A FIXED POINT THEOREM FOR GENERALIZED WEAKLY CONTRACTIVE MAPPINGS IN b-METRIC SPACES

A PROPOSAL SUBMITTED TO THE DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS

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March, 2019
Jimma, Ethiopia

Acknowledgment

First of all, I am indebted to Allah who gave me long life and helped me to pass through different ups and downs to reach this time. Next, my Special heartfelt thanks go to my advisor, Dr.Kidane Koyas and co-advisor Aynalem Girma for their unreserved support, advice and guidance throughout the preparation of this research proposal. Lastly, I would like to express my deepest gratitude to my friend Mifta Jebel for his great contribution in all directions while I was preparing this research proposal.

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Chapter 1

Introduction

1.1 Background of the study

Fixed point theory is an important tool in the study of nonlinear analysis. It is considered to be the key connection between pure and applied mathematics. It is also widely applied in different fields of study such as Economics, Chemistry, Physics and almost all engineering fields.
In 1922, Banach proved the following famous fixed point theorem.
(Banach, 1922) Let (X,d) be a complete metric space, $T: X \to X$ be a contraction, there exists a unique fixed-point $x_0 \in X$ of T. This theorem, called the Banach contraction principle is a forceful tool in nonlinear analysis.
Another category of contraction which is separate from Banach contraction, and does not imply continuity, was proposed by (Kannan, 1968) who also established in the same work that such mappings necessarily have unique fixed points in complete metric spaces. Mappings belonging to this category are known as Kannan type.
In 1972, a new concept which is different from that of (Bannach, 1922) and (Kannan, 1968) for contraction type mapping was introduced by (Chatterjea, 1972) which gives a new direction to the study of fixed-point theory. There are a class of contractive mappings which are different from Banach contraction and have unique fixed point in complete metric spaces.

The family of contractive mappings in metric spaces is a great interest and has already been studied in the literature since long time.
In 1997, Alber *et al.* generalized Banach's contraction principle by introducing the concept of weakly contractive mappings in Hilbert spaces. Every weakly contractive mapping on a complete Hilbert space has a unique fixed point.
Rhoades, (2001) showed that every weakly contractive mapping has a unique fixed point in complete metric spaces. Since then, many authors obtained generalizations and extensions of the weakly contractive mappings.

In particular, (Choudhury *et al*., 2011) generalized fixed point results on weakly contractive mappings by using altering distance functions. In 1993, Czerwik introduced the concept of b-metric spaces and proved the Banach contraction mapping principle in the setting of b-metric spaces. Afterwards, several research papers were published on the existence of fixed point results for single-valued and multi-valued mappings in the setting of b-metric spaces.
Very recently, (Cho, 2018) introduced the notion of generalized weakly contractive mappings in metric spaces and proved a fixed point theorem for generalized weakly contractive mappings defined on complete metric spaces.
Inspired and motivated by the results of (Cho, 2018) the purpose of this research is to extend the main theorem of (Cho, 2018) in the setting of b-metric spaces.

1.2 Statements of the problem

This study will focus on establishing a fixed point theorem for generalized weakly contractive mappings in the setting of b-metric spaces.

1.3 Objectives of the study

1.3.1 General objective

The main objective of this study is to establish a fixed point theorem for generalized weakly contractive mappings in the setting of b-metric spaces.

1.3.2 Specific objectives

This study has the following specific objectives:

- To prove the existence of fixed point for generalized weakly contractive self-mappings in the setting of complete b-metric space.
- To verify the uniqueness of the fixed point.
- To verify the applicability of the main results obtained using specific example.

1.4 Significance of the study

The study may have the following importance:

- The outcome of this study may contribute to research activities on study area
- It may provide basic research skills to the researcher.
- It may help to show existence and uniqueness of problems involving integral and differential equations.

1.5 Delimitation of the Study

This study will be delimited to finding the existence of a fixed point for generalized weakly contractive self-mappings in the setting of complete b-metric spaces.

Note: Throughout this proposal, we assume that R^{+} is the set of non-negative real numbers.

Chapter 2

Review of Related Literatures

Let X be a non empty set and $T: X \to X$ be a self map. We say that x is a fixed point of T if $T(x) = x$ and denoted by $Fix(T)$. Fixed point theory has been studied extensively, which can be seen from the works of many authors.
(Banach, 1922) Banach contraction principle was introduced as follows:

Let (X,d) be a metric space and let $T : X \longrightarrow X$. Then T is called a Banach contraction mapping if there exists $k \in [0,1)$ such that $d(Tx,Ty) \leq kd(x,y)$ for all $x,y \in X$. If (X,d) is a complete metric space, then T has a unique fixed point.
Kannan, (1968) The concept of Kannan mapping was introduced in 1968 as follow:

Let (X,d) be a metric space and let $T : X \longrightarrow X$. Then T is called a Kannan mapping if there exists $k \in [0,1/2)$ such that $d(Tx,Ty) \leq k[d(x,Tx) + d(y,Ty)]$ for all $x,y \in X$. If (X,d) is a complete metric space, then T has a unique fixed point.
(Chatterjea,1972) The concept of Chatterjea type mapping was introduced in 1972 as follow:
T is called Chatterjea mapping if there exists $k \in [0,1/2)$ such that
$d(Tx,Ty) \leq k[d(x,Ty) + d(y,Tx)]$ for all $x,y \in X$. If (X,d) is a complete metric space, then T has a unique fixed point.

Definition 2.0.1 *(Khan, 1984) A function $\psi : R^+ \longrightarrow R^+$ is called altering distance function if the following properties holds.*
ψ is continuous and non-decreasing function and $\psi(t) = 0$ if and only if $t = 0$.

Definition 2.0.2 *A function $f : X \to R^+$, where X is a metric space, is called lower semi-continuous if for all $x \in X$ and $\{x_n\} \in X$ with*

$$f(x) \leq \liminf_{n\to\infty} f(x_n).$$

Definition 2.0.3 *(Czerwik 1993) Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \longrightarrow R^+$ is a $b-$metric if and only if for all $x,y,z \in X$, the following conditions are satisfied:*
(a) $d(x,y) = 0$ if and only if $x = y$;
(b) $d(x,y) = d(y,x)$;
(c) $d(x,z) \leq s[d(x,y) + d(y,z)]$.

The pair(X,d) is called a b-metric space.
It should be noted that, the class of $b-$metric spaces is effectively larger than that of metric spaces, since a b-metric is a metric when $s = 1$. but,in general, the converse is not true.

Example 1.1. (Roshan *et al.*,2014) Let $X = R$ and $d : X \times X \longrightarrow R^+$ be given by $d(x,y) = (x-y)^2$ for all $x,y \in X$, then d is a b-metric on X with $s = 2$ but it is not a metric on X:
for all $x,y,z \in R$ where, $x = 2, y = 4$ and $z = 6$ we have $d(2,6) \nleq 2[d(2,4)+d(4,6)]$
Hence the triangle inequality for a metric does not hold.

Definition 2.0.4 *(Boriceanu, 2009) Let(X,d) be a b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$ be a given mapping. We say that T is continuous at $x_o \in X$ if and only if for every sequence $x_n \in X$, we have $x_n \to x_o$ as $n \to \infty$ then $Tx_n \to Tx_o$ as $n \to \infty$. If T is continuous at each point of $x_0 \in X$ then we say that T is continuous on X.*

Definition 2.0.5 *(Sintunavarat et al., 2016) Let X be a b-metric space and $\{x_n\}$ be a sequence in X we say that*
a. b-converges to $x \in X$ if $d(x_n,x) \to 0$ as $n \to \infty$
b. x_n is a b-Cauchy sequence if $d(x_n,x_m) \to 0$ as $n,m \to \infty$
c. (X,d) is b-complete if every b-Cauchy sequence in X is b -convergent.

Definition 2.0.6 *(Alber et al.,1997) Let (X,d) be a metric space. A self-mapping f on X is said to be weakly contractive if, $d(fx, fy) \leq d(x,y) - \phi(d(x,y))$ for all $x,y \in X$, where ϕ is an altering distance function.*

Theorem 2.0.1 *(Rhoades, 2001) Let* (X,d) *be a complete metric space. If* $f\colon X \to X$ *is a weakly contractive mapping, then* f *has a unique fixed point.*

Theorem 2.0.2 *(Dutta et al., 2008) Let* (X,d) *be a complete metric space. If* $f\colon X \to X$ *satisfies* $\psi(d(fx,fy)) \leq \psi(d(x,y)) - \varphi(d(x,y))$*for all* $x,y \in X$*, where* $\psi,\varphi\colon R^+ \to R^+$ *are altering distance functions. Then* f *has a unique fixed point.*

Theorem 2.0.3 *(Choundery et al., 2011) Suppose that a mapping* $g: X \to X$ *where* X *is a metric space with metric d, satisfies the following condition:*
$\psi(d(gx,gy)) \leq \psi(max\{d(x,y),d(x,gx),d(y,gy),\frac{1}{2}[d(x,gy)+d(y,gx)]\}) - \varphi(max\{d(x,y),d(y,gy)\})$
for all $x,y \in X$*, where*
$\varphi\colon R^+ \to R^+$ *is a continuous function and*
$\psi\colon R^+ \to R^+$ *is an altering distance function.*
Then T *has a unique fixed point.*

Theorem 2.0.4 *(Hamid and Kourosh, 2017) Let* (X,d) *be a complete b-metric space with parameter* $s \geq 1, T: X \to X$ *be a self-mapping satisfaying the* (ψ,φ)*-weakly contractive condition*
$\psi(s(Tx,Ty)) \leq \psi(\frac{d(x,y)}{S^2}) - \varphi(d(x,y))$ *for all* $x,y \in X$*,where*
$\varphi\colon R^+) \to R^+$ *is a continuous function,* $\varphi(t) = 0$ *if and only if* $t = 0$*, and*
$\psi\colon R^+ \to R^+$ *is an altering distance function.*
Then T *has a unique fixed point.*

Recently, (Cho, 2018) introduced the notion of generalized weakly contractive mappings in metric spaces and proved a fixed point theorem for generalized weakly contractive mappings defined on complete metric spaces.

Let X be a metric space with metric d, let $T: X \to X$ and let $\phi: X \to R^+$ be a lower semi-continuous function. Then T is called a generalized weakly contractive mapping if it satisfies the following condition.

$$\psi(d(Tx,Ty)) + \varphi(Tx) + \varphi(Ty) \leq \psi(m(x,y,d,T,\varphi)) - \phi(l(x,y,d,T,\varphi))$$

where,

$$\begin{aligned} m(x,y,d,T,\varphi) &= max\{d(x,y) + \varphi(x) + \varphi(y), d(x,Tx) + \varphi(x) + \varphi(Tx), \\ &\quad d(y,Ty) + \varphi(y) + \varphi(Ty), \\ &\quad \frac{1}{2}[d(x,Ty) + \varphi(x) + \varphi(Ty) + d(y,Tx) + \varphi(y) + \varphi(Tx)]\} \end{aligned}$$

and $l(x,y,d,T,\varphi) = max\{d(x,y) + \varphi(x) + \varphi(y), d(y,Ty) + \varphi(y) + \varphi(Ty)\}$
for all $x, y \in X$, where
$\psi\colon [0,\infty) \to [0,\infty)$ is a continuous with $\psi(t) = 0$ if and only if $t = 0$
$\varphi\colon [0,\infty) \to [0,\infty)$ is a lower semi-continuous function with
$\varphi(t) = 0$ if and only if $t = 0$.

Theorem 2.0.5 *(Cho, 2018) Let X be complete. If T is a generalized weakly contractive mapping, then there exists a unique $z \in X$ such that $z = Tz$ and $\varphi(z) = 0$.*

Chapter 3

Methodology

3.1 Study area and period

The study will be conducted at Jimma University under the department of mathematics from September, 2018 G.C. to June, 2019 G.C.

3.2 Study Design

This study will employ analytical method of design.

3.3 Source of Information

The relevant sources of information for this study are books and published articles related to the area of the study.

3.4 Mathematical Procedure of the Study

In this study we will follow the procedures stated below:

- Establishing a theorem.
- Constructing a sequence.
- Showing the constructed sequence is Cauchy .
- Showing the convergence of the sequence.
- Proving the existence of a fixed point.
- Showing uniqueness of fixed point.
- Giving an example in support of our main result.

Chapter 4

STUDY PLAN AND BUDGET

4.1 Time Plan

Time given below illustrates the activities beginning from the selection of title to presentation of the thesis.

Table 1: Time plan

No.	Activities	Oct	Nov	Dec	Jan	Feb	Mar	Apr	May	Jun	Rem
1	Reading different literatures/books to	X									
2	Title selection and reading relevant	X	X								
3	Preparation of the proposal			X	X						
4	Submission of first draft of proposal				x						
5	Modifying the proposal based on the points of view				X						
6	Submission of the final proposal					X					
7	Proposal presentation						X				
8	Analyzing and writing the whole						X	X			
9	Submission of the first draft of the								X		
10	Editing the draft of research paper								X		
11	Submission of the final draft of the research paper									x	
12	Presentation of the research paper									x	

4.2 Budget Plan

The total budget for running this study will be analyzed as follows

Table 2: Budget plan

No	*Description of items*		*Unit*	*Quantity*	*Unit price*	*Total cost*
1	Perdim for	The researcher to gather documentary information's at Addis Ababa	Days	4x5	206	4120
		Supervisor/Assistant	Days	5	206	1030
	A researcher Transport cost for Addis Ababa		Days	4x2	127	1016
2	Transport cost for advisor consultancy (taxes)		Local agreement per month	3	530	1590
3	Stationary	Flash	Number	3(32GB)	400	1200
		Notebook	Number	5	100	500
		Paper	Rim	7	120	840
		hard disc	Number	1	3000	3000
		Pen	Number	8	5	40
		CD Rewrite	Number	4	26	104
		Kangaroo stapler	Number	1	233	233
4	print of different journal and necessary books		Pages	4000	1	4000
	Photo copy		Page	2000	1	2000
	Binding necessary books and journals		Number	25	50	1250
5	Thesis binding cost		Number	5	150	750
	Transport cost for thesis binding in Addis Ababa		Days	2	100	200
	Perdim for thesis binding in A.A		Day	6	206	1236
6	Communication with advisor (mobile card)		Number	7	100	700
7	Miscellaneous expenditure		5% of total	-		1194.75
8	Total expenditure					25,003.75

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