(x^*) = \min_{\overline{\Omega}} \psi(x)$. Suppose that $\psi(x^*) < 0$. It is clear that $x^* \notin \{0, 1\} \Rightarrow \psi'(x^*) = 0$ and $\psi''(x^*) \geq 0$ from elementary calculus.

consequently,

$$L_\varepsilon \psi(x^*) = -\varepsilon \psi''(x^*) + b(x)\psi(x^*) < 0 \quad \text{which is false.}$$

Therefore, it follows that $\psi(x^*) \geq 0$ and $\psi(x) \geq 0$, for all $x \in \overline{\Omega}$.

The following lemma gives a bound on the solutions of eqn(4.1) with the boundary condition eqn(4.2) and it's derivatives.

Lemma 2 *Let Ω be the interval (0,1) and $u \in C^2(\overline{\Omega})$ be the solution of the problem in eqn(4.1) with the boundary conditions eqn(4.2). Then for $0 \leq k \leq 4$, the following holds.*

$$||u^k(x)|| \leq C(1 + \varepsilon^{-\frac{k}{2}})$$

Proof: We handle the first case when $k = 0$. Consider the following functions:

$$\psi^{\pm}(x) = \frac{1}{\beta}||f|| + \max\{|u_0|, |u_1|\} \pm u(x)$$

when $x = 0$, we have

$$\begin{aligned}
\psi^{\pm}(0) &= \frac{1}{\beta}||f|| + \max\{|u_0|, |u_1|\} \pm u(0) \\
&\geq \frac{1}{\beta}||f||, \quad \text{since} \quad \max\{|u_0|, |u_1|\} \geq u(0) \\
&\geq 0
\end{aligned}$$

Similarly

$$\begin{aligned}
\psi^{\pm}(1) &= \frac{1}{\beta}||f|| + \max\{|u_0|, ||u_1|\} \pm u(1) \\
&\geq \frac{1}{\beta}||f||, \quad \text{since} \quad \max\{|u_0|, |u_1|\} \geq u(1) \\
&\geq 0
\end{aligned}$$

Now

$$\begin{aligned}
L_\varepsilon \psi^{\pm}(x) &= -\varepsilon(\psi^{\pm}(x))'' + b(x)\psi^{\pm}(x) \\
&= \mp(\varepsilon u''(x)) + \frac{b}{\beta}||f|| + b\max\{|u_0|, |u_1|\} \pm bu(x). \\
&= \pm f(x) + \frac{b}{\beta}||f|| + b\max\{|u_0|, |u_1|\} \\
&\geq b\max\{|u_0|, |u_1|\}, \quad \text{since} \quad \frac{b}{\beta}||f|| \geq f(x) \\
&\geq 0
\end{aligned}$$

Applying the maximum principle, it follows that $\psi^{\pm}(x) \geq 0$, for all $x \in \overline{\Omega}$.

Therefore:

$$|u(x)| \leq \frac{1}{\beta}||f|| + \max\{|u_0|, |u_1|\}, \quad \text{for all} \quad x \in \overline{\Omega}.$$

We now handle the case when $k = 1$. Let $x \in \Omega$ and construct an associated neighborhood $N_x = (p, p + \sqrt{\varepsilon})$, such that $x \in N_x$ and $N_x \subset \Omega$.

Then by mean value theorem, for some $q \in N_x$,

$$\begin{aligned}
u'(q) &= \frac{u(p + \sqrt{\varepsilon}) - u(p)}{\sqrt{\varepsilon}} \\
|u'(q)| &= \frac{|u(p + \sqrt{\varepsilon}) - u(p)|}{\sqrt{\varepsilon}} \\
&\leq \frac{1}{\sqrt{\varepsilon}}\{|u(p + \sqrt{\varepsilon})| + |u(p)|\} \\
&\leq \frac{1}{\sqrt{\varepsilon}}\{||u|| + ||u||\} \\
&\leq \frac{2}{\sqrt{\varepsilon}}||u||
\end{aligned}$$

which can be rewritten as

$$|u'(q)| \leq 2\varepsilon^{\frac{-1}{2}}||u|| \leq C\varepsilon^{\frac{-1}{2}}$$

but

$$\int_p^x u''(z)dz = u'(x) - u'(p)$$

$$u'(x) = u'(p) + \int_p^x u''(z)dz$$

$$= u'(p) + \int_p^x (\frac{bu(x) - f}{\varepsilon})(z)dz$$

hence,

$$|u'(x)| \leq \frac{2}{\sqrt{\varepsilon}}||u|| + |\int_p^x (\frac{bu(x) - f}{\varepsilon})(z)dz|$$

$$|u'(x)| \leq \frac{2}{\sqrt{\varepsilon}}||u|| + |(\frac{bu(x) - f}{\varepsilon})(\xi)| \int_p^x dz, \quad \xi \in (p, x)$$

$$\leq \frac{2}{\sqrt{\varepsilon}}||u|| + \frac{1}{\varepsilon}(||b||||u|| + ||f||)\sqrt{\varepsilon}$$

$$\leq \frac{2}{\sqrt{\varepsilon}}||u|| + \frac{1}{\sqrt{\varepsilon}}(||b||||u|| + ||f||)$$

$$\leq \frac{1}{\sqrt{\varepsilon}}(2||u|| + ||b||||u|| + ||f||) = C\varepsilon^{\frac{-1}{2}}$$

Therefore,

$$|u'(x)| \leq C(1 + \varepsilon^{\frac{-1}{2}})$$

The bounds on $|u^{(k)}|$ for $= 2, 3, 4$ is obtained from the differential equation and the bounds on u, u'.

In the proof of error estimates, sharper bounds on the solution and its derivatives are required. To find these, the solution u is decomposed in to a regular component v and a singular component w as follows.

$$u = v + w$$

Here let $v = v_0 + \varepsilon v_1$, where v_0 is the solution of the reduced problem, w is the solution of the homogeneous problem

$$L_\varepsilon w = 0, \quad w(0) = u_0 - v_0(0), \quad w(1) = u_1 - v_0(1)$$

consequently, v_1 satisfies the following.

$$L_\varepsilon v_1 = v_0'', \quad v_1(0) = 0, \quad v_1 = 0$$

since,

$$v_1 = \varepsilon^{-1}(v - v_0) = \varepsilon^{-1}(u - w - v_0)$$

implies

$$\begin{aligned} L_\varepsilon v_1 &= \varepsilon^{-1}(L_\varepsilon u - L_\varepsilon w - L_\varepsilon v_0) \\ &= \varepsilon^{-1}(L_\varepsilon u - L_\varepsilon v_0) \\ &= \varepsilon^{-1}(f - L_\varepsilon v_0) \\ &= \varepsilon^{-1}(f - (-\varepsilon v_0'' + b v_0)) \\ &= \varepsilon^{-1}(\varepsilon v_0'' + (f - b v_0)) = v_0'' \end{aligned}$$

hence,

$$L_\varepsilon v_1 = v_0''$$

Now

$$v_1 = \varepsilon^{-1}(u - v_0 - w)$$

$$v_1(x) = \varepsilon^{-1}(u(x) - v_0(x) - w(x))$$

$$v_1(0) = \varepsilon^{-1}(u(0) - v_0(0) - w(0))$$

$$= \varepsilon^{-1}((u(0) - v_0(0)) - w(0))$$

$$= 0$$

Similarly,

$$v_1(1) = \varepsilon^{-1}(u(1) - v_0(1) - w(1))$$

$$v_1(1) = \varepsilon^{-1}(u(1) - v_0(1) - w(1))$$

$$= \varepsilon^{-1}((u(1) - v_0(1)) - w(1))$$

$$= 0$$

Thus, because of the bound on v_0'', v_1 is the solution of a problem similar to eqn(4.1). This implies that, for $0 \leq k \leq 4$,

$$|v_1^{(k)}(x)| \leq C(1 + \varepsilon^{\frac{-k}{2}})$$

The singular component w of the solution is also bounded as shown below. Decompose the singular component into left layer and right layer as:

$$w = w_L + w_R$$

where the boundary layer functions, w_L and w_R are defined as the solution of the problems.

$$L_\varepsilon w_L = 0, \quad w_L(0) = w(0), \quad w_L(1) = 0$$

$$L_\varepsilon w_R = 0, \quad w_R(0) = 0, \quad w_R(1) = w(1)$$

Now define the functions,

$$\psi^{\pm}(x) = Ce^{-x\sqrt{\frac{\beta}{\varepsilon}}} \pm w_L(x)$$

where the constant C is chosen sufficiently large that the inequalities $\psi^{\pm}(0) \geq 0$, $\psi^{\pm}(1) \geq 0$ holds.

Thus,

$$\begin{aligned}
L_\varepsilon \psi^{\pm}(x) &= -\varepsilon(\psi^{\pm}(x))'' + b(x)\psi^{\pm}(x) \\
&= -\varepsilon[\frac{C\beta}{\varepsilon}e^{-x\sqrt{\frac{\beta}{\varepsilon}}} \pm w_L''(x)] + b(x)[Ce^{-x\sqrt{\frac{\beta}{\varepsilon}}} \pm w_L(x)] \\
&= (b(x) - \beta)Ce^{-x\sqrt{\frac{\beta}{\varepsilon}}} \pm (-\varepsilon w_L''(x) + b(x)w_L(x)) \\
&= (b(x) - \beta)Ce^{-x\sqrt{\frac{\beta}{\varepsilon}}} \geq 0 \\
&\geq 0
\end{aligned}$$

which gives, $\psi^{\pm}(x) \geq 0$ by maximum principle.

So, it follows that

$$|w_L(x)| \leq Ce^{-x\sqrt{\frac{\beta}{\varepsilon}}}, \quad \text{for all}, x \in \Omega$$

Using similar procedure for the right boundary layer, gives

$$|w_R(x)| \leq Ce^{-(1-x)\sqrt{\frac{\beta}{\varepsilon}}}, \quad \text{for all}, x \in \Omega$$

which implies that the boundary layer solution is bounded. To bound the first derivative w'_L, we use the same technique as in the proof of lemma 2. For each $x \in \overline{N}_x = (y, y+\sqrt{\varepsilon})$, such that $|w'_L(y)| \leq 2\varepsilon^{\frac{-1}{2}}||w_L||_{N_x}$

Hence,

$$\begin{aligned}
w'_L(x) &= w'_L(y) + \int_y^x w''_L(m)dm \\
&= w'_L(y) + \varepsilon^{-1}\int_y^x (-bw_L)(m)dm \\
|w'_L(x)| &\leq ||w'_L(y)||_{N_x} + \varepsilon^{-1}||bw_L||_{N_x}\int_y^x dm \\
&\leq 2\varepsilon^{\frac{-1}{2}}||w_L||_{N_x} + \varepsilon^{\frac{-1}{2}}||bw_L||_{N_x} \\
&\leq C\varepsilon^{\frac{-1}{2}}||w_L||_{N_x}
\end{aligned}$$

but $||w_L||_{N_x} = \sup_{x\in N_x}|w_L(x)| \leq Ce^{-y\sqrt{\frac{\beta}{\varepsilon}}}$, because w_L is monotonically decreasing.

$$\begin{aligned}
||w_L||_{N_x} \leq Ce^{-y\sqrt{\frac{\beta}{\varepsilon}}} &= Ce^{(x-y)\sqrt{\frac{\beta}{\varepsilon}}}e^{-x\sqrt{\frac{\beta}{\varepsilon}}} \\
&= Ce^{\frac{\sqrt{\varepsilon}}{\sqrt{\varepsilon}}\sqrt{\beta}}e^{-x\sqrt{\frac{\beta}{\varepsilon}}}, \quad \text{since} \quad x-y \leq \sqrt{\varepsilon} \\
&= Ce^{-x\sqrt{\frac{\beta}{\varepsilon}}}
\end{aligned}$$

Therefore,

$$|w'_L(x)| \leq C\varepsilon^{\frac{-1}{2}}e^{-x\sqrt{\frac{\beta}{\varepsilon}}}$$

Lemma 3 *The solution u of eqn(4.1) with the boundary conditions eqn(4.2) has the form*

$$u = v + w_L + w_R$$

where, for $0 \leq k \leq 4$, the regular component, v satisfies,

$$|v^{(k)}(x)| \leq C(1+\varepsilon^{\frac{-(k-2)}{2}}), \textit{for all} \quad x \in \overline{\Omega}$$

and the singular components w_L and w_R satisfy,

$$|w_L^k(x)| \leq C\varepsilon^{\frac{-k}{2}} e^{-x\sqrt{\frac{\beta}{\varepsilon}}}, \quad \textit{for all}, x \in \overline{\Omega}$$
$$|w_R^k(x)| \leq C\varepsilon^{\frac{-k}{2}} e^{-(1-x)\sqrt{\frac{\beta}{\varepsilon}}}, \quad \textit{for all}, x \in \overline{\Omega}$$

Proof: Noting that $v = v_0 + \varepsilon v_1$, from the above lemma, we have

$$|v_0^k(x)| \leq C$$
$$|v_1^k(x)| \leq C(1 + \varepsilon^{\frac{-k}{2}}), \quad \text{for all}, x \in \Omega$$

so that the regular component v satisfies,

$$|v^k(x)| \leq C(1 + \varepsilon^{\frac{-(k-2)}{2}}), \quad \text{for all}, x \in \Omega$$

The bounds on w_L and w_R is also given by the above lemma. To obtain the bounds for higher derivatives we simply differentiate it as follows.

$$|w_L(x)| \leq Ce^{-x\sqrt{\frac{\beta}{\varepsilon}}}$$
$$|w_L'(x)| \leq C\varepsilon^{\frac{-1}{2}} e^{-x\sqrt{\frac{\beta}{\varepsilon}}}$$
$$|w_L''(x)| \leq C\varepsilon^{-1} e^{-x\sqrt{\frac{\beta}{\varepsilon}}}$$

and so does for $k = 3, 4$

The same procedure is used for right side singular component as follows.

$$|w_R(x)| \leq Ce^{-(1-x)\sqrt{\frac{\beta}{\varepsilon}}}$$
$$|w_R'(x)| \leq C\varepsilon^{\frac{-1}{2}} e^{-(1-x)\sqrt{\frac{\beta}{\varepsilon}}}$$
$$|w_R''(x)| \leq C\varepsilon^{-1} e^{-x\sqrt{\frac{\beta}{\varepsilon}}}$$

and so does for $k = 3, 4$

The next theorem gives an ε-uniform estimates of $\overline{u} - u$ in the maximum norm, where $\overline{u}$ is the numerical solution of the problem eqn(4.1) with the homogeneous form of the boundary condition eqn(4.2).

Theorem *let $\overline{u}$ be the numerical solution of the problem in eqn(4.1) on the fitted mesh Ω_τ^N. Then*

$$\sup_{0<\varepsilon\leq 1} ||\overline{u} - u||_{\overline{\Omega}^N} \leq CN^{-2}(lnN)^2$$

where C is a constant independent of ε.

Proof:

The estimate is obtained separately on each element $\Omega_i = (x_{i-1}, x_i)$. Note that for any function g on Ω_i, $\quad \overline{g} = g_{i-1}\varphi_{i-1} + g_i\varphi_i$, so that on Ω_i we have

$$|\overline{g}(x)| \leq \max_{\Omega_i} |g(x)|$$

by appropriate Taylor expansion we have

$$|\overline{g}(x) - g(x)| \leq Ch_i^2 \max_{\Omega_i} |g''(x)| \tag{4.27}$$

From eqn(4.27) and lemma 2, on Ω_i

$$\begin{aligned} |\overline{u}(x) - u(x)| &\leq Ch_i^2 \max_{\Omega_i} |u''(x)| \\ &\leq Ch_i^2\big(1 + \varepsilon^{-1}\big) \\ &\leq \frac{Ch_i^2}{\varepsilon} \end{aligned} \tag{4.28}$$

Also, using lemma 3, it follows that

$$\begin{aligned}|\overline{u}(x) - u(x)| &= |\overline{v}(x) + \overline{w}_R(x) + \overline{w}_L(x) - v(x) - w_R(x) - w_L(x)| \\ &\leq |\overline{v}(x) - v(x)| + |\overline{w}_L(x) - w_L(x)| + |\overline{w}_R(x) - w_R(x)| \\ &\leq Ch_i^2 \max_{\Omega_i} |v''(x)| + 2\max_{\Omega_i} |w_L(x)| + 2\max_{\Omega_i} |w_R(x)| \\ &\leq C\left[h_i^2 + e^{-x\sqrt{\frac{\beta}{\varepsilon}}} + e^{-(1-x)\sqrt{\frac{\beta}{\varepsilon}}}\right]\end{aligned}$$

$$|\overline{u}(x) - u(x)| \leq C\left[h_i^2 + e^{-x\sqrt{\frac{\beta}{\varepsilon}}} + e^{-(1-x)\sqrt{\frac{\beta}{\varepsilon}}}\right] \tag{4.29}$$

Now, the argument depends on the layer resolving parameter $\tau = \min\{\frac{1}{4}, 2\sqrt{\frac{\varepsilon}{\beta}}\ln N\}$.
That is, whether $\tau = \frac{1}{4}$ or $\tau = 2\sqrt{\frac{\varepsilon}{\beta}}\ln N$,
In the first case, we have $\frac{1}{4} \leq 2\sqrt{\frac{\varepsilon}{\beta}}\ln N$ which implies $\frac{1}{\varepsilon} \leq \frac{C}{\beta}(lnN)^2$.

Here since the mesh is uniform, we have $h_i = N^{-1}$ implying that $h_i^2 = N^{-2}$

Thus using eqn(4.28), it follows that

$$|\overline{u}(x) - u(x)| \leq \frac{Ch_i^2}{\varepsilon} \leq CN^{-2}(lnN)^2$$

In the second case, we have $\frac{1}{4} \geq 2\sqrt{\frac{\varepsilon}{\beta}}\ln N$, which implies $\frac{1}{\varepsilon} \geq \frac{C}{\beta}(lnN)^2$.

And we consider the cases for the layer regions and the outer layer region.

Case 1: For the layer regions

That is, if $1 \leq i \leq \frac{N}{4}$ and $\frac{3N}{4} + 1 \leq i \leq N$, then

$$h_i = \frac{4\tau}{N} = 4N^{-1}\sqrt{\frac{\varepsilon}{\beta}}(lnN) \quad \text{Hence,} \quad \frac{h_i^2}{\varepsilon} = CN^{-2}(lnN)^2$$

Now using eqn(4.28), it follows that

$$|\overline{u}(x) - u(x)| \leq CN^{-2}(lnN)^2$$

Case 2: For the outer layer region

If i satisfies $\frac{N}{4}+1 \leq i \leq \frac{3N}{4}$, then $\tau \leq 1-x_i$, so that

$$e^{-(1-x_i)\sqrt{\frac{\beta}{\varepsilon}}} \leq N^{-2}$$

Similarly

$$e^{-x_i\sqrt{\frac{\beta}{\varepsilon}}} \leq N^{-2}, \text{since} \quad \tau \leq x_i$$

Which implies that

$$|\overline{u}(x)-u(x)| \leq C\left[h_i^2 + e^{-x_i\sqrt{\frac{\beta}{\varepsilon}}} + e^{-(1-x_i)\sqrt{\frac{\beta}{\varepsilon}}}\right] \leq CN^{-2}$$

Then combining the outer layer , left layer and right layer bounds, eqn(4.29) becomes

$$|\overline{u}(x)-u(x)| \leq CN^{-2} + CN^{-2}(lnN)^2 + CN^{-2}(lnN)^2 \leq CN^{-2}(lnN)^2$$

Therefore,

$$\sup_{0<\varepsilon\leq 1} ||\overline{u}-u||_{\overline{\Omega}^N} \leq CN^{-2}(lnN)^2$$

4.3 Numerical Example

To confirm the established theoretical results in this study, we have performed an experiment using the proposed numerical scheme on the problem of the form given in eqn(4.1) - eqn(4.2)

Example : Consider the singularly perturbed problem.

$$-\varepsilon u''(x) + u(x) = 1 + 2\sqrt{\varepsilon}\left[e^{\left(\frac{-x}{\sqrt{\varepsilon}}\right)} + e^{\left(\frac{x-1}{\sqrt{\varepsilon}}\right)}\right]$$

subjected to the boundary conditions

$$u(0) = 0, \quad u(1) = 0$$

its exact solution is given by

$$u(x) = 1 - xe^{\left(\frac{x-1}{\sqrt{\varepsilon}}\right)} - (1-x)e^{\left(\frac{-x}{\sqrt{\varepsilon}}\right)}$$

The numerical solution for the given example is expressed interms of maximum absolute error and it is obtained by:

$$E^N = \max_{1 \leq i \leq N} |\overline{u}_i^N - u_i^N|$$

Similarlly, the ε-uniform error estimate is obtained by:

$$E_\varepsilon^N = \sup_{0<\varepsilon<<1} |E^N|$$

Table 4.1: Maximum absolute error E_N for different values of ε and N using uniform mesh.

ε	N=16	N=32	N=64	N=128	N=256	N=512	N=1024
2^{-2}	9.425e-04	2.353e-04	5.879e-05	1.470e-05	3.674e-06	9.185e-07	2.296e-07
2^{-4}	2.240e-03	5.576e-04	1.393e-04	3.480e-05	8.700e-06	2.175e-06	5.438e-07
2^{-8}	2.015e-02	4.664e-03	1.145e-03	2.848e-04	7.113e-05	1.778e-05	4.446e-06
2^{-12}	1.583e-01	6.638e-02	1.770e-02	4.107e-03	1.009e-03	2.511e-04	6.271e-05
2^{-16}	2.593e-01	2.299e-01	1.538e-01	6.425e-02	1.708e-02	3.968e-03	9.749e-04
2^{-20}	2.678e-01	2.658e-01	2.578e-01	2.285e-01	1.526e-01	6.372e-02	1.693e-02
2^{-25}	2.680e-01	2.680e-01	2.677e-01	2.667e-01	2.627e-01	2.471e-01	1.962e-01
E^N	2.680e-01	2.680e-01	2.677e-01	2.667e-01	2.627e-01	2.471e-01	1.962e-01

Table 4.2: Maximum absolute error E_N for different values of ε and N using shishkin mesh.

ε	N=16	N=32	N=64	N=128	N=256	N=512	N=1024
2^{-2}	9.425e-04	2.353e-04	5.879e-05	1.470e-05	3.674e-06	9.185e-07	2.296e-07
2^{-4}	2.240e-03	5.576e-04	1.393e-04	3.480e-05	8.700e-06	2.175e-06	5.438e-07
2^{-8}	2.015e-02	4.664e-03	1.145e-03	2.848e-04	7.113e-05	1.778e-05	4.446e-06
2^{-12}	1.925e-02	6.759e-03	3.375e-03	8.311e-04	2.712e-04	8.576e-05	2.647e-05
2^{-16}	1.859e-02	6.540e-03	3.259e-03	8.028e-04	2.621e-04	8.289e-05	2.558e-05
2^{-20}	1.842e-02	6.486e-03	3.230e-03	7.957e-04	2.599e-04	8.218e-05	2.725e-05
2^{-25}	1.838e-02	6.471e-03	3.222e-03	7.937e-04	2.592e-04	8.198e-05	2.923e-05
E^N	2.015e-02	6.759e-03	3.375e-03	8.311e-04	2.712e-04	8.576e-05	2.923e-05

The computed rate of convergence of the developed scheme is obtained by:

$$R^N = \frac{\log E^N - \log E^{2N}}{\log 2}$$

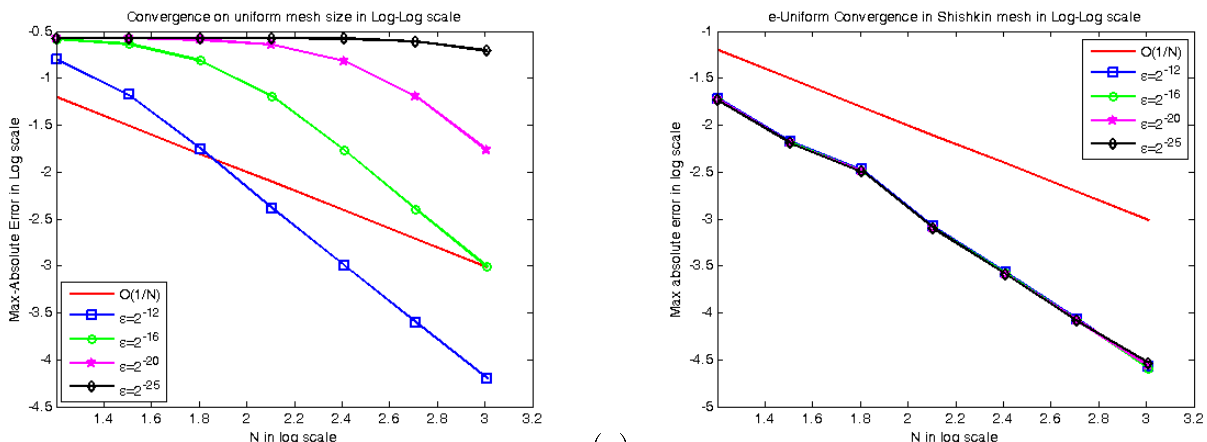
and the ε-uniform rate of convergence of the developed scheme is obtained by:

$$R_\varepsilon^N = \frac{\log E_\varepsilon^N - \log_\varepsilon E^{2N}}{\log 2}$$

The following table shows the rate of convergence of the present methods for different values of the mesh element.

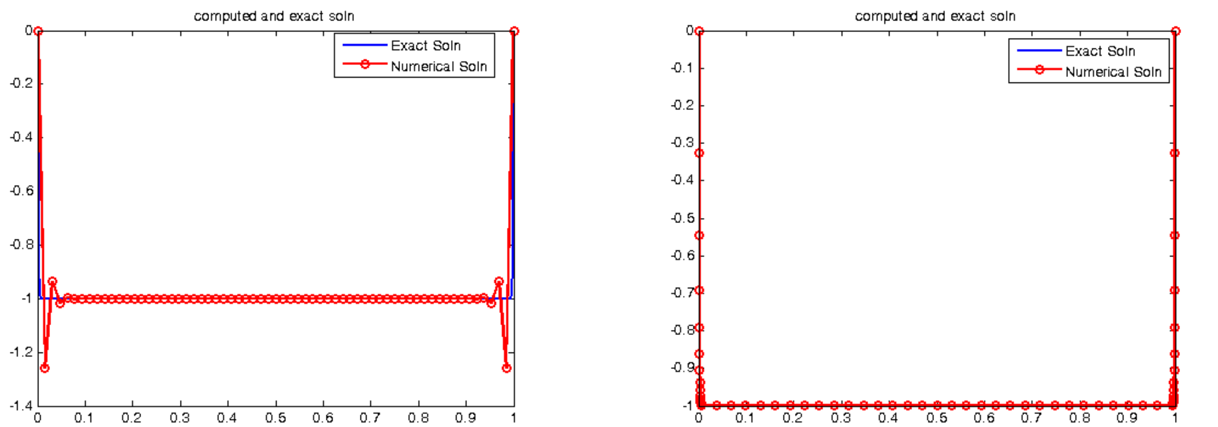
Table 4.3: A Rate of convergence for the developed scheme

ε	N=16	N=32	N=64	N=128	N=256	N=512
2^{-2}	2.0020	2.0009	1.9998	2.0004	2.0000	2.0002
2^{-4}	2.0062	2.0010	2.0010	2.0000	2.0000	1.9999
2^{-8}	2.1111	2.0262	2.0073	2.0014	2.0002	1.9997
2^{-12}	1.5100	1.0019	2.0218	1.6157	1.6610	1.6959
2^{-16}	1.5072	1.0049	2.0213	1.6149	1.6608	1.6962
2^{-20}	1.5059	1.0058	2.0212	1.6143	1.6611	1.5925
2^{-25}	1.5061	1.0060	2.0213	1.6145	1.6607	1.4878
R_ε^N	1.5759	1.0019	2.0218	1.6157	1.6610	1.5529



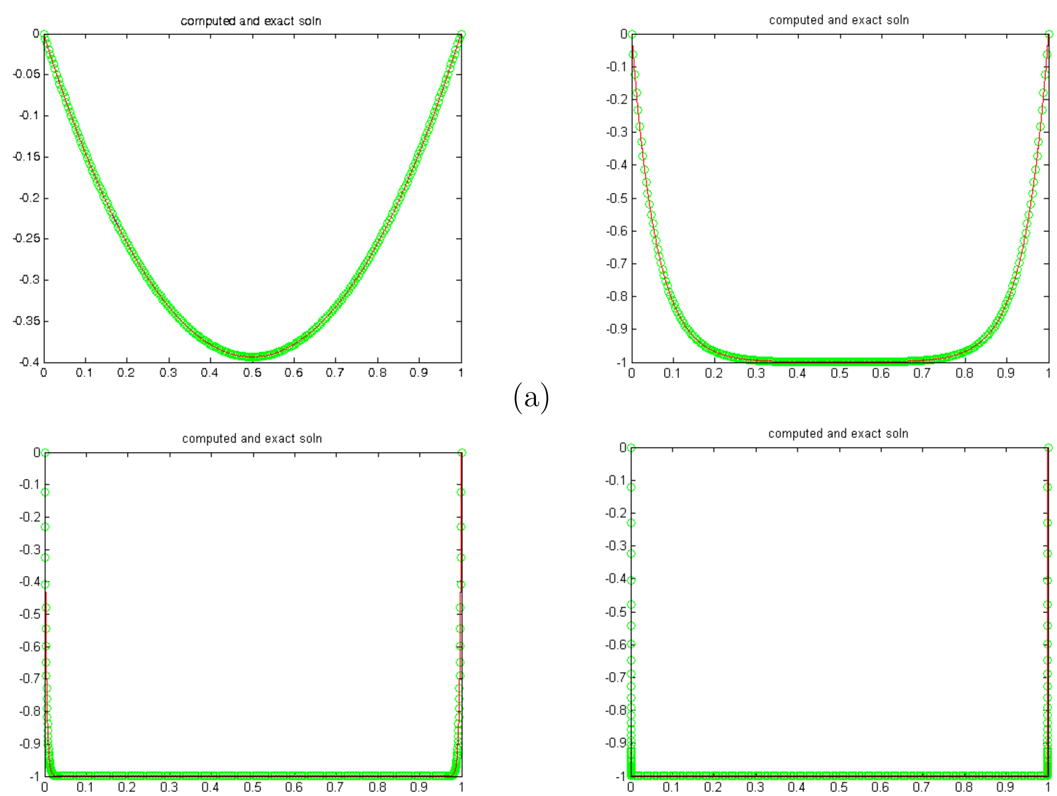
(a) (b)

Figure 4.2: The graph of maximum absolute error versus the number of mesh for a uniform mesh on a and shishkin mesh on b respectively using log-log scale plot.



(a) (b)

Figure 4.3: Computed solution of example 1 when $\varepsilon = 2^{-20}$, $N = 64$, using uniform mesh on (a) and shishkin mesh in (b).



(a) (b) (c) (d)

Figure 4.4: Computed solution of example 1 when $N = 256$ and $\varepsilon = 1, 2^{-8}, 2^{-16}$ and 2^{-25} respectively.

4.4 Discussion

In this thesis, fitted mesh finite element method is presented for solving singularly perturbed one dimensional reaction-diffusion problems. First the given differential equation is transformed to it's weak form and the given domain is discretized in to a mesh elements. By defining a linear base functions, the approximate solution is represented by a linear combination of the base functions and obtained a system of equation on each element by using Galerkin's method which is assembled to obtain system of algebraic equation. The numerical results is presented in tables for different values of the perturbation parameter ε and different number of mesh elements N.

Table (4.1) and (4.2) indicates the maximum absolute error for the numerical solution of the present method for the given model problem using uniform and shishkin mesh respectively. As it can be observed from these tables, the present method with uniform meshes is not convergent because as the perturbation parameter varies the maximum absolute error also vary so that the method is dependent on the perturbation parameter. Hence, in order to overcome this, we have applied shishkin mesh so that the mesh is constructed in such way that the boundary layer regions contains more mesh elements that enables us to capture the layer regions. As a result, the maximum absolute error is shown to be independent of the effect of the perturbation parameter ε implying that the method is ε-uniform or uniformly convergent. The result from the two tables also indicate that, computed result using shishkin mesh shows greater agreement with the exact solution. Moreover as the mesh number increases the maximum absolute error decreases along a row which shows the convergence of the present method.

Figure (4.2) indicates the graph of maximum absolute error versus the number of meshes for a uniform mesh and shishkin mesh respectively using log log plot. As it can be observed from graph (a), whenever the perturbation parameter decreases the graphs diverges implying that the method is not convergent and the ε-uniform convergence of the method also deteriorate due to the fact that the layers are not resolved. But from (b), it can be observed that, as the perturbation parameter decreases the graphs converges uniformly implying that ε-uniform converges of the method is not deteriorated so that the layers are resolved. Figure (4.3) indicates the computed solution with uniform mesh oscillates in the boundary layer regions. To control this disturbances, we have used shishkin mesh discretization technique and the results are far better than using uniform mesh. It is also possible to deduce from figure (4.4) that applying shishkin mesh discretization technique resolves the oscillation of the numerical solution in the boundary layers as the perturbation decreases.

Table (4.3) indicates that the the rate of convergence for the present method is almost second order which is in agreement with the theoretical expectation.

Chapter 5

Conclusion and Scope of the Future work

5.1 Conclusion

In this thesis, fitted mesh finite element method is presented for solving singularly perturbed reaction-diffusion equation. The scheme is shown to be ε-uniform theoretically. The study is implemented on a model example by taking different values for the perturbation parameter ε and the computational results are presented in a tables and graphs. The obtained result indicates that the present method is ε-uniform and approximate the exact solution very well in agreement with the theoretical result. The result from the rate of convergence indicate the method is almost second-order which is the theoretical expectation.

5.2 Scope of the Future work

In the present thesis, the numerical methods based on fitted mesh finite element method using linear basis functions is constructed for solving singularly perturbed reaction-diffusion problems. The scheme proposed in this thesis can also be extended to quadratic basis functions for solving singularly perturbed reaction-diffusion problems.

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