FIXED POINT RESULTS OF ALMOST (ϕ, ψ) CONTRACTIONS INVOLVING RATIONAL EXPRESSIONS IN PARTIALLY ORDERED METRIC SPACES

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS, JIMMA UNIVERSITY IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS

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Declaration

I, the undersigned declare that, this thesis entitled "Fixed point results of almost (ϕ, ψ) contractions involving rational expressions in partially ordered metric spaces" is original and it has not been submitted to any institution elsewhere for the award of any academic degree or like, where other sources of information that have been used, they have been acknowledged .

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Last but not least, I would like to thank both Differential Equations and Numerical Analysis Stream Post Graduate Students for their constructive comments and provision of some references while I was doing this research.

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Abstracts

This research dealt with some fixed point results of almost (ϕ, ψ) contractions involving rational expressions in the framework of metric spaces endowed with a partial order by extending the works of S. Chandok et.al. [8]. Our results extend and improves the results of S. Chandok et.al. [8]. The researcher followed analytical design in this research work. Secondary source of data such as journal articles and books which are found in different libraries and internet were used for the study. The procedures employed for the analysis of this study were techniques used by S.Chandok et.al. [8] We provided examples in support of our main findings. This study was conducted from October 2014 to June 2015.

Keywords*:* Partially ordered metric space, fixed point, Rational type contraction, Altering distance function, Almost (ϕ, ψ) contraction, partially ordered set.

Unit one

1. Introduction

1.1 Background of the study

Let X be a nonempty set. A map $T: X \longrightarrow X$ is said to a self-map of X. An element x in X is called a fixed point of T if $Tx = x$. Let X be the set of all real numbers then the fixed points of $T: X \longrightarrow X$ defined by $Tx = x^2 - 3x + 3$ are 1 and 3.

Let (X, d) be a metric space. A self-map $T: X \longrightarrow X$ is said to be a contraction if there is a real number $k \ in$ $[0,1)$ such that:

$$d(Tx, Ty) \leq kd(x, y) \text{ for all } x, y \text{ in } X. \quad (1)$$

In this case k is called a contraction constant.

A theory of fixed point is one of the most powerful and popular tools of modern mathematics. Its use is not only confined to pure and applied mathematics but also it serves as a bridge between analysis and topology besides facilitating a very fruit full area of interaction between analysis and topology and also to examine the quantitative problems involving certain mappings and space structures required in various areas such as: economics, chemistry, biology, computer science, engineering, and others. For more details one can refer [5, 6, 8, 15].

The first most significant result of metric fixed point theory was given by the Polish mathematician Stefan Banach, in 1922, which is known as Banach contraction principle. The famous Banach contraction principle [5] states that if (X, d) is a complete metric space and a self map $T: X \longrightarrow X$ is a contraction, then T has a unique fixed point x in X. Banach contraction principle is one of the cornerstones in the development of nonlinear analysis. [1,5].

There are a number of extensions and generalizations of Banach contraction principle by many researchers who have obtained fixed point and common fixed point results in metric spaces, cone-metric spaces, partially ordered metric spaces and others spaces. $[1 - 18]$.

In 1968, Kannan [19] introduced a different contraction condition where the map $T: X \longrightarrow X$ considered need not be continuous.

Theorem 1.1(Kannan, 1968) [19] Let(X, d) be a complete metric space and T be a self- map of on X. .Suppose there exists $\alpha \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \ in \ X, \quad (2)$$

then T has a unique fixed point.

In 1972, Chatterjea [12] gave the dual of Kannan fixed point theorem as follows:

Theorem 1.2 (Chatterjea, 1972) [12] Let (X, d) be a complete metric space and T be a self-map on X. Suppose there exists $\alpha \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \alpha[d(x, Ty) + d(y, Tx)] \text{ for all } x, y \in X. \quad (3)$$

Then T has a unique fixed point.

In 1977, Rhoades [28] showed that Banach contraction principle, Kannan.R mapping and Chateerijea are independent.

In 1972, Zamfirescu [33] proved the following fixed point theorem by combining (1), (2) and (3) as follows.

Theorem 1.3 (Zamfirescu, 1972)[33] Let (X, d) be a complete metric space and T be a self - map of X for which there exist real numbers a, b *and* c satisfying $0 \leq a < 1, 0 \leq b, c < \frac{1}{2}$ such that for each pair $x, y \in X$ at least one of the following holds:

$$(Z_1)\ d(Tx, Ty) \leq ad(x, y),$$
$$(Z_2)\ d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)], \quad (4)$$
$$Z_3)\ d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$$

then T has a unique fixed point. Therefore Zamfirescu's theorem [33] is a unification of (Z_1) Banach's theorem, (Z_2) Kannan's theorem [19] and (Z_3) Chatterjea's theorem [12].

Notation: Throughout this paper we denote $\mathbb{R}^+ = [0, \infty)$ (The set of non- negative real numbers)

In [1] weakly contraction mapping is defined as follows:

Definition 1.4 [1] Let (X, d) be a metric space. A mapping $T: X \to X$ is said to be weakly contraction if $d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y))$, for all $x, y \in X$, where $\varphi: \mathbb{R}^+ \to \mathbb{R},^+$ is a continuous and non-decreasing function with $\varphi(t) = 0$ if and only if $t = 0$. (5)

In 1997, Alber and Guerre-Delbariere [1] introduced the concept of weakly contractive maps in a complete Hilbert spaces as a generalization of contractive maps and proved that any weakly contractive mapping defined on complete Hilbert spaces has a unique fixed point. Rhoades [29] extended this concept to the Banach spaces and proved the existence of fixed points of weakly contractive maps in the setting of metric space.

One of the generalizations of Banach contraction principle is through the method of altering distances between the points with the help of a continuous control function. In 1977 Delbosco [14] and Skof [31] initiated the technique of altering distances between the points to establish the existence of fixed points simultaneously. The method of altering distances became famous by khan, Swaleh and Sessa[20]. Some works in this line of research can be referred in [14,20] and reference there in.

In 2004, Berinde [7] introduced weak contraction maps which are named as almost contractions as a generalization of contraction maps and proved fixed point results in complete metric spaces.

Definition 1.5: Berinde, 2004 [7] Let (X, d) be a metric space then a map $T: X \longrightarrow X$ is called almost contraction or (δ, L) contraction if there exist a constant $\delta \in (0,1)$ and a constant $L \geq 0$ such that:

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx) \quad \text{for all } x, y \in X. \tag{6}$$

So almost contraction form is a class of generalized contractions that includes several contractive mappings like usual contraction, kanan mappings, Zamfirescu mappings etc. For more works on almost contraction refer [4,7,22,23].

Since the early days of metric fixed point theory, numerous authors attempted to vary the contraction conditions by improving the existing contraction conditions and replacing with various types of the general conditions.

For example in 1975, B.K. Dass & S. Gupta [13] extended Banach's contraction principle through rational expression as follows:

Theorem 1.6: (Dass and Gupta [13]) Let(X, d) be a complete metric space and $T: X \longrightarrow X$ a mapping such that there exist $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$ satisfying the contraction condition:

$$d(Tx, Ty) \leq \alpha \left(\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}\right) + \beta d(x, y) \tag{7}$$

for all $x, y \in X$, Then T has a unique fixed point.

Since the notion of metric spaces was introduced in 1906 by Maurice Frechet several authors worked in metric spaces endowed with partial order.

Ran and Reurings [27] extended the Banach contraction principle in partially ordered sets with some applications to linear and nonlinear matrix equations. While Nieto and Rodriguez-Lopez [25] extended the result of Ran and Reurings [27] and applied their main theorems to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions, Hence the Banach's contraction mapping principle is the most versatile elementary results of mathematical analysis which is widely applied in different branches of mathematics and it is regarded as the source of metric fixed point theory $[1-18]$.

In 2012, Chandok S. and Kim J.K. [9] proved the following fixed point theorem.

Theorem 1.7: (Chandok S. and Kim J.K. [9]) Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exist a metric d on X such that (X, d) is a complete metric space. Suppose that T is a continuous self- mapping on X, T is monotone non-decreasing mapping and

$$d(Tx, Ty) \leq \alpha\left(\frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)}\right)+\beta d(x,y) \text{ for all } x, y \in X, \tag{8}$$

with $x \succcurlyeq y$ and for some $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$, if there exists $x_\circ \in X$ with $x_\circ \preccurlyeq Tx_\circ$, then T has a fixed point.

In 2013 M.Arshad et al. [3] proved some unique fixed point theorems for rational type contractions in partially ordered metric spaces and in the year 2013 S. Chandok et al. [10] proved some common fixed point results in partially ordered metric paces for generalized rational type contraction mappings using auxiliary functions. So the rational type contractions have been improved by many researchers in various ways.

In 2015 S. Chandok et.al [8] proved fixed point result in partially ordered metric spaces with rational type contraction using some auxiliary functions. So the researcher motivated to extend and improves this work.

The purpose of this study was to prove the existence of fixed point results of almost(ϕ, ψ) contractions involving rational expressions in partially ordered metric spaces by extending the work`s of S. Chandok et al. [8]. We provided illustrative examples which support the results of the study.

1.2 Statements of the Problems

In this study the researcher concentrated on fixed point results of almost (ϕ, ψ) contractions involving rational expressions in partially ordered metric spaces.

This study answered the following questions:

1. How can we prove the existence of fixed points of almost (ϕ, ψ) contractions involving rational expressions in partially ordered metric spaces?
2. What additional conditions are required to obtain a unique fixed point for almost (ϕ, ψ) Contractions involving rational expressions in partially ordered metric spaces?
3. How can the researcher support the results by providing applicable example?

1.3 Objectives of the study

1.3.1General Objective of the study

The main objectives of this study was to establish some fixed point results of almost (ϕ, ψ) contractions involving rational expressions in partially ordered metric spaces by extending the work`s of S. chandok et.al. [8].

1.3.2 Specific objectives

The specific objectives of this study are:

1. To prove the existence of fixed point results of almost (ϕ, ψ) contractions involving rational expressions in partially ordered metric spaces.
2. To discuss additional conditions required to obtain a unique fixed point for almost $(\phi, \boldsymbol{\psi})$ Contractions involving rational expression in partially ordered metric spaces.
3. To provide examples in the support of the result of the study.

1.4 Significance of the study

The study would have the following importance:

1. The results obtained in this study may contribute to research activities in this area.
2. It may help the researcher to develop scientific research writing skills and scientific communication in Mathematics.

1.5 Delimitation of the study

This study was delimited to prove the existence of fixed point results of almost (ϕ, ψ) contractions involving rational expressions in partially ordered metric spaces.

Unit 2

2. Literature Review

Fixed point theory is one of the famous and traditional theories in mathematics and has a broad set of applications. The applications of fixed point theory are very important in diverse disciplines of mathematics .It can be applied for solving various problems, for instance, equilibrium problems, variation problems, and optimization problems. In 1922, Stefan Banach [5] stated his celebrated theorem on the existence and uniqueness of fixed point of contraction of selfmaps defined on complete metric spaces for the first time, which is known as the Banach contraction mapping principle.

The Banach's contraction mapping principle is one of the cornerstones in the development of fixed point theory. In particular, this principle is used to demonstrate the existence and uniqueness of a solution of differential equations, integral equations, functional equations, partial differential equations and others. Due to the importance, generalizations of Banach's contraction mapping principle have been investigated heavily by many authors. Consequently, a number of generalizations of this celebrated principle have appeared in the literature (see [5,6, 25]).

Since then many researchers have obtained fixed point and common fixed point results in metric spaces, cone metric spaces, partially ordered metric spaces and other spaces.

In the theory of fixed point, contraction is one of the main tools to prove the existence and uniqueness of a fixed point. Banach's contraction principle, which gives an answer on the existence and uniqueness of a solution of an operator equation,$Tx = x$ is the most widely used fixed point theorem in all of analysis. This principle is constructive in nature and is one of the most useful tools in the study of nonlinear equations. There have been a number of generalizations of metric spaces such as, cone metric spaces, cone b metric spaces, partially ordered metric spaces and other spaces[8, 24 − 33].

Alber and Guerre-Delbariere [1] introduce the concept of weakly contractive maps in a complete Hilbert spaces as a generalization of contraction maps. Rhoades [29] extended this concept to the Banach spaces and proved the existence of fixed points of weakly contractive maps in the setting of metric space.

One of the generalizations of Banach contraction principle is through the method of altering distances between the points with the help of a continuous control function. In 1977, Delbosco [14] and Skof [31] initiated the technique of altering distances between the points to establish the existence of fixed points simultaneously and the method of altering distances become famous by khan, Swaleh and Sessa [20], $s$ome works in this line of research can be referred in [14,20].
In 2004, Berinde [7] introduced weak contraction maps which are named as almost contractions as a generalization of contraction maps and proved fixed point results in complete metric spaces and almost contractions are defined as follows:
Let (X, d) be a metric space. Then a self map $T: X \to X$ is called almost contraction or $(\delta, L)-$ contraction (weak contraction) if there exists a constant $\delta \in (0, 1)$ and a constant $L \geq 0$ such that:

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx) \quad \text{for all } x, y \in X . \tag{9}$$

So almost contraction form is a class of generalized contractions that includes several contractive mappings like the usual contraction, kanan mappings etc. For more works on almost contraction refer [4,7,22,23]. Since the early days of metric fixed point theory, numerous authors attempted to vary the contraction conditions by improving the existing contraction conditions and replacing with various types of the general conditions.

For example in 1975, Dass & Gupta [13] extend Banach's contraction principle through rational expression as follows.
Let (X, d) be a complete metric space and $T: X \to X$ a mapping such that there exist $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$ satisfying;

$$d(Tx, Ty) \leq \alpha \left(\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}\right)+\beta d(x, y) \text{ for all } x, y \in X \tag{10}$$

then T has a unique fixed point.
Hence the Banach's contraction mapping principle is the most versatile elementary results of mathematical analysis which is widely applied in different branches of mathematics and is regarded as the source of metric fixed point theory $[1-17]$.
In recent times, fixed point theory has developed rapidly in partially ordered metric spaces, that is, metric spaces endowed with a partial ordering. The triple $(X, d, \preccurlyeq)$ is called partially ordered metric spaces if$(X, \preccurlyeq)$ is a partially ordered set and (X, d) is a metric space. [6, 4]

Application of fixed point results in partially ordered metric spaces was made subsequently, for example, by Ran and Reurings [27] in solving matrix equations and by Nieto and Rodriguez-Lopez [25] to obtain solutions of certain partial differential equations with periodic boundary conditions.

In 2012 Chandok and Kim [9] proved the following fixed point theorem.

Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exist a metric d on X such that (X, d) is a complete metric space. Suppose that T is a continuous self mapping on X, T is monotone non decreasing mapping and

$$d(Tx, Ty) \leq \alpha\left(\frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)}\right)+\beta d(x, y) \text{ for all } x, y \in X \text{ such that} \qquad (11)$$

$x \succcurlyeq y$ and for some $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$, if there exists $x_\circ \in X$ with $x_\circ \preccurlyeq Tx_\circ$,then T has a fixed point.

Recently, many researchers have obtained fixed point and common fixed point results in partially ordered metric spaces. In 2013 M.Arshad et al. [3] proved some unique fixed point theorems for rational type contractions in partially ordered metric spaces and in the year 2013 S.chandok et al. [11] proved some common fixed point results in partially ordered metric paces for generalized rational type contraction mappings. In 2015, Sumit chandok [8] proved fixed point results in partially ordered metric spaces involving rational type expressions using some auxiliary functions.

Unit 3

3. Methodology

3.1 Study site and period

This study was conducted from October 2014 G.C to June 2015 G.C. in Jimma University under Mathematics Department.

3.2 Study Design

In order to achieve the objectives of the study, Analytical design method was used.

3.3 Source of information

This study mostly depended on document materials, so the available source of information for the study were Books, Journals, different study related to the topic and internet services. So, the researcher collected different documents that were listed which support the study and discussed about the collected materials and other activities with advisor.

3.4 Procedure of the study

The procedure the researcher followed for analysis were the standard technique used by S. Chandok et al. [8]

3.5 Ethical issue

The researcher has taken a cooperation request letter from Mathematics Department of Jimma University to get consent from the institute(s) where Books, Journals, internet and other related materials were available for this study to collect related information. Moreover; kept rules and regulations of the institute(s) from where the researcher got these materials.

Unit 4
4. Discussion and Result

4.1 Preliminaries

Definition.4.1.1 Let X be a non-empty set and $d: X \times X \longrightarrow \mathbb{R}^+$ be a mapping satisfying the following conditions for all $x, y, z \in X$:

i. $d(x,y) \geq 0$ and $d(x,y) = 0$ ifand only $x = y$;

ii. $d(x,y) = d(y,x)$, (symmetry);

iii. $d(x,z) \leq d(x,y) + d(y,z)$ (Triangular inequality),

then d is called a metric on X. Then the pair (X, d) is called a metric space.

Example.1 Let $X = \mathbb{R}$ (the set of real numbers) define $d: X \times X \longrightarrow \mathbb{R}^+$ by $d(x,y) = |x - y|$, for all $x, y \in X$then clearly the pair (X, d) is a metric space.

Example.2. For any set X, Define $d: X \times X \longrightarrow \mathbb{R}^+$ by $(x,y) = \begin{cases} 0 & if\ x = y \\ 1 & if\ x \neq y \end{cases}$. Then the pair (X, d) is a metric space and we call this metric discrete metric.

Definition.4.1.2 A partially ordered set (poset) is a system $(X, \preccurlyeq)$ where X is non-empty set and $\preccurlyeq$ is a binary relation of X satisfying for all $x, y, z \in X$:

i. $x \preccurlyeq x$ (reflexivity);

ii. $if\ x \preccurlyeq y\ and\ y \preccurlyeq x\ \ then\ x = y$ (anti symmetry);

iii. $if\ x \preccurlyeq y\ and\ y \preccurlyeq\ z\ \ then\ x \preccurlyeq\ z$ (transitivity).

Example

(1) If X is any set $(P\ (X), \subseteq)$ is a partially ordered set. Where $P(X)$= the power set of X.

(2) On the set on natural numbers N, define $m \preccurlyeq n$ if m divides n then $(N, \preccurlyeq)$ is a Partially ordered set.

Definition 4.1.3 Let X is a non-empty set**.** Then $(X, d, \preccurlyeq)$ is called partially ordered metric spaces if:

i. (X, d) is a metric space and

ii. $(X, \preccurlyeq)$ is a partially ordered set.

Definition 4.1.4 [8, 4] Let $(X, \preccurlyeq)$ be a partially ordered set and $T: X \rightarrow X$ is a self- mapping, we say T is monotone non-decreasing with respect to $\preccurlyeq$ if for $x, y \in X, x \preccurlyeq y \Longrightarrow Tx \preccurlyeq Ty$.

Definition: 4.1.5 [6] Let (X, $\preccurlyeq$) be a partially ordered set and $x, y \in X$ then x and y are said to be comparable elements of X if $x \preccurlyeq y$ or $y \preccurlyeq x$.

Definition 4.1.6 [8]: A function $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ is called an altering distance function if :

i. φ is non-decreasing, continuous and
ii. $\varphi(t) = 0$ if and only if $t = 0$.

Notation:

We denote: $\boldsymbol{\Phi} = \{\boldsymbol{\phi}: \mathbb{R}^+ \to \mathbb{R}^+,$ **such that** $\boldsymbol{\phi}$ **is continuous, non-decreasing and** $\boldsymbol{\phi(t)} = 0$ **if and only if t=0}.**

$\boldsymbol{\Psi} = \{\boldsymbol{\psi}: \mathbb{R}^+ \to \mathbb{R}^+,$ **such that for any sequence** $\{\boldsymbol{x_n}\}$ **in** $\mathbb{R}^+$ **with** $\boldsymbol{x_n} \to \boldsymbol{t}, (\boldsymbol{t} > 0)$, $\underline{\lim}\, \psi(x_n) > 0\}$**.**

Theorem 4.1.7 [8] Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $T: X \longrightarrow X$ be a continuous and non decreasing mapping. Suppose that there exist $\phi \in \Phi, \psi \in \Psi$ such that

$$\phi\big(d(Tx, Ty)\big) \le \phi\big(M(x, y)\big) - \psi\big(N(x, y)\big)$$

for all $x, y \in X$ with $x \preccurlyeq y$ where

$$M(x, y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, \frac{d(y,Tx)[1+d(x,Ty)]}{1+d(x,y)}, d(x, y)\right\} \text{ and}$$

$$N(x, y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, d(x, y)\right\}.$$

If there exists $x_\circ \in X$ with $x_\circ \preccurlyeq Tx_\circ$, then T has a fixed point in X.

Theorem 4.1.8 [8] Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Assume that if $\{x_n\}$ a non-decreasing sequence in X such that $x_n \longrightarrow x$, then $x_n \preccurlyeq x$, for all $n \in N$. Let $T: X \to X$ be a nondecreasing mapping suppose that there exist $\phi \in \Phi, \psi \in \Psi$ such that

$$\phi\big(d(Tx, Ty)\big) \le \phi\big(M(x, y)\big) - \psi\big(N(x, y)\big)$$

for all $x, y \in X$ with $x \preccurlyeq y$ where

$$M(x, y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, \frac{d(y,Tx)[1+d(x,Ty)]}{1+d(x,y)}, d(x, y)\right\} \text{ and}$$

$$N(x, y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, d(x, y)\right\}.$$

If there exists $x_\circ \in X$ with $x_\circ \preccurlyeq Tx_\circ$, then T has a fixed point X.

Theorem 4.1.9 [8] In addition to the hypotheses of Theorem 4.1.7 or Theorem 4.1.8 suppose that for every $x, y \in X$, there exists $u \in X$ such that $u \preccurlyeq x$ and $u \preccurlyeq y$. Then T has a unique fixed point in X.

Definition 4.1.10 Let $(X, \preccurlyeq)$ be a partially ordered set. Suppose that there exist a metric d on X such that (X, d) is a metric space. Let $T: X \to X$ be a self-map of X. If there exist functions $\phi \in \Phi, \psi \in \Psi$ and a constant $L \geq 0$ such that

$$\phi\big(d(Tx, Ty)\big) \leq \phi\big(M(x, y)\big) - \psi\big(N(x, y)\big) + Lm(x, y)$$

for all $x, y \in X$ with $x \preccurlyeq y$

Where: $M(x, y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, \frac{d(y,Tx)[1+d(x,Ty)]}{1+d(x,y)}, d(x, y)\right\}$

$N(x, y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, d(x, y)\right\}$ and

$m(x, y) = min\left\{\frac{d(x,Ty)d(y,Tx)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\}$.

Then we say that T satisfies almost (ϕ, ψ) contraction condition involving rational expression.

4.2 Main Result

Theorem: 4.2.1 Let $(X, \preccurlyeq)$ be a partially ordered set. Suppose that there exist a metric d on X such that (X, d) is a complete metric space. Let $T: X \to X$ be a continuous and non decreasing mapping. Assume that there exist functions $\phi \in \Phi, \psi \in \Psi$ and a constant $L \geq 0$ such that

$$\phi\big(d(Tx, Ty)\big) \leq \phi\big(M(x,y)\big) - \psi\big(N(x,y)\big) + Lm(x,y) \qquad (4.2.1.1)$$

for all $x, y \in X$ with $x \preccurlyeq y$

where: $M(x,y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, \frac{d(y,Tx)[1+d(x,Ty)]}{1+d(x,y)}, d(x,y)\right\}$

$N(x,y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, d(x,y)\right\}$ and

$m(x,y) = min\left\{\frac{d(x,Ty)d(y,Tx)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\}$.

If there exists $x_\circ \in X$ with $x_\circ \preccurlyeq Tx_\circ$ then T has a fixed point.

***Proof*:** Let $x_\circ \in X$. If $Tx_\circ = x_\circ$, then $x_\circ$ is a fixed point of T.
Suppose that $x_\circ \preccurlyeq Tx_\circ$, constructing a sequence $\{x_n\}$ in X, such that

$$x_{n+1} = Tx_n \text{ for every } n \geq 0. \qquad (4.2.1.2)$$

Since T is non-decreasing mapping and $x_\circ \preccurlyeq Tx_\circ$ we have:

$$x_\circ \preccurlyeq Tx_\circ = x_1 \preccurlyeq Tx_1 = x_2 \preccurlyeq Tx_2 = x_3 \ldots \preccurlyeq Tx_{n-1} = x_n \preccurlyeq Tx_n = x_{n+1} \preccurlyeq \cdots$$

Hence $x_n \preccurlyeq x_{n+1}$ for each$n = 0, 1, 2, \ldots,$ then we have

$$x_\circ \preccurlyeq x_1 \preccurlyeq x_2 \preccurlyeq x_3 \preccurlyeq \cdots \preccurlyeq x_{n-1} \preccurlyeq x_n \preccurlyeq x_{n+1} \preccurlyeq \cdots \qquad (4.2.1.3)$$

Now, if there exists $n_\circ \geq 1$ such that $x_{n_\circ}$=$x_{n_\circ+1}$then from (4.2.1.2)

We have $x_{n_\circ} = x_{n_\circ+1} = Tx_{n_\circ}$, hence $x_{n_\circ}$ is a fixed point of T.

We now suppose $x_n \neq x_{n+1}$ for each $n = 0,1,2 \ldots$. Then $d(x_{n+1}, x_n) \neq 0$ for all $n \geq 0$ and

Let $R_n = d(x_{n+1}, x_n), \forall n \geq 0$.

Since$x_{n-1} \preccurlyeq x_n, \forall n \geq 1$, from (4.2.1.1) and (4.2.1.3) we have:

$$\phi\big(d(x_n, x_{n+1})\big) = \phi\big(d(Tx_{n-1}, T x_n)\big) \leq \phi\big(M(x,y)\big) - \psi\big(N(x,y)\big) + Lm(x,y) \qquad (4.2.1.4)$$

$$\leq \phi\left(max\left\{\frac{d(x_n,Tx_n)[1+\mathrm{d}(x_{n-1},\mathrm{T}x_{n-1})]}{1+d(x_{n-1},x_n)}, \frac{d(x_n,Tx_{n-1})[1+\mathrm{d}(x_{n-1},\mathrm{T}x_n)]}{1+d(x_{n-1},x_n)}, d(x_{n-1}, x_n)\right\}\right)$$

$$-\psi\left(max\left\{\frac{d(x_n, Tx_n)\big[1 + \mathrm{d}\big(x_{n-1}, \mathrm{T}x_{n-1}\big)\big]}{1 + d(x_{n-1}, x_n)}, d(x_{n-1}, x_n)\right\}\right)$$

$$+ L(min\left\{\frac{d(x_{n-1},\mathrm{T}x_n)d(x_n,\mathrm{T}x_{n-1})}{1+d(x_{n-1},x_n)}, \frac{d(x_{n-1},\mathrm{T}x_{n-1})d(x_n,\mathrm{T}x_n)}{1+d(x_{n-1},x_n)}\right\})$$

$$= \phi\left(max\left\{\frac{d(x_n, x_{n+1})[1 + \mathrm{d}(x_{n-1,}x_n)]}{1 + d(x_{n-1}, x_n)}, \frac{d(x_n, x_n)[1 + \mathrm{d}(x_{n-1,}x_{n+1})]}{1 + d(x_{n-1}, x_n)}, d(x_{n-1}, x_n)\right\}\right)$$

$$-\psi\left(max\left\{\frac{d(x_n, x_{n+1})[1+\mathrm{d}(x_{n-1,}x_n)]}{1+d(x_{n-1},x_n)}, d(x_{n-1}, x_n)\right\}\right)$$

$$+ L(min\left\{\frac{d(x_{n-1},x_{n+1})d(x_n,x_n)}{1+d(x_{n-1},x_n)}, \frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{1+d(x_{n-1},x_n)}\right\})$$

$$= \phi(max\{d(x_n, x_{n+1}), 0, d(x_{n-1}, x_n)\})_{-\psi(max\{d(x_n,x_{n+1}),d(x_{n-1},x_n)\})}$$

$$+ L(min\left\{0, \frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{1+d(x_{n-1},x_n)}\right\})$$

$$= \phi(max\{d(x_n, x_{n+1}), 0, d(x_{n-1}, x_n)\}) - \psi(max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\}) + L(0)$$

$$= \phi(max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\}) - \psi(max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\}$$

This gives: $\phi(R_n) = \phi(d(x_{n+1}, x_n)) \leq \phi(max\{R_n, R_{n-1}\}) - \psi(max\{R_n, R_{n-1}\})$ (4.2.1.5)

Where $R_{n-1} = d(x_{n-1}, x_n)$.

Now, if $max\{R_n, R_{n-1}\} = R_n$ then from (4.2.1.5 we have

$$\phi(R_n) \leq \phi(R_n) - \psi(R_n).$$

It follows that $\psi(R_n) \leq 0$, which is a contradiction to the definition of ψ. So the maximum, $max\{R_n, R_{n-1}\}$ is R_{n-1}. Thus $R_n \leq R_{n-1}$ for each n=1, 2, 3, and hence $\{R_n\}$ is a non-increasing sequence of positive real numbers. Thus from the inequality (4.2.1.5) we have

$$\phi(R_n) \leq \phi(R_{n-1}) - \psi(R_{n-1}). \qquad (4.2.1.6)$$

Now, since $\{R_n\}$ is a non-increasing sequence of positive real numbers which is bounded below, there exist $\alpha \geq 0$ such that,

$$R_n = d(x_{n+1}, x_n) \to \alpha \text{ as n} \to \infty . \qquad (4.2.1.7)$$

We want to show that (i) $\alpha = 0$.

Suppose $\alpha > 0$, and then taking the limit supremum in both sides of (4.2.1.6) and using the continuity of ϕ and the property of ψ we have:

$$\overline{\lim}\ (\phi(R_n)) \leq \overline{\lim}\ (\phi(R_{n-1})) + \overline{\lim}\ (-\psi(R_{n-1}))$$

which gives $\phi(\boldsymbol{\alpha}) \leq \phi(\boldsymbol{\alpha}) - \underline{\lim}\ (\psi(R_{n-1}))$, (Where $\overline{\lim}\ (-\psi(R_{n-1})) = -\underline{\lim}\ (\psi(R_{n-1}))$.

This implies that $\underline{\lim}\ (\psi(R_{n-1})) \leq 0$, a contradiction to the property of ψ unless $\alpha = 0$.

So $R_n = d(x_{n+1}, x_n) \to 0 \ as \ n \to \infty$.

Therefore, we have

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \qquad (4.2.1.8)$$

(ii) Now we need to show that $\{x_n\}$ is a Cauchy sequence in *X*.

Suppose $\{\boldsymbol{x_n}\}$ is not a Cauchy sequence in *X*. Then there exists an $\varepsilon > 0$, such that for which we can find two sequences of positive integers $\{x_{m(k)}\}$ *and* $\{x_{n(k)}\}$ such that for all positive integers k with

$$n(k) > m(k) > k, \quad d(x_{m(k)}, x_{n(k)}) \geq \varepsilon\ . \tag{4.2.1.9}$$

Assuming $n(k)$ is the smallest integer we get

$$n(k) > m(k) > k, d\big(x_{m(k)}, x_{n(k)}\big) \geq \varepsilon \text{ and} \tag{4.2.1.10}$$

$$d\big(x_{m(k)}, x_{n(k)-1}\big) < \varepsilon. \tag{4.2.1.11}$$

Now by the triangle inequality and using (4.2.1.9) and (4.2.1.11) we have

$$\varepsilon \leq d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}))$$

$$< \varepsilon + \ d(x_{n(k)-1}, x_{n(k)})).$$

Using (4.2.1.8) and letting $k \to \infty, we\ have$

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon, \tag{4.2.1.12}$$

From the triangular inequality, the method we have

$$d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{m(k)-1}) + d\big(x_{m(k)-1}, x_{n(k)-1}\big) + d(x_{n(k)-1}, x_{n(k)})) \quad \text{and}$$

$$d\big(x_{m(k)-1}, x_{n(k)-1}\big) \leq d\left(x_{m(k)-1}, x_{m(k)} + d\big(x_{m(k)}, x_{n(k)}\big) + d\big(x_{n(k)}, x_{n(k)-1}\big)\right). \tag{4.2.1.13}$$

Using (4.2.1.8), (4.2.1.12) and letting k⟶∞ in (4.2.1.13) we get

$$\lim_{k\to\infty} d\big(x_{m(k)-1}, x_{n(k)-1}\big) = \varepsilon \tag{4.2.1.14}$$

Similarly: $\lim_{k\to\infty} d\big(x_{m(k)}, x_{n(k)-1}\big) = \varepsilon$

and $\lim_{k\to\infty} d\big(x_{m(k)-1}, x_{n(k)}\big) = \varepsilon\ .$ (4.2.1.15)

Now, we have,

$$\phi\left(d\big(x_{m(k)}, x_{n(k)}\big)\right) = \phi\big(d(Tx_{m(k)-1}, Tx_{n(k)-1})\big)$$

$$\leq \phi\big(M(x_{m(k)-1}, x_{n(k)-1})\big) - \psi\big(N(x_{m(k)-1}, x_{n(k)-1})\big)$$

$$+L\, m\big(x_{m(k)-1}, x_{n(k)-1}\big) \tag{4.2.1.16}$$

Where $M_k = M(x_{m(k)-1}, x_{n(k)-1})$

$$= max\left\{\frac{d(x_{n(k)-1}, Tx_{n(k)-1})[1+d(x_{m(k)-1}, Tx_{m(k)-1})]}{1+d(x_{m(k)-1}, x_{n(k)-1})}, \frac{d(x_{n(k)-1}, Tx_{m(k)-1})[1+d(x_{m(k)-1}, Tx_{n(k)-1})]}{1+d(x_{m(k)-1}, x_{n(k)-1})}, d(x_{m(k)-1}, x_{n(k)-1})\right\}$$

$$= max\left\{\frac{d\big(x_{n(k)-1}, x_{n(k)}\big)\big[1+d\big(x_{m(k)-1}, x_{m(k)}\big)\big]}{1+d(x_{m(k)-1}, x_{n(k)-1})}, \frac{d\big(x_{n(k)-1}, x_{m(k)}\big)\big[1+d\big(x_{m(k)-1}, x_{n(k)}\big)\big]}{1+d(x_{m(k)-1}, x_{n(k)-1})}, d(x_{m(k)-1}, x_{n(k)-1})\right\} \tag{4.2.1.17}$$

Letting $K \to \infty$ in (4.2.1.17) and using (4.2.1.8),(4.2.1.14), (4.2.1.15), we have:

$$\lim_{k\to\infty} M_k = max\,\{0,\varepsilon,\varepsilon\} = \varepsilon. \qquad (4.2.1.18)$$

Similarly: $N_k = N\big(x_{m(k)-1}, x_{n(k)-1}\big)$

$$= max\left\{\frac{d\big(x_{n(k)-1},Tx_{n(k)-1}\big)\big[1+d\big(x_{m(k)-1},Tx_{m(k)-1}\big)\big]}{1+d(x_{m(k)-1},x_{n(k)-1})}, d(x_{m(k)-1}, x_{n(k)-1})\right\}$$

$$= max\left\{\frac{d\big(x_{n(k)-1},x_{n(k)}\big)\big[1+d\big(x_{m(k)-1},x_{m(k)}\big)\big]}{1+d(x_{m(k)-1},x_{n(k)-1})}, d(x_{m(k)-1}, x_{n(k)-1})\right\}. \qquad (4.2.1.19)$$

Letting $k \to \infty$ in (4.2.1.19) and using (4.2.1.8) and (4.2.1.15) ,we get

$$\lim_{k\to\infty} N_k = max\,\{0,\varepsilon\} = \varepsilon. \qquad (4.2.1.20)$$

And let $m_k = m\big(x_{m(k)-1}, x_{n(k)-1}\big)$

$$= min\left\{\frac{d\big(x_{m(k)-1},Tx_{n(k)-1}\big)d\big(x_{n(k)-1},Tx_{m(k)-1}\big)}{1+d\big(x_{m(k)-1},x_{n(k)-1}\big)}, \frac{d\big(x_{m(k)-1},Tx_{m(k)-1}\big)d\big(,x_{n(k)-1},Tx_{n(k)-1}\big)}{1+d\big(x_{m(k)-1},x_{n(k)-1}\big)}\right\}$$

$$= min\left\{\frac{d\big(x_{m(k)-1},x_{n(k)}\big)d\big(x_{n(k)-1},x_{m(k)}\big)}{1+d\big(x_{m(k)-1},x_{n(k)-1}\big)}, \frac{d\big(x_{m(k)-1},x_{m(k)}\big)d\big(,x_{n(k)-1},x_{n(k)}\big)}{1+d\big(x_{m(k)-1},x_{n(k)-1}\big)}.\right\} \qquad (4.2.1.21)$$

Letting K→∞ in (4.2.1.21) and using (4.2.1.8), (4.2.1.12),(4.2.1.14) and (4.2.1.15) we have:

$$\lim_{k\to\infty} m\big(x_{m(k)-1}, x_{n(k)-1}\big) = min\left\{\frac{\varepsilon^2}{1+\varepsilon}, 0\right\} = 0. \qquad (4.2.1.22)$$

Using (4.2.1.18), (4.2.1.20) and (4.2.1.22) in (4.2.1.16) we have

$$\phi\left(d\big(x_{m(k)}, x_{n(k)}\big)\right) = \phi\big(d(Tx_{m(k)-1}, Tx_{n(k)-1})\big)$$

$$\leq \phi(M_k) - \psi(N_k) + L\min(m_k). \qquad (4.2.1.23)$$

Now by taking the limit supremum of both sides of (4.2.1.23) and using (4.2.1.12), (4.2.1.18), (4.2.1.20) and (4.2.1.22) and by the continuity of ϕ and property of ψ, we get

$$\phi(\varepsilon) \leq \phi(\varepsilon) + \overline{\lim}\,\big(-\psi(N_k)\big).$$

This implies that

$$\phi(\varepsilon) \leq \phi(\varepsilon) - \underline{\lim}\ \psi(N_k\,).$$

That is $\underline{\lim}\,\psi(N_k) \leq 0$.

But this is a contradiction to the property of ψ. Hence $\{x_n\}$ is a Cauchy sequence in a complete metric space X. So, there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u \qquad (4.2.1.24)$$

And by the continuity of T, $u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T(\lim_{n\to\infty}(x_n)) = Tu$.

Thus, u is a fixed point of T.

Remark 1. By choosing L=0 in Theorem 4.2.1 we get Theorem 4.1.7 as a corollary to Theorem 4.2.1.

The following is an example in support of Theorem 4.2.1.

Example 4.2.1.1 Let $X = \{1, 2, 3, 4, 5\}$ with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$. We define a partial order on X by $\preccurlyeq := \{(1,1), (2,2), (3,3), (4,4), (5,5), (3,4), (3,5), (4,5)\}$.

Then $(X, \preccurlyeq)$ is a partially ordered set.

We define $T: X \to X$ by $T=\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 2 & 2 \end{pmatrix}$. T is clearly continuous.

We show that T is non-decreasing.

Since $1 \preccurlyeq 1 \Longrightarrow 1 = \mathrm{T}(1) \preccurlyeq 1 = \mathrm{T}(1)$;

$2 \preccurlyeq 2 \Longrightarrow 1 = \mathrm{T}(2) \preccurlyeq 1 = \mathrm{T}(2)$;

$3 \preccurlyeq 3 \Longrightarrow 2 = \mathrm{T}(3) \preccurlyeq 2 = \mathrm{T}(3)$;

$4 \preccurlyeq 4 \Longrightarrow 2 = \mathrm{T}(4) \preccurlyeq 2 = \mathrm{T}(4)$;

$5 \preccurlyeq 5 \Longrightarrow 2 = \mathrm{T}(5) \preccurlyeq 2 = \mathrm{T}(5)$;

$3 \preccurlyeq 4 \Longrightarrow 2 = \mathrm{T}(3) \preccurlyeq 2 = \mathrm{T}(4)$;

$3 \preccurlyeq 5 \Longrightarrow 2 = \mathrm{T}(3) \preccurlyeq 2 = \mathrm{T}(5)$;

$4 \preccurlyeq 5 \Longrightarrow 2 = \mathrm{T}(4) \preccurlyeq 2 = \mathrm{T}(5)$.

Thus T is a non-decreasing mapping.

Now we verify the inequality (4.2.1.1) with $L = 1$, $\phi, \psi: \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$\phi(t) = \frac{3t}{10}$, $\psi(t) = \frac{t}{3}$ then clearly $\phi \in \Phi, \psi \in \Psi$. .

We note that the case $x = y$ follows trivially, so let $x, y \in X$ such that $x \preccurlyeq y$ and $x \neq y$.

We consider the following three cases:

<u>Case I:</u> Let us take $x = 3$ *and* $y = 5$ then we have $T(3) = 2$, $T(5) = 2$

1. $d(Tx, Ty) = d\big(T(3), T(5)\big) = d(2,2) = 0$
2. $M(x, y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx]}{1+d(x,y)}, \frac{d(y,Tx)[1+d(x,Ty]}{1+d(x,y)}, d(x, y)\right\}$

$$= max\left\{\frac{d\big(5,T(5)\big)[1+d(3,T(3)]}{1+d(3,5)}, \frac{d\big(5,T(3)\big)[1+d(3,T(5)]}{1+d(3,5)}, d(3,5)\right\}$$

$$= max\left\{\frac{d(5,2)[1+d(3,2)]}{1+d(3,5)}, \frac{d(5,2)[1+d(3,2)]}{1+d(3,5)}, d(3,5)\right\}$$

$$= max\left\{\frac{|5-2|[1+|3-2|]}{1+|3-5|}, \frac{|5-2|[1+|3-2|]}{1+|3-5|}, |3-5|\right\}$$

$$= max\left\{\frac{(3)(2)}{3}, \frac{(3)(2)}{3}, 2\right\} = max\{2,2,2\} = 2.$$

3. $N(x,y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx]}{1+d(x,y)}, d(x,y)\right\}$.

$$= max\left\{\frac{d\big(5,T(5)\big)[1+d(3,T(3)]}{1+d(3,5)}, d(3,5)\right\}$$

$$= max\left\{\frac{d(5,2)[1+d(3,2)]}{1+d(3,5)}, d(3,5)\right\}$$

$$= max\left\{\frac{|5-2|[1+|3-2|]}{1+|3-5|}, |3-5|\right\}$$

$$= max\left\{\frac{(3)(2)}{3}, 2\right\} = max\{2,2\} = 2.$$

4. $m(x,y) = min\left\{\frac{d(x,Ty)d(y,Tx)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\}$

$$= min\left\{\frac{d(3,T(5))d(5,T(3))}{1+d(3,5)}, \frac{d(3,T(3))d(5,T(5))}{1+d(3,5)}\right\}$$

$$= min\left\{\frac{d(3,2)d(5,2))}{1+d(3,5)}, \frac{d(3,2))d(5,2))}{1+d(3,5)}\right\}$$

$$= min\left\{\frac{|3-2||5-2|}{1+|3-5|}, \frac{|3-2||5-2|}{1+|3-5|}\right\}$$

$$= min\left\{\frac{(1)(3)}{3}, \frac{(1)(3)}{3}\right\} = min\{1,1\} = 1.$$

So $\phi\big(d(Tx,Ty)\big) \le \phi\big(M(x,y)\big) - \psi\big(N(x,y)\big) + Lm(x,y)$ becomes

$$\frac{3}{10}(0) \le \frac{3}{10}(2) - \frac{1}{3}(2) + 1(1)$$

$$0 \le \frac{6}{10} - \frac{2}{3} + 1$$

$$0 \le \frac{-2}{30} + 1$$

$$0 \le \frac{14}{15}.$$

<u>Case II.</u> Let $x = 3$ and $y = 4$ then $T(3) = 2$ and $T(4) = 2$.

1. $d(Tx,Ty) = d\big(T(3),T(4)\big) = d(2,2) = 0.$
2. $M(x,y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx]}{1+d(x,y)}, \frac{d(y,Tx)[1+d(x,Ty]}{1+d(x,y)}, d(x,y)\right\}$

$$= max\left\{\frac{d\big(4,T(4)\big)[1+d(3,T(3)]}{1+d(3,4)}, \frac{d\big(4,T(3)\big)[1+d(3,T(4)]}{1+d(3,4)}, d(3,4)\right\}$$

$$= max\left\{\frac{d(4,2)[1+d(3,2)]}{1+d(3,4)}, \frac{d(4,2)[1+d(3,2)]}{1+d(3,4)}, d(3,4)\right\}$$

$$= max\left\{\frac{|4-2|[1+|3-2|]}{1+|3-4|}, \frac{|4-2|[1+|3-2|]}{1+|3-4|}, |3-4|\right\}$$

$$= max\left\{\frac{(2)(2)}{2}, \frac{(2)(2)}{2}, 1\right\} = max\{2,2,1\} = 2.$$

3. $N(x,y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx]}{1+d(x,y)}, d(x,y)\right\}$

$= max\left\{\frac{d(4,T(4))[1+d(3,T(3)]}{1+d(3,4)}, d(3,4)\right\}$

$= max\left\{\frac{d(4,2)[1+d(3,2)]}{1+d(3,4)}, d(3,4)\right\}$

$= max\left\{\frac{|4-2|[1+|3-2|]}{1+|3-4|}, |3-4|\right\}$

$= max\left\{\frac{(2)(2)}{2}, 1\right\}$

$= max\{2,1\} = 2.$

4. $m(x,y) = min\left\{\frac{d(x,Ty)d(y,Tx)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\}$

$= min\left\{\frac{d(3,T(4))d(4,T(3))}{1+d(3,4)}, \frac{d(3,T(3))d(4,T(4))}{1+d(3,4)}\right\}$

$= min\left\{\frac{d(3,2)d(4,2))}{1+d(3,4)}, \frac{d(3,2))d(4,2))}{1+d(3,4)}\right\}$

$= min\left\{\frac{|3-2||4-2|}{1+|3-4|}, \frac{|3-2||4-2|}{1+|3-4|}\right\}$

$= min\left\{\frac{(1)(2)}{2}, \frac{(1)(2)}{2}\right\} = min\{1,1\} = 1.$

So $\phi\big(d(Tx,Ty)\big) \leq \phi\big(M(x,y)\big) - \psi\big(N(x,y)\big) + Lm(x,y)$ becomes

$0 \leq \frac{3}{10}(2) - \frac{1}{3}(2) + 1(1)$

$0 \leq -\frac{2}{30} + 1$

This gives $0 \leq \frac{14}{15}$.

<u>Case III</u>: **Let** $x = 4$ *and* $y = 5$ then $T(4) = 2$ and $T(5) = 2$

1. $d(Tx,Ty) = d\big(T(4),T(5)\big) = d(2,2) = 0.$
2. $M(x,y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx]}{1+d(x,y)}, \frac{d(y,Tx)[1+d(x,Ty]}{1+d(x,y)}, d(x,y)\right\}$

$= max\left\{\frac{d(5,T(5))[1+d(4,T(4)]}{1+d(4,5)}, \frac{d(5,T(4))[1+d(4,T(5)]}{1+d(4,5)}, d(4,5)\right\}$

$= max\left\{\frac{d(5,2)[1+d(4,2)]}{1+d(4,5)}, \frac{d(5,2)[1+d(4,2)]}{1+d(4,5)}, d(4,5)\right\}$

$= max\left\{\frac{|5-2|[1+|4-2|]}{1+|4-5|}, \frac{|5-2|[1+|4-2|]}{1+|4-5|}, |4,5|\right\}$

$= max\left\{\frac{(3)(3)}{2}, \frac{(3)(3)}{2}, 1\right\}$

$= max\left\{\frac{9}{2}, \frac{9}{2}, 1\right\} = \frac{9}{2}.$

3. $N(x,y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx]}{1+d(x,y)}, d(x,y)\right\}$

$= max\left\{\frac{d(5,T(5))[1+d(4,T(4)]}{1+d(4,5)}, d(4,5)\right\}$

$= max\left\{\frac{d(5,2)[1+d(4,2)]}{1+d(4,5)}, d(4,5)\right\}$

$= max\left\{\frac{|5-2|[1+|4-2|]}{1+|4-5|}, |4,5|\right\}$

$= max\left\{\frac{(3)(3)}{2}, 1\right\} \ = \frac{9}{2}.$

4. $m(x,y) = min\left\{\frac{d(x,Ty)d(y,Tx)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\}$

$= min\left\{\frac{d(4,T(5))d(5,T(4))}{1+d(4,5)}, \frac{d(4,T(4))d(5,T(5))}{1+d(4,5)}\right\}$

$= min\left\{\frac{d(4,2)d(5,2)}{1+d(4,5)}, \frac{d(4,2)d(5,2)}{1+d(4,5)}\right\}$

$= min\left\{\frac{|4-2||5-2|}{1+|4-5|}, \frac{|4-2||5-2|}{1+|4-5|}\right\}$

$= min\left\{\frac{(2)(3)}{2}, \frac{(2)(3)}{2}\right\} = 3$

So $\phi(d(Tx,Ty)) \leq \phi(M(x,y)) - \psi(N(x,y)) + Lm(x,y)$

becomes: $\frac{3}{10}(0) \leq \frac{3}{10}\left(\frac{9}{2}\right) - \frac{1}{3}\left(\frac{9}{2}\right) + 1(3)$

$0 \leq \frac{27}{20} - \frac{9}{6} + 3$

$0 \leq \ -\frac{3}{20} + 3$

$0 \leq \frac{57}{20}.$

From the Cases (I) - (III) considered above T satisfies the inequality (4.2.1.1) and hence T satisfies all the hypotheses of the Theorem 4.2.1 for the ϕ and ψ chosen in example 4.2.1.1 and T has a fixed point $x_\circ = 1$.If we choose L=0 in the inequality (4.2.1.1), from examples 4.2.1.1 in Cases (I-III) we observe that the inequality (4.2.1.1) fails to hold. This indicates the importance of L in Theorem 4.2.1.

The following is also an example in support of Theorem 4.2.1.

Example 4.2.1.2 Let $X = \{2, 3, 4, 5, 6\}$ be endowed with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$, and we define the partial order as follows

$\preccurlyeq:=\{(2,2), (3,3), (4,4), (5,5), (6,6), (4,5), (4,6), (5,6)\}$ then $(X, \preccurlyeq)$ is a partially ordered set..

Now we define the mapping $T: X \to X$ by $T=\begin{pmatrix} 2 & 3 & 4 & 5 & 6 \\ 2 & 2 & 3 & 3 & 3 \end{pmatrix}$ clearly T is continuous.

Since $2 \preccurlyeq 2 \Longrightarrow 2 = \mathrm{T}(2) \preccurlyeq 2 = \mathrm{T}(2);$

$3 \preccurlyeq 3 \Longrightarrow 2 = \mathrm{T}(3) \preccurlyeq 2 = \mathrm{T}(3);$

$4 \preccurlyeq 4 \Longrightarrow 3 = \mathrm{T}(4) \preccurlyeq 3 = \mathrm{T}(4);$

$5 \preccurlyeq 5 \Longrightarrow 3 = \mathrm{T}(5) \preccurlyeq 3 = \mathrm{T}(5);$

$6 \preccurlyeq 6 \Longrightarrow 3 = \mathrm{T}(6) \preccurlyeq 3 = \mathrm{T}(6);$

$4 \preccurlyeq 5 \Longrightarrow 3 = \mathrm{T}(4) \preccurlyeq 3 = \mathrm{T}(5);$

$4 \preccurlyeq 6 \Longrightarrow 3 = \mathrm{T}(4) \preccurlyeq 3 = \mathrm{T}(6);$

$5 \preccurlyeq 6 \Longrightarrow 3 = \mathrm{T}(5) \preccurlyeq 3 = \mathrm{T}(6);$

Thus T is non-decreasing mapping.

Now we verify the inequality (4.2.1.1) with $L = \frac{1}{2}$, ϕ, ψ: $\mathbb{R}^+ \to \mathbb{R}^+$ defined by

$\phi(t) = \frac{t}{4}$, $\psi(t) = \frac{t}{2}$, then clearly $\phi \in \Phi, \psi \in \Psi$.

We note that the case $x = y$ follows trivially, so let $x, y \in X$ such that $x \preccurlyeq y$ and $x \neq y$.

We also consider the following three cases:

<u>Case I:</u> Let us take $x = 4 \; and \; y = 5$ then

1. $d(Tx, Ty) = d\big(T(4), T(5)\big) = d(3,3) = 0$

2. $M(x, y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, \frac{d(y,Tx)[1+d(x,Ty)]}{1+d(x,y)}, d(x, y)\right\}$

$$= max\left\{\frac{d\big(5,T(5)\big)[1+d(4,T(4)]}{1+d(4,5)}, \frac{d\big(5,T(4)\big)[1+d(4,T(5)]}{1+d(4,5)}, d(4,5)\right\}$$

$$= max\left\{\frac{d(5,3)[1+d(4,3)]}{1+d(4,5)}, \frac{d(5,3)[1+d(4,3)]}{1+d(4,5)}, d(4,5)\right\}$$

$$= max\left\{\frac{|5-3|[1+|4-3|]}{1+|4-5|}, \frac{|5-3|[1+|4-3|]}{1+|4-5|}, |4-5|\right\}$$

$$= max\left\{\frac{(2)(2)}{2}, \frac{(2)(2)}{2}, 1\right\}$$

$$= max\{2,2,1\} = 2.$$

3. $N(x,y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx]}{1+d(x,y)}, d(x,y)\right\}$

$= max\left\{\frac{d(5,T(5))[1+d(4,T(4)]}{1+d(4,5)}, d(4,5)\right\}$

$= max\left\{\frac{d(5,3)[1+d(4,3)]}{1+d(4,5)}, d(4,5)\right\}$

$= max\left\{\frac{|5-3|[1+|4-3|]}{1+|4-5|}, |4-5|\right\}$

$= max\left\{\frac{(2)(2)}{2}, 1\right\} = max\{2,1\} = 2.$

4. $m(x,y) = min\left\{\frac{d(x,Ty)d(y,Tx)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\}$

$= min\left\{\frac{d(4,T(5))d(5,T(4))}{1+d(4,5)}, \frac{d(4,T(4))d(5,T(5))}{1+d(4,5)}\right\}$

$= min\left\{\frac{d(4,3)d(5,3))}{1+d(4,5)}, \frac{d(4,3))d(5,3))}{1+d(4,5)}\right\}$

$= min\left\{\frac{|4-3||5-3|}{1+|4-5|}, \frac{|4-3||5-3|}{1+|4-5|}\right\}$

$= min\left\{\frac{(1)(2)}{2}, \frac{(1)(2)}{2}\right\} = min\{1,1\} = 1.$

So $\phi(d(Tx,Ty)) \leq \phi(M(x,y)) - \psi(N(x,y)) + Lm(x,y)$ becomes

$\frac{1}{4}(0) \leq \frac{1}{4}(2) - \frac{1}{2}(2) + \frac{1}{2}(1)$

$0 \leq \frac{1}{2} - 1 + \frac{1}{2}$

$0 \leq 0.$

<u>Case II</u> Let $x = 4\ and\ y = 6$ then

1. $d(Tx,Ty) = d(T(4),T(6)) = d(3,3) = 0.$
2. $M(x,y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx]}{1+d(x,y)}, \frac{d(y,Tx)[1+d(x,Ty]}{1+d(x,y)}, d(x,y)\right\}$

$= max\left\{\frac{d(6,T(6))[1+d(4,T(4)]}{1+d(4,6)}, \frac{d(6,T(4))[1+d(4,T(6)]}{1+d(4,6)}, d(4,6)\right\}$

$= max\left\{\frac{d(6,3)[1+d(4,3)]}{1+d(4,6)}, \frac{d(6,3)[1+d(4,3)]}{1+d(4,6)}, d(4,6)\right\}$

$= max\left\{\frac{|6-3|[1+|4-3|]}{1+|4-6|}, \frac{|6-3|[1+|4-3|]}{1+|4-6|}, |4-6|\right\}$

$= max\left\{\frac{(3)(2)}{3}, \frac{(3)(2)}{3}, 2\right\}$

$= max\{2,2,2\} = 2.$

3. $N(x,y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx]}{1+d(x,y)}, d(x,y)\right\}$

$= max\left\{\frac{d(6,T(6))[1+d(4,T(4)]}{1+d(4,6)}, d(4,6)\right\}$

$= max\left\{\frac{d(6,3)[1+d(4,3)]}{1+d(4,6)}, d(4,6)\right\}$

$= max\left\{\frac{|6-3|[1+|4-3|]}{1+|4-6|}, |4-6|\right\}$

$= max\left\{\frac{(3)(2)}{3}, 2\right\} = max\{2,2\} = 2.$

4. $m(x,y) = min\left\{\frac{d(x,Ty)d(y,Tx)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\}$

$= min\left\{\frac{d(4,T(6))d(6,T(4))}{1+d(4,6)}, \frac{d(4,T(4))d(6,T(6))}{1+d(4,6)}\right\}$

$= m\,in\left\{\frac{d(4,3)d(6,3))}{1+d(4,6)}, \frac{d(4,3))d(6,3))}{1+d(4,6)}\right\}$

$= min\left\{\frac{|4-3||6-3|}{1+|4-6|}, \frac{|4-3||6-3|}{1+|4-6|}\right\}$

$= min\left\{\frac{(1)(3)}{3}, \frac{(1)(3)}{3}\right\} = min\{1,1\} = 1.$

So $\phi(d(Tx,Ty)) \le \phi(M(x,y)) - \psi(N(x,y)) + Lm(x,y)$ becomes

$\frac{1}{4}(0) \le \frac{1}{4}(2) - \frac{1}{2}(2) + \frac{1}{2}(1)$

$\Longrightarrow 0 \le \frac{1}{2} - 1 + \frac{1}{2}$

$\Longrightarrow 0 \le 0.$

<u>Case III</u>: **Let** $x = 5$ and $y = 6$ then

1. $d(Tx,Ty) = d(T(5),T(6)) = d(3,3) = 0$
2. $M(x,y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx]}{1+d(x,y)}, \frac{d(y,Tx)[1+d(x,Ty]}{1+d(x,y)}, d(x,y)\right\}$

$= max\left\{\frac{d(6,T(6))[1+d(5,T(5)]}{1+d(5,6)}, \frac{d(6,T(5))[1+d(5,T(6)]}{1+d(5,6)}, d(5,6)\right\}$

$= max\left\{\frac{d(6,3)[1+d(5,3)]}{1+d(5,6)}, \frac{d(6,3)[1+d(5,3)]}{1+d(5,6)}, d(5,6)\right\}$

$= max\left\{\frac{|6-3|[1+|5-3|]}{1+|5-6|}, \frac{|6-3|[1+|5-3|]}{1+|5-6|}, |5-6|\right\}$

$= max\left\{\frac{(3)(3)}{2}, \frac{(3)(3)}{2}, 1\right\}$

$= max\left\{\frac{9}{2}, \frac{9}{2}, 1\right\} = \frac{9}{2}.$

3. $N(x,y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx]}{1+d(x,y)}, d(x,y)\right\}$

$$= max\left\{\frac{d\big(6,T(6)\big)[1+d(5,T(5)]}{1+d(5,6)}, d(5,6)\right\}$$

$$= max\left\{\frac{d(6,3)[1+d(5,3)]}{1+d(5,6)}, d(5,6)\right\}$$

$$= max\left\{\frac{|6-3|[1+|5-3|]}{1+|5-6|}, |5-6|\right\}$$

$$= max\left\{\frac{(3)(3)}{2}, 1\right\} = max\left\{\frac{9}{2}, 1\right\} = \frac{9}{2}.$$

4. $m(x,y) = min\left\{\frac{d(x,Ty)d(y,Tx)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\}$

$$= min\left\{\frac{d(5,T(6))d(6,T(5))}{1+d(5,6)}, \frac{d(5,T(5))d(6,T(6))}{1+d(5,6)}\right\}$$

$$= min\left\{\frac{d(5,3)d(6,3))}{1+d(5,6)}, \frac{d(5,3))d(6,3))}{1+d(5,6)}\right\}$$

$$= min\left\{\frac{|5-3||6-3|}{1+|5-6|}, \frac{|5-3||6-3|}{1+|5-6|}\right\}$$

$$= min\left\{\frac{(2)(3)}{2}, \frac{(2)(3)}{2}\right\} = min\{3,3\} = 3.$$

So $\phi\big(d(Tx,Ty)\big) \le \phi\big(M(x,y)\big) - \psi\big(N(x,y)\big) + Lm(x,y)$ becomes

$$\frac{1}{4}(0) \le \frac{1}{4}\left(\frac{9}{2}\right) - \frac{1}{2}\left(\frac{9}{2}\right) + \frac{1}{2}(3)$$

$$0 \le -\frac{9}{8} + \frac{3}{2}$$

$$0 \le \frac{3}{8}.$$

From the Cases (I) - (III) considered above T satisfies the inequality (4.2.1.1) and hence T satisfies all the hypotheses of the Theorem 4.2.1 for the ϕ and ψ chosen in example 4.2.1.2 and T has a fixed point $x_\circ = 2$. If we choose L=0 in the inequality (4.2.1.1), from examples 4.2.1.1 in Cases (I-III) we observe that the inequality (4.2.1.1) fails to hold. This indicates the importance of L in Theorem 4.2.1.

In the following, we prove fixed point results by relaxing the continuity assumption of T in Theorem 4 .2.1.

Theorem 4.2.2: Let (X, $\preccurlyeq$) be a partially ordered set. Suppose that there exists a metric d on X such that (X,d) is a complete metric space. Assume that if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \longrightarrow x$, then $x_n \preccurlyeq x$ for all $n \in N$. Let $T: X \rightarrow X$ be a non- decreasing mapping. Assume that there exist functions $\phi \in \Phi, \psi \in \Psi$ and a constant $L \geq 0$ such that

$$\phi\big(d(Tx,Ty)\big) \leq \phi\big(M(x,y)\big) - \psi\big(N(x,y)\big) + Lm(x,y) \quad (4.2.2.1)$$

for all $x, y \in X$ with $x \preccurlyeq y$

where: $M(x,y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, \frac{d(y,Tx)[1+d(x,Ty)]}{1+d(x,y)}, d(x,y)\right\}$

$N(x,y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, d(x,y)\right\}$ and

$m(x,y) = min\left\{\frac{d(x,Ty)d(y,Tx)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\}$.

If there exists $x_\circ \in X$ with $x_\circ \preccurlyeq Tx_\circ$ then T has a fixed point.

Proof: Let $\{x_n\}$ be a sequence in X. As in the proof of Theorem 4.2.1 we have for some $x_\circ \in X$ with $x_\circ \preccurlyeq Tx_\circ$ and

$$x_\circ \preccurlyeq T_{x_\circ} = x_1 \preccurlyeq Tx_1 = x_2 \preccurlyeq Tx_2 = x_3 \preccurlyeq \cdots \preccurlyeq Tx_{n-1} = x_n \preccurlyeq Tx_n = x_{n+1} \preccurlyeq \cdots$$

$$\Longrightarrow x_\circ \preccurlyeq x_1 \preccurlyeq x_2 \preccurlyeq x_3 \preccurlyeq \cdots x_{n-1} \preccurlyeq x_n \preccurlyeq x_{n+1} \preccurlyeq \cdots \quad (4.2.2.2)$$

Thus, $\{x_n\}$ is a non-decreasing sequence and converges to u, then $x_n \preccurlyeq u, for\ all\ n \in N$.

Now we show that $u = Tu$. (4.2.2.3)

Suppose that $u \neq Tu$. Then $d(u,Tu) > 0$.

Now consider

$$\phi\big(d(x_{n+1},Tu)\big) = \phi\big(d(Tx_n,Tu)\big) \leq \phi(M_n) - \psi(N_n) + Lm(x,y)$$

where

(i) $M_n = M(x_n,u)$

$$= max\left\{\frac{d(u,Tu)[1+d(x_n,Tx_n)]}{1+d(x_n,u)}, \frac{d(u,Tx_n)[1+d(x_n,Tu)]}{1+d(x_n,u)}, d(x_n,u)\right\}$$

$$= max\left\{\frac{d(u,Tu)[1+d(x_n,x_{n+1})]}{1+d(x_n,u)}, \frac{d(u,x_{n+1})[1+d(x_n,Tu)]}{1+d(x_n,u)}, d(x_n,u)\right\}. \quad (4.2.2.4)$$

Letting $n \to \infty$ in (4.2.2.4) and using the fact that $x_n \to u\ as\ n \to \infty$, we have

$$\lim_{n\to\infty} M_n = max\{d(u,Tu),0,0\} = d(u,Tu) > 0 \quad (4.2.2.5)$$

(ii) $N_n = N(x_n,u) = max\left\{\frac{d(u,Tu)[1+d(x_n,Tx_n)]}{1+d(x_n,u)}, d(x_n,u)\right\}$

$$= max\left\{\frac{d(u,Tu)[1+d(x_n,x_{n+1})]}{1+d(x_n,u)}, d(x_n,u)\right\} \quad (4.2.2.6)$$

Letting $n \to \infty$ in (4.2.2.6) and using the fact that $x_n \to u$ we have,

$$\lim_{n\to\infty} N_n = max\{d(u,Tu), 0\} = d(u,Tu) > 0 \quad (4.2.2.7)$$

$$(iii) \quad m(x,y) = m(x_n,u) = min\left\{\frac{d(x_n,Tu)d(u,Tx_n)}{1+d(x_n,u)}, \frac{d(x_n,Tx_n)d(u,Tu)}{1+d(x_n,u)}\right\}$$

$$= min\left\{\frac{d(x_n,Tu)d(u,x_{n+1})}{1+d(x_n,u)}, \frac{d(x_n,x_{n+1})d(u,Tu)}{1+d(x_n,u)}\right\} \quad (4.2.2.8)$$

and letting $n\to\infty$ in (4.2.2.8) we have

$$= min\{0,0\} = 0. \quad (4.2.2.9)$$

Since $x_n \preccurlyeq u$ for all n, then by using (4.2.2.5) ,(4.2.2.7) and (4.2.2.9) in (4.2.2.1)

We have $\phi(d(\{x_{n+1}, Tu\}) = \phi\big(d(Tx_n, Tu)\big) \le \phi(M_n) - \psi(N_n) + Lm(x,y)$

$$\le \phi(M_n) - \psi(N_n) + Lmin\left\{\frac{d(x_n,Tu)d(u,x_{n+1})}{1+d(x_n,u)}, \frac{d(x_n,x_{n+1})d(u,Tu)}{1+d(x_n,u)}\right\}.$$

Thus $\phi(d(x_{n+1}, Tu)) = \phi(d(Tx_n, Tu)) \le \phi(M_n) - \psi(N_n) + Lmin\{0,0\}$

Hence, $\phi(d(x_{n+1}, Tu)) = \phi(d(Tx_n, Tu)) \le \phi(M_n) - \psi(N_n)$ (4.2.2.10)

By taking the limit supremum of (4.2.2.10), using (4.2.2.5) and (4.2.2.7) and the property of ψ and the continuity of ϕ we have

$$\phi(d(u,Tu)) \le \phi(d(u,Tu)) + \overline{\lim}\left(-\psi(N_n)\right),$$

That is, $\phi(d(u,Tu)) \le \phi(d(u,Tu)) - \underline{\lim}\,(\psi(N_n))$

This gives $\underline{\lim}\,(\psi(N_n) \le 0$.

But this is a contradiction by (4.2.2.7) and the definition of ψ.

Hence $u = Tu$ and u is the fixed point of T.

Remark 2. By choosing L=0 in Theorem 4.2.2 we get Theorem 4.1.8 as a corollary to Theorem 4.2.2.

We now illustrate an example in the support of theorem 4.2.2

Example 4.2.2.1: *Let* $X = [0,2]$ with the usual metric.

We define the partial order $''\preccurlyeq''$ on X by

$$\preccurlyeq = \{(x,y): x,y \in [0,1), x = y\} \cup \{(x,y): x,y \in [1,2], x \leq y\}.$$

Then $(X,\preccurlyeq)$ is a partially ordered set..

We define $T: X \longrightarrow X$ by $Tx = \begin{cases} \frac{x}{3} & if\ 0 \leq x < 1 \\ \frac{2}{3} & if\ 1 \leq x \leq 2. \end{cases}$

Also, we define $\phi, \psi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ *by* $\phi(t) = \frac{t^2}{2,}$, $\psi(t) = t^2$, then clearly $\phi \in \Phi, \psi \in \Psi$.

and let $L = 10$.

Now we verify the inequality 4.2.2.1

Case I: Let $x, y \in [0,1), then\ x = y$ so $Tx = \frac{x}{3} = Ty$

Then we have

1. $d(Tx,Ty) = d\left(\frac{x}{3},\frac{x}{3}\right) = \left|\frac{x}{3} - \frac{x}{3}\right| = 0$, Since $x = y \in [0,1)$

2. $M(x,y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, \frac{d(y,Tx)[1+d(x,Ty)]}{1+d(x,y)}, d(x,y)\right\}$

$$= max\left\{\frac{d(y,\frac{y}{3})\left[1+d(x,\frac{x}{3})\right]}{1+d(x,y)}, \frac{d(y,\frac{x}{3})\left[1+d(x,\frac{y}{3})\right]}{1+d(x,y)}, d(x,y)\right\}$$

$$= max\left\{\frac{\left|y-\frac{y}{3}\right|\left[1+\left|x-\frac{x}{3}\right|\right]}{1+|x-y|}, \frac{\left|y-\frac{x}{3}\right|\left[1+\left|x-\frac{y}{3}\right|\right]}{1+|x-y|}, |x-y|\right\}$$ Since $x = y \in [0,1)$we have

$$= max\left\{\frac{\left|\frac{2x}{3}\right|\left[1+\left|\frac{2x}{3}\right|\right]}{1+|x-x|}, \frac{\left|\frac{2x}{3}\right|\left[1+\left|\frac{2x}{3}\right|\right]}{1+|x-x|}, |x-x|\right\}$$

$$= max\left\{\frac{2x}{3}\left(1+\frac{2x}{3}\right), \frac{2x}{3}\left(1+\frac{2x}{3}\right), 0\right\}$$

$$= max\left\{\frac{2x}{3}\left(1+\frac{2x}{3}\right), 0\right\}$$

$$= \frac{2x}{3}\left(1+\frac{2x}{3}\right).$$

3. $N(x,y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, d(x,y)\right\}$

$$= max\left\{\frac{d(y,\frac{y}{3})\left[1+d(x,\frac{x}{3})\right]}{1+d(x,y)}, d(x,y)\right\}$$

$$= max\left\{\frac{\left|y-\frac{y}{3}\right|\left[1+\left|x-\frac{x}{3}\right|\right]}{1+|x-y|}, |x-y|\right\}$$

$$= max\left\{\frac{\left|\frac{2x}{3}\right|\left[1+\left|\frac{2x}{3}\right|\right]}{1+|x-x|}, |x-x|\right\}$$ Since $x = y \in [0,1)$we have

$$= max\left\{\frac{2x}{3}\left(1+\frac{2x}{3}\right),0\right\} = \frac{2x}{3}\left(1+\frac{2x}{3}\right).$$

$$4.\, m(x,y) = min\left\{\frac{d(x,Ty)d(y,Tx)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\}$$

$$= min\left\{\frac{d\left(x,\frac{y}{3}\right)d\left(y,\frac{x}{3}\right)}{1+d(x,y)}, \frac{d\left(x,\frac{x}{3}\right)d\left(y,\frac{y}{3}\right)}{1+d(x,y)}\right\}$$

$$= min\left\{\frac{\left|x-\frac{y}{3}\right|\left|y-\frac{x}{3}\right|}{1+|x-y|}, \frac{\left|x-\frac{x}{3}\right|\left|y-\frac{y}{3}\right|}{1+|x-y|}\right\}$$

$$= min\left\{\left\{\frac{\left|x-\frac{x}{3}\right|\left|x-\frac{x}{3}\right|}{1+|x-x|}, \frac{\left|x-\frac{x}{3}\right|\left|x-\frac{x}{3}\right|}{1+|x-x|}\right\}\right\} \text{ since } x = y \in [0,1)$$

$$= min\left\{\frac{4x^2}{9}, \frac{4x^2}{9}\right\} = \frac{4x^2}{9}.$$

Then we have the following

$$\phi\big(d(Tx,Ty)\big) \leq \phi\big(M(x,y)\big) - \psi\big(N(x,y)\big) + Lm(x,y)$$

$$0 \leq \frac{\frac{4x^2}{9}\left(1+\frac{4x}{3}+\frac{4x^2}{9}\right)}{2} - \left(\frac{4x^2}{9}\left(1+\frac{4x}{3}+\frac{4x^2}{9}\right)\right) + L.\frac{4x^2}{9}$$

$$0 \leq \frac{-\frac{4x^2}{9}\left(1+\frac{4x}{3}+\frac{4x^2}{9}\right)}{2} + L.\frac{4x^2}{9}$$

$$\frac{4x^2}{9}\left(1+\frac{4x}{3}+\frac{4x^2}{9}\right) \leq 2L.\frac{4x^2}{9} \tag{1}$$

Here if $x = 0$, (1) holds clearly.

If $x \neq 0$, we have $1 + \frac{4x}{3} + \frac{4x^2}{9} \leq 2.L$ (2)

Since the left hand side of (2) is less than or equal to $\frac{25}{9}$ for any $x \in [0,1)$ we observe that (2) holds for L=10.

<u>Case II:</u> Let $x, y \in [1,2]$ such that $x \leq y$, then we have the following

$Tx = \frac{2}{3}$, $Ty = \frac{2}{3}$

$$1.\, d(Tx,Ty) = d\left(\frac{2}{3},\frac{2}{3}\right) = \left|\frac{2}{3}-\frac{2}{3}\right| = 0.$$

$$2.\, M(x,y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, \frac{d(y,Tx)[1+d(x,Ty)]}{1+d(x,y)}, d(x,y)\right\}$$

$$= max\left\{\frac{d\left(y,\frac{2}{3}\right)\left[1+d\left(x,\frac{2}{3}\right)\right]}{1+d(x,y)}, \frac{d\left(y,\frac{2}{3}\right)\left[1+d\left(x,\frac{2}{3}\right)\right]}{1+d(x,y)}, d(x,y)\right\}$$

$$= max\left\{\frac{\left|y-\frac{2}{3}\right|\left[1+\left|x-\frac{2}{3}\right|\right]}{1+|x-y|}, \frac{\left|y-\frac{2}{3}\right|\left[1+\left|x-\frac{2}{3}\right|\right]}{1+|x-y|}, |x-y|\right\}$$

$$= max\left\{\frac{\left|y-\frac{2}{3}\right|\left[1+\left|x-\frac{2}{3}\right|\right]}{1+|x-y|}, |x-y|\right\}$$

$$= max\left\{\frac{\left(y-\frac{2}{3}\right)\left[1+\left(x-\frac{2}{3}\right)\right]}{1+y-x}, y-x\right\}$$

3. $$N(x,y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx]}{1+d(x,y)}, d(x,y)\right\}$$

$$= max\left\{\frac{d(y,\frac{2}{3})\left[1+d(x,\frac{2}{3})\right]}{1+d(x,y)}, d(x,y)\right\}$$

$$= max\left\{\frac{\left|y-\frac{2}{3}\right|\left[1+\left|x-\frac{2}{3}\right|\right]}{1+|x-y|}, |x-y|\right\}$$

$$= max\left\{\frac{\left(y-\frac{2}{3}\right)\left[1+\left(x-\frac{2}{3}\right)\right]}{1+y-x}, y-x\right\} \text{ Since } y \geq x$$

4. $$m(x,y) = min\left\{\frac{d(x,Ty)d(y,Tx)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\}$$

$$= min\left\{\frac{d\left(x,\frac{2}{3}\right)d\left(y,\frac{2}{3}\right)}{1+d(x,y)}, \frac{d\left(x,\frac{2}{3}\right)d\left(y,\frac{2}{3}\right)}{1+d(x,y)}\right\}$$

$$= min\left\{\frac{\left|x-\frac{2}{3}\right|\left|y-\frac{2}{3}\right|}{1+|x-y|}, \frac{\left|x-\frac{2}{3}\right|\left|y-\frac{2}{3}\right|}{1+|x-y|}\right\} = \frac{\left(x-\frac{2}{3}\right)\left(y-\frac{2}{3}\right)}{1+y-x}$$

(a). If $M(x,y) = y-x$

$N(x,y) = y-x$

$m(x,y) = \frac{\left(x-\frac{2}{3}\right)\left(y-\frac{2}{3}\right)}{1+y-x}$

Now consider the following:

$$\phi\big(d(Tx,Ty)\big) \leq \phi\big(M(x,y)\big) - \psi\big(N(x,y)\big) + Lm(x,y) \qquad (3)$$

$$0 \leq \frac{(y-x)^2}{2} - (y-x)^2 + L\left(\frac{\left(x-\frac{2}{3}\right)\left(y-\frac{2}{3}\right)}{1+y-x}\right)$$

$$0 \leq -\frac{(y-x)^2}{2} + L\left(\frac{\left(x-\frac{2}{3}\right)\left(y-\frac{2}{3}\right)}{1+y-x}\right)$$

$$\frac{(y-x)^2}{2} \leq L\left(\frac{\left(x-\frac{2}{3}\right)\left(y-\frac{2}{3}\right)}{1+y-x}\right) \qquad (4)$$

$$\leq 2.L\left(x-\frac{2}{3}\right)\left(y-\frac{2}{3}\right).$$

Since the left hand side of (4) is at most $\frac{1}{2}$ for any $x, y \in [1,2]$ with $x \leq y$ it is clear that (4) holds for L= 10.

(b) .If $M(x,y) = \frac{\left(y-\frac{2}{3}\right)\left[1+\left(x-\frac{2}{3}\right)\right]}{1+y-x}$

$$N(x,y) = \frac{\left(y-\frac{2}{3}\right)\left[1+\left(x-\frac{2}{3}\right)\right]}{1+y-x}$$

$$m(x,y) = \frac{\left(x-\frac{2}{3}\right)\left(y-\frac{2}{3}\right)}{1+y-x}$$

We have the following

$$\phi\big(d(Tx,Ty)\big) \leq \phi\big(M(x,y)\big) - \psi\big(N(x,y)\big) + Lm(x,y)$$

$$0 \leq \left(\frac{\left(y-\frac{2}{3}\right)\left[1+\left(x-\frac{2}{3}\right)\right]}{2(1+y-x)^2}\right)^2 - \left(\frac{\left(y-\frac{2}{3}\right)\left[1+\left(x-\frac{2}{3}\right)\right]}{1+y-x}\right)^2 + L\left(\frac{\left(x-\frac{2}{3}\right)\left(y-\frac{2}{3}\right)}{1+y-x}\right)$$

$$0 \leq -\left(\frac{\left(y-\frac{2}{3}\right)\left[1+\left(x-\frac{2}{3}\right)\right]}{2(1+y-x)^2}\right)^2 + L\left(\frac{\left(x-\frac{2}{3}\right)\left(y-\frac{2}{3}\right)}{1+y-x}\right)$$

$$\left(\frac{\left(y-\frac{2}{3}\right)\left[1+\left(x-\frac{2}{3}\right)\right]}{2(1+y-x)^2}\right)^2 \leq L\left(\frac{\left(x-\frac{2}{3}\right)\left(y-\frac{2}{3}\right)}{1+y-x}\right)$$

$$\frac{\left(y-\frac{2}{3}\right)\left(1+\left(x-\frac{2}{3}\right)\right)^2}{2(1+y-x)} \leq L\left(x-\frac{2}{3}\right). \tag{6}$$

Here also, (6) holds with L= 10.

From the Cases (I) - (II) considered above T satisfies the inequality (4.2.2.1) for the ϕ and ψ chosen in example 4.2.2.1 and hence T satisfies all the hypotheses of the Theorem 4.2.2 and T has a fixed point $x_\circ = 0$. If we choose L=0 in the inequality (4.2.2.1), from examples 4.2.2.1 in Cases (I-II) we observe that the inequality (4.2.2.1) fails to hold. This indicates the importance of L in Theorem 4.2.2.

The following is also an example in support of Theorem 4.2.2.

Example 4.2.2.2:*Let* $X = [0,2]$ with the usual metric.

We define the partial order "$\preccurlyeq$" on X by

$$\preccurlyeq = \{(x,y): x,y \in [0,1), x = y\} \cup \{(x,y): x,y \in [1,2],\ X \leq y\}.$$

then $(X,\preccurlyeq)$ is a partially ordered set.

Let $T: X \longrightarrow X$ be defined by $Tx = \begin{cases} 0 & if\ 0 \leq x < 1 \\ 1 & if\ 1 \leq x \leq 2 \end{cases}$

$\phi, \psi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ *be defined by*: $\phi(t) = 2t,\ \psi(t) = 2t$,

then clearly $\phi \in \Phi, \psi \in \Psi$. *and Let* $L = 1$,

Now we verify the inequality 4.2.2.1

Case I: Let $x, y \in [0,1)$ then $Tx = 0$

$Ty = 0$ then we have

1. $d(Tx, Ty) = d(0,0) = |0 - 0| = 0.$

. 2 . $M(x,y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx]}{1+d(x,y)}, \frac{d(y,Tx)[1+d(x,Ty]}{1+d(x,y)}, d(x,y)\right\}$

$= max\left\{\frac{d(y,0)[1+d(x,0)]}{1+d(x,y)}, \frac{d(y,0)[1+d(x,0)]}{1+d(x,y)}, d(x,y)\right\}$

$= max\left\{\frac{|y|[1+|x|]}{1+|x-y|}, \frac{|y|[1+|x|]}{1+|x-y|}, |x-y|\right\}$ since $x = y \in [0.1)$

$= max\{x(1+x), x(1+x), 0\}$

$=max\{x^2 + x, x^2 + x, 0\}$

$= x^2 + x.$

.3. $N(x,y) = max\left\{\frac{d(y,0)[1+d(x,0)]}{1+d(x,y)}, d(x,y)\right\}$

$= max\left\{\frac{d(y,0)[1+d(x,0)]}{1+d(x,y)}, d(x,y)\right\}$

$= max\left\{\frac{|y|[1+|x|]}{1+|x-y|}, |x-y|\right\}$ since $x = y \in [0.1)$

$= max\{x(1+x), 0\}$

$=max\{x^2 + x, 0\}$

$=x^2 + x.$

4. $m(x,y) \quad = min\left\{\frac{d(x,Ty)d(y,Tx)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\}$

$= min\left\{\frac{d(x,0)d(y,0)}{1+d(x,y)}, \frac{d(x,0)d(y,0)}{1+d(x,y)}\right\}$

$= min\left\{\frac{|x||y|}{1+|x-y|}, \frac{|x||y|}{1+|x-y|}\right\}$

$= min\left\{\left\{\frac{|x||x|}{1+|x-x|}, \frac{|x||x|}{1+|x-x|}\right\}\right\}$ since $x = y \in [0,1)$

$= min\{x^2, x^2\}$

$=x^2.$

1. Suppose: $M(x,y) = x^2 + x$

$N(x,y) = x^2 + x$

$m(x,y) = x^2$

Then we have the following.

$$\phi(d(Tx,Ty)) \leq \phi(M(x,y)) - \psi(N(x,y)) + Lm(x,y)$$

$$\phi(0) \leq 2(x^2 + x) - 2(x^2 + x) + x^2$$

$$0 \leq x.^2$$

This is true for all $x \in X = [0,1)$

<u>Case II:</u> Let $x, y \in [1,2]$ then we have the following

$$Tx = 1, Ty = 1$$

1. $d(Tx, Ty) = d(1,1) = |1-1| = 0.$

2. $M(x,y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx]}{1+d(x,y)}, \frac{d(y,Tx)[1+d(x,Ty]}{1+d(x,y)}, d(x,y)\right\}$

$$= max\left\{\frac{d(y,1)[1+d(x,1]}{1+d(x,y)}, \frac{d(y,1)[1+d(x,1]}{1+d(x,y)}, d(x,y)\right\}$$

$$= max\left\{\frac{|y-1|[1+|x-1|]}{1+|x-y|}, \frac{|y-1|[1+|x-1|]}{1+|x-y|}, |x-y|\right\}.$$

3. $N(x,y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx]}{1+d(x,y)}, d(x,y)\right\}$

$$= max\left\{\frac{d(y,1)[1+d(x,1]}{1+d(x,y)}, d(x,y)\right\}$$

$$= max\left\{\frac{|y-1|[1+|x-1|]}{1+|x-y|}, |x-y|\right\}.$$

4. $m(x,y) = min\left\{\frac{d(x,Ty)d(y,Tx)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\}$

$$= min\left\{\frac{d(x,1)d(y,1)}{1+d(x,y)}, \frac{d(x,1)d(y,1)}{1+d(x,y)}\right\}$$

$$= min\left\{\frac{|x-1||y-1|}{1+|x-y|}, \frac{|x-1||y-1|}{1+|x-y|}\right\}.$$

Now consider the following:

1. Suppose: $M(x,y) = \frac{|y-1|[1+|x-1|]}{1+|x-y|}$

$$N(x,y) = \frac{|y-1|[1+|x-1|]}{1+|x-y|}$$

$$m(x,y) = \frac{|x-1||y-1|}{1+|x-y|}$$

Then we have the following

$$\phi\big(d(Tx,Ty)\big) \le \phi\big(M(x,y)\big) - \psi\big(N(x,y)\big) + Lm(x,y) \qquad (1)$$

$$(1)(0) \le (2)\left(\frac{|y-1|[1+|x-1|]}{1+|x-y|}\right) -2\left(\frac{|y-1|[1+|x-1|]}{1+|x-y|}\right) +1\left(\frac{|x-1||y-1|}{1+|x-y|}\right)$$

$$0 \le \left(\frac{|x-1||y-1|}{1+|x-y|}\right).$$

This holds for any $x \preccurlyeq y$ *such that* $x, y \in [1,2]$.

2. Suppose: $M(x,y) = |x-y|$

$$N(x,y) = |x-y|$$

$$m(x,y) = \frac{|x-1||y-1|}{1+|x-y|}$$

So, consider the following

$$\phi\big(d(Tx,Ty)\big) \leq \phi\big(M(x,y)\big) - \psi\big(N(x,y)\big) + Lm(x,y)$$

$$0 \leq 2(|x-y|) - 2(|x-y|) + L\left(\frac{|x-1||y-1|}{1+|x-y|}\right)$$

$$0 \leq L\left(\frac{|x-1||y-1|}{1+|x-y|}\right).\,.$$

This holds for any $x, y \in [1,2]$ such that $x \leq y$.

From the Cases (I) - (II) considered above T satisfies the inequality (4.2.2.1) and hence T satisfies all the hypotheses of the Theorem 4.2.2 and T has two fixed points $x_\circ = 0$ *and* $x_\circ = 1$.

The following is a theorem to demonstrate the uniqueness of fixed point for the mapping T.

Theorem 4.2.3 In addition to the hypotheses of Theorem 4.2.1 or Theorem 4.2.2 suppose that for every $x, y \epsilon X$, there exists $u \in X$ such that $u \preccurlyeq x$ *and* $u \preccurlyeq y$. Then T has a unique fixed point.

Proof: Following from theorem 4.2.1 the sets of fixed points of T is non- empty. Now we shall show that if x and y are two distinct fixed points of T, that is,

$$\text{if } x = Tx \text{ and } y = Ty \text{, then } x = y\,. \qquad (4.2..3.1)$$

Assume that there exist $u_\circ \in X$ such that $u_\circ \preccurlyeq x$ and $u_\circ \preccurlyeq y$, then as in the proof of Theorem 4.2.1, we define the sequence $\{u_n\}$ such that:

$$u_{n+1} = Tu_n = T^{n+1}u_\circ, \quad n = 1,2,\ldots \qquad (4.2.3.2)$$

Monotoncity of T implies $T^n u_\circ = u_n \preccurlyeq x = T^n x$ and $T^n u_\circ = u_n \preccurlyeq y = T^n y$

If there exist a positive integer m such that $x = u_m$, then $x = Tx = Tu_n = u_{n+1}$ for all $n \geq m$, then $u_n \to x$ as $n \to \infty$.

Now suppose that $x \neq u_n$ for all $n \geq 0$, so $u_n \preccurlyeq x$ for all $n \geq 0$, then $d(u_n, x) \neq 0$ for all $n \geq 0$.

Let $p_n = d(u_n, x)$ for all $n \geq 0$. As $u_n \preccurlyeq x$ for all $n \geq 0$, by applying the inequality (4.2.1.1) we have

$$\phi(d(u_{n+1}, x)) = \phi(d(Tu_n, Tx))$$

$$\leq \phi\left(max\left\{\frac{d(x,Tx)[1+d(u_n,Tu_n)]}{1+d(u_n,x)}, \frac{d(x,Tu_n)[1+d(u_n,Tx)]}{1+d(u_n,x)}, d(u_n,x)\right\}\right) \qquad (4.2.3.3)$$

$$-\psi\left(\left(max\left\{\frac{d(x,Tx)[1+d(u_n,Tu_n)]}{1+d(u_n,x)}, d(u_n,x)\right\}\right)\right) + Lmin\left\{\frac{d(u_n,Tx)d(x,Tu_n)}{1+d(u_n,x)}, \frac{d(u_n,Tu_n)d(x,Tx)}{1+d(u_n,x)}\right\}$$

$$= \phi\left(\left(max\left\{\frac{d(x,x)[1+d(u_n,u_{n+1})]}{1+d(u_n,x)}, \frac{d(x,u_{n+1})[1+d(u_n,x)]}{1+d(u_n,x)}, d(u_n,x)\right\}\right)\right)$$

$$-\psi\left(\left(max\left\{\frac{d(x,x)[1+d(u_n,u_{n+1})]}{1+d(u_n,x)}, d(u_n,x)\right\}\right)\right) + Lmin\left\{\frac{d(u_n,x)d(x,u_{n+1})}{1+d(u_n,x)}, \frac{d(u_n,u_{n+1})d(x,x)}{1+d(u_n,x)}\right\}$$

$$= \phi\big((max\{0, d(x,u_{n+1}), d(u_n,x)\})\big) - \psi\big((max\{0, d(u_n,x)\})\big) + Lmin\left\{\frac{d(u_n,x)d(x,u_{n+1})}{1+d(u_n,x)}, 0\right\}$$

$$= \phi\big((max\{d(x,u_{n+1}), d(u_n,x)\})\big) - \psi\{d(u_n,x)\}$$

Let $d(x,u_{n+1}) = p_{n+1}$ and $d(u_n,x) = p_n$

Then $\phi(p_{n+1}) \le \phi(max\{p_{n+1}, p_n\}) - \psi(p_n)$ (4.2.3.4)

If $p_{n+1} > p_n$ then from (4.2.3.4) we have

$\phi(p_{n+1}) \le \phi(p_{n+1}) - \psi(p_n)$ that is $\psi(p_n) \le 0$, which is a contradiction to the definition of ψ.

So $p_n > p_{n+1}$ and $\{p_n\}$ is a decreasing sequence, then from (4.2.3.4)

We have $\phi(p_{n+1}) \le \phi(p_n) - \psi(p_n)$ (4.2.3.5)

Since $\{p_n\}$ is a decreasing sequence of positive real numbers which is bounded below, there exist $r \ge 0$ such that $p_n = d(u_n,x) \to r$ as $n \to \infty$. (4.2.3.6)

Similarly as shown in the proof of theorem 4.2.1 we can show that $r = 0$, then we have $p_n = d(u_n,x) \to 0$ as $\to \infty$. (4.2.3.7)

Then $u_n \to x$ as $n \to \infty$. (4.2.3.8)

Similarly: $u_n \to y$ as $n \to \infty$.

Finally, the uniqueness of the limit implies $x = y$.

Thus T has a unique fixed point.

Remark 3. By choosing L=0 in Theorem 4.2.3 we get Theorem 4.1.9 as a corollary to Theorem 4.2.3.

The following is an example in the support of Theorem 4.2.3.

Example 4.2.3.1 Let $X = \{3,4,5,6,7\}$ be endowed with the usual metric $d(x,y) = |x-y|$ for all $x, y \in X$, and We define a partial order on X as follows.

$\preccurlyeq := \{(3,3), (4,4), (5,5), (6,6), (7,7), (5,6), (5,7), (6,7)\}$ then $(X, \preccurlyeq)$ is a partially ordered set.

Now we define the mapping $T: X \to X$ by $T = \begin{pmatrix} 3 & 4 & 5 & 6 & 7 \\ 3 & 3 & 4 & 4 & 4 \end{pmatrix}$, Clearly T is continuous.

Since $3 \preccurlyeq 3 \Longrightarrow 3 = T(3) \preccurlyeq 3 = T(3)$;

$4 \preccurlyeq 4 \Longrightarrow 3 = T(4) \preccurlyeq 3 = T(4)$;

$5 \preccurlyeq 5 \Longrightarrow 4 = \mathrm{T}(5) \preccurlyeq 4 = \mathrm{T}(5);$

$6 \preccurlyeq 6 \Longrightarrow 4 = \mathrm{T}(6) \preccurlyeq 4 = \mathrm{T}(6);$

$7 \preccurlyeq 7 \Longrightarrow 4 = \mathrm{T}(7) \preccurlyeq 4 = \mathrm{T}(7);$

$5 \preccurlyeq 6 \Longrightarrow 4 = \mathrm{T}(5) \preccurlyeq 4 = \mathrm{T}(6);$

$5 \preccurlyeq 7 \Longrightarrow 4 = \mathrm{T}(5) \preccurlyeq 4 = \mathrm{T}(7);$

$6 \preccurlyeq 7 \Longrightarrow 4 = \mathrm{T}(6) \preccurlyeq 4 = \mathrm{T}(7);$

Thus T is non-decreasing mapping.

By defining the functions ϕ, ψ: $\mathbb{R}^+ \to \mathbb{R}^+$ by: $\phi(t) = \frac{t}{6}$, $\psi(t) = \frac{t}{3}$, and Let L= 1.

then clearly $\phi \in \Phi, \psi \in \Psi$.

We note that the case $x = y$ follows trivially, so let $x, y \in X$ such that $x \preccurlyeq y$ and $x \neq y$.

We also consider the following three cases to verify the inequality (4.2.3.1)

<u>Case I:</u> Let us take $x = 5 \; and \; y = 6$ then

1. $d(Tx, Ty) = d\big(T(5), T(6)\big) = d(4,4) = 0.$

2. $M(x,y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, \frac{d(y,Tx)[1+d(x,Ty)]}{1+d(x,y)}, d(x,y)\right\}$

$$= max\left\{\frac{d\big(6,T(6)\big)[1+d(5,T(5)]}{1+d(5,6)}, \frac{d\big(6,T(5)\big)[1+d(5,T(6)]}{1+d(5,6)}, d(5,6)\right\}$$

$$= max\left\{\frac{d(6,4)[1+d(5,4)]}{1+d(5,6)}, \frac{d(6,4)[1+d(5,4)]}{1+d(5,6)}, d(5,6)\right\}$$

$$= max\left\{\frac{|6-4|[1+|5-4|]}{1+|5-6|}, \frac{|6-4|[1+|5-4|]}{1+|5-6|}, |5-6|\right\}$$

$$= max\left\{\frac{(2)(2)}{2}, \frac{(2)(2)}{2}, 1\right\}$$

$$= max\{2,2,1\} = 2.$$

3. $N(x,y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx]}{1+d(x,y)}, d(x,y)\right\}.$

$$= max\left\{\frac{d\big(6,T(6)\big)[1+d(5,T(5)]}{1+d(5,6)}, d(5,6)\right\}$$

$$= max\left\{\frac{d(6,4)[1+d(5,4)]}{1+d(5,6)}, d(5,6)\right\}$$

$$= max\left\{\frac{|6-4|[1+|5-4|]}{1+|5-6|}, |5-6|\right\}$$

$$= max\left\{\frac{(2)(2)}{2}, 1\right\}$$

$$=max\{2,1\} = 2.$$

4. $m(x,y) = min\left\{\frac{d(x,Ty)d(y,Tx)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\}$

$= min\left\{\frac{d(5,T(6))d(6,T(5))}{1+d(5,6)}, \frac{d(5,T(5))d(6,T(6))}{1+d(5,6)}\right\}$

$= min\left\{\frac{d(5,4)d(6,4))}{1+d(5,6)}, \frac{d(5,4))d(6,4))}{1+d(5,6)}\right\}$

$= min\left\{\frac{|5-4||6-4|}{1+|5-6|}, \frac{|5-4||6-4|}{1+|5-6|}\right\}$

$= min\left\{\frac{(1)(2)}{2}, \frac{(1)(2)}{2}\right\}$

$= min\{1,1\} = 1.$

So $\phi\big(d(Tx,Ty)\big) \leq \phi\big(M(x,y)\big) - \psi\big(N(x,y)\big) + Lm(x,y)$ becomes

$\frac{1}{6}(0) \leq \frac{1}{6}(2) - \frac{1}{3}(2) + 1(1)$

$0 \leq -\frac{1}{3} + 1$

$0 \leq \frac{2}{3}.$

<u>Case II</u> Let $x = 5$ *and* $y = 7$ then

1. $d(Tx,Ty) = d\big(T(5),T(7)\big) = d(4,4) = 0.$
2. $M(x,y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx]}{1+d(x,y)}, \frac{d(y,Tx)[1+d(x,Ty]}{1+d(x,y)}, d(x,y)\right\}$

$= max\left\{\frac{d\big(7,T(7)\big)[1+d(5,T(5)]}{1+d(5,7)}, \frac{d\big(7,T(5)\big)[1+d(5,T(7)]}{1+d(5,7)}, d(5,7)\right\}$

$= max\left\{\frac{d(7,4)[1+d(5,4)]}{1+d(5,7)}, \frac{d(7,4)[1+d(5,4)]}{1+d(5,7)}, d(5,7)\right\}$

$= max\left\{\frac{|7-4|[1+|5-4|]}{1+|5-7|}, \frac{|7-4|[1+|5-4|]}{1+|5-7|}, |5-7|\right\}$

$= max\left\{\frac{(3)(2)}{3}, \frac{(3)(2)}{3}, 2\right\}$

$= max\{2,2,2\} = 2.$

3. $N(x,y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx]}{1+d(x,y)}, d(x,y)\right\}$

$= max\left\{\frac{d\big(7,T(7)\big)[1+d(5,T(5)]}{1+d(5,7)}, d(5,7)\right\}$

$= max\left\{\frac{d(7,4)[1+d(5,4)]}{1+d(5,7)}, d(5,7)\right\}$

$= max\left\{\frac{|7-4|[1+|5-4|]}{1+|5-7|}, |5-7|\right\}$

$= max\left\{\frac{(3)(2)}{3}, 2\right\}$

$=max\{2,2\} = 2.$

4.$m(x,y) = min\left\{\frac{d(x,Ty)d(y,Tx)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\}$

$= min\left\{\frac{d(5,T(7)d(7,T(5))}{1+d(5,7)}, \frac{d(5,T(5))d(7,T(7))}{1+d(5,7)}\right\}$

$= min\left\{\frac{d(5,4)d(7,4))}{1+d(5,7)}, \frac{d(5,4))d(7,4))}{1+d(5,7)}\right\}$

$= min\left\{\frac{|5-4||7-4|}{1+|5-7|}, \frac{|5-4||7-4|}{1+|5-7|}\right\}$

$= min\left\{\frac{(1)(3)}{3}, \frac{(1)(3)}{3}\right\}$

$= min\{1,1\} = 1.$

So $\phi\big(d(Tx,Ty)\big) \leq \phi\big(M(x,y)\big) - \psi\big(N(x,y)\big) + Lm(x,y)$ becomes

$\frac{1}{6}(0) \leq \frac{1}{6}(2) - \frac{1}{3}(2) + 1(1)$

$0 \leq -\frac{1}{3} + 1$

$0 \leq \frac{2}{3}.$

<u>Case III</u>: **Let** $x = 6\ and\ y = 7$ then

1. $d(Tx,Ty) = d\big(T(6),T(7)\big) = d(4,4) = 0.$
2. $2. M(x,y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx]}{1+d(x,y)}, \frac{d(y,Tx)[1+d(x,Ty]}{1+d(x,y)}, d(x,y)\right\}$

$= max\left\{\frac{d\big(7,T(7)\big)[1+d(6,T(6)]}{1+d(6,7)}, \frac{d\big(7,T(6)\big)[1+d(6,T(7)]}{1+d(6,7)}, d(6,7)\right\}$

$= max\left\{\frac{d(7,4)[1+d(6,4)]}{1+d(6,7)}, \frac{d(7,4)[1+d(6,4)]}{1+d(6,7)}, d(6,7)\right\}$

$= max\left\{\frac{|7-4|[1+|6-4|]}{1+|6-7|}, \frac{|7-4|[1+|6-4|]}{1+|6-7|}, |6-7|\right\}$

$= max\left\{\frac{(3)(3)}{2}, \frac{(3)(3)}{2}, 1\right\}$

$= max\left\{\frac{9}{2}, \frac{9}{2}, 1\right\} = \frac{9}{2}.$

3. $N(x,y) = max\left\{\frac{d(y,Ty)[1+d(x,Tx]}{1+d(x,y)}, d(x,y)\right\}$

$= max\left\{\frac{d\big(7,T(7)\big)[1+d(6,T(6)]}{1+d(6,7)}, d(6,7)\right\}$

$= max\left\{\frac{d(7,4)[1+d(6,4)]}{1+d(6,7)}, d(6,7)\right\}$

$= max\left\{\frac{|7-4|[1+|6-4|]}{1+|6-7|}, |6-7|\right\}$

$= max\left\{\frac{(3)(3)}{2}, 1\right\}$

$=max\left\{\frac{9}{2}, 1\right\} = \frac{9}{2}.$

$$4.m(x,y) = min\left\{\frac{d(x,Ty)d(y,Tx)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\}$$

$$= min\left\{\frac{d((6,T(7))d(7,T(6))}{1+d(6,7)}, \frac{d((6,T(6))d(7,T(7))}{1+d(6,7)}\right\}$$

$$= min\left\{\frac{d((6,4)d(7,4))}{1+d(6,7)}, \frac{d(6,4)d(7,4))}{1+d(6,7)}\right\}$$

$$= min\left\{\frac{|6-4||7-4|}{1+|6-7|}, \frac{|6-4||7-4|}{1+|6-7|}\right\}$$

$$= min\left\{\frac{(2)(3)}{2}, \frac{(2)(3)}{2}\right\} = min\{3,3\} = 3.$$

So $\phi\big(d(Tx,Ty)\big) \leq \phi\big(M(x,y)\big) - \psi\big(N(x,y)\big) + Lm(x,y)$ becomes

$$\frac{1}{6}(0) \leq \frac{1}{6}\left(\frac{9}{2}\right) - \frac{1}{3}\left(\frac{9}{2}\right) + 1(3)$$

$$0 \leq -\frac{6}{8} + 3$$

$$0 \leq \frac{18}{8}.$$

From the Cases (I) - (III) considered above T satisfies the inequality (4.2.3.1) for the ϕ and ψ chosen in example 4.2.3.1 and hence T satisfies all the hypotheses of the Theorem 4.2.3 and T has a unique fixed point $x_\circ = 3$. If we choose L=0 in the inequality (4.2.3.1), from examples 4.2.3.1 in Cases (I-III) we observe that the inequality (4.2.3.1) fails to hold. This indicates the importance of L in Theorem 4.2.3.

Unit 5

5. Conclusion and future scope

5.1 Conclusion

In this Thesis, we proved two fixed point theorems namely Theorem 4.2.1 and Theorem 4.2.2 on the existence of fixed points for almost (ϕ, ψ) contractions involving rational expressions in partially ordered metric space. By imposing additional conditions we also proved uniqueness of fixed points of almost (ϕ, ψ)-contractions involving rational expressions in partially ordered metric space.

1. By Remark 1 and Examples (4.2.1.1), we conclude that Theorem 4.2.1 is more general than Theorem 4.1.7.
2. By Remark 2 and Examples (4.2.2.1), we conclude that Theorem 4.2.2 is more general than Theorem 4.1.8.

Our result extends and improves the results of S.Chandok, B.S.Choudhury and N.Metiya [8].

5.2 Future scopes

The existence of fixed point of almost (ϕ, ψ)-contractions involving rational expressions in partially ordered metric space is new area of study. Recently there are a number of published research papers related to this area of study. So the student researcher recommends the upcoming post graduate students of the department and other researchers who are interested to do their research work in this area of study.

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