# FOURTH-ORDER STABLE CENTRAL DIFFERENCE METHOD FOR SELF-ADJOINT SINGULAR PERTURBATION PROBLEMS

**A Thesis Submitted to the Department of Mathematics, Jimma University, for Partial Fulfillment of the Requirements for the Degree of Masters of Science in Mathematics.**

**(Numerical Analysis)**


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**Jimma, Ethiopia**

## Declaration

I here with submit the dissertation entitled "**Fourth Order Stable Central Difference Method for Self-adjoint Singularly Perturbed Two Point Boundary Value Problems"** for the award of degree of Master of Science in Mathematics. I, undersigned, declare that this study is original and it has not been submitted to any institution elsewhere for the award of any academic degree or like, where other sources of information have been used, they have been acknowledged.

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The work has been done under the supervision of;

1. Gemechis File (Ph.D.)

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2. Tesfaye Aga (MSC)

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*Terefe Asrat*

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## Abstract

*In this thesis, fourth order stable central difference method has been presented for solving self-adjoint singular perturbation problems for small values of perturbation parameter $\varepsilon$. First, the given interval is discritized and the derivative of the given differential equation is replaced by the finite difference approximations. Then, the given differential equation is transformed to linear system of algebraic equations. Further, these algebraic equations are transformed into a three-term recurrence relation, which can easily be solved by using Thomas Algorithm. To validate the applicability of the proposed method, four model examples with and without exact solution have been considered and solved for different values of perturbation parameter and mesh sizes. Both theoretical error bounds and numerical rate of convergence have been established for the method. As it can be observed from the numerical results presented in tables compared to the numerical solution* by *Kadalbajoo and Kumar [17], Kumar and Kadalbajoo [19] and Patidar and Kadalbajoo [29] from literature and graphs, the present method approximates the exact solution very well.*

# Chapter One: Introduction

## 1.1. Background of the Study

Due to the advancement in the field of computational mathematics numerical methods are most widely utilized to solve the equation arising in the field of applied medical science, engineering and technology. Numerical analysis is the branch of mathematics that deals with the computational methods which helps to find approximate solutions for difficult problems such as finding the roots of non-linear equations, integration involving complex expressions and solving differential equations for which analytical solutions does not exist. The theory of singular perturbations has been with us, in one form or other, for a little over a century (although the term 'singular perturbation' dates from the 1940s). The subjects, and the techniques associated with it, have evolved over this period as a response to need to find approximate solutions (in an analytical form) to complex problems. Numerical analysis plays a significant role when difficulties encountered in finding the exact solution of an equation using a direct method and when it becomes very difficult to apply theoretical methods to find the exact solution. In real life, we often encounter many problems which are described by parameter dependent differential equations. The behaviors of the solutions of these types of differential equations depend on the magnitude of the parameters. Any differential equation in which the highest order derivative is multiplied by a small positive parameter $\varepsilon\,(0<\varepsilon<<1)$ is called Singular Perturbation Problem (SPP) and the parameter is known as the perturbation parameter. In fact, any differential equation whose solution changes rapidly in some parts of the interval/domain is generally known as Singular Perturbation Problem and also called Boundary Layer Problem (BLP). A singular perturbation problem is best defined as one in which no single asymptotic expansion is uniformly valid throughout the interval. Classical numerical methods which have been known to be effective for solving most problems that arises in application have failed when applied to singular perturbation problems, so most of these methods are not effective for solving singular perturbation problems because, as $\varepsilon\to 0$, the error in numerical solutions increases and often becomes not comparable in magnitude to the exact solutions Farell et al.[8]. There are so many authors who have worked in the field of Self-adjoint SPP. One can refer Boglave [3] ,Niijima [26,27], Miller [22] ,Mishra.et.al [24],  Kumar and Kadalbajoo [19],Gupta and Pankaj [14] and Gupta.et.al [15] . Patidar [28] was presented a fitted operator finite difference method (FOFDM) that derived via Numerov's method and shown to be fourth order accurate for moderate value of $\varepsilon$ and second order accurate for very small values of $\varepsilon$ . On the other hand, Patidar and kadalbajoo [29], exponentially fitted spline approximation method for solving self-adjoint singularly perturbed problem and also Vulanovic [46] presented a higher order-uniformly convergent method for a non-linear problem and also the defect-correction method has been investigated by Frank [10], Frank and Ueberhuber [9] and Stetter [38] to combine the stability of the upwind method and the accuracy of central difference method. However, according to segal's [36] report the methods are not useful for $\varepsilon << h$ because of their slow convergence. Riordan and Stynes [33] gave a method using finite elements with uniform mesh. This method is second order accurate in L-norm. The numerical results obtained by this method indicate that scheme with uniform mesh is not uniformly convergent for sufficiently small value of $\varepsilon$ and the maximal nodal error increases as $\varepsilon$ decreases. Dekema and Schultz [5] presented higher order methods for solving singular perturbation problems. These methods rewrite higher derivatives in Taylor's expansions in terms of first and second derivative term by differentiating their problem. They obtained up-to a fourth order method for two dimensional problems. However, these methods dealt with problems for only constant coefficients and they also faced instability as $\varepsilon$ becomes very small. Pearson [30], was perhaps the first to solve numerically by taking net adjustments in finite difference method. This idea was further developed by Abrahamsson et al.[1] in their study of difference methods for a general class of singular perturbation problems. Their aim was to devise numerical schemes with constant mesh spacing $h \geq \varepsilon$ to yield accurate solutions. However, they were able to show in general, that the accuracy of the scheme cannot be better than $o(h)$ . To overcome this drawback, we have presented suitable numerical method that is accurate and easy method for solving self-adjoint singularly perturbed two-point boundary value problems for small values of perturbation parameter $\varepsilon$ and also established both theoretical errors bound and computational rate of convergence of this method for reasonable step size h.

## 1.2. Statement of the Problem

A finite element method for solving self-adjoint singularly perturbed boundary value problem was presented by Vukoslavcevic and Surla[45]. It is well-known that standard discretization methods for solving self-adjoint singular perturbation problems are unstable and fail to give accurate results when the perturbation parameter $\varepsilon$ is very small. Classical computational approaches to singularly perturbed problems are known to be inadequate as they require extremely large numbers of mesh points to produce satisfactory computed solutions Farell, et al. [8] and Roos, et al. [34]. So, the treatment of singularly perturbed problems presents severe difficulties that have to be addressed to ensure accurate numerical solutions Doolan, et al. [6], Kadalbajoo, et al. [17] and Roos, et al. [34]. Thus, existing numerical methods produce good results only when we take step size $h < \varepsilon$. This is very costly and time consuming process. Schultz and Joshua [37], presented stable higher order central difference methods for differential equations with small coefficients for second order terms and the idea was further developed by Tilak [40] ,who was proved that the stability of three-points in higher order stable central difference methods were $\varepsilon$-independent. Their aim was to devise numerical schemes with constant mesh spacing $h \geq \varepsilon$ to yield accurate solutions. However, they were able to show in general, that the accuracy of the scheme cannot be better than $o(h^2)$.

Hence, this research aims to present suitable numerical method (*i.e.* method which is more efficient and requires simpler computational techniques) which can work with a reasonable step size $h$.

Owing to this, the present study attempts to answer the following questions:

- How does the fourth order stable central difference method can be described for solving self-adjoint singularly perturbed problems?
- To what extent the present method approximate the exact solution?
- To what extent the proposed method converges?

## 1.3. Objectives of the Study

### 1.3.1 General Objective

The general objective of this study is to present numerical method which is simple, efficient and easily adaptable for solving self-adjoint singularly perturbed problems.

### 1.3.2 Specific Objectives

The specific objectives of the present study are:

- To describe fourth order stable central difference method for solving self-adjoint singularly perturbed problems.
- To compare the numerical solutions obtained by the present method with the exact solution.
- To establish the stability and convergence of the proposed method.

## 1.4. Significance of the Study

The result obtained in this research may:

- Serve as a reference material for scholars who works on this area.
- Give an idea about the application of numerical methods in different field of studies.
- Help the graduate students to acquire research skills and scientific procedures.

## 1.5. Delimitation of the Study

The singular perturbation problems are perhaps the most important differential equations in all applied mathematics that contribute for the advantage of science and technology. Though singular perturbation problems are vast topics and have many applications in the real world, this study was delimited to a class of self-adjoint singularly perturbed equation of the form:

$$-\varepsilon(p(x)y')' + q(x)y(x) = f(x); \quad 0 < x < 1 \tag{1.1}$$

with the Dirichlet boundary conditions,

$$y(0) = \alpha,\ y(1) = \beta \tag{1.2}$$

where, $\varepsilon\,(0 < \varepsilon << 1)$ is a small parameter; $\alpha,\ \beta$ are given constants and $p(x)$ , $q(x)$ and $f(x)$ are assumed to be sufficiently continuous differentiable functions. Further, the study was delimited to fourth order stable central difference method though there are varieties of methods for solving the problem under the study.

## 1.6 Definition of Terms

**Definition 1.1:** Boundary value problem is a problem, typically an ODE or a PDE, which has values assigned on the physical boundary of the domain in which the problem is specified, is called a boundary value problem (BVP).

**Definition 1.2**: Any differential equation (DE) obtained from a given DE and having property that solution is an integrating factor of the other is known as adjoint DE.

**Definition 1.3**: Any DE that has the same solution as its adjoint DE is known as Self-Adjoint DE and if it's highest order derivative is multiplied by a small positive parameter $\varepsilon, 0 < \varepsilon << 1$ and which has the form $-\varepsilon(p(x)y')' + q(x)y = f(x)$ is called second order self-adjoint SPP Byme [4] and Mishra.et.al[23] .

# Chapter Two: Review of Related Literature

## 2.1. Singular Perturbation Theory

The term singular perturbation was first introduced by Friedrichs, et al. [11]. An excellent survey of the historical development of singular perturbations is found in a recent book by O'Malley [21] and Vasil'eva [43]. Other historical surveys concerning the research activity in singular perturbation theory at Moscow State University and elsewhere are found in Vasileva, et al. [44], and [42]. In recent decades this is a field of increasing interest to applied mathematicians and numerical analysts in view of the challenges the problems there in pose to the researchers. In Mathematics, more precisely in perturbation theory, a singular perturbation problem is a problem containing a small parameter that cannot be approximated by setting the parameter value to zero. This is in contrast to regular perturbation problems, for which an approximation can be obtained by simply setting the small parameter to zero. More precisely, the solution cannot be uniformly approximated by an asymptotic expansion as $\varepsilon \to 0$ . Here $\varepsilon$ is the small parameter of the problem. Next we will focus on some methods used to solve self-adjoint singular perturbation problems.

## 2.2 An Initial Value Technique

Here self-adjoint singularly perturbed two-point boundary value problem of the form $L_y = -\varepsilon\left(p(x)y'\right)' + q(x)y(x) = f(x)$ , $0 < x < 1$ is considered with the Dirichlet boundary conditions $y(0) = \alpha,\ y(1) = \beta$ where, $\varepsilon\,(0 < \varepsilon << 1)$ is a small parameter; $\alpha$ and $\beta$ are given constants $p(x)$ , $q(x)$ and $f(x)$ are assumed to be sufficiently continuous differentiable functions. $L_y$ admits maximum principle Protter and Weinberger[32]. This problem depends on a small positive parameter in such a way that the solution varies rapidly in some parts and varies slowly in some other parts. Mishra .K, et.al [23] have studied an initial value technique for self-adjoint singular perturbation boundary value problems. Natesan and Ramanujam [25] have studied initial-value technique for singularly perturbed boundary-value problems for second-order ordinary differential equations arising in chemical reactor theory .The required approximate solution is obtained by combining solutions of two terminal-value problems and one initial- value problem which are obtained from the original boundary-value problem through asymptotic expansion procedures. Error estimates for approximate solutions of this method are not satisfactory.

### 2.3 Singular Perturbation Methods

The methods used to tackle problems in this field are many. Let us see two methodologies for investigating solutions to singularly perturbed differential equations. The choice of technique to be applied depends on the form of the problem and also on the desired properties to be studied. The more basic of these include the method of matched asymptotic expansions, and the method of multiple scales Roos.et.al [34], Keller [18] and Miller [22].

#### 2.3.1. Boundary Layers and Matched Asymptotic Expansions

Singularly perturbed differential equations can yield solutions containing regions of rapid variation. The regions, which may be apparent in the solution or in its derivatives, are called ‘layers’ and often appear at the boundary of the domain. Constructing a solution to a differential equation or system involves several steps: identifying the locations of layers (boundary or internal), deriving asymptotic approximations to the solution in the different regions (corresponding to different distinguished limits in the equations), and ultimately, forming a uniformly valid solution over the entire domain. Solutions obtained for the layers (singular distinguished limits) are usually termed as inner solutions while the slowly varying solutions for the regular distinguished limits are referred to as outer solutions. The uniformly valid solution can be constructed through asymptotic matching of the inner and an outer solution, which relies on the fundamental assumption that the different solution forms overlap at on some identifiable region.

#### 2.3.2. Multiple Scales Analysis for Long-Time Dynamics

For finite, bounded times, solutions can be asymptotically approximated by application of the regular expansion, with the leading order solution. However, at large times, the naive regular expansion breaks-down due to the appearance of secular terms (terms which grow with time). This failure of the inner and outer expansion can be traced to the fact that the limits do not commute and which indicates that the expansion is not uniformly valid in time.

## 2.4. Numerical versus Analytical Methods

Suppose we have a differential equation and we want to find a solution of the differential equation. The best is when we can find out the exact solution using calculus, trigonometry and other techniques. The techniques used for calculating the exact solution are known as analytic methods because we used the analysis to figure it out. Analytical solution is continuous. The exact solution is also referred to as a closed form solution or analytical solution. But this tends to work only for simple differential equations with simple coefficients, but for higher order or non-linear differential equations with complex coefficient, it becomes very difficult to find exact solution. Therefore, we need numerical methods for solving these equations. In this thesis, fourth order stable central difference method has been presented for solving self adjoint singular perturbation problems.

## 2.5. Finite Difference Methods

The finite difference methods are always a convenient choice for solving boundary value problems because of their simplicity. Finite difference methods are one of the most widely used numerical schemes to solve differential equations. In finite difference methods, derivatives appearing in the differential equations are replaced by finite difference approximations obtained by Taylor series expansions at the grid points. This gives a large algebraic system of equations to be solved by any iterative methods in place of the differential equation to give the solution value at the grid points and hence the solution is obtained at grid points. Some of the finite difference methods include forward difference method, backward difference method, central difference method, etc. The general concepts of finite difference methods in details such as stability, convergence conditions, etc have been presented in literature for solving differential equations. The challenge in analyzing finite difference methods for new classes of problems is often to find an appropriate definition of stability that allow one to prove convergence and to estimate the error in approximation. The finite-difference method as cited by Vasil'eva [41] and Prandtl [31] is widely used by the scientific community for the numerical solution of reaction–diffusion equations; however, there are comparatively few studies that give stability and convergence results see Beckett, et al. [2], Hoff [16], Gartland [12], Lubuma and Patidar[20] and Vukoslavcevic and Surla[45]. For a unified treatment of how and when the finite-difference method for reaction–diffusion equations breaks down see Stuart [39], Elliott, et al. [7], and Ruuth [35].

# Chapter Three: Methodology

This chapter consists of the following methods and materials that were used to carry out the study.

## 3.1. Study Area and Period

The study was conducted at Jimma University, which is Ethiopia's first innovative community based education institution of higher learning, department of Mathematics from September 2014 to September 2015. Conceptually, the study has been focused on numerical method particularly on fourth order stable central difference method for solving self-adjoint singularly perturbed two point boundary value problems with Dirichlet boundary conditions.

## 3.2. Study Design

This study was employed documentary review design and experimental design on self-adjoint singularly perturbed two-point boundary value problems.

## 3.3. Source of Information

The relevant sources of information for this study were books, published articles & related studies from internet and the numerical results obtained by writing MATLAB code for the present numerical method.

## 3.4. Study Procedures

In order to achieve the stated objectives, the study has been followed the following steps:

1. Defining the problem.
2. Discretizing the given interval.
3. Replacing differential equation by finite difference approximations.
4. Obtaining the tri-diagonal system (TDS) which can be easily solved by Thomas Algorithm.
5. Writing MATLAB code for the tri-diagonal system obtained.
6. Validating the schemes using numerical examples.

## 3.5. Ethical Considerations

To be legal for collecting all the above materials it is important to have a permission letter. So, a letter of permission was taken from Research and Postgraduate studies of the College of Natural Science so as to make easy the collection of materials.

# Chapter Four: Description of the Method, Results and Discussion

## 4.1 Description of the method

In this section, the description of fourth order stable central difference method (SCD4) and its stability and convergence analysis have been given. To describe the scheme, we divide the interval [0 1] into $N$ equal subintervals with uniform step length $h$. Let $0 = x_0$, $x_1, \ldots, x_N = 1$ be the mesh points, then we have $x_i = x_0 + ih$, $i = 0, 1, 2, \ldots, N$ Denote $a(x_i) = a_i$, $b(x_i) = b_i$ $c(x_i) = c_i$ $f(x_i) = f_i$ and $y(x_i) = y_i$.

From Eq. (1.1) we have:

$$-\varepsilon(p(x_i)y_i')' + q(x_i)y(x_i) = f(x_i); \quad \text{for} \quad 0 < x < 1 \quad , \quad i = 0,1,2,\ldots.N$$

with boundary condition :

$$x_0 = 0 \text{ and } x_N = 1 \qquad y(x_o = 0) = \alpha \text{ and } y(x_N = 1) = \beta$$

which can be rewritten in the form:

$$-y_i'' + a(x_i)y_i' + b(x_i)y_i = c(x_i) \tag{4.1}$$

where, $a_i = a(x_i) = \dfrac{-p'(x_i)}{p(x_i)}$, $b_i = b(x_i) = \dfrac{q(x_i)}{\varepsilon p(x_i)}$ and $c_i = c(x_i) = \dfrac{f(x_i)}{\varepsilon p(x_i)}$

To find a description of fourth order stable central difference scheme, we use Taylor's series expansion in-order to get central difference formula for $y_i''$ and $y_i'$. Assume that $y_i$ has continuous fourth derivatives in the interval [0 1].

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!}y_i'' + \frac{h^3}{3!}y_i''' + \frac{h^4}{4!}y_i^{(4)} + \frac{h^5}{5!}y_i^{(5)} + \frac{h^6}{6!}y_i^{(6)} + O(h^7) \tag{4.2}$$

$$y_{i-1} = y_i - hy_i' + \frac{h^2}{2!}y_i'' - \frac{h^3}{3!}y_i''' + \frac{h^4}{4!}y_i^{(4)} - \frac{h^5}{5!}y_i^{(5)} + \frac{h^6}{6!}y_i^{(6)} + O(h^7) \tag{4.3}$$

Then, subtracting Eq. (4.3) from Eq. (4.2) we get:

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6}y_i''' - \frac{2h^4 y_i^{(5)}}{5!} + O(h^6) \tag{4.4}$$

Thus, the central difference approximation for the first derivative of $y_i$ is given by:

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6} y_i''' + \tau_1 \qquad (4.5)$$

where , $\tau_1 = -\frac{h^4 y_i^{(5)}}{120} + 0(h^6)$

Similarly, adding Eqs. (4.2) and (4.3), we obtain central difference approximation for the second derivative of $y_i$ as:

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{h^2}{12} y_i^{(4)} + \tau_2 \qquad (4.6)$$

where, $\tau_2 = -\frac{h^4}{360} y_i^{(6)} + 0(h^6)$

Substituting Eqs. (4.5) and (4.6) into Eq. (4.1) at $x = x_i$, we obtain

$$\left(\frac{1}{h^2} - \frac{a_i}{2h}\right) y_{i-1} + \left(b_i - \frac{2}{h^2}\right) y_i + \left(\frac{1}{h^2} + \frac{a_i}{2h}\right) y_{i+1} - \frac{h^2}{6} a_i y_i''' + \frac{h^2}{12} y_i^{(4)} + \tau_3 = c_i \qquad (4.7)$$

where, $\tau_3 = a_i \tau_1 - \tau_2 = a_i \frac{h^4 y_i^{(5)}}{120} + \frac{h^4}{360} y_i^{(6)}$

Differentiating Eq. (4.1) successively, we obtain

$$y_i''' = a_i y_i'' + (a_i' + b_i) y_i' + b_i' y_i - c_i' \qquad (4.8)$$

and

$$y_i^{(4)} = \left(a_i^2 + 2a_i' + b_i\right) y_i'' + \left(a_i a_i' + a_i b_i + a_i'' + 2b_i'\right) y_i' + \left(b_i'' + a_i b_i'\right) y_i - a_i c_i' - c_i'' \qquad (4.9)$$

Substituting Eqs. (4.8) and (4.9) into Eq. (4.7) we obtain:

$$\left(-\frac{1}{h^2} - \frac{a_i}{2h}\right) y_{i-1} + \left(\frac{2}{h^2} + b_i + \frac{h^2}{12}\left(b_i'' - a_i b_i'\right)\right) y_i + \left(-\frac{1}{h^2} + \frac{a_i}{2h}\right) y_{i+1} + \frac{h^2}{12}\left(-a_i^2 + 2a_i' + b_i\right) y_i''$$
$$+ \frac{h^2}{12}\left(-a_i a_i' - a_i b_i + a_i'' + 2b_i'\right) y_i' = c_i - \tau_3 - \frac{h^2 a_i c_i'}{12} + \frac{h^2 c_i''}{12} \qquad (4.10)$$

Approximating the converted error term, which has a stabilizing effect, in Eq. (4.10) and by using the central difference formulas given in Eqs. (4.5) and (4.6) for $y_i''$ and $y_i'$ we obtain:

$$\left( -\frac{1}{h^2} - \frac{a_i}{2h} + \frac{1}{12}\left(-a_i^{\,2} + 2a_i' + b_i\right) - \frac{h}{24}\left(-a_i a_i' - a_i b_i + a_i'' + 2b_i'\right)\right) y_{i-1} +$$

$$\left(\frac{2}{h^2} + b_i + \frac{h^2}{12}\left(b_i'' - a_i b_i'\right) - \frac{1}{6}\left(-a_i^{\,2} + 2a_i' + b_i\right)\right) y_i + \qquad (4.11)$$

$$\left(-\frac{1}{h^2} + \frac{a_i}{2h} + \frac{1}{12}\left(-a_i^{\,2} + 2a_i' + b_i\right) + \frac{h}{24}\left(-a_i a_i' - a_i b_i + a_i'' + 2b_i'\right)\right) y_{i+1}$$

$$= c_i - \frac{h^2 a_i c_i'}{12} + \frac{h^2 c_i''}{12} + \tau_4$$

where, $\tau_4$ is local truncation error of SCD4 and given by:

$$\tau_4 = \left(-a_i' a_i + a_i'' - a_i b_i + 2b_i'\right)\frac{h^4 y_i'''}{72} + \left(-a_i^{\,2} + 2a_i' + b_i\right)\frac{h^4 y_i^{(4)}}{144} + a_i \frac{h^4 y_i^{(5)}}{120} - \frac{h^4}{360} y_i^{(6)} \qquad (4.12)$$

where, $\tau_3 = \tau_2 + a_i \tau_1 = O(h^4)$

Rearranging Eq. (4.11), we have the following three term recurrence relation of the form:

$$-E_i y_{i-1} + F_i y_i - G_i y_{i+1} = H_i \qquad (4.13)$$

where,

$$E_i = \frac{1}{h^2} + \frac{a_i}{2h} - \frac{1}{12}\left(-a_i^{\,2} + 2a_i' + b_i\right) + \frac{h}{24}\left(-a_i a_i' - a_i b_i + a_i'' + 2b_i'\right)$$

$$F_i = b_i + \frac{2}{h^2} - \frac{1}{6}\left(-a_i^{\,2} + 2a_i' + b_i\right) + \frac{h^2}{12}\left(b_i'' - a_i b_i'\right)$$

$$G_i = \frac{1}{h^2} - \frac{a_i}{2h} - \frac{1}{12}\left(-a_i^{\,2} + 2a_i' + b_i\right) - \frac{h}{24}\left(-a_i a_i' - a_i b_i + a_i'' + 2b_i'\right)$$

$$H_i = c_i - \frac{h^2}{12}\left(a_i c_i' - c_i''\right), \qquad \text{for } i = 1,2,3,\ldots N-1$$

Eq. (4.13) gives N by N tri-diagonal systems, which can easily be solved by Thomas Algorithm.

## 4.2 Stability and Convergence Analysis

**Remark:** Here we shall use the definition of the stability of the difference operator given in keller [18].

**Definition:** The linear difference operator $L_h(.)$ is stable if for sufficiently small $h$, there exists a constant $k$, independent of $h$, such that

$$|\omega_j| \le k\left\{\max(|\omega_o|,|\omega_N|) + \max_{1\le i\le N-1}|L_h\omega_i|\right\},\ j = 0,1,2,....N \text{ for any mesh function, } \{\omega_j\}^N{}_{j=0}.$$

**Theorem**

Under the assumption $b(x_i) = \theta$ >0, for $\theta$ is constant, $-a_i{}^2 + 2a_i' + b_i \ge 0$ and

$$h < \min \frac{2\left(-a_i{}^2 + 2a_i' + b_i\right)}{\left|-a_i\left(a_i' + b_i\right) + a_i'' + 2b_i'\right|},\ \text{for } i = 1,2,3...N-1.$$

The difference operator defined on Eq. (4.13) is stable with $k = \max\left\{1, \dfrac{1}{\theta}\right\}$

**Proof**: Let $L_h(.)$ denote the difference operator on left hand side of Eq. (4.13) and $\omega_i$ be any mesh function satisfying:

$$L_h(\omega_i) = H_i \tag{4.14}$$

If $\max|\omega_i|$ occurs for i=0 or i=N, then the definition holds, since $k \ge 1$.

So, assume that $\max|\omega_i|$ occurs for $i = 1,2,3,...N-1$

Under the given assumption:

$E_i > 0, G_i > 0, F_i \ge E_i + G_i$ and $|E_i| \le |G_i|$

This implies that the tri-diagonal system in Eq.(4.13) is diagonally dominant and its solution exists , is unique Greenspan and Casulli [13].Then by rearranging the difference Eq.(4.13) and using the non-negativity of the coefficients, we have:

$$F_i|\omega_i| \le E_i|\omega_{i-1}| + G_i|\omega_{i+1}| + |H_i|$$

$$\Rightarrow \quad F_i|\omega_i| \le E_i|\omega_{i-1}| + G_i|\omega_{i+1}| + |L_h\omega_i| \tag{4.15}$$

since $b(x_i) = \theta$ is a constant, by assumption $b'(x_i) = 0$ .Thus, we have :

$$F_i = \frac{2}{h^2} - \frac{1}{6}\left(-a_i^{\,2} + 2a_i' - b_i\right) + \theta \tag{4.16}$$

Now using the fact, $E_i + G_i = \frac{2}{h^2} - \frac{1}{6}\left(-a_i^{\,2} + 2a_i' + b_i\right)$ and Eqs. (4.15) and (4.16), we get,

$$\begin{aligned}\left(\frac{2}{h^2} - \frac{1}{6}\left(-a_i^{\,2} + 2a_i' + b_i\right) + \theta\right)|\omega_i| &\le E_i\,|\omega_{i-1}| + G_i\,|\omega_{i+1}| + |L_h\omega_i| \\ &\le (E_i + G_i)\max_{1\le k\le N-1}|\omega_k| + \max_{1\le k\le N-1}|L_h\omega_k|\end{aligned} \tag{4.17}$$

Since the inequalities in Eq. (4.17) holds for every i , it follows that

$$\left(\frac{2}{h^2} - \frac{1}{6}\left(-a_i^{\,2} + 2a_i' + b_i\right) + \theta\right)\max_{1\le i\le N-1}|\omega_i| \le \left(\frac{2}{h^2} - \frac{1}{6}\left(-a_i^{\,2} + 2a_i' + b_i\right)\right)\max_{1\le k\le N-1}|\omega_k| + \max_{1\le k\le N-1}|L_h\omega_k| \;,$$

this implies that:

$$\theta\max_{1\le i\le N-1}|\omega_i| \le \max_{1\le k\le N-1}|L_h\omega_k|$$

Hence,

$$\max_{1\le i\le N-1}|\omega_i| \le \frac{1}{\theta}\max_{1\le k\le N-1}|L_h\omega_k| \le \frac{1}{\theta}\left\{\max\left(|\omega_0|,|\omega_N|\right) + \max_{1\le k\le N-1}|L_h\omega_k|\right\}$$

Therefore, $|\omega_i| \le k\left\{\max\left(|\omega_0|,|\omega_N|\right) + \max_{1\le k\le N-1}|L_k\omega_k|\right\}$

Hence, $L_h(.)$ is stable and this implies that the solution to the system of the difference equation Eq.(4.13) are uniformly bounded ,independent of mesh size $h$ and the parameter $\varepsilon$ .Hence, the scheme is stable for all step size $h$ .

**Corollary**: Under the conditions for the above theorem, the error $e_i = y(x_i) - y_i$ , between the solution $y(x)$ of the continuous problem and the solution $y_i$ of the discrete problem with the boundary condition satisfies the estimate:

$$|e_i| \le k\max_{1\le i\le N-1}|\tau_i| \tag{4.18}$$

where, $\tau_4$ in Eq.(4.12) denoted by $\tau_i$ which is the truncation error given by:

$$|\tau_i| \le \max_{x_{i-1}\le x\le x_{i+1}} \left\{ \left(-a_i' a_i + a_i'' - a_i b_i + 2b_i'\right) \frac{h^4 |y_i'''|}{72} \right\} + \max_{x_{i-1}\le x\le x_{i+1}} \left\{ \left(-a_i^{\,2} + 2a_i' + b_i\right) \frac{h^4 |y_i^{(4)}|}{144} \right\} +$$

$$\max_{x_{i-1}\le x\le x_{i+1}} \left\{ a_i \frac{h^4 |y_i^{(5)}|}{120} \right\} + \max_{x_{i-1}\le x\le x_{i+1}} \left\{ \frac{h^4 |y_i^{(6)}|}{360} \right\}$$

**Proof**: Under the given condition error $e_i$ satisfies:

$L_h(e(x_i)) = L_h(y(x_i) - y_i) = \tau_i$ for, $i = 1,2,3,\ldots N-1$ and $e_0 = e_N = 0.$

Then, from the above theorem stability of $L_h(.)$ implies that

$$|y(x_i) - y_i| = |e_i| \le k \max_{1\le i\le N-1} |\tau_i| \tag{4.19}$$

Hence, the estimate in Eq. (4.18) establishes the convergence of the scheme for fixed values of the perturbation parameter $\varepsilon$ .

**4.3. Thomas Algorithm**

The tri-diagonal matrix algorithm, also known as Thomas Algorithm, is a specific form of Gauss elimination that can be used to solve tri-diagonal system of equations /three-term recurrence relation. The Thomas Algorithm is based on the Gauss elimination procedure and consists of two parts: a forward elimination phase and a backward substitution phase. Our goal is to find unknown vector $y_i$ . A brief description for solving the tri-diagonal system which is called Thomas Algorithm is presented as follows.

Consider the scheme (4.13):

$$-E_i y_{i-1} + F_i y_i - G_i y_{i+1} = H_i\,, \qquad i = 1,\ 2,\ \ldots.,\ N \tag{4.20}$$

subject to the boundary conditions,

$$y_0 = y(0) = \alpha \tag{4.21}$$

$$y_N = y(1) = \beta \tag{4.22}$$

assume that the solution to Eq.(1.1) be:

$$y_i = W_i y_{i+1} + T_i\,, \qquad i = N, N-1, \ldots, 2, 1 \tag{4.23}$$

where, $W_i = W(x_i)$ and $T_i = T(x_i)$ which are to be determined.

Solving Eq. (4.23) at $x = x_{i-1}$, we have

$$y_{i-1} = W_{i-1} y_i + T_{i-1} \tag{4.24}$$

Now by substituting Eq. (4.24) into Eq. (4.20) yields

$$y_i = \frac{G_i}{F_i - E_i W_{i-1}} y_{i+1} + \frac{H_i + E_i T_{i-1}}{F_i - E_i W_{i-1}} \tag{4.25}$$

Then comparing Eq. (4.23) with Eq. (4.25) we get the recurrence relations:

$$W_i = \frac{G_i}{F_i - E_i W_{i-1}} \tag{4.26}$$

$$T_i = \frac{H_i + E_i T_{i-1}}{F_i - E_i W_{i-1}} \tag{4.27}$$

To solve these recurrence relations for $i = 1,2,3,\ldots,N-1$ we need the initial conditions for $W_0$ and $T_0$. For this, we take $y_0 = y(0) = W_0 y_1 + T_0$. Choose, $W_0 = 0$, then the value of $T_0 = y(0) = \alpha$. With these initial values, we compute $W_i$ and $T_i$ for $i = 1,2,\ldots,N-1$ from Eq. (4.26) and Eq. (4.27) in forward process, and then we obtain $y_i$ in the backward process from Eq. (4.22) and Eq. (4.23).

**4.4 Numerical Examples**

To demonstrate the applicability of the method, we have solved four examples: a non-homogeneous SPP and a SPP with variable coefficients. For each ε and N, the maximum absolute errors at nodal points are approximated by the formula, $\|E\| = \max|y(x_i) - y_i|$ , for $i = 0,1,2,...N$ and where, $y(x_i)$ and $y_i$ are the exact and computed solution of the given problem and nodal point $x_i$.

**Example 4.1:**

$$-\varepsilon y'' + y = f(x),$$

with boundary condition, $y(0) = y(1) = 0$

where , $f(x)$ is chosen such that the exact solution of the problem is given by:

$$y(x) = \exp(x) + e^{\left(\frac{-x}{\sqrt{\varepsilon}}\right)} - x\left(e^1 + e^{\left(\frac{-1}{\sqrt{\varepsilon}}\right)}\right) - 2 + 2x$$

The numerical (solution in terms of maximum absolute errors ($\|E\|$)) is given in table 4.1.

**Example 4.2:**

$$-\varepsilon y'' + \left(1 + x - x^2\right) y = f(x)$$

with boundary condition, $y(0) = y(1) = 0$

where , $f(x) = 1 + x - x^2 + \left(2\sqrt{\varepsilon} - x^2 + x^3\right) e^{\left(-\frac{1-x}{\sqrt{\varepsilon}}\right)} + \left(2\sqrt{\varepsilon} - x(1-x)^2\right) e^{\left(\frac{-x}{\sqrt{\varepsilon}}\right)}$

The exact solution is given by:

$$y(x) = 1 + (x-1)\, e^{\left(\frac{-x}{\sqrt{\varepsilon}}\right)} - x e^{\left(-\frac{1-x}{\sqrt{\varepsilon}}\right)}$$

The numerical (solution in terms of maximum absolute errors ($\|E\|$)) is given in table 4.2.

**Example 4.3:**

$$-\varepsilon\left(\left(1 + x^2\right) y'\right)' + \left(1 + x - x^2\right) y = f(x),$$

with boundary condition, $y(0) = y(1) = 0;$

where , $f(x)$ is chosen such that the exact solution of the problem is given by:

$$y(x) = 1 + (x-1) e^{\left(\frac{-x}{\sqrt{\varepsilon}}\right)} - x\, e^{\left(\frac{x-1}{\sqrt{\varepsilon}}\right)}$$

The numerical (solution in terms of maximum absolute errors ($\|E\|$)) is given in table 4.3.

**Example** 4.4:

$$-\varepsilon\left(\left(1 + x^2\right) y'\right)' + \left(\frac{\cos x}{(3-x)^3}\right) y = 4\left(3\mathrm{x}^2 - 3\mathrm{x} + 1\right)\left((\mathrm{x} - 1/2)^2 + 2\right)$$

with boundary condition, $y(0) = -1, \quad y(1) = 0$

The exact solution for this problem is not available. The numerical solution for this problem is obtained by using the Double-Mesh Principle [6].

The numerical (solution in terms of maximum absolute errors ($\|E\|$)) is given in table 4.4.

## 4.5 Numerical Results

**Table 4.1**: Maximum Absolute Error ($\|E\|$) of **Example 4.1**

| $\varepsilon$ | $N=100$ | $N=500$ | $N=1000$ | $N=1500$ |
|---|---|---|---|---|
| Our Method | | | | |
| $2^{-4}$ | 2.0028E-09 | 3.2068E-12 | 2.2249E-13 | 1.3678E-13 |
| $2^{-6}$ | 3.1415E-08 | 5.0189E-11 | 3.1462E-12 | 6.1062E-13 |
| $2^{-8}$ | 5.0141E-07 | 8.0365E-10 | 5.0230E-11 | 9.9172E-12 |
| $2^{-10}$ | 7.9979E-06 | 1.2853E-08 | 8.0358E-10 | 1.5874E-10 |
| $2^{-12}$ | 1.2252E-04 | 2.0554E-07 | 1.2853E-08 | 2.5393E-09 |
| $2^{-14}$ | 1.8953E-04 | 3.2823E-06 | 2.0554E-07 | 4.0615E-08 |
| $2^{-16}$ | 1.6249E-03 | 5.2132E-05 | 3.2823E-06 | 6.4929E-07 |
| Kadalbajoo and Kumar [17] using variable mesh | | | | |
| $2^{-4}$ | 2.380E-05 | 9.440E-07 | 2.360E-07 | 1.0500E-07 |
| $2^{-6}$ | 5.310E-05 | 2.100E-06 | 5.260E-07 | 2.3400E-07 |
| $2^{-8}$ | 1.070E-04 | 4.260E-06 | 1.060E-06 | 4.7300E-07 |
| $2^{-10}$ | 2.150E-04 | 8.530E-06 | 2.130E-06 | 9.4800E-07 |
| $2^{-12}$ | 4.310E-04 | 1.710E-05 | 4.260E-06 | 1.9000E-06 |
| $2^{-14}$ | 8.610E-04 | 3.410E-05 | 8.530E-06 | 3.7900E-06 |
| $2^{-16}$ | 1.700E-03 | 6.820E-05 | 1.710E-05 | 7.5800E-06 |
| Kadalbajoo and Kumar [17] using Uniform mesh | | | | |
| $2^{-4}$ | 2.480E-05 | 9.900E-07 | 2.480E-07 | 1.1000E-07 |
| $2^{-6}$ | 9.810E-05 | 3.930E-06 | 9.820E-07 | 4.3600E-07 |
| $2^{-8}$ | 3.910E-04 | 1.570E-05 | 3.920E-06 | 1.7400E-06 |
| $2^{-10}$ | 1.600E-03 | 6.270E-05 | 1.570E-05 | 6.9800E-06 |
| $2^{-12}$ | 5.900E-03 | 2.510E-04 | 6.270E-05 | 2.7900E-05 |
| $2^{-14}$ | 2.150E-02 | 9.980E-04 | 2.510E-04 | 1.1200E-04 |
| $2^{-16}$ | 4.120E-02 | 3.900E-03 | 9.980E-04 | 4.4500E-04 |

**Table 4.2**: Maximum Absolute Error ($\|E\|$) of **Example 4.2.**

| $\varepsilon$ | $N=16$ | $N=32$ | $N=64$ | $N=128$ | $N=256$ | $N=512$ |
|---|---|---|---|---|---|---|
| Our Method | | | | | | |
| 1/8 | 0.1424E-05 | 0.8806E-07 | 0.5486E-08 | 0.3429E-09 | 0.2141E-10 | 0.1325E-11 |
| 1/16 | 0.4148E-05 | 0.2577E-06 | 0.1617E-07 | 0.1017E-08 | 0.6322E-10 | 0.3933E-11 |
| 1/32 | 0.9622E-05 | 0.5962E-06 | 0.3721E-07 | 0.2324E-08 | 0.10452E-09 | 0.9143E-11 |
| 1/64 | 0.3074E-04 | 0.1927E-05 | 0.1207E-06 | 0.7502E-08 | 0.2443E-08 | 0.2930E-10 |
| 1/128 | 0.1301E-03 | 0.8424E-05 | 0.5255E-06 | 0.3280E-07 | 0.2053E-08 | 0.1283E-09 |
| 1/256 | 0.5910E-03 | 0.3704E-04 | 0.2319E-05 | 0.1450E-06 | 0.9072E-08 | 0.5671E-09 |
| 1/512 | 0.1331E-02 | 0.1444E-03 | 0.9916E-05 | 0.6241E-06 | 0.3905E-07 | 0.2443E-08 |
| 1/1024 | 0.1521E-02 | 0.6190E-03 | 0.4110E-04 | 0.2633E-05 | 0.1640E-06 | 0.1030E-07 |
| Patidar and Kadalbajoo [29] using fitting factor | | | | | | |
| 1/8 | 0.320E-03 | 0.800E-04 | 0.200E-04 | 0.500E-05 | 0.120E-05 | 0.310E-06 |
| 1/16 | 0.350E-03 | 0.860E-04 | 0.210E-04 | 0.530E-05 | 0.130E-05 | 0.330E-06 |
| 1/32 | 0.400E-03 | 0.990E-04 | 0.250E-04 | 0.620E-05 | 0.150E-05 | 0.390E-06 |
| 1/64 | 0.530E-03 | 0.130E-03 | 0.330E-04 | 0.820E-05 | 0.210E-05 | 0.510E-06 |
| 1/128 | 0.830E-03 | 0.190E-03 | 0.460E-04 | 0.120E-04 | 0.290E-05 | 0.720E-06 |
| 1/256 | 0.130E-02 | 0.260E-03 | 0.660E-04 | 0.160E-04 | 0.410E-05 | 0.100E-05 |
| 1/512 | 0.180E-02 | 0.420E-03 | 0.950E-04 | 0.230E-04 | 0.580E-05 | 0.140E-05 |
| 1/1024 | 0.250E-02 | 0.620E-03 | 0.130E-03 | 0.330E-04 | 0.810E-05 | 0.200E-05 |
| Patidar and Kadalbajoo [29] without using fitting factor | | | | | | |
| 1/8 | 0.150E-02 | 0.360E-03 | 0.910E-04 | 0.230E-04 | 0.57E-05 | 0.140E-05 |
| 1/16 | 0.200E-02 | 0.490E-03 | 0.120E-03 | 0.310E-04 | 0.77E-05 | 0.190E-05 |
| 1/32 | 0.290E-02 | 0.730E-03 | 0.180E-03 | 0.450E-04 | 0.11E-04 | 0.280E-05 |
| 1/64 | 0.510E-02 | 0.130E-02 | 0.310E-03 | 0.780E-04 | 0.20E-04 | 0.490E-05 |
| 1/128 | 0.950E-02 | 0.230E-02 | 0.580E-03 | 0.150E-03 | 0.36E-04 | 0.910E-05 |
| 1/256 | 0.190E-01 | 0.450E-02 | 0.110E-02 | 0.280E-03 | 0.69E-04 | 0.170E-04 |
| 1/512 | 0.380E-01 | 0.850E-02 | 0.210E-02 | 0.530E-03 | 0.13E-03 | 0.330E-04 |
| 1/1024 | 0.670E-01 | 0.180E-01 | 0.420E-02 | 0.100E-02 | 0.26E-03 | 0.640E-04 |

**Table 4.3:** Maximum Absolute Error ($\|E\|$) of **Example 4.3**.

| $\varepsilon$ | $N=8$ | $N=16$ | $N=32$ | $N=64$ | $N=128$ | $N=256$ | $N=512$ |
|---|---|---|---|---|---|---|---|
| $2^{-3}$ | 1.44E-03 | 3.64E-04 | 9.06E-05 | 2.26E-05 | 5.66E-06 | 1.41E-06 | 3.53E-07 |
| $2^{-4}$ | 2.95E-03 | 5.85E-04 | 1.46E-04 | 3.67E-05 | 9.17E-06 | 2.29E-06 | 5.73E-07 |
| $2^{-5}$ | 3.71E-03 | 8.57E-04 | 2.08E-04 | 5.09E-05 | 1.26E-05 | 3.16E-06 | 7.89E-07 |
| $2^{-6}$ | 3.12E-03 | 9.58E-04 | 2.12E-04 | 6.44E-05 | 1.51E-05 | 3.70E-06 | 9.45E-07 |
| $2^{-7}$ | 2.81E-03 | 1.29E-03 | 2.23E-04 | 6.92E-05 | 1.65E-05 | 4.17E-06 | 1.03E-06 |
| $2^{-8}$ | 4.63E-03 | 1.65E-03 | 2.33E-04 | 6.09E-05 | 1.73E-05 | 4.37E-06 | 1.08E-06 |
| $2^{-12}$ | 6.76E-02 | 3.76E-02 | 7.40E-03 | 6.17E-04 | 3.54E-05 | 4.74E-06 | 1.21E-06 |

**Table 4.4**: Maximum Absolute Error ($\|E\|$) of **Example 4.4**

| $\varepsilon$ | $N=32$ | $N=64$ | $N=128$ | $N=256$ | $N=512$ | $N=1024$ |
|---|---|---|---|---|---|---|
| Our Method | | | | | | |
| $2^{-2}$ | 3.7103E-08 | 2.3160E-09 | 1.4471E-10 | 9.0275E-12 | 6.7535E-13 | 8.0491E-14 |
| $2^{-4}$ | 2.7916E-07 | 1.7470E-08 | 1.0916E-09 | 6.8258E-11 | 4.9849E-12 | 4.9531E-12 |
| $2^{-6}$ | 2.8983E-06 | 1.8098E-07 | 1.1309E-08 | 7.0665E-10 | 4.6779E-11 | 7.6206E-12 |
| $2^{-8}$ | 2.8429E-05 | 1.7762E-06 | 1.1100E-07 | 6.9378E-09 | 4.3499E-10 | 2.8159E-11 |
| $2^{-10}$ | 2.2479E-04 | 1.4080E-05 | 8.7980E-07 | 5.4984E-08 | 3.4341E-09 | 2.2168E-10 |
| $2^{-12}$ | 3.1128E-03 | 1.9481E-04 | 1.2190E-05 | 7.6239E-07 | 4.7650E-08 | 2.9734E-09 |
| $2^{-14}$ | 5.0533E-02 | 3.3041E-03 | 2.0803E-04 | 1.3042E-05 | 8.1533E-07 | 5.0976E-08 |
| $2^{-16}$ | 1.7521E-01 | 3.3893E-02 | 3.5055E-03 | 2.2167E-04 | 1.3873E-05 | 8.6815E-07 |
| Kadalbajoo and Kumar [19] using fitting factor | | | | | | |
| $2^{-2}$ | 1.310E-03 | 3.280E-04 | 8.210E-05 | 2.050E-05 | 5.130E-06 | 1.280E-06 |
| $2^{-4}$ | 4.930E-03 | 1.230E-03 | 3.080E-04 | 7.710E-05 | 1.930E-05 | 4.820E-06 |
| $2^{-6}$ | 1.600E-02 | 4.000E-03 | 1.000E-03 | 2.500E-04 | 6.260E-05 | 1.560E-05 |
| $2^{-8}$ | 3.710E-02 | 9.270E-03 | 2.320E-03 | 5.790E-04 | 1.450E-04 | 3.620E-05 |
| $2^{-10}$ | 6.190E-02 | 1.540E-02 | 3.860E-03 | 9.650E-04 | 2.410E-04 | 6.030E-05 |
| $2^{-12}$ | 9.390E-02 | 2.340E-02 | 5.830E-03 | 1.460E-03 | 3.640E-04 | 9.100E-05 |
| $2^{-14}$ | 1.340E-01 | 3.290E-02 | 8.150E-03 | 2.030E-03 | 5.080E-04 | 1.270E-04 |
| $2^{-16}$ | 1.900E-01 | 4.310E-02 | 1.050E-02 | 2.600E-03 | 6.500E-04 | 1.620E-04 |

The computational rate of convergence can also be obtained by using the double mesh principle Doolan.et.al [6], defined below. Let

$$Z_h = \max_i \left| y_i^h - y_i^{h/2} \right|, \qquad i = 1, 2, \ldots, N-1$$

where $y_i^h$ is the numerical solution on the mesh $\{x_i\}_1^{N-1}$ at the nodal point $x_i$ and $x_i = x_0 + ih$, $i = 1,2,\ldots, N-1$ and where $y_i^{h/2}$ is the numerical solution at the nodal point $x_i$ on the mesh $\{x_i\}_1^{2N-1}$ where, $x_i = x_0 + ih/2$, $i = 1,2,\ldots,2N-1$.

In the same way one can define $Z_{h/2}$ by replacing $h$ by $h/2$ and $N-1$ by $2N-1$ , that is,

$$Z_{h/2} = \max_i \left| y_i^{h/2} - y_i^{h/4} \right| , \text{ for } i = 1, 2, \ldots, 2N-1.$$

The computed rate of convergence is defined as:

$$\text{Rate} = \log \left( \frac{Z_h - Z_{h/2}}{2} \right)$$

Also maximum absolute error based on the double-mesh principle Doolan.et.al [6] is given by:

$\|E\| = \max_i \left| y_i^{N} - y_{2i}^{2N} \right|$ , for $i = 0,1,2,\ldots N$. and $y_i^{h/2}$ denotes the values of $y_i$ for mesh length $h/2$.

Tables 4.5, 4.6 and 4.7 show that the sample examples of rate of convergence for examples 4.1, 4.2 and 4.4 respectively.

**Table 4.5**: Rate of convergence of **Example 4.1**

| $\varepsilon$ | $h$ | $h/2$ | $Z_h$ | $h/4$ | $Z_{h/2}$ | Rate |
|---|---|---|---|---|---|---|
| $2^{-4}$ | 1/100 | 1/200 | 1.877660E-09 | 1/400 | 1.173327E-10 | 4.00006 |
| | 1/200 | 1/400 | 1.173327E-10 | 1/800 | 7.318690E-12 | 4.00008 |
| | 1/400 | 1/800 | 7.318690E-12 | 1/1600 | 4.35097E-13 | 4.00010 |
| $2^{-8}$ | 1/100 | 1/200 | 4.700047E-07 | 1/400 | 1.83953E-09 | 3.999227 |
| | 1/200 | 1/400 | 2.940009E-08 | 1/800 | 1.14926E-10 | 3.998824 |
| | 1/400 | 1/800 | 1.839530E-09 | 1/1600 | 1.14926E-10 | 4.000056 |
| $2^{-12}$ | 1/100 | 1/200 | 1.145221E-04 | 1/400 | 7.49654E-06 | 3.933614 |
| | 1/200 | 1/400 | 7.496540E-06 | 1/800 | 4.70000E-07 | 3.995819 |
| | 1/400 | 1/800 | 4.700000E-07 | 1/1600 | 2.93981E-08 | 3.999216 |

**Table 4.6**: Rate of convergence of **Example 4.2**

| $\varepsilon$ | $h$ | $h/2$ | $Z_h$ | $h/4$ | $Z_{h/2}$ | Rate |
|---|---|---|---|---|---|---|
| 1/16 | 1/16 | 1/32 | 3.9350E-06 | 1/64 | 2.4357E-07 | 4.00019 |
| | 1/32 | 1/64 | 2.4357E-07 | 1/128 | 1.5185E-08 | 4.00036 |
| | 1/64 | 1/128 | 1.5185E-08 | 1/256 | 9.4847E-10 | 4.00009 |
| | 1/128 | 1/256 | 9.4847E-10 | 1/512 | 5.9290E-11 | 3.99988 |
| 1/32 | 1/16 | 1/32 | 9.0256E-06 | 1/64 | 5.5941E-07 | 4.00012 |
| | 1/32 | 1/64 | 5.5941E-07 | 1/128 | 3.4888E-08 | 4.00031 |
| | 1/64 | 1/128 | 3.4888E-08 | 1/256 | 2.1795E-09 | 4.00007 |
| | 1/128 | 1/256 | 2.1795E-09 | 1/512 | 1.3614E-10 | 4.00009 |
| 1/128 | 1/16 | 1/32 | 1.2249E-04 | 1/64 | 7.8992E-06 | 3.95647 |
| | 1/32 | 1/64 | 7.8992E-06 | 1/128 | 4.9271E-07 | 4.00029 |
| | 1/64 | 1/128 | 4.9271E-07 | 1/256 | 3.0786E-08 | 4.00004 |
| | 1/128 | 1/256 | 3.0786E-08 | 1/512 | 1.9253E-09 | 3.99911 |
| 1/512 | 1/16 | 1/32 | 2.1873E-03 | 1/64 | 1.3947E-04 | 3.97111 |
| | 1/32 | 1/64 | 1.3947E-04 | 1/128 | 9.3471E-06 | 3.89931 |
| | 1/64 | 1/128 | 9.3471E-06 | 1/256 | 5.8546E-07 | 3.99691 |
| | 1/128 | 1/256 | 5.8546E-07 | 1/512 | 3.6610E-08 | 3.99931 |

**Table 4.7**: Rate of convergence of **Example 4.4**

| $\varepsilon$ | $h$ | $h/2$ | $Z_h$ | $h/4$ | $Z_{h/2}$ | Rate |
|---|---|---|---|---|---|---|
| $2^{-2}$ | 1/32 | 1/64 | 3.4787E-08 | 1/128 | 2.1713E-09 | 4.00019 |
| | 1/64 | 1/128 | 2.1713E-09 | 1/256 | 1.3568E-10 | 4.0002 |
| | 1/128 | 1/256 | 1.3568E-10 | 1/512 | 8.3521E-12 | 4.00029 |
| | 1/256 | 1/512 | 8.3521E-12 | 1/1024 | 5.9486E-13 | 3.81125 |
| $2^{-4}$ | 1/32 | 1/64 | 2.6169E-07 | 1/128 | 1.6378E-08 | 3.99870 |
| | 1/64 | 1/128 | 1.6378E-08 | 1/256 | 1.0233E-09 | 4.00041 |
| | 1/128 | 1/256 | 1.0233E-09 | 1/512 | 6.3273E-11 | 4.00016 |
| $2^{-8}$ | 1/32 | 1/64 | 2.6653E-05 | 1/128 | 1.6652E-06 | 4.00054 |
| | 1/64 | 1/128 | 1.6652E-06 | 1/256 | 1.0406E-07 | 4.00023 |
| | 1/128 | 1/256 | 1.0406E-07 | 1/512 | 6.5028E-09 | 4.00021 |
| | 1/256 | 1/512 | 6.5028E-09 | 1/1024 | 4.0683E-10 | 3.99862 |
| $2^{-10}$ | 1/32 | 1/64 | 2.1071E-04 | 1/128 | 1.3200E-05 | 3.99663 |
| | 1/64 | 1/128 | 1.3200E-05 | 1/256 | 8.2482E-07 | 4.00031 |
| | 1/128 | 1/256 | 8.2482E-07 | 1/512 | 5.1550E-08 | 4.00002 |
| | 1/256 | 1/512 | 5.1550E-08 | 1/1024 | 3.2124E-09 | 4.00043 |
| $2^{-12}$ | 1/32 | 1/64 | 0.0029000 | 1/128 | 1.8262E-04 | 3.99812 |
| | 1/64 | 1/128 | 1.8262E-04 | 1/256 | 1.1428E-05 | 3.99812 |
| | 1/128 | 1/256 | 1.1428E-05 | 1/512 | 7.1474E-07 | 3.99901 |
| | 1/256 | 1/512 | 7.1474E-07 | 1/1024 | 4.4677E-08 | 3.99984 |
| $2^{-14}$ | 1/32 | 1/64 | 0.0472000 | 1/128 | 0.0031000 | 3.93121 |
| | 1/64 | 1/128 | 0.0031000 | 1/256 | 1.9499E-04 | 3.98910 |
| | 1/128 | 1/256 | 1.9499E-04 | 1/512 | 1.2227E-05 | 3.99523 |
| | 1/256 | 1/512 | 1.2227E-05 | 1/1024 | 7.6435E-07 | 3.99936 |

The following graphs (figure 4.1-4.8) show the numerical solutions obtained by the present method for $h \geq \varepsilon$

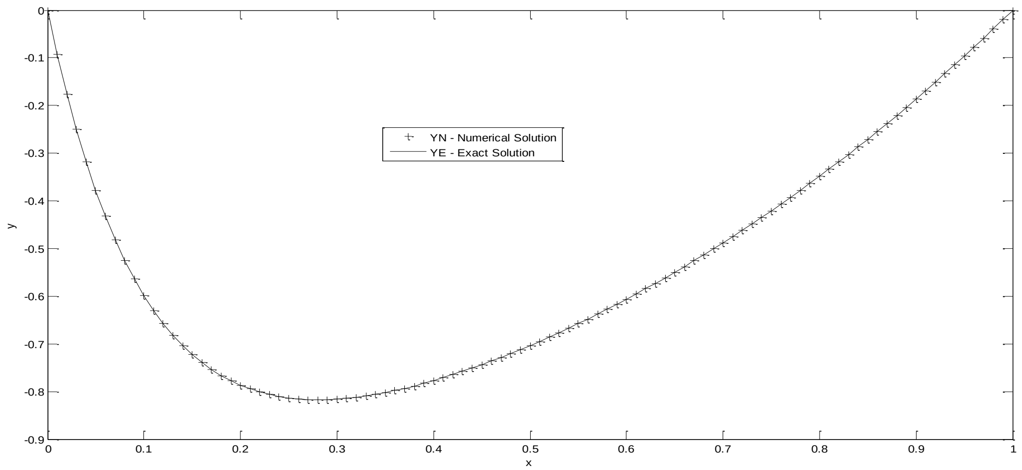


Figure 4.1 Numerical solution of Example 4.1 with $\varepsilon = 0.01$ and h=0.01

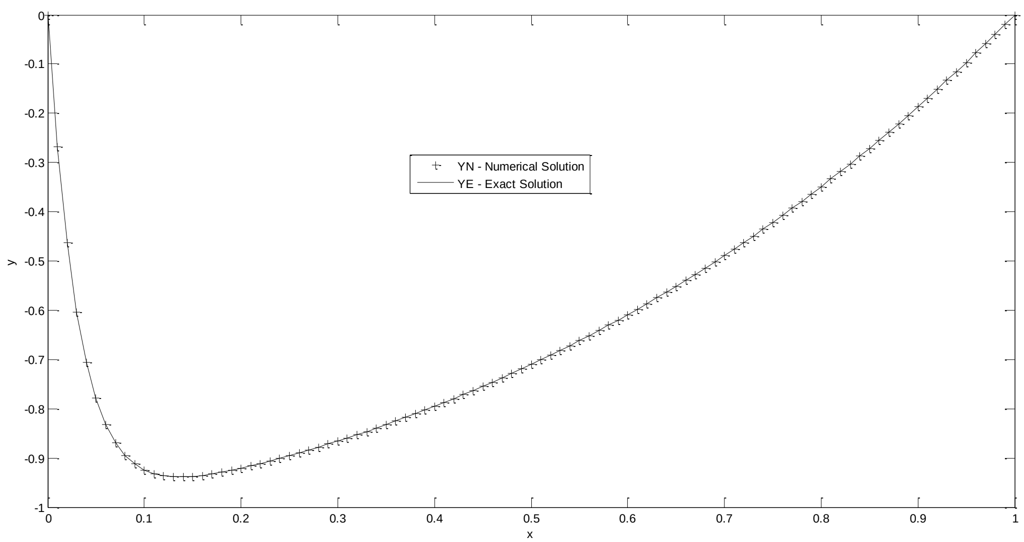


Figure 4.2 Numerical solution of Example 4.1 with $\varepsilon = 0.001$ and h=0.01

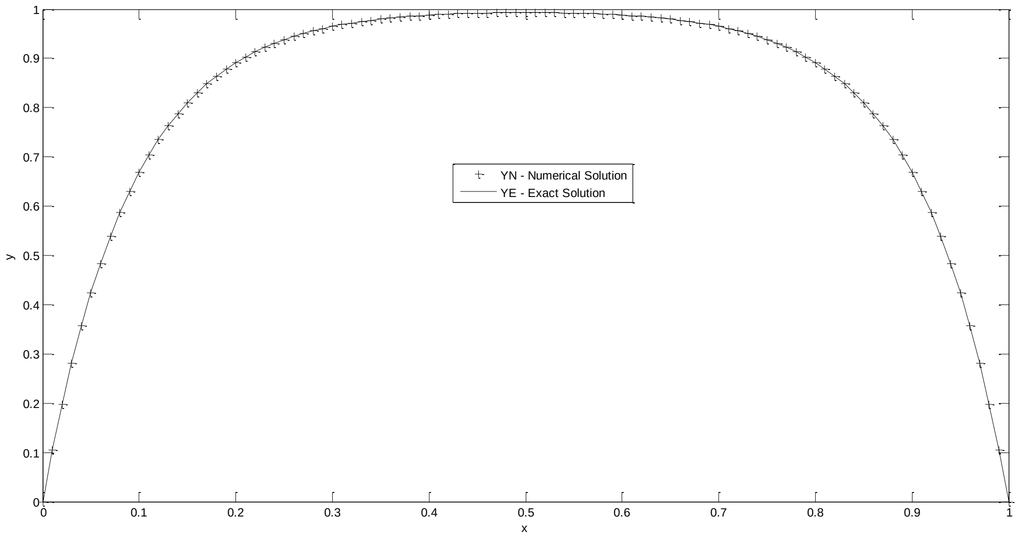


Figure 4.3: Numerical solution of Example 4.2 with $\varepsilon = 0.01$ and h=0.01

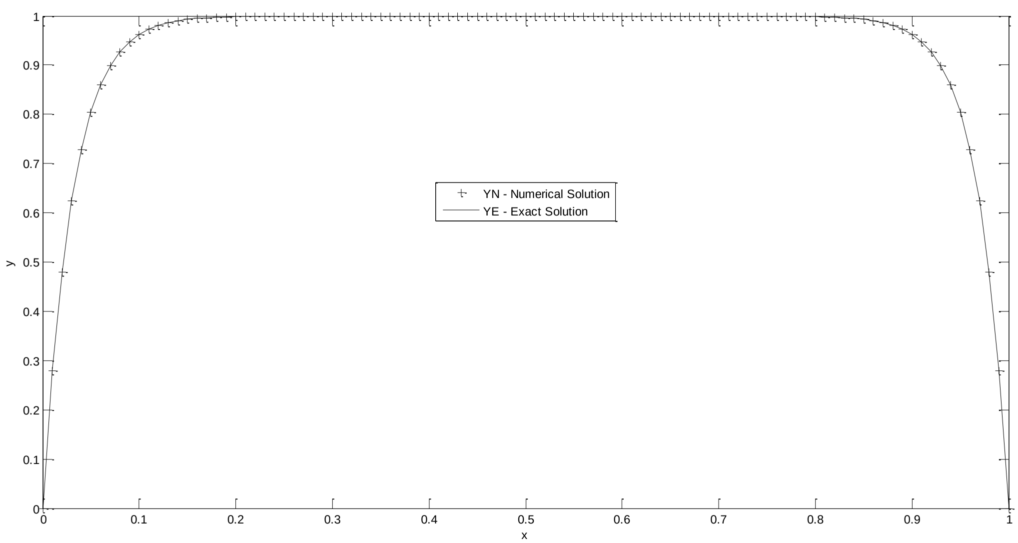


Figure 4.4: Numerical solution of Example 4.2 with $\varepsilon = 0.001$ and h=0.01

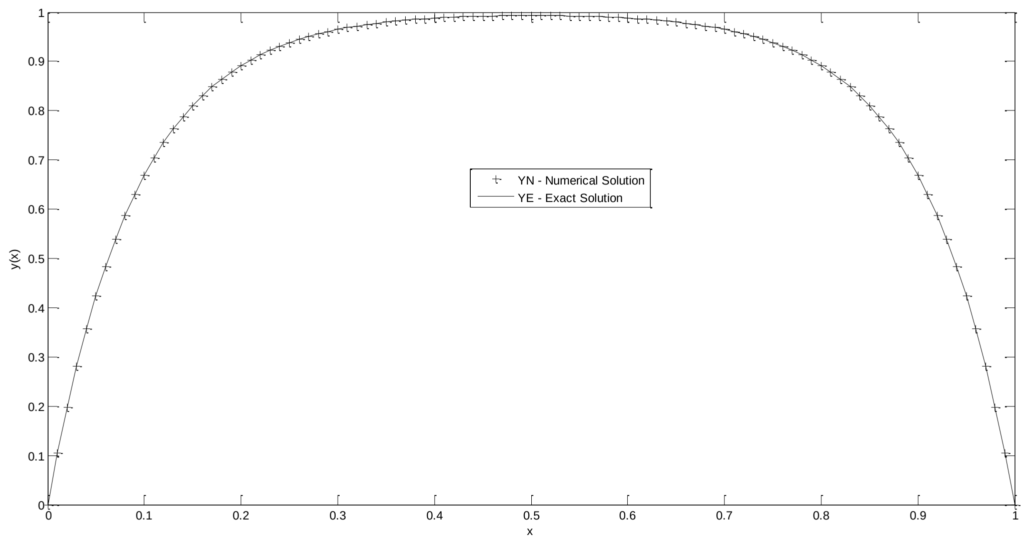


Figure 4.5: Numerical solution of Example 4.3 with $\varepsilon = 0.01$ and h=0.01

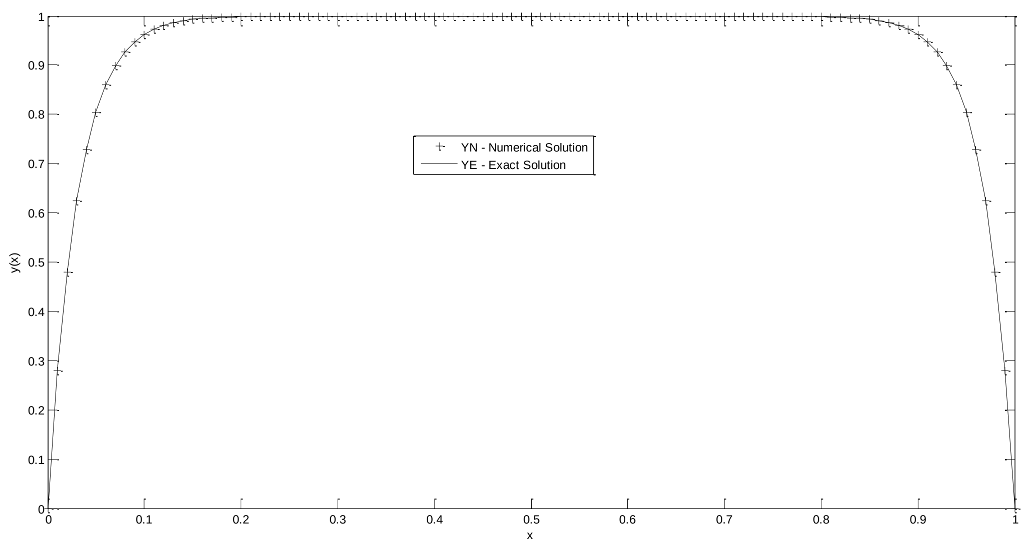


Figure 4.6: Numerical solution of Example 4.3 with $\varepsilon = 0.001$ and h=0.01

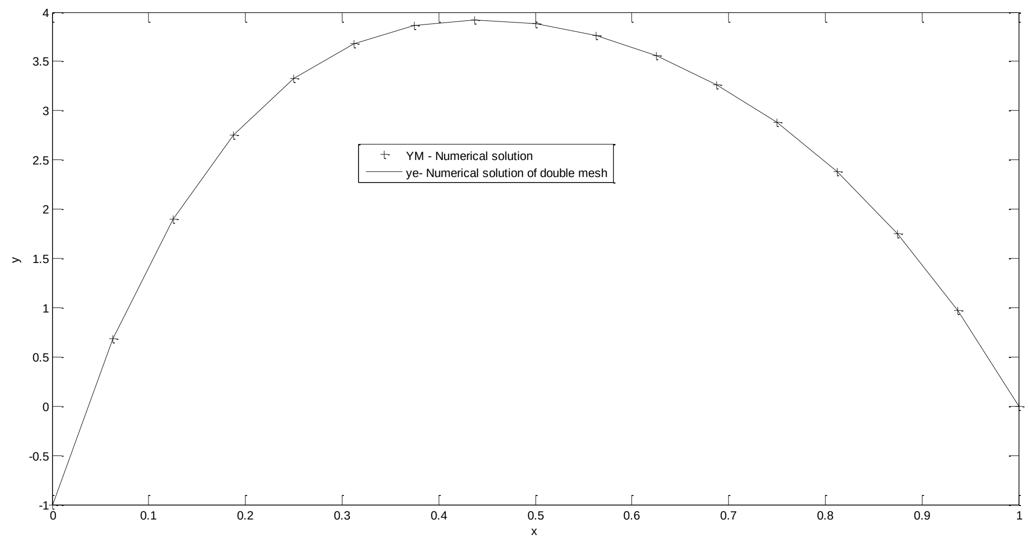


Figure 4.7: Numerical solution of Example 4.4 with $\varepsilon = 1/16$ and $h = 1/16$

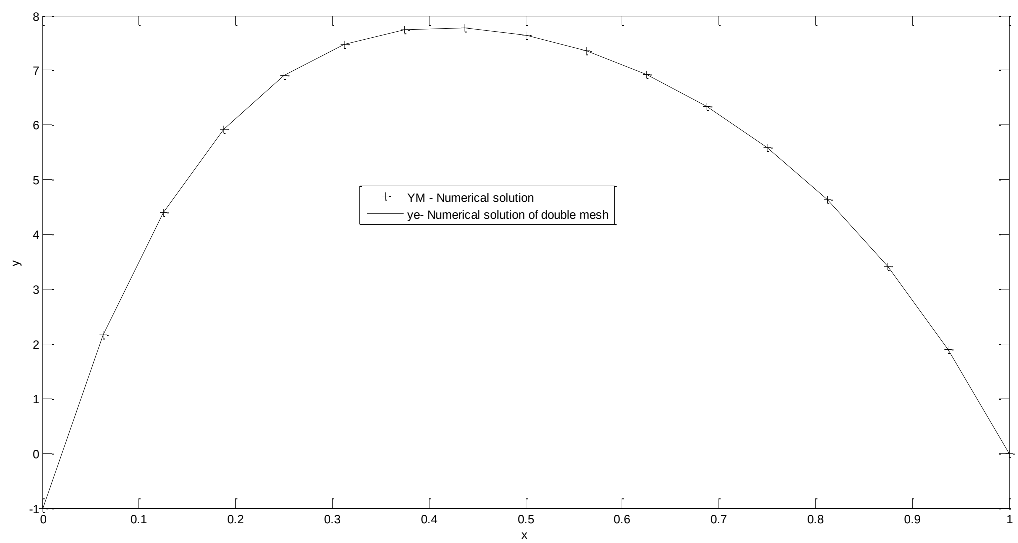


Figure 4.8: Numerical solution of Example 4.4 with $\varepsilon = 1/32$ and $h = 1/16$

### 4.6 Discussion

In this thesis, fourth order stable central difference method has been presented for solving self-adjoint singular perturbation problems. First, the given interval is discritized and the derivative of the given differential equation is replaced by the finite difference approximations. Then, the given differential equation is transformed to linear system of algebraic equations. Further, these algebraic equations are transformed into a three-term recurrence relation, which can easily be solved by using Thomas Algorithm. The numerical results have been presented in tables 4.1- 4.4 for different values of the perturbation parameter $\varepsilon$ and number of mesh points N. The results obtained by the present method have been compared with Kadalbajoo and Kumar [17], Kumar and Kadalbajoo [19] and Patidar and Kadalbajoo [29] from literature and the results are summarized in the tables 4.1, 4.2 and 4.4. The stability and convergence of the method have been established.

It can be observed from the tables that the present method gives better results than the methods by Kadalbajoo and Kumar [17], Kumar and Kadalbajoo [19] and Patidar and Kadalbajoo [29]. Further, it can be observed from the graphs that the present method approximates the exact solution very well for $h \geq \varepsilon$ for which most of the existing methods fails to give good results Segal [36]. Moreover, it is significant that all of the maximum errors decrease rapidly as $N$ increases.

To further corroborate the applicability of the proposed method, graphs have been plotted for Examples 4.1, 4.2, 4.3 and 4.4 for values of $x \in [0,1]$ versus the numerical solution obtained at different values of $\varepsilon$ for a fixed step size $h$. Figures 4.1 – 4.8 provide a good agreement of results presenting exact as well as numerical solutions, which proves the reliability of the stable central difference method. That means the exact solution and numerical solution graphs are overlapped.

# Chapter Five: Conclusion and Scope of Future Work

## 5.1. Conclusion

Fourth order stable central difference method was proposed to solve self-adjoint singularly perturbed problem. This method is conceptually simple, easy to use and readily adaptable for computer implementation. This study has been implemented on four linear examples those have exact solutions and without exact solutions by taking different small values for the perturbation parameter $\varepsilon$ and the computational results are presented in the tables. The results observed from the tables demonstrate that the present method approximates the exact solution very well. It has been shown that stability and convergence of the method were established well. Numerical results presented in this thesis show the superiority of the proposed method over some existing methods reported in the literature. The existence and uniqueness of the method along with stability estimates are discussed.

It may be noted that computational order of convergence as well as theoretical estimates indicates that fourth-order method is a fourth order convergent. In concise manner, the present method is conceptually simple, easy to use and readily adaptable for computer implementation for solving self-adjoint singularly perturbed two points boundary value problems.

## 5.2. Scope for future Work

In the present thesis, the numerical method based on fourth order stable central difference method was constructed for solving self-adjoint singularly perturbed problems. Hence, the scheme proposed in this thesis can also be extended to sixth-order and higher- order stable central difference methods for solving self-adjoint singularly perturbed problems. And also, this thesis considered the uniform mesh length. So, one can extended this to non-uniform mesh length.

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