A FIXED POINT THEOREM FOR CYCLIC CIRIC MAPPING IN COMPLETE DISLOCATED QUASI b-METRIC SPACES

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Declaration

I herewith submit the dissertation entitled "a fixed point theorem for Ciric type cyclic contraction in complete dislocated quasi b-metric spaces" for the award of degree of Master of Science in Mathematics. I undersigned declare that this study is original and it has not been submitted to any institution elsewhere for the award of any academic degree or like, where other sources of information have been used, they have been acknowledged.

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Abstract

The aim of this research is to establish the existence and uniqueness of fixed point of a dqb-cyclic-Ciric mapping in complete dislocated quasi b-metric spaces (dqb-metric space). In this study we followed analytical design method and secondary source of data such as journal articles, books etc. from the internet and JU library used.

Keywords: Fixed points; dislocated quasi b-metric spaces, dqb-cyclic-contraction, dqb-cyclic-Kannan mapping; dqb-cyclic-Ciric mapping.

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CHAPTER 1: INTRODUCTION

1.1 Back ground of the study

Let X be a nonempty set and $T: X \to X$ a self-map. We say that $x \in X$ is a fixed point of T if $\mathrm{T(x) = x}$ and denote by $\mathrm{F(T)}$ or $\mathrm{Fix(T)}$ the set of all fixed points of T.

The study of fixed point theorems and common fixed point theorems satisfying contractive type conditions has been a very active field of research activity during the last decades. In 1922, the Polish mathematician, Stefan Banach [2] proved a theorem which ensures, under approximation conditions, the existence and uniqueness of the fixed point. His result is called Banach fixed point theorem or the Banach contraction principle. This theorem provides techniques for solving the various types of applied problems in mathematical sciences and engineering. Banach contraction principle was introduced by Banach [2] as follows:

Definition 1.1: Let $\mathrm{(X, d)}$ be a metric space and let $\mathrm{T: X \to X}$. Then T is called a contraction mapping if there exists $\mathrm{k} \in [0, 1)$ such that

$$\mathrm{d(Tx, Ty) \leq kd(x, y), \forall x, y \in X.} \tag{1.1}$$

Banach Contraction Principle: Let $\mathrm{(X, d)}$ be a non-empty complete metric space with a contraction mapping $T : X \to X$. Then T admits a unique fixed-point x^* in X. Furthermore, x^* can be found as follows: start with an arbitrary element x_0 in X and define a sequence $\{x_n\}$ by $x_n = T(x_{n-1})$, then $x_n \to x^*$.

Inequality (1.1) implies continuity of T. A natural question is that whether we can find contractive conditions which will imply existence of fixed point in a complete metric space but will not imply continuity.

In 1968, Kannan [10] gave a new idea for the contractive type mapping which is very useful in the study of fixed point theory. Also he omitted the continuity assumption of the self-map T.

Definition 1.2: T is called a **Kannan mapping** if there exists $k \in (0, \frac{1}{2})$ such that

$$\mathrm{d(Tx, Ty) \leq k[d(Tx, x) + d(Ty, y)], \forall x, y \in X.}$$

Remark 1.1: Let $\mathrm{(X, d)}$ be a non-empty complete metric space with a Kannan mapping $T : X \to X$. Then T admits a unique fixed-point x^* in X. Furthermore, x^* can be found as follows: start with an arbitrary element x_0 in X and define a sequence $\{x_n\}$ by $x_n = T(x_{n-1})$, then $x_n \to x^*$.

In 1972, Chatterjea [6] introduced the following concept which gives a new direction to the study of the fixed point theory as follow:

Definition 1.3: T is called a **Chatterjea mapping** if there exists $k \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq k[\mathrm{d}(\mathrm{Tx}, \mathrm{y}) + \mathrm{d}(\mathrm{Ty}, \mathrm{x})], \forall x, y \in X.$$

Remark 1.1: Let (X, d) be a non-empty complete metric space with a Chatterjea mapping $T : X \to X$. Then T admits a unique fixed-point x^* in X. Furthermore, x^* can be found as follows: start with an arbitrary element x_0 in X and define a sequence $\{x_n\}$ by $x_n = T(x_{n-1})$, then $x_n \to x^*$.

Fixed point theory has been studied extensively, which can be seen from the works of many authors ([4], [8], [12], [14], [15]). In 2003, Kirk et al. [13] introduced the concept of cyclic mapping and established a fixed point theorem for mapping satisfying cyclic contraction condition.

Definition 1.4: Let A and B be non-empty closed subsets of a metric space (X, d) and $T: A \cup B \to A \cup B$. T is called a **cyclic map** if and only if $T(A) \subseteq B$ and $T(B) \subseteq A$.

In 2013, George [16] reviewed a cyclic Banach contraction mapping, cyclic Kannan mapping, cyclic Chatterjee mapping and cyclic Ciric mapping contraction as follows:

Definition 1.5: Let *A* and *B* be non-empty subsets of a metric space (*X*, *d*). A cyclic map $T: A \cup B \to A \cup B$ is said to be a

i. **Cyclic contraction** if there exists $b \in (0, 1)$ such that

$$d(Tx, Ty) \leq bd(x, y), \forall x \in A \text{ and } \forall y \in B.$$

ii. **Kannan type cyclic contraction** if there exists $b \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq b[d(Tx, x) + d(Ty, y)], \forall x \in A \text{ and } \forall y \in B.$$

iii. **Chatterjea type cyclic contraction** if there exists $b \in (0, \frac{1}{2})$such that

$$d(Tx, Ty) \leq b\,[d(Tx, y) + d(Ty, x)],\ \forall x \in A \text{ and } \forall y \in B.$$

iv. **Ciric type cyclic contraction** if there exists $b \in (0,1)$such that

$$d(Tx, Ty) \leq bmax\,[d(x, y), d(Tx, x), d(Ty, y)], \forall x \in A \text{ and } \forall y \in B.$$

Remark 1.3: In Definition 1.5, if (X, d) is a complete metric space and at least one of (i), (ii), (iii) and (iv) holds, then the self-map T has a unique fixed point. ([2], [8], [11], [30]) In 1931, Wilson [19] introduced the concept of quasi-metric spaces as a generalization of metric spaces. Also, in 2000, Hitzler and Seda [9] introduced dislocated metric spaces as a generalization of metric space. Zeyada et al. [28] introduced the concept of dislocated quasi-metric space which generalizes the result of Hitzler, Seda [9] and Wilson [19].

In 1989, Bakhtin [3] introduced b-metric space as a generalization of metric space. Also, many other authors introduced ambient spaces which generalized b-metric spaces such as quasi b-metric spaces by Hussain et.al. [17], b-metric-like spaces by Alghamdi et al. [1] and quasi b-metric-like spaces by Włodarczyk et al. [29].

Recently Klin-eam and Suanoom [5] proved the existence unique fixed point of a dqb-cyclic-Banach contraction mapping and a dqb-cyclic-Kannan mapping in complete dislocated quasi b-metric spaces. They also support their results by appropriate examples. However, the proof for Theorem 2.12 in [5] needs some modification.

In this study, inspired by the recent works of George et al. [16] and Klin-eam et al. [5], we introduce a dqb-cyclic-Ciric mapping in dislocated quasi b-metric spaces; and establish the existence and uniqueness of fixed point of a dqb-cyclic-Ciric mapping in complete dislocated quasi b-metric spaces. We also provide examples in support of our main result.

1.2. Statement of the problems

This study will focus on establishing a fixed point theorem for the existence and uniqueness of fixed point of a dqb-cyclic-Ciric mapping in complete dislocated quasi b-metric spaces. We shall support our main finding with example.

This study answered the following questions:

1. How can we prove the existence of fixed point of a dqb-cyclic-Ciric mapping in complete dislocated quasi b-metric spaces?
2. How can we show the uniqueness of fixed point of a dqb-cyclic-Ciric mapping in complete dislocated quasi b-metric spaces?
3. How can we support the main result of this study with an example?

1.3. Objective of the study

1.3.1 General objective

The main Objective of this study was to establish the existence and uniqueness of fixed point a dqb-cyclic-Ciric mapping in complete dislocated quasi b-metric spaces.

1.3.2. Specific objectives

This study has the following specific objectives:

1. To prove the existence of fixed point of a dqb-cyclic-Ciric mapping in complete dislocated quasi b-metric spaces.
2. To show the uniqueness of fixed point of a dqb-cyclic-Ciric mapping in complete dislocated quasi b-metric spaces.
3. To provide an example in support of the main result of this study.

1.4. Significance of the study

Fixed point theory has been a subject of growing interest of many researchers for various types of well-known contractions in these spaces. We hope that the results obtained in this study may contribute to research activities in this area. Further, collaboration in this research will be useful for the graduate program of the department. The research may also beneficial from this study since it uses to develop scientific research writing skill, scientific literature collecting skill and scientific communication in mathematics.

1.5. Delimitation of the study

This study focuses only on proving the existence and uniqueness of fixed point theorem of a Ciric type cyclic contraction in complete dislocated quasi b-metric spaces.

CHAPTER 2: METHODOLOGY OF THE STUDY

2.1 Study site and period

Our study focused on metric fixed point theory which one of an interesting topic in a functional analysis. The goal of this research is to prove the existence and uniqueness of fixed point of a dqb-cyclic-Ciric mapping in dislocated quasi b-metric spaces. This study was conducted from September, 2014 to September, 2015 in Mathematics Department of Jimma University.

2.2. Study design

In order to achieve the objective of this study analytical design was used.

2.3 Source of information

This study mostly depends on documented materials. So, the available source of information are different mathematics reference books, unpublished mathematics MSc. Thesis in the department and Journals and Published research works from the internet. The researcher collected different documents that are listed above which supports the study and discuss about the collected material and other activities with an advisor.

2.4 Procedure of the study

The procedures we shall use for analysis is the standard iteration technique used in

- George et al. [15] and
- Klin-eam and Suanoom [5].

2.6 Ethical consideration

Ethical consideration has to be considered in all stages of the research process. This study needs books, published journal articles and other related materials was collected from different sources. But there may be a problem in collecting all the above listed materials without any permission letters. So to make the study legal, permission was taken from a research review and ethical committee of college of Natural science of Jimma University.

CHAPTER THREE: DISCUSSION AND RESULT

3.1. PRELIMINERIES

We begin with the following definition as a recall from [8].

Definition 3.1.1 [8] let X be a nonempty set. Suppose that the mapping $d: X \times X \rightarrow [0, \infty)$ satisfies the following conditions:

(d1) $d(x, x) = 0$ for all $x \in X$;

(d2) $d(x, y) = d(y, x) = 0$ implies $x = y$ for all$x, y \in X$;

(d3) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d4) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

If d satisfies conditions $(d1)$, $(d2)$ and $(d4)$, then d is called a quasi-metric on X. If d satisfies conditions (d2), (d3) and (d4), then d is called a dislocated metric on X. If d satisfies conditions $(d1) - (d4)$, then d is called a metric on X.

In 2005, the concept of dislocated quasi-metric spaces [27], which is a new generalization of quasi b-metric spaces and dislocated b-metric spaces, was introduced. By Definition 2.1, if setting conditions (d2) and (d4) hold true, then d is called a dislocated quasi metric on X.

Remark 3.1.2 [27] Metric spaces are quasi metric spaces and dislocated metric spaces, but the converse is not true.

In 1989, Bakhtin [3] introduced the concept of b-metric spaces and investigated some fixed point theorems in such spaces.

Definition 3.1.3 [3] let X be a nonempty set and s$\geq$ 1 be a Constant. Suppose that the mapping $b: X \times X \rightarrow [0, \infty)$ satisfies the following conditions:

(b1) $b(x, y) = b(y, x) = 0 \Leftrightarrow x = y$ for all $x, y \in X$;

(b2) $b(x, y) = b(y, x)$ for all $x, y \in X$;

(b3) $b(x, y) \leq s[b(x, z) + b(z, y)]$ for all $x, y, z \in X$.

The pair (X, b) is then called a b-metric space.

Remark 3.1.4[3] Metric spaces are b-metric spaces, but the converse is not true.

In 2012, Shah and Hussain [17] introduced the concept of quasi b-metric spaces and verified some fixed point theorems in quasi b-metric spaces.

Definition 3.1.5 [16] let X be a nonempty set and s≥ 1 be a Constant. Suppose that the mapping $q: X \times X \to [0, \infty)$ Satisfies the following conditions:

(q1) $q(x, y) = q(y, x) = 0 \Leftrightarrow x = y$ for all $x, y \in X$;

(q2) $q(x, y) \leq s[q(x, z) + q(z, y)]$ for all $x, y, z \in X$.

The pair (X, q) is then called a quasi b-metric space.

Remark 3.1.6 [16] b-metric spaces are quasi b-metric spaces, but the converse is not true.

Recently, the concept of b-metric-like spaces, which is a new generalization of b-metric spaces, was introduced by Alghamdi et al. [1].

Definition 3.1.7 [1] let X be a nonempty set and s≥ 1 be a Constant. Suppose that the mapping $D: X \times X \to [0, \infty)$ Satisfies the following conditions: (D1): $D(x, y) = 0 \Rightarrow x = y$ for all $x, y \in X$;

(D2) $D(x, y) = D(y, x)$ for all $x, y \in X$;

(D3) $D(x, y) \leq s[D(x, z) + D(z, y)]$ for all $x, y, z \in X$.

The pair (X, D) is then called a b-metric-like space (or a dislocated b-metric space).

Remark 3.1.8 [1] b-metric spaces are b-metric-like spaces, but the converse is not true.

Definition 3.1.9$[5]$ Let X be a nonempty set and constant $s \geq 1$. Suppose that the mapping $d: X \times X \to [0, \infty)$ satisfies the following conditions:

(d1) $d(x, y) = d(y, x) = 0$ implies $x = y$ for all $x, y \in X$;

(d2) $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

The pair (X, d) is then called a dislocated quasi b-metric space (or simply dqb-metric). The number s is called the coefficient of (X, d).

Remark 3.1.10$[5]$ b-metric spaces, quasi b-metric spaces and b-metric-like spaces are dislocated quasi b-metric spaces, but the converse is not true.

Example 3.1.11$[5]$ Let $X = \mathbb{R}$ and let

$$d(x, y) = |x - y|^2 + \frac{|x|}{n} + \frac{|y|}{m},$$

where $n, m \in \mathbb{N} \setminus \{1\}$ with $n \neq m$.

Here since $d(1,1) \neq 0$ and $d(1,2) \neq d(2,1)$,, (X, d) is a dislocated quasi b-metric space with the coefficient $s = 2$. . It is obvious that (X, b) is not a dislocated quasi-metric space.

Example 3.1.12 [5] Let $X = \{0,1,2\}$, and let $d: X \times X \to \mathbb{R}^+$ be defined by

$$d(x,y) = \begin{cases} 2; & \mathrm{x = y = 0,} \\ \frac{1}{2}; & \mathrm{x = 0, y = 1,} \\ 2; & \mathrm{x = 1, y = 0,} \\ \frac{1}{2}; & \text{otherwise.} \end{cases}$$

Here also since $d(1,1) \neq 0$ and $d(1,2) \neq d(2,1)$, the space (X,d) is a dislocated quasi-b-metric space with the coefficient $s = 2$, Also, it is obvious that (X,b) is not a dislocated quasi-metric space.

Definition3.1.13 [**5**] Let A and B be nonempty closed subsets of a dislocated quasi b-metric space (X,d) and $\mathrm{s} \geq 1$ and $sk \leq 1$be a Constants. A cyclic map $T: A \cup B \to A \cup B$ is said to be a dqb cyclic Banach contraction if there exists $k \in (0,1)$ such that

$$d(Tx,Ty) \leq kd(x,y), \forall x \in A, \forall y \in B$$

Theorem 3.1.14 [**5**] Let A and B be nonempty closed subsets of a complete dislocated quasi b-metric space (X,d). Let T be a cyclic mapping that satisfies the condition of a dqb-cyclic-Banach contraction. Then T has a unique fixed point in $A \cap B$.

3. 2 Main results

In this section, we begin with introducing a dislocated quasi b-convergent sequence, a Cauchy sequence and a complete space according to Zoto et al. [29].

Definition 3.2.1 [29]

(1) A sequence $\{x_n\}$ in a dqb-metric space (X,d) dislocated quasi b-**converges** (for short, dqb-converges) to $x \in X$ if $\lim_{n\to\infty} d(x_n, x) = 0 = \lim_{n\to\infty} d(x, x_n)$.

In this case, we say x is called a dqb-limit of$\{x_n\}$, and we write $(x_n \to x)$.

(2) A sequence $\{x_n\}$ in a dqb-metric space (X,d) is called **Cauchy** if

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0 = \lim_{n\to\infty} d(x_m, x_n)$$

(3) A dqb-metric space (X,d) is **complete** if every Cauchy sequence in it is dqb-convergent in X.

Definition 3.2.2 Let A and B be nonempty closed subsets of a dislocated quasi b-metric space (X,d) and $\mathrm{s} \geq 1$ and $sk \leq 1$be a Constants , $T: A \cup B \to A \cup B$ is called a dqb-cyclic-Kannan mapping if there exists $r \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq r\big(d(x, Tx) + d(x, Ty)\big), \forall\, x \in A, \forall\, y \in B. \qquad (3.1)$$

The following theorem is Theorem 2.12 of Klin-eam and Suanoom [5]. We state the theorem as it is given in [5] and make some modification in the proof part of it.

Theorem 3.2.3 Let A and B be nonempty subsets of a complete dislocated quasi b-metric space (X,d). Let T be a cyclic mapping that satisfies the condition of a dqb-cyclic-Kannan mapping. Then T has a unique fixed point in $A \cap B$.

Proof Let $x \in A$ (fix) and, using the contractive condition of the theorem, we have

$$\begin{aligned} d(Tx, T^2x) &= d\big(Tx, T(Tx)\big) \\ &\leq rd(x, Tx) + rd(Tx, T^2x). \end{aligned}$$

So

$$d(Tx, T^2x) \leq \frac{r}{1-r} d(x, Tx). \qquad (3.2)$$

And from (3.2), we have

$$
\begin{aligned}
d(T^2x, Tx) = d(T(Tx), Tx) \\
&\le rd(Tx, T^2x) + rd(x, Tx) \\
&\le \tfrac{r}{1-r} rd(x, Tx) + rd(x, Tx) \\
&\le r\left[\tfrac{r}{1-r} d(x, Tx) + d(x, Tx)\right] \\
&= \tfrac{r}{1-r} d(x, Tx).
\end{aligned}
$$

So

$$d(Tx, T^2x) \le \tfrac{r}{1-r}\beta, \tag{3.3}$$

where $\beta = d(x, Tx)$.

By using (3.2) and (3.3), we have

$$d(T^3x, T^2x) \le (\tfrac{r}{1-r})^2\beta$$

and

$$d(T^2x, T^3x) \le (\tfrac{r}{1-r})^2\beta.$$

For all $n \in \mathbb{N}$, we get

$$d(T^{n+1}x, T^nx) \le (\tfrac{r}{1-r})^n\beta$$

and

$$d(T^nx, T^{n+1}x) \le (\tfrac{r}{1-r})^n\beta.$$

Let $n, m \in \mathbb{N}$, with $m > n$. By using the triangular inequality, we have

$$
\begin{aligned}
d(T^mx, T^nx) &\le s^{m-n}d(T^mx, T^{m-1}x) + s^{m-n-1}d(T^{m-1}x, T^{m-2}x) + \cdots + sd(T^{n+1}x, T^nx) \\
&\le (s^{m-n}k^{m-1} + s^{m-n-1}k^{m-2} + s^{m-n-2}k^{m-3} + \cdots + s^2k^{n+1} + sk^n)\beta \\
&\le \left(\left(\frac{r}{1-r}\right)^{n-1} + \left(\frac{r}{1-r}\right)^{n-1} + \left(\frac{r}{1-r}\right)^{n-1} + \cdots + \left(\frac{r}{1-r}\right)^{n-1} + \left(\frac{r}{1-r}\right)^{n-1}\right)\beta
\end{aligned}
$$

$$= \left(\frac{r}{1-r}\right)^{n-1} (m-n+1)\beta$$

$$< (\frac{r}{1-r})^{n-1}\xi\beta$$

for some $\xi > m-n+1$. Taking $n \to \infty$, we get $d(T^m x, T^n x) \to 0$.

Similarly, let $n, m \in N$ with $m > n$, by using the triangular inequality, we have

$$d(T^n x, T^m x) < (\frac{r}{1-r})^{n-1}\xi\beta.$$

Taking $n \to \infty$, we get $d(T^n x, T^m x) \to 0$. Thus $\{T^n x\}$ is a Cauchy sequence in X.

Since (X, d) is complete, the sequence $\{T^n x\}$ converges to some $z \in X$.

We note that $\{T^{2n} x\}$ is a sequence in A and $\{T^{2n-1} x\}$ is a sequence in B in a way that both sequences tend to the same limit z.

Since A and B are closed, we have $z \in A \cap B$, and then $A \cap B \neq \emptyset$.

Now, we will show that $Tz = z$.

By using (3.1), consider

$$d(T^n x, Tz) = d(T(T^{n-1}x), Tz) \leq rd(T^{n-1}x, T^n x) + rd(z, Tz).$$

Taking limit as $n \to \infty$ in the above inequality, we have

$$d(z, Tz) \leq rd(z, Tz).$$

Since $0 \leq r < \frac{1}{2}$, we have $d(z, Tz) = 0$.

Similarly, considering from (3.1), we get

$$d(Tz, T^n x) = d(Tz, T(T^{n-1}x)) \leq rd(z, Tz) + rd(T^{n-1}x, T^n x).$$

Taking limit as $n \to \infty$ in the above inequality, we have

$$d(Tz, z) \leq rd(z, Tz).$$

Since $d(z, Tz) = 0$, we have $d(z, Tz) = 0$.

Hence $d(z, Tz) = d(Tz, z) = 0$

Therefore $Tz = z$ and hence z is a fixed point of T.

Finally, to prove the uniqueness of the fixed point z, let $z^* \in X$ be another fixed point of T.

Then from the give contractive condition (3.1) we obtain $d(z, z) = d(z^*, z^*) = 0$.

Now consider

$$\begin{aligned} d(z, z^*) &= d(Tz, Tz^*) \\ &\leq rd(z, Tz) + rd(z^*, Tz^*) \qquad (3.4) \\ &= rd(z, z) + rd(z^*, z^*) \\ &= 0. \end{aligned}$$

On the other hand,

$$\begin{aligned} d(z^*, z) &= d(Tz^*, Tz) \\ &\leq rd(z^*, Tz^*) + rd(z, Tz) \qquad (3.5) \\ &= rd(z^*, z^*) + rd(z, z) \\ &= 0. \end{aligned}$$

From (3.4) and (3.5), we obtain

$$d(z, z^*) = d(z^*, z) = 0$$

which implies that

$$z^* = z.$$

Therefore, z is a unique fixed point of T.

Next, we begin with introducing the concept of a dqb-cyclic-Ciric mapping.

Definition 3.2.4 Let A and B be nonempty closed subsets of a dislocated quasi b-metric space (X, d) and s ≥ 1 and $sk \leq 1$be a Constants . A cyclic map $T: A \cup B \to A \cup B$ is said to be a dqb-cyclic-Ciric mapping if there exists $k \in (0,1)$such that

$$d(Tx, Ty) \leq k\ max\ \{d(x, y), d(Tx, x), dd(Ty, y)\}. \forall\ x \in A \text{ and } \forall\ y \in B. \tag{3.6}$$

Remark: In Definition 3.2.4, if $A = B = X$, then the map $T: X \longrightarrow X$ is a dqb-Ciric mapping.

Lemma 3.2.5. Let (X, d) be a dislocated quasi b-metric space and $T: X \longrightarrow X$ be a dqb-Ciric mapping. If z is a fixed point of T, then $d(z, z) = 0$.

Proof: Let z be a fixed point of T. Then

$$d(z, z) = d(Tz, Tz) \leq k \max\{d(z, z), d(Tz, z), d(Tz, z)\} = kd(z, z).$$

$$\Rightarrow (1 - k)d(z, z) \leq 0.$$

Since $1 - k > 0$, $d(z, z) \geq 0$ and hence $d(z, z) = 0$.

Theorem 3.2.6 Let A and B be nonempty closed subsets of a complete dislocated quasi b-metric space (X, d). let $T: A \cup B \to A \cup B$ be a dqb-cyclic-Ciric mapping. Then T has a unique fixed point in $A \cap B$.

Proof Let $x \in A$ (fix) and, using the contractive condition (3.6) of the theorem, we have

$$d(T^2x, Tx) \leq \text{k} \max\{\text{d}(\text{Tx}, \text{x}), \text{d}(\text{T}^2\text{x}, \text{Tx})\}.$$

If $\max\{\text{d}(\text{Tx}, \text{x}), \text{d}(\text{T}^2\text{x}, \text{Tx})\}$ is $d(T^2x, Tx)$, then

$$d(T^2x, Tx) \leq kd(T^2x, Tx) < d(T^2x, Tx),$$

which is not possible. So, we have

$$d(T^2x, Tx) \leq kd(Tx, x). \tag{3.7}$$

In similar way, we have

$$d(Tx, T^2x) \leq \text{k} \max\{\text{d}(\text{x}, \text{Tx}), \text{d}(\text{Tx}, \text{T}^2\text{x})\} \tag{3.8}$$

From (3.7) using (3.8), we obtain

$$d(T^2x, Tx) \leq kmax\{\text{d}(\text{x}, \text{Tx}), \text{d}(\text{Tx}, \text{x})\}. \tag{3.9}$$

Now let α = $k\ max\{d(x, Tx), d(Tx, x)\}$. Then

$$d(T^2x, Tx) \leq k\alpha, \tag{3.10}$$

and

$$d(Tx, T^2x) \leq k\alpha. \tag{3.11}$$

Using (3.10) and (3.11), we have

$$d(T^3x, T^2x) \leq k^2\alpha, \tag{3.12}$$

and

$$d(T^2x, T^3x) \leq k^2\alpha. \tag{3.13}$$

Inductively, for each $n \in N$, we have

$$d(T^{n+1}x, T^nx) \leq k^n\alpha, \tag{3.14}$$

and

$$d(T^nx, T^{n+1}x) \leq k^n\alpha\ . \tag{3.15}$$

Letting $n \to \infty$ in (3.13) and (3.14), we have

$d(T^{n+1}x, T^nx) \to 0$ And $d(T^nx, T^{n+1}x) \to 0$. (3.16)

Now, let $n, m \in N$with$m > n$, by using the triangular inequality, we have

$d(T^mx, T^nx) \leq sd(T^mx, T^{n+1}1x) + sd(T^{n+1}x, T^nx).$

$d(T^mx, T^nx) \leq s^2d(T^mx, T^{n+2}1x) + s^2d(T^{n+2}x, T^{n+1}x) + sd(T^{n+1}x, T^nx).$

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$$\begin{aligned}
&\leq s^{m-n}d(T^mx, T^{m-1}1x) + s^{m-n-1}d(T^{m-1}x, T^{m-2}x) + \cdots + sd(T^{n+1}x, T^nx).\\
&\leq (s^{m-n}k^{m-1} + s^{m-n-1}k^{m-2} + s^{m-n-2}k^{m-3} + \cdots + s^2k^{n+1} + sk^n)\alpha.\\
&= \begin{pmatrix} (sk)^{m-n}k^{n-1} + (sk)^{m-n-1}k^{m-2} + (sk)^{m-n-2}k^{n-1} + \ldots \\ +(sk)^2k^{n-1} + (sk)k^{n-1}. \end{pmatrix}\alpha\\
&\leq sk\alpha((sk)^{m-n-1} + (sk)^{m-n-2} + (\cdots + (sk)^2 + sk + 1)\\
&= sk\alpha \textstyle\sum_{j=0}^{m-n-1}(sk)^j\\
&= sk\alpha \textstyle\sum_{j=0}^{\infty}(sk)^j\\
&\leq (sk^n\alpha)\frac{1}{1-sk} \to 0 \text{ and } n \to \infty.
\end{aligned} \tag{3.17}$$

Similarly, let $n, m \in N$with$m > n$, by using the triangular inequality, we have

$d(T^nx, T^mx) \leq sd(T^nx, T^{n+1}1x) + sd(T^{n+1}x, T^mx).$

$d(T^nx, T^mx) \leq s^2d(T^{n+1}x, T^{n+2}1x) + s^2d(T^{n+2}x, T^mx) + sd(T^{n+1}x, T^nx).$

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$$\leq s^{m-n}d(T^{m-1}x,T^{m}x)+s^{m-n-1}d(T^{m-2}x,T^{m-1}x)+\cdots+sd(T^{n}x,T^{n+1}x).$$
$$\leq (s^{m-n}k^{m-1}+s^{m-n-1}k^{m-2}+s^{m-n-2}k^{m-3}+\cdots+s^{2}k^{n+1}+sk^{n})\alpha.$$
$$=\begin{pmatrix}(sk)^{m-n}k^{n-1}+(sk)^{m-n-1}k^{m-2}+(sk)^{m-n-2}k^{n-1}+\ldots\\ +(sk)^{2}k^{n-1}+(sk)k^{n-1}.\end{pmatrix}\alpha$$
$$\leq sk\alpha((sk)^{m-n-1}+(sk)^{m-n-2}+(\cdots+(sk)^{2}+sk+1)$$
$$= sk\alpha\sum_{j=0}^{m-n-1}(sk)^{j}$$
$$\leq sk\alpha\sum_{j=0}^{\infty}(sk)^{j}$$
$$\leq (sk^{n}\alpha)\frac{1}{1-sk}\rightarrow 0 \text{ as } n\rightarrow 0 \qquad (3.18)$$

Thus, from (3.17) and (3.18), we conclude that the sequence $\{T^{n}x\}_{n=0}^{\infty}$ is Cauchy.

Since (X,d) is complete, we have $\{T^{n}x\}$ converges to some $z\in X$.

$$\Rightarrow \lim_{n\to\infty}d(T^{n}z,z)=0=\lim_{n\to\infty}d(z,T^{n}z)). \qquad (3.19)$$

We note that $\{T^{2n}x\}$ is a sequence in A and $\{T^{2n-1}x\}$ is a sequence in B in a way that both sequences tend to the same limit z.

Hence, A and B are closed, we have $z\in A\cap B$, and then $A\cap B\neq\emptyset$.

Now, we will show that $Tz=z$.

Now writing z for x and $T^{n-1}(x)$ for y in (3.6), we get

$$d(Tz,T^{n}z)=d\big(Tz,T(T^{n-1}x)\big)$$
$$\leq kmax\{d\big(z,T^{n-1}(x)\big),d(Tz,z),d\big(T^{n}(x),T^{n-1}(x)\big)\}. \qquad (3.20)$$

Letting $n\to\infty$ in (3.20), using (3.19) and (3.15), we have

$$d(Tz,z)\leq k\ max\{0,d(Tz,z),0\}$$
$$\Rightarrow d(Tz,z)\leq kd(Tz,z).$$

This implies that $(1-k)d(z,Tz)\leq 0$, where $0\leq k<1$,which follows that

$$d(Tz,z)=0. \qquad (3.21)$$

Similarly, putting $T^{n-1}(x)$ for x and z for y in (3.6), we get

$$d(T^n z, Tz\ ) = d(T(T^{n-1}(x), Tz)$$
$$\leq kmax\{d(\ T^{n-1}z, z), d(T^{n-1}z, T^n z), d(z, Tz)\}. \qquad (3.22)$$

Taking limit as $n \to \infty$ in (3.22) using (3.18) and (3.15), we obtain

$$d(z, Tz) \leq kd(z, Tz).$$

This implies that $(1-k)d(z, Tz) \leq 0$, where $0 \leq k < 1$, which follows that

$$d(z, Tz) = 0. \qquad (3.23)$$

Now, using (3.21) and (3.23), we get

$$d(Tz, z) = 0 = d(z, Tz),$$

which follows that $Tz = z$.

Thus, z is a fixed point of T.

To show the uniqueness, let us assume that there exists two fixed points z and w of T. That is, $Tz = z$ and $Tw = w$

Then, by Lemma 3.2.5, we have $d(z, z) = 0$ and $d(w, w) = 0$.

Now, writing z for x and w for y in (3.6), we have

$$d(z, w) = d(Tz, Tw) \leq\ kmax\{d(z, w), d(Tz, z), d(Tw, w)\}.$$
$$\Longrightarrow d(z, w) \leq kmax\{d(z, w), d(z, z), d(w, w)\} = k\ d(z, w).$$
$$\Longrightarrow d(z, w) \leq k\ d(z, w).$$
$$\Longrightarrow\ (1-k)d(z, w)\ \leq 0.$$

Hence, $d(z, w) = 0$.

Similarly, we can show d (w, z)= 0.

Thus, d (z, w) $= 0 = d(z, w)$ and hence $z = w$.

Therefore, z is a unique fixed point of T.

Next we give example in support of Theorem 3.2.5

Example 3.2.6 Let $X = \mathbb{R}$ and and $T: X \to X$ define the function $d: X \times X \to \mathbb{R}$ given by

$$d(x, y) = |x - y|^2 + |y|, \forall x, y \in \mathbb{R}.$$

Then d(x, y)=0 $\Rightarrow |x-y|^2 = 0\ and\ |y| = 0$

$$\Rightarrow x = y \text{ and } y = 0 \Rightarrow x = y = 0.$$

And also d(x, y) = $|x-y|^2+|y|$

$$\leq (|x-z|+|z-y|)^2+|y|$$

$$\leq |x-z|^2+|z-y|^2+2|x-z||z-y|+|y|$$

$$\leq 2(|x-z|^2+|z-y|^2)+|y|$$

$$\leq 2(|x-z|^2+|z|)+2(|z-y|^2+|y|)$$

$$=2(d(x, z) + d(z, y))\ \forall x,y \in \mathbb{R}.$$

Therefore, d is a dislocated quasi b-metric with the coefficient $s=2$.

Note that Since d(1,1)= $1 \neq 0$. This implies (x, d) is dislocated metric space on $\mathbb{R}$ and since 3=d(1,2)$\neq d(2,1)=2$ .This implies (x, d) is not a quasi b-metric space on $\mathbb{R}$. These imply (x, d) is not dislocated quasi metric space on $\mathbb{R}$.

Now, let $A=(-\infty,0]$ and $B=[0,\infty)$. Define T: R→ R by $Tx=\begin{cases} 0 \text{ if } x\leq 1 \\ -1 \text{ if } x>1 \end{cases}$.

Thus, $T(A)=\{0\}\subset B$ and $T(B)=\{-1,0\}\subset A$. Hence the map T is cyclic on $\mathbb{R}$.

Next, we show that the map T satisfying the contractive condition (3.6). We shall the following cases.

Case-1: x, y $\in(-\infty,1]$.

Here d(Tx, Ty) = 0 and hence in these case for any $k\in(0,1)$ the condition (3.6) is satisfied.

Case-2: x, y $\in(1,\infty)$

Here d(Tx, Ty) = $|Tx-Ty|^2+|Ty|$=1;

d(x, y) = $|x-y|^2+|y|>1$;

d(Tx, x) = $|-1-x|^2+|x|=(1+x)^2+x>5$ and

$d(\mathrm{Ty},\mathrm{y}) = |-1-y|^2+|y|=(1+y)^2+y>5$.

Case-3:- x $\in(-\infty,1]$ and y $\in(1,\infty)$.

Here d(Tx, Ty) = $|0+1|^2+|-1|=2$;

d(x, y) = $|x-y|^2 + |y| > 1$;

d(Tx, x) = $|0-x|^2 + |x| = x^2 + |x| \geq 0$;

and

$d(Ty, y) = |-1-y|^2 + |y| = (1 + y)^2 + y > 5$.

<u>Case-4:-</u> y $\in (-\infty, 1]$ and x $\in (1, \infty)$.

Here d(Tx, Ty) = $|-1 - 0|^2 + |0| = 1$;

d(x, y) = $|x-y|^2 + |y| > 0$;

d(Tx, x) = $|-1-x|^2 + |x| = x^2 + |x| > 5$,

and

d(Ty, y) = $|0-y|^2 + |y| = y^2 + |y| \geq 0$.

So, in all the four cases the contractive condition is satisfied for $\propto = \frac{2}{5}$. Thus, T satisfies the dqb-cyclic-ciric mapping condition Theorem 3.2.5 and 0 is the unique fixed point of T.

Remark: Theorem 3.1.14 follows as a corollary to Theorem 3.2.5. Of course, Theorem 3.2.5 generalizes Theorem 3.1.14, since Theorem 3.1.14 does not hold for any $k \in (0, 1)$ if $x = \frac{11}{10}$ and $y = \frac{1}{2}$.

CHAPTER FOUR: - CONCLUSIONS AND FUTURE SCOPES

4.1. CONCLUSIONS

In this Thesis, we proved two fixed point theorems namely Theorem 3.2.4and Theorem 3.2.6 on the existence and uniqueness of fixed points for dqb-cyclic-Kannan mapping and dqb-cyclic-Ciric mappings in complete dqb dislocated quasi metrics. Theorem 3.2.4 is merely making some line of modification on Theorem 2.12 of [5]. We also gave example in support of our main result.

Theorem 3.2.5 generalizes Theorem 2.9 in Klin-eam and Suanoom [5] as shown in Example 3.2.6 and the remark following it.

4.2. FUTURE SCOPE

The search for the existence and uniqueness of fixed points of different type of mappings defined on dislocated quasi b-metric is an active area of study as it is exhibited from a number of recent published scientific communications in the literature. So the researcher recommends the upcoming post graduate students and other researchers who have interest in the area to do their research work in this area of study.

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