GENERAL SOLUTION OF THE GENERALIZED BURGER-FISHER EQUATION USING THE IMPROVED (G'/G)-EXPANSION METHOD.

BY

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DECLARATION

I, the undersigned, declare that the theses entitled "**More General Solution of the Generalized Burger-Fisher's Equation Using the Improved $\left(\frac{G'}{G}\right)$- Expansion Method**" is original and it has not been submitted to any institution elsewhere for the award of any academic degree or like, where other sources of information have been used, they have been acknowledged.

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ABSTRACT

The main purpose of this study was to construct more general solution and to find some new exact traveling wave solutions of the Generalized Burger-Fisher's equation. To achieve this purpose, the improved $\left(\frac{G'}{G}\right)$*-expansion method was proposed to construct more general solution and exact traveling wave solutions of Burger-Fisher's equation .As a result, more general solution was constructed with free parameters as compared to the solution obtained in the existing literature Abdollah Boharfani and Reza Abazari [3]; moreover, some exact traveling wave solutions in terms of hyperbolic, trigonometric and rational functions were found which were not obtained in the work of Abdollah Boharfani and Reza Abazari [3].*

The study outlined that the improved $\left(\frac{G'}{G}\right)$ *-expansion method is very effective, simple and a powerful mathematical tool for finding exact traveling wave solutions of the Generalized Burger-Fisher's equation. Therefore, it was concluded that the proposed method can be applied to find exact traveling wave solutions of nonlinear partial differential equations (which are nonlinear evolution equations) which can arise in physics, engineering science, and other related areas.*

CHAPTER ONE

1 INTRODUTION

1.1 Background of the study

Nonlinear partial differential equations (NLPDEs) are widely used to describe complex physical phenomena in different fields of study like mathematical physics, engineering science and other areas of natural science [1]. In Particular, the generalized Burgers-Fisher equation has a wide range of applications in mathematical-physics, engineering science, physics, chemistry etc.

General form of NLPDEs: has

The general form of NLPDEs can be expressed as $P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \ldots) = 0$,

where P is a polynomial in its arguments, which includes nonlinear terms and the highest order derivatives, the subscript stands for partial derivatives and $u(x,t)$ is the unknown function [3].

Nonlinear evolution equation (NLEE)

NLEE is a NPDE which is dependent of a time t.

EXAMPLE

1. $u_t = uu_x + u_{xx}$ (Burger's equation)
2. $u_t = uu_x + u_{xx} + u(1-u))$ (Burger-Fisher equation)
3. $u_t + pu^n u_x + qu_{xx} + ru(1-u^n) = 0$,(Generalized Burger-Fisher equation), and so on [3].

The traveling wave transformations

Combining the real variables x and t by a wave variable $\eta = kx + \omega t$ and $u(x,t) = U(\eta)$ where ω is the speed of the traveling wave and k is a constant to be determined.

The traveling wave transformations $\eta = kx + \omega t$ & $u(x,t) = U(\eta)$ converts into an ordinary differential equation (ODE) $Q(U, kU', \omega U', k\omega U'', k^2 U'', \ldots) = 0$,

where Q is a polynomial in U and the derivatives of U; the superscripts indicate the ordinary derivatives of U with respect to η.

Exact travelling wave solutions of NLPDEs play an important role in the study of physical phenomena. Looking for exact solutions to nonlinear evolution equations (NLEEs) has long been a major concern for both mathematicians and physicists. These solutions may well describe various phenomena in physics and other fields [1].But unlike linear partial differential equations (LPDE), NPDEs are difficult to study because there are almost no general techniques that work for all NPDEs, and usually each individual equation has to be studied as a separate problem. Therefore, many authors have been introducing different techniques to obtain exact traveling wave solutions for nonlinear evolution equations (NLEEs) for the past many years.

In 2008, Wang et al. [2] introduced effective method to construct exact traveling wave solutions of some nonlinear evolution equations (NLEEs) called the $\left(\frac{G'}{G}\right)$ - expansion method. The $\left(\frac{G'}{G}\right)$ -expansion method is based on the assumptions that the travelling wave solutions $U(\eta)=\sum_{i=0}^{m}\alpha_i\left(\frac{G'}{G}\right)^i, \alpha_m \neq 0$ can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$, and that G = G (η), ($\eta = kx + \omega t$, is a wave variable) satisfies a second order ordinary differential equations (ODE) and the degree of this polynomial can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear term appearing in this second order ODE . That is, first the given NLEE could be put into the corresponding ODE that can further be transformed into systems of algebraic polynomials with constants to be determined. The exact traveling solutions and rational solutions of the NLEE can be obtained from the transformed second order ODE.

After wards, several authors applied this method to obtain exact traveling wave solutions of some NPDEs. In 2011, Bhajan lal in [1] used the $\left(\frac{G'}{G}\right)$ - expansion method to find travelling wave solutions of Sinh-Gordon equation, Huber’s equation, and Boussinesq equation. As a result this author found travelling wave solutions of Sinh- Gordon equation, and Huber’s equation in terms of hyperbolic and trigonometric functions and a travelling wave solution of Boussinesq equation in terms of trigonometric function. In 2012, Abdollah Borhanifar and Reza Abazari [3] used the $\left(\frac{G'}{G}\right)$ - expansion method for General Burger-Fisher equation. As a result they obtained exact traveling wave solutions in terms of hyperbolic, trigonometric and rational functions.

In 2012, Xiaohua Liu et al. [4] used improved expansion method. In this method the solution of the given NPDE can be expressed in the form $U(\eta)=\sum_{i=0}^{m}\alpha_i\left(\frac{G'}{G}\right)^i$, $\alpha_m \neq 0$ where $G=G(\eta)$ satisfies second order nonlinear ordinary differential equation, $GG''=AG^2+BGG'+C(G')^2$, A, B and C are real parameters to construct the general solution of the generalized regularized long wave (RLW) and symmetric regularized long wave (SRLW). In 2012, Hassibun Naher et al. [16] used the improved $\left(\frac{G'}{G}\right)$-expansion method in the form of $v(\eta)=\sum_{j=-n}^{n}\alpha_j\left(\frac{G'}{G}\right)^n$

$\alpha_j(j=0,\pm1,\pm2,\pm,\cdots,\pm n)$ to find the exact solution of The Modified KdV-Zakharov-Kusnetsev equation. As a result they could obtain some new exact traveling wave solutions which were not found by other researcher. In general much work has been done on developing this method to construct more general solution of some NPDEs by appropriate choice of the auxiliary equation. But in the generalized cases, a small amount of work has been done, see [3 and the references therein]. Having been Motivated by the work of [4&16], the purpose of this study was to generalize the main results of Abdollah Borhanifar and Reza Abazari [3] using the improved $\left(\frac{G'}{G}\right)$-expansion method.

1.2 Statement of the Problem

This study focused on the generalized Burger–Fisher equation which arises in some physical phenomena such as applications in plasma physics, fluid physics, capillary-gravity waves, nonlinear optics and chemical physics etc.

In 2012, Abdollah Borhanifar and Reza Abazari [3] used the $\left(\frac{G'}{G}\right)$- expansion method for General Burger-Fisher equation where G satisfies the auxiliary equation, $G''+\lambda G'+\mu G=0$ Recently, much work has been done on developing the $\left(\frac{G'}{G}\right)$- expansion method to construct more general solution of some nonlinear partial differential equations by appropriate choice of the auxiliary equation.

But in the generalized cases, only small amount of work has been done, see [3] and their references therein. As a result, the researcher is interested to conduct this research.

To this end the researcher is intended to answer the following basic questions:

1. How can we transform generalized Burger–Fisher's equation into ordinary differential equation?
2. How can we find the positive integer m in $U(\eta)=\sum_{i=0}^{m}\alpha_i\left(\frac{G'}{G}\right)^{i}+\sum_{i=1}^{m}\beta_i\left(\frac{G'}{G}\right)^{-i}$?
3. How can we find some new exact traveling wave solutions of the generalized Burger-Fisher equation which were not found by Abdollah Borhanifar and Reza Abazari [3]?
4. How can we find more general solution of the generalized Burger-Fisher equation as compared to the general solution obtained by Abdollah Borhanifar and Reza Abazari [3]?

1.3 Objectives of the Study

1.3.1 General objective

The general objective of this study is to obtain more general solution of the generalized Burger – Fisher equation using the $\left(\frac{G'}{G}\right)$ -expansion method.

1.3.2 Specific Objective

- ❖ To transform the generalized Burger-Fisher equation into ordinary differential equation.
- ❖ To obtain the value of m in $U(\eta)=\sum_{i=0}^{m}\alpha_i\left(\frac{G'}{G}\right)^{i}+\sum_{i=1}^{m}\beta_i\left(\frac{G'}{G}\right)^{-i}$
- ❖ To find some new exact traveling wave solutions of the generalized Burger-Fisher equation which were not found by Abdollah Borhanifar et al. in [3]
- ❖ To find more general solution of the generalized Burger-Fisher equation as compared to the general solution obtained Abdollah Borhanifar et al. in [3].

1.4 Significance of the Study

Nonlinear partial differential equations have been the subject of the study in different branches of mathematical-physical sciences such as physics, biology, chemistry, etc. The solutions of such equations are of fundamental importance since a lot of mathematical-physical models are described by NLPDEs.

The results of this research, on the general solution of the generalized Burger-Fisher equation by the $\left(\frac{G'}{G}\right)$ –expansion method will have a vital importance for the following reasons

- ❖ It has a contribution on developing the $\left(\frac{G'}{G}\right)$ –expansion method for its wide-range of applicability on other generalized form of some NPDEs.
- ❖ It can be used as a reference material for anyone who wants to work on this area. In particular, for teachers and graduate students of our department.
- ❖ It can be used as a base for the next researcher.
- ❖ The researcher will be beneficial since it enhances how to develop scientific research writing skills in practical.

1.5 Delimitation of the Study

This study will be done under the solution of nonlinear partial differential equations. In particular, the solution of nonlinear partial differential equation namely the Generalized Burger – Fisher's equation with constant coefficients by the improved $\left(\frac{G'}{G}\right)$-expansion method.

CHAPTER TWO

2 Literatures Review

The exact solutions of nonlinear evolution equations play an important role in the study of nonlinear physical phenomena. Therefore, the powerful and efficient methods to find exact solutions of nonlinear equations still have drawn a lot of interest by diverse group of scientists [5].

Looking for exact solutions to nonlinear evolution equations (NLEEs) has long been a major concern for both mathematicians and physicists. These solutions may well describe various phenomena in physics and other fields [1]. Since long, many authors have been introducing different techniques to obtain exact traveling wave solutions for non-linear evolution equations (NLEEs) such as: the sine-cosine method [6], the First integral method [7], expanding tanh-function method [8], and others. Every method has some restrictions in their implementations. Basically there is no integrated method which could be utilized to handle all types of nonlinear PDEs [3].After wards, several authors applied this method to obtain exact traveling wave solutions of some NLPDEs.

M.M Kabir and R.Bagherzadeh in [9], applied the $\left(\frac{G'}{G}\right)$ - expansion method, to the non-linear variants of the (2+1)-Dimensional Camassa-Holm-KP equations. El. Sayed, M.E. Zayed and Shorog Al-Joudi [10], used $\left(\frac{G'}{G}\right)$ - expansion method, solve Benjamin-Bona Mahony; Kdv coupled equations and Ostrovsky equation. As a result they obtained traveling wave solutions which are expressed by hyperbolic, trigonometric and rational functions.

In 2008, Wang et al. [2] introduced effective method to construct exact traveling wave solutions of some NLEEs and called the $\left(\frac{G'}{G}\right)$- expansion method. In this method $U(\eta)=\sum_{i=0}^{m}\alpha_i\left(\frac{G'}{G}\right)^i$, where m is positive integer and α_i (i=0,1,…,m) is a wave solution which is a polynomial in $\left(\frac{G'}{G}\right)$, where G(η) satisfies the second order linear ordinary differential equation with constant coefficients $G''+\lambda G'+\mu G=0$ is used, as an auxiliary equation to transform the given nonlinear

PDE to nonlinear ordinary differential equation (NLODE) where λ and μ are arbitrary constants whereas G=G(η) $,\eta = x - kt$ is traveling wave variable. The degree of this polynomial (the positive integer m) can be determined by considering the homogeneous balance between the highest order derivative and the non-linear terms with highest power appearing in the nonlinear ordinary differential equation. The coefficients of this polynomial ($\alpha_0, \alpha_1, ...\alpha_m$) can be obtained by solving a set of algebraic equations resulted from the process of using the $\left(\frac{G'}{G}\right)$- expansion method by the help of such as Maple and Mathematica. After wards, several authors applied this method to obtain exact traveling wave solutions for different nonlinear PDEs. M.M Kabir and Bagional [10], they applied the $\left(\frac{G'}{G}\right)$ - expansion method, to the non-linear variants of the (2+1)-Dimensional Camassa-Holm-KP equations where G satisfies $G'' + \lambda G + \mu G = 0$. El. Sayed, M.E. Zayed and Shorog Al-Joudi [5], used $\left(\frac{G'}{G}\right)$ - expansion method, where G satisfies the second order linear ODE, $G'' + \lambda G + \mu G = 0$ to solve Benjamin-Bona Mahony; Kdv coupled equations and Ostrovsky equation. As a result they obtained traveling wave solutions which are expressed by hyperbolic, trigonometric and rational functions.

In 2012, Abdollah Borhanifar and Reza Abazari [3] also used the $\left(\frac{G'}{G}\right)$- expansion method for General Burger-Fisher equation where G satisfies Second Order linear ODE, $G'' + \lambda G + \mu G = 0$ where λ and μ are real parameters. As a result they obtained exact Traveling wave solutions in terms of hyperbolic, trigonometric functions by taking particular values for free parameters. And see also [1, 11 & 12].

In 2013, M.M Zanan and Sayeda Sultan [13] used $\left(\frac{G'}{G}\right)$- expansion method, to find traveling wave solutions for the non-linear PDE, $u_x + 2ku_x + u_{xx} + \beta u^2 u_x = 0, k, \beta \in \mathbb{R}$, where G satisfies

$G' = c + aG + bG^2$ Where c, a, and b are real parameters. In 2013, Muhammad Shakeel et al. [14] applied the $\left(\frac{G'}{G}\right)$-expansion method with generalized Riccati equation to Boussinesq equation.

In 2012, Xiaohua Liu et al. [4], used the improved $\left(\frac{G'}{G}\right)$- expansion method. In this method the solution of given NPDE can be expressed in the form $U(\eta) = \sum_{i=0}^{m} \alpha_i \left(\frac{G'}{G}\right)^i, \alpha_m \neq 0$ where $G = G(\eta)$ satisfies second order nonlinear ordinary differential equation, $GG'' = AG^2 + BGG' + C(G')^2$, A, B and C are real parameters used to construct the general solution of the regularized long wave (RLW) and symmetric regularized long wave (SRLW).

In 2012 Hassibun Naher et al [16] used the improved $\left(\frac{G'}{G}\right)$-expansion method in the form of $v(\eta) = \sum_{j=-n}^{n} \alpha_j \left(\frac{G'}{G}\right)^n \alpha_j (j = 0, \pm 1, \pm 2, ..., \pm n)$, either α_n *or* α_{-n} can be zero but both of them can't be zero at the same time and $G = G(\eta)$ satisfies second order linear ODE with constant coefficients to find the exact solution of The Modified KdV-Zakharov-Kusnetsev equation. As a resulted they could obtain some new exact traveling wave solutions which were not found by other researcher

In general much work has been done on developing this method to construct more general solution of some NPDEs by appropriate choice of the auxiliary equation. But in generalized cases, a small amount of work has been done see for [3] and their references therein.

The purpose of this study was to construct more general solution of the generalized Burger-Fisher equation in [3] by using the improved (G′/G) – expansion.

CHAPTER THREE

3 METHODOLGY

3.1 Study site and Period

The study was focused on finding the general solution of the generalized Burger–Fisher equation using the improved $\left(\frac{G'}{G}\right)$ -expansion method**;** and was conducted in Jimma University under Mathematics department, from November, 2006 E.C – June, 2006 E.C

3.2 Study Design

The study designs used were Analytical and Numerical designs.

3.3 Source of information

The information or data that was used to conduct this study was collected from:

- Books or related reference,
- Internets and
- Some related published Journals

3.4 Administration and Instrumentation of Information or Data

- o Collecting of information or data from secondary sources such as reference books, internet and published research articles (or Journals) were administered by the researcher, and
- o Solving systems of multi-variables of algebraic equations was administered by skilled person (assistant) on Mathematica.

3.5 Procedures of the study

In order to achieve the objectives of this study, the standard techniques /procedures have been used.

1. The given NLPDE is converted to an ODE.
2. The obtained ODE is solved by the improved $\left(\frac{\mathrm{G}'}{\mathrm{G}}\right)$- expansion method.

3.6 Ethical Issue

To collect related data at the place where they were available and to process other related supports, cooperation request letters were written by officials of Jimma University Natural Science College. So, the cooperation request letter from Mathematics department of Jimma University was taken to the institute(s) where these materials are available by the researcher to get consent from them. Moreover, rules and regulations of the institute(s), from which information was collected, were kept by the researcher.

CHAPTER FOUR

4 Result and Discussion

4.1 Main Result

Consider NLPDE

$$P(u,u_t,u_x,u_{tt},u_{xt},u_{xx},\ldots)=0, \quad (1)$$

which includes nonlinear terms and the highest order derivatives, the subscripts stand for partial derivatives and $u(x,t)$ is the unknown function.

Combining the real variables x and t by a wave variable

$$\eta = kx+\omega t \text{ and } u(x,t)=V^{\frac{1}{n}}(\eta), \quad (2)$$

where ω is a speed of the traveling wave and k is a constant to be determine, can be used for the transformation of the non-linear partial differential equation in to an ODE.

The traveling wave transformations eq. (2) converts eq. (1) into an ordinary differential equation (ODE) $Q(V,V',V'',V''',\ldots)=0$, (3)

where Q is a function of V and any order derivatives V of , the superscripts indicate the ordinary derivatives of V with respect to η.

Consider

$$AGG''=BGG'+EG^2+C(G')^2 \quad (A\neq 0), \quad (4)$$

where the prime stands for derivative with respect to η and $A,B,C,$ and E are real parameters.

Suppose $F=\left(\frac{G'}{G}\right)$

$$F'=\frac{G''G-(G')^2}{G^2}$$

From equation (4) we have

$$F' = \frac{1}{A}(E + BF - WF^2) \quad \text{where} \quad W = A - C \tag{5}$$

Assume that the solution of eq. (3) can be expressed in the form:

$$V(\eta) = \sum_{i=0}^{m} \alpha_i (F)^i + \sum_{i=1}^{m} \beta_i (F)^{-i}, \tag{6}$$

Where $\alpha_i (i = 0,1,2,...,m)$, $\beta_i (i = 1,2,...,m)$ are arbitrary constants to be determined and either α_m *or* β_m can be zero but both can't be zero at the same time [16].

Note that from eq. (4) - (6) it follows that:

$$\begin{aligned} V'(\eta) &= \sum_{i=0}^{m} i\alpha_i F^{i-1} F' - \sum_{i=1}^{m} i\beta_i F^{-i-1} F' \\ &= F' \left(\sum_{i=0}^{m} i\alpha_i F^{i-1} - \sum_{i=1}^{m} i\beta_i F^{-i-1} \right) \\ &= \frac{1}{A}\left(E + BF - WF^2\right)\left(\sum_{i=0}^{m} i\alpha_i F^{i-1} - \sum_{i=1}^{m} i\beta_i F^{-i-1} \right) \end{aligned} \tag{7}$$

$$V''(\eta) = \left(\sum_{i=0}^{m} i\alpha_i F^{i-1} - \sum_{i=1}^{m} i\beta_i F^{-i-1} \right) F'' + \left(\sum_{i=0}^{m} i(i+1)\alpha_i F^{i-2} + \sum_{i=1}^{m} i(i+1)\beta_i F^{-i-2} \right) (F')^2 \tag{8}$$

And so forth,

where the prime denotes derivative/derivatives of V with respect to η .

Now, to determine $u(x,t)$ explicitly we follow the following steps:

Step 1: transforming eq. (1) into eq. (3) Using traveling wave variables given in eq. (2).

Step 2: substituting eq. (6), (7) & (8) into eq.(3) to determine the positive integer m, taking the homogeneous balance between the highest order nonlinear term and the derivative of the highest order appearing in eq. (3).

Step 3: using the value of m (obtained in Step 2) to obtain polynomials in $\left(\frac{G'}{G}\right)^{i}$ i=0, 1, 2 ...4 subsequently, we collect each coefficient of the resulted polynomials to zero, yields a set of algebraic equations for $\alpha_0, \alpha_1, ...\alpha_m, , \beta_1, \beta_2, ...\beta_m$ and ω.

Step 4: Suppose that the value of the constants $\alpha_i (i = 0,1,2,...m), \beta_i (i = 1,2,...m)$ and ω can be found by solving the algebraic equations obtained in Step 3. Since the general solutions of eq. (4) are known to us, inserting the values of $\alpha_i (i = 0,1,2,...m)$, $\beta_i (i = 1,2,...m)$, and ω into eq. (6), we obtain more general type and new exact traveling wave solutions of the nonlinear partial differential equation of eq. (1).

REMARK: Using the general solution of eq. (4), we have the following solutions of

$F = \left(\frac{G'}{G}\right)$ [16].

Case1. When $B \neq 0, \Omega = B^2 + 4EW > 0$, then

$$F = \left(\frac{G'}{G}\right) = \frac{B}{2W} + \frac{\sqrt{\Omega}}{2W} \frac{c_1 \sinh(\frac{\sqrt{\Omega}}{2A}) + c_2 \cosh(\frac{\sqrt{\Omega}}{2A})}{c_1 \cosh(\frac{\sqrt{\Omega}}{2A}) + c_2 \sinh(\frac{\sqrt{\Omega}}{2A})} \tag{9}$$

Case2. When $B \neq 0, \Omega = B^2 + 4EW < 0$ then

$$F = \left(\frac{G'}{G}\right) = \frac{B}{2W} + \frac{\sqrt{\Omega}}{2W} \frac{-c_1 \sin(\frac{\sqrt{-\Omega}}{2A}) + c_2 \cos(\frac{\sqrt{-\Omega}}{2A})}{c_1 \cos(\frac{\sqrt{-\Omega}}{2A}) + c_2 \sin(\frac{\sqrt{-\Omega}}{2A})} \tag{10}$$

Case3.When $B = 0, \Delta = EW > 0$, then (11)

$$F=\left(\frac{G'}{G}\right)=\frac{\sqrt{\Delta}}{W}\frac{c_1\sinh(\frac{\sqrt{\Delta}}{A})+c_2\cosh(\frac{\sqrt{\Delta}}{A})}{c_1\cosh(\frac{\sqrt{\Delta}}{A})+c_2\sinh(\frac{\sqrt{\Delta}}{A})}$$

Case4. When $B=0 and \Delta=EW<0$

$$F=\left(\frac{G'}{G}\right)=\frac{\sqrt{-\Delta}}{W}\frac{-c_1\sin(\frac{\sqrt{-\Delta}}{A})+c_2\cos(\frac{\sqrt{-\Delta}}{A})}{c_1\cos(\frac{\sqrt{-\Delta}}{A})+c_2\sin(\frac{\sqrt{-\Delta}}{A})} \tag{12}$$

Case5. When $B=0$ *and* Ω=EW=0

$$F=\left(\frac{G'}{G}\right)=\frac{B}{2W}+\frac{c_2}{c_1+c_2\eta} \tag{13}$$

Next, we will apply the improved $\left(\frac{G'}{G}\right)$ expansion method to construct many new and more general traveling wave solutions of the generalized Burger-fisher equation [3].

$$u_t+pu^n u_x+qu_{xx}+ru\left(1-u^n\right)=0 \ , \tag{14}$$

$n>1$, and p, r, and q are real parameters.

4.1.1 Transforming the Generalized Burger-Fisher's equation into ODE

Using traveling wave variables**,**

Let $u(x,t)=V^{\frac{1}{n}}(\eta), \quad \eta=kx+\omega t$ (15)

Such that: $V(\eta)=\alpha_0+...+\alpha_m F^m+\beta_1 F^{-1}+...+\beta_m F^{-m}$

By eq. (14) & eq. (15) it follows that

$$u_t = \frac{1}{n} V^{\frac{1}{n}} V^{-1} V' \omega$$

$$u_x = \frac{1}{n} V^{\frac{1}{n}} V^{-1} V' k \qquad (16)$$

$$u_{xx} = \frac{1}{n^2}(1-n) V^{\frac{1}{n}} V^{-2} (V')^2 k^2 + \frac{1}{n} V^{\frac{1}{n}} V^{-1} V'' k^2$$

By substituting eq. (17) in eq. (14) and after simplification we found that:

$$\omega n V V' + pnkV^2 V' + qk^2(1-n)(V')^2 + qk^2 n V V'' + rn^2 V^2 - rn^2 V^3 = 0,$$

which is the required ODE (17)

4.1.2 Determining the value of m by homogenous balance

By step2, the positive integer m can be determined by considering the homogenous balance between VV'' and V^2V' appears in eq. (17). To do this,

we need to balance the power of VV'' and V^2V' .

Now, From $V(\eta) = \alpha_0 + \ldots + \alpha_m F^m + \beta_1 F^{-1} + \ldots + \beta_m F^{-m}$, (18)

where $F = \left(\frac{G'}{G}\right)$ and $\mathrm{F}' = \frac{1}{\mathrm{A}}(\mathrm{E} + \mathrm{BF} - \mathrm{WF}^2)$ (by using eq.(4)) we found

$$V'(\eta) = \frac{1}{A}(E(\alpha_1 + \ldots + m\alpha_m F^{m-1} - \beta_1 F^{-2} - \ldots - m\beta_m F^{-m-1}) + B(\alpha_1 + \ldots + m\alpha_m F^m - \beta_1 F^{-1} - \ldots - m\beta_m F^{-m}) - W(\alpha_1 + \ldots + m\alpha_m F^{m+1} - \beta_1 + \ldots m\beta_m F^{-m+1}))$$

$$= \frac{1}{A}(E\alpha_1 + \ldots + m\alpha_m EF^{m-1} - \beta_1 EF^{-2} - \ldots - m\beta_m EF^{-m-1} + B\alpha_1 + \ldots + m\alpha_m BF^m - \beta_1 BF^{-1} - \ldots - m\beta_m BF^{-m} - W\alpha_1 + \ldots - \mathrm{m}\alpha_m WF^{m+1} + \beta_1 W + \ldots + m\beta_m WF^{-m+1})$$

$$= \frac{1}{A} E\alpha_1 + \ldots - \frac{mW\alpha_m}{A} F^{m+1} + W\beta_1 + \ldots + W\beta_m F^{-m+1}$$

$$V''(\eta)=\frac{1}{A}(BF'-2WFF')(\alpha_1+2\alpha_2F+m\alpha_mF^{m-1}-\beta_1F^{-2}-2\beta_2F^{-3}-m\beta_mF^{-m-1})+$$
$$\frac{1}{A}(E+BF-WF^2)(2\alpha_2F'+m(m-1)\alpha_mF^{m-2}F'+2\beta_1F^{-3}F'+6\beta_2F^{-4}F'+m(m+1)\beta_mF^{-m-2})$$

$$=\frac{1}{A^2}(E+BF-WF^2)[(B-2WF)(\alpha_1+2\alpha_2F+m\alpha_mF^{m-1}-\beta_1F^{-2}-2\beta_2F^{-3}-m\beta_mF^{-m-1})+$$
$$(E+BF-WF^2)(2\alpha_2+m(m-1)\alpha_mF^{m-2}+2\beta_1F^{-3}+6\beta_2F^{-4}+m(m+1)\beta_mF^{-m-2})]$$
$$=\ldots+\ldots+\frac{2W^2m^2\alpha_mF^{m+2}}{A^2}$$

Now the homogeneous balance between VV'' and V^2V' in eq. (17) was found as

$$\Rightarrow F^mF^{m+2}=F^{2m}F^{m+1}$$
$$\Rightarrow 2m+2=3m+1$$
$$\Rightarrow m=1$$

$\therefore$ The solution of eq. (18) becomes

$$V(\eta)=\alpha_0+\alpha_1\left(\frac{G'}{G}\right)+\beta_1\left(\frac{G'}{G}\right)^{-1} \tag{19}$$

Using eq. (6) ($AGG''=BGG'+EG^2+C(G')^2$) $A\neq 0$ & (19) we found

$$V'(\eta)=\frac{1}{A}(\alpha_1-\beta_1F^{-2})(E+BF-WF^2) \tag{20}$$
$$V''(\eta)=\frac{1}{A^2}\begin{pmatrix}B(E\alpha_1-W\beta_1)+\alpha_1(B^2-2EW)F-3BW\alpha_1F^2+\\ 2W^2\alpha_1F^3+\beta_1(B^2-2EW)F^{-1}+3BE\beta_1F^{-2}+2E^2\beta_1F^{-3}\end{pmatrix} \tag{21}$$

By step 3: Now inserting eq. (19) − (21) in eq.(17) we obtain polynomials in $\left(\frac{G'}{G}\right)^i$ (,i=0,1,2,3,…8)Subsequently, we collected each coefficient of the resulted polynomials to zero, yields a set of algebraic equations for α_0,α_1, A, B, C, E k, W and ω as follows:

$$F^0:-Ekq\beta^2-Eknq\beta^2+Anp\beta^3=0$$

$$F^1: -2BEk^2q\beta^2 - BEk^2nq\beta^2 + A^2n^2r\beta^2 + ABknp\beta^3 + AEn\beta^2\omega - 2E^2k^2nq\beta\alpha_0 +$$
$$2AEknp\beta^2\alpha_0 = 0$$

$$F^2: -B^2k^2q\beta^2 - A^2n^2r\beta^2 + 2Ek^2qW\beta^2 - AknpW\beta^3 + ABn\beta^2\omega - 3BEk^2nq\beta\alpha_0 +$$
$$2ABknp\beta^2\alpha_0 + 3A^2n^2r\beta^2\alpha_0 + AEn\beta\omega\alpha_0 + AEknp\beta\alpha_0^2 + 2E^2k^2q\beta\alpha_1 - 4E^2k^2nq\beta\alpha_1 +$$
$$AEknp\beta^2\alpha_1 = 0$$

$$F^3: 2Bk^2qW\beta^2 - Bk^2nqW\beta^2 - AnW\beta^2\omega - B^2k^2nq\beta\alpha_0 - 2A^2n^2r\beta\alpha_0 + 2Ek^2nqW\beta\alpha_0 -$$
$$2AknpW\beta^2\alpha_0 + ABn\beta\omega\alpha_0 + ABknp\beta\alpha_0^2 + 3A^2n^2r\beta\alpha_0^2 + 4BEk^2q\beta\alpha_1 - 8BEk^2nq\beta\alpha_1 +$$
$$ABknp\beta^2\alpha_1 + 3A^2n^2r\beta^2\alpha_1 = 0$$

$$F^4: k^2qW^2\beta - k^2nqW^2\beta - Bk^2nqW\beta\alpha_0 + AnW\beta\omega\alpha_0 + A^2n^2r\alpha_0^2 + AknpW\beta\alpha_0^2 - A^2n^2r\alpha_0^3 -$$
$$2B^2k^2q\beta\alpha_1 + 4Ek^2qW\beta\alpha_1 - 8Ek^2nqW\beta\alpha_1 + AknpW\beta^2\alpha_1 + BEk^2nq\alpha_0\alpha_1 - 6A^2n^2r\beta\alpha_0\alpha_1 +$$
$$AEn\omega\alpha_0\alpha_1 + AEknp\alpha_0^2\alpha_1 + E^2k^2q\alpha_1^2 - E^2k^2nq\alpha_1^2 + AEknp\beta\alpha_1^2 + 8B^2k^2n^3qr\beta^2\alpha_1^2 = 0$$

$$F^5: 4Bk^2qW\beta\alpha_1 - 8Bk^2nqW\beta\alpha_1 + B^2k^2nq\alpha_0\alpha_1 + 2A^2n^2r\alpha_0\alpha_1 - 2Ek^2nqW\alpha_0\alpha_1 + ABn\omega\alpha_0\alpha_1 +$$
$$ABknp\alpha_0^2\alpha_1 - 3A^2n^2r\alpha_0^2\alpha_1 + 2BEk^2q\alpha_1^2 - BEk^2nq\alpha_1^2 + ABknp\beta\alpha_1^2 - 3A^2n^2r\beta\alpha_1^2 +$$
$$AEn\omega\alpha_1^2 + 2AEknp\alpha_0\alpha_1^2 = 0$$

$$F^6: -2k^2qW^2\beta\alpha_1 + 4k^2nqW^2\beta\alpha_1 - 3Bk^2nqW\alpha_0\alpha_1 - AnW\omega\alpha_0\alpha_1 - AknpW\alpha_0^2\alpha_1 + B^2k^2q\alpha_1^2 +$$
$$A^2n^2r\alpha_1^2 - 2Fk^2qW\alpha_1^2 - AknpW\beta\alpha_1^2 + ABn\omega\alpha_1^2 + 2ABknp\alpha_0\alpha_1^2 - 3A^2n^2r\alpha_0\alpha_1^2 + AEknp\alpha_1^3 = 0$$

$$F^7: 2k^2nqW^2\alpha_0\alpha_1 - 2Bk^2qW\alpha_1^2 - Bk^2nqW\alpha_1^2 - AnW\omega\alpha_1^2 - 2AknpW\alpha_0\alpha_1^2 + ABknp\alpha_1^3 -$$
$$A^2n^2r\alpha_1^3 = 0$$

$$F^8: kqW^2\alpha_1{}^2 + knqW^2\alpha_1{}^2 - AnpW\alpha_1^3 = 0$$

Where $F=\left(\frac{G'}{G}\right)$

4.1.3 Exact traveling wave solutions of Burger-Fisher's equation

By step4: Now, to achieve our objectives, solving the above algebraic equations using Mathematica 7 software, we found that:

a) . For $W = A = 1$ (i.e C=0 in eq.(4)) and $\beta_1=0$ we get

$$\left\{ -E = \frac{B^2}{4} - \frac{n^2p^2}{4q^2k^2(n+1)^2}, \alpha_0 = \frac{1}{2} - \frac{qkB(n+1)}{2pn}, \alpha_1 = \frac{qk(n+1)}{pn}, \quad \omega = -\frac{k(qr(n+1)^2+p^2)}{p(n+1)} \right\} \tag{22}$$

Thus using eq. (9) - (13) and eq. (22) we found the following solutions of eq. (19)

Case1: When $B \neq 0$ and $\Omega = B + 4EW > 0,$ then

$$F = \left(\frac{G'}{G}\right) = \frac{B}{2} + \frac{\sqrt{\Omega}}{2}\frac{c_1 \sinh\left(\frac{\sqrt{\Omega}}{2}\eta\right) + c_2 \cosh\left(\frac{\sqrt{\Omega}}{2}\eta\right)}{c_1 \cos\left(\frac{\sqrt{\Omega}}{2}\eta\right) + c_2 \sinh\left(\frac{\sqrt{\Omega}}{2}\eta\right)}$$

$$V_1(\eta) = \alpha_0 + \alpha_1\left(\frac{G'}{G}\right)$$

$$V_1(\eta) = \frac{qk(n+1)}{pn}\left(\frac{n}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\frac{c_1 \sinh\left(\frac{n}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\eta\right) + c_2 \cosh\left(\frac{n}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)}{c_1 \cosh\left(\frac{n}{2(n+1)}\eta\right) + c_2 \sinh\left(\frac{n}{2(n+1)}\eta\right)} + \frac{B}{2}\right) + \frac{1}{2} - \frac{qkB(n+1)}{2pn}$$

$$=\left(\frac{qk}{2p}\sqrt{\frac{p^2}{q^2k^2}}\,\frac{c_1\sinh\left(\frac{n}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)+c_2\cosh\left(\frac{n}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)}{c_1\cosh\left(\frac{n}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)+c_2\sinh\left(\frac{n}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)}\right)+\frac{1}{2} \qquad (23)$$

Case 2: When B $\neq$ 0 and $\Omega = B + 4EW < 0$, then

$$F=\left(\frac{G'}{G}\right)=\frac{B}{2}+\frac{\sqrt{-\Omega}}{2}\,\frac{-c_1\sin\left(\frac{\sqrt{-\Omega}}{2}\eta\right)+c_2\cos\left(\frac{\sqrt{-\Omega}}{2}\eta\right)}{c_1\cos\left(\frac{\sqrt{-\Omega}}{2}\eta\right)+c_2\sin\left(\frac{\sqrt{-\Omega}}{2}\eta\right)}$$

$$V_2(\eta)=\alpha_0+\alpha_1\left(\frac{G'}{G}\right)$$

$$V_2(\eta)=\frac{qk(n+1)}{pn}\left(\frac{ni}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\,\frac{-c_1\sin\left(\frac{ni}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)+c_2\cos\left(\frac{ni}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)}{c_1\cos\left(\frac{n}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)+c_2\sin\left(\frac{n}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)}+\frac{B}{2}\right)+$$

$$\frac{1}{2}-\frac{qkB(n+1)}{2pn}$$

$$=\left(\frac{qki}{2p}\sqrt{\frac{p^2}{q^2k^2}}\,\frac{-c_1\sin\left(\frac{ni}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)+c_2\cos\left(\frac{ni}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)}{c_1\cos\left(\frac{ni}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)+c_2\sin\left(\frac{ni}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)}\right)+\frac{1}{2} \qquad (24)$$

Case 3: When B = 0 and $\Delta = EW > 0$, then

$$F=\left(\frac{G'}{G}\right)=\sqrt{\Delta}\,\frac{c_1\sinh(\sqrt{\Delta})+c_2\cosh(\sqrt{\Delta})}{c_1\cosh(\sqrt{\Delta})+c_2\sinh(\sqrt{\Delta})}$$

$$V_3(\eta)=\alpha_0+\alpha_1\left(\frac{G'}{G}\right)$$

$$V_3(\eta)=\frac{qk(n+1)}{pn}\left(\frac{n}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\,\frac{c_1\sinh\left(\frac{n}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)+c_2\cosh\left(\frac{n}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)}{c_1\cosh\left(\frac{n}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)+c_2\sinh\left(\frac{n}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)}\right)+\frac{1}{2}$$

$$=\left(\frac{qk}{2p}\sqrt{\frac{p^2}{q^2k^2}}\,\frac{c_1\sinh\left(\frac{n}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)+c_2\cosh\left(\frac{n}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)}{c_1\cosh\left(\frac{n}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)+c_2\sinh\left(\frac{n}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)}\right)+\frac{1}{2} \qquad (25)$$

$$\therefore V_3(\eta)=V_1(\eta) \qquad (26)$$

Similarly, Case 4: When B = 0 and $\Delta = EW < 0$, then

we have $V_4(\eta)=V_2(\eta)$ (27)

Where, $\eta=kx-\frac{k(qr(n+1)^2+p^2)}{p(n+1)}t$, k, c_1 & c_2 are arbitrary constants and $i^2=-1$

Case 5: When B $\neq$ 0 and $\Omega = 4EW=0$, then

$$F=\left(\frac{G'}{G}\right)=\frac{B}{2}+\frac{c_2}{c_1+c_2\eta}$$

$$V_5(\eta)=\alpha_0+\alpha_1\left(\frac{G'}{G}\right)$$

$$=\frac{qk(n+1)}{pn}\left(\pm\frac{n}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}+\frac{c_2}{c_1+c_2\eta}\right)+\frac{1}{2}\mp\frac{qk}{2p}\sqrt{\frac{p^2}{q^2k^2}} \tag{28}$$

Some exact traveling wave solutions of eq. 23, 24 & 28

Now, by setting $c_2=0$ & $c_1\neq 0$ in eq.23, 24 & 28 we found respectively:

$$V_{1_1}(\eta)=\left(\frac{qk}{2p}\sqrt{\frac{p^2}{q^2k^2}}\tanh\left(\frac{n}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)\right)+\frac{1}{2} \tag{29}$$

$$V_{2_1}(\eta)=\left(\frac{-qki}{2p}\sqrt{\frac{p^2}{q^2k^2}}\tan\left(\frac{ni}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)\right)+\frac{1}{2} \tag{30}$$

$$V_{5_1}(\eta)=\frac{qk(n+1)}{p}\left(\pm\frac{n}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\right)+\frac{1}{2}\mp\frac{qk}{2p}\sqrt{\frac{p^2}{q^2k^2}} \tag{31}$$

Where, $\eta=kx-\dfrac{k(qr(n+1)^2+p^2)}{p(n+1)}t$, k is arbitrary constant, $i^2=-1$

Again by setting $c_2\neq 0$ & $c_1=0$ in eq. 24, 25 & 28 then we found respectively:

$$V_{1_2}(\eta)=\left(\frac{qk}{2p}\sqrt{\frac{p^2}{q^2k^2}}\coth\left(\frac{n}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)\right)+\frac{1}{2} \tag{32}$$

$$V_{2_2}(\eta)=\left(\frac{qki}{2p}\sqrt{\frac{p^2}{q^2k^2}}\cot\left(\frac{ni}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)\right)+\frac{1}{2} \tag{33}$$

$$V_{5_2}(\eta)=\frac{qk(n+1)}{pn}\left(\pm\frac{n}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\right)+\frac{1}{2}\mp\frac{qk}{2p}\sqrt{\frac{p^2}{q^2k^2}} \tag{34}$$

Where, $\eta=kx-\frac{k(qr(n+1)^2+p^2)}{p(n+1)}t$, k is arbitrary constant, $i^2=-1$

In particular, for $n=\frac{3}{2}$

$$V_{1_1}(\eta)=\left(\frac{qk}{2p}\sqrt{\frac{p^2}{q^2k^2}}\tanh\left(\frac{3}{10}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)\right)+\frac{1}{2}$$

$$V_{1_2}(\eta)=\left(\frac{qk}{2p}\sqrt{\frac{p^2}{q^2k^2}}\tan\left(\frac{3i}{10}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)\right)+\frac{1}{2}$$

$$V_{2_1}(\eta)=\left(\frac{qk}{2p}\sqrt{\frac{p^2}{q^2k^2}}\coth\left(\frac{3}{10}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)\right)+\frac{1}{2}$$

b). when $\alpha_0=0$ we found that

$$\alpha_1=\frac{qkW(n+1)}{Anp},\beta_1=\frac{qkE(n+1)}{Anp}$$

$$\begin{aligned}&\omega=(kq(A^7BE(B+E)kn^5p^{10}-A^5Ekn^3p^7(2B^2k(1+n)q-ABEkn(-7+n+8n^2)p^2q+\\&A^2Ep(-4(-1+n)n^2p^2+3(n+n^2)^2qr))W+Anp^2(k(-A^5E(-1+n)n^3p^6+\\&k(1+n)(B^2+ABE(7-8n)np^2-7A^4E^2(-1+n)n^2p^5)q)+\\&A^2n^2(1+n)(1+Ek(1+n)pq(-3+8B^2Ek^3n(1+n)p^2q^2))r)W^2-\\&k(1+n)q(kp(B-4AE(-1+n)np)+An(1+n)r)W^3))/\\&(Anp(A^7E(B+E)n^5p^9-A^6E^2kn^4(1+n)p^8qW-Akn(1+n)p(B+AEnp^2)qW^2+k(1+n)qW^3))\end{aligned}$$

Case1: When B $\neq$ 0 and Ω = B+ 4EW > 0, then

$$F=\left(\frac{G'}{G}\right)=\frac{B}{2W}+\frac{\sqrt{\Omega}}{2W}\frac{c_1\sinh\left(\frac{\sqrt{\Omega}}{2A}\eta\right)+c_2\cosh\left(\frac{\sqrt{\Omega}}{2A}\eta\right)}{c_1\cos\left(\frac{\sqrt{\Omega}}{2A}\eta\right)+c_2\sinh\left(\frac{\sqrt{\Omega}}{2A}\eta\right)}$$

$$V(\eta)=\alpha_1\left(\frac{G'}{G}\right)+\beta_1\left(\frac{G'}{G}\right)^{-1}$$

$$\mathrm{V}_1(\eta)=\alpha_1\left(\frac{B}{2W}+\frac{\sqrt{\Omega}}{2W}\frac{c_1\sinh\left(\frac{\sqrt{\Omega}}{2A}\eta\right)+c_2\cosh\left(\frac{\sqrt{\Omega}}{2A}\eta\right)}{c_1\cosh\left(\frac{\sqrt{\Omega}}{2A}\eta\right)+c_2\sinh\left(\frac{\sqrt{\Omega}}{2A}\eta\right)}\right)+$$

$$\beta_1\left(\frac{B}{2W}+\frac{\sqrt{\Omega}}{2W}\frac{c_1\sinh\left(\frac{\sqrt{\Omega}}{2A}\eta\right)+c_2\cosh\left(\frac{\sqrt{\Omega}}{2A}\eta\right)}{c_1\cosh\left(\frac{\sqrt{\Omega}}{2A}\eta\right)+c_2\sinh\left(\frac{\sqrt{\Omega}}{2A}\eta\right)}\right)^{-1}$$

Case 2: When $B \neq 0$ and $\Omega = B + 4EW < 0$, then

$$F=\left(\frac{G'}{G}\right)=\frac{B}{2W}+\frac{\sqrt{-\Omega}}{2W}\frac{-c_1\sin\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)+c_2\cos\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)}{c_1\cos\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)+c_2\sin\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)}$$

$$V(\eta)=\alpha_1\left(\frac{G'}{G}\right)+\beta_1\left(\frac{G'}{G}\right)^{-1}$$

$$V_2(\eta)=\alpha_1\left(\frac{B}{2W}+\frac{\sqrt{\Omega}}{2W}\frac{-c_1\sin\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)+c_2\cos\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)}{c_1\cos\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)+c_2\sin\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)}\right)+$$

$$\beta_1\left(\frac{B}{2W}+\frac{\sqrt{\Omega}}{2W}\frac{-c_1\sin\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)+c_2\cos\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)}{c_1\cos\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)+c_2\sin\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)}\right)^{-1}$$

Case 3: When B = 0 and $\Omega = E$W > 0, then

$$F=\left(\frac{G'}{G}\right)=\frac{\sqrt{\Delta}}{W}\frac{c_1\sinh\left(\frac{\sqrt{\Delta}}{A}\right)+c_2\cosh\left(\frac{\sqrt{\Delta}}{A}\right)}{c_1\cosh\left(\frac{\sqrt{\Delta}}{A}\right)+c_2\sinh\left(\frac{\sqrt{\Delta}}{A}\right)}$$

$$V(\eta)=\alpha_1\left(\frac{G'}{G}\right)+\beta_1\left(\frac{G'}{G}\right)^{-1}$$

$$V_3(\eta)=\alpha_1\left(\frac{\sqrt{\Omega}}{W}\frac{c_1\sinh\left(\frac{\sqrt{\Omega}}{A}\eta\right)+c_2\cosh\left(\frac{\sqrt{\Omega}}{A}\eta\right)}{c_1\cos\left(\frac{\sqrt{\Omega}}{A}\eta\right)+c_2\sinh\left(\frac{\sqrt{\Omega}}{A}\eta\right)}\right)+$$

$$\beta_1\left(\frac{\sqrt{\Omega}}{W}\frac{c_1\sinh\left(\frac{\sqrt{\Omega}}{A}\eta\right)+c_2\cosh\left(\frac{\sqrt{\Omega}}{A}\eta\right)}{c_1\cos\left(\frac{\sqrt{\Omega}}{A}\eta\right)+c_2\sinh\left(\frac{\sqrt{\Omega}}{A}\eta\right)}\right)^{-1}$$

Where, $\eta=kx-t,$ k is arbitrary constant, $i^2=-1$

Similarly, Case 4: When B = 0 and $\Omega = \mathrm{EW} < 0$, then

$$F=\left(\frac{G'}{G}\right)=\frac{\sqrt{-\Omega}}{W}\frac{-c_1\sin\left(\frac{\sqrt{-\Omega}}{A}\eta\right)+c_2\cos\left(\frac{\sqrt{-\Omega}}{A}\eta\right)}{c_1\cos\left(\frac{\sqrt{-\Omega}}{A}\eta\right)+c_2\sin\left(\frac{\sqrt{-\Omega}}{A}\eta\right)}$$

$$V(\eta)=\alpha_1\left(\frac{G'}{G}\right)+\beta_1\left(\frac{G'}{G}\right)^{-1}$$

$$V_4(\eta)=\alpha_1\left(\frac{\sqrt{-\Omega}}{W}\frac{-c_1\sin\left(\frac{\sqrt{-\Omega}}{A}\eta\right)+c_2\cos\left(\frac{\sqrt{-\Omega}}{A}\eta\right)}{c_1\cos\left(\frac{\sqrt{-\Omega}}{A}\eta\right)+c_2\sin\left(\frac{\sqrt{-\Omega}}{A}\eta\right)}\right)+$$

$$\beta_1\left(\frac{\sqrt{\Omega}}{W}\frac{-c_1\sin\left(\frac{\sqrt{-\Omega}}{A}\eta\right)+c_2\cos\left(\frac{\sqrt{-\Omega}}{A}\eta\right)}{c_1\cos\left(\frac{\sqrt{-\Omega}}{A}\eta\right)+c_2\sin\left(\frac{\sqrt{-\Omega}}{A}\eta\right)}\right)^{-1}$$

Case 5: When B $\neq$ 0 and $\Omega = \mathrm{EW}=0$, then

$$F=\left(\frac{G'}{G}\right)=\frac{B}{2}+\frac{c_2}{c_1+c_2\eta}$$

$$V(\eta)=\alpha_1\left(\frac{G'}{G}\right)+\beta_1\left(\frac{G'}{G}\right)^{-1}$$

$$V(\eta)=\alpha_1\left(\frac{B}{2}+\frac{c_2}{c_1+c_2\eta}\right) \text{ ,if } E=0 \text{ \& W} \neq 0$$

$$V(\eta)=\beta_1\left(\frac{B}{2}+\frac{c_2}{c_1+c_2\eta}\right)^{-1} \text{ ,if } E\neq 0 \text{ \& W=0}$$

$$V_5(\eta)=\alpha_1\left(\frac{B}{2}+\frac{c_2}{c_1+c_2\eta}\right) \text{ ,if } E=0 \text{ \& W} \neq 0$$

$$V_5(\eta)=\beta_1\left(\frac{B}{2}+\frac{c_2}{c_1+c_2\eta}\right)^{-1} \text{ ,if } E\neq 0 \text{ \& W=0}$$

Where $\eta = kx+\omega t$

Note: - for case 5 if $W=0\,\&E\neq 0\, then\, \alpha_1=0; if\ E=0\ \&W=0\, then\, \beta_1=0, and$ the value of ω is Changed accordingly.

:- for cases 3 &4 the value of B is zero in ω

Some exact traveling wave solutions of w1-w5 in above solutions by setting

$c_2=0\&c_1\neq 0$ *and* $c_2\neq 0\&c_1=0$ *respectively, we found that* :

$$V_{1,a}(\eta)=\alpha_1\left(\frac{B}{2W}+\frac{\sqrt{\Omega}}{2W}\tanh\left(\frac{\sqrt{\Omega}}{2A}\eta\right)\right)+\beta_1\left(\frac{B}{2W}+\frac{\sqrt{\Omega}}{2W}\tanh\left(\frac{\sqrt{\Omega}}{2A}\eta\right)\right)^{-1}$$

$$V_{1,b}(\eta)=\alpha_1\left(\frac{B}{2W}+\frac{\sqrt{\Omega}}{2W}\coth\left(\frac{\sqrt{\Omega}}{2A}\eta\right)\right)+\beta_1\left(\frac{B}{2W}+\frac{\sqrt{\Omega}}{2W}\coth\left(\frac{\sqrt{\Omega}}{2A}\eta\right)\right)^{-1}$$

$$V_{2,a}(\eta)=\alpha_1\left(\frac{B}{2W}-\frac{\sqrt{-\Omega}}{2W}\tan\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)\right)+\beta_1\left(\frac{B}{2W}-\frac{\sqrt{-\Omega}}{2W}\tan\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)^{-1}\right)$$

$$V_{2,b}(\eta)=\alpha_1\left(\frac{B}{2W}+\frac{\sqrt{-\Omega}}{2W}\cot\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)\right)+\beta_1\left(\frac{B}{2W}+\frac{\sqrt{-\Omega}}{2W}\cot\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)\right)^{-1}$$

$$V_{3,a}(\eta)=\alpha_1\left(\frac{\sqrt{\Omega}}{W}\tanh\left(\frac{\sqrt{\Omega}}{A}\eta\right)\right)+\beta_1\left(\frac{\sqrt{\Omega}}{W}\tanh\left(\frac{\sqrt{\Omega}}{A}\eta\right)\right)^{-1}$$

$$V_{3,b}(\eta)=\alpha_1\left(\frac{\sqrt{\Omega}}{W}\coth\left(\frac{\sqrt{\Omega}}{A}\eta\right)\right)+\beta_1\left(\frac{\sqrt{\Omega}}{W}\coth\left(\frac{\sqrt{\Omega}}{A}\eta\right)\right)^{-1}$$

$$V_{4,a}(\eta)=\alpha_1\left(-\frac{\sqrt{-\Omega}}{W}\tan\left(\frac{\sqrt{-\Omega}}{A}\eta\right)\right)+\beta_1\left(-\frac{\sqrt{-\Omega}}{W}\tan\left(\frac{\sqrt{-\Omega}}{A}\eta\right)\right)^{-1}$$

$$\mathrm{V}_{4,b}(\eta)=\alpha_1\left(\frac{\sqrt{\Omega}}{W}\cot\left(\frac{\sqrt{\Omega}}{A}\eta\right)\right)+\beta_1\left(\frac{\sqrt{\Omega}}{W}\cot\left(\frac{\sqrt{\Omega}}{A}\eta\right)\right)^{-1}$$

$$\mathrm{V}_{5,a_1}(\eta)=\alpha_1\left(\frac{B}{2}\right),\ \mathrm{c}_2=0\ \&\ \mathrm{c}_1\neq 0,\mathrm{E}=0\ \&\ \mathrm{W}\neq 0$$

$$\mathrm{V}_{5,a_2}(\eta)=\alpha_1\left(\frac{B}{2}+\frac{1}{\eta}\right),\mathrm{c}_2\neq 0\ \&\ \mathrm{c}_1=0,\mathrm{E}=0\ \&\ \mathrm{W}\neq 0$$

$$\mathrm{V}_{5,b_1}(\eta)=\beta_1\left(\frac{B}{2}\right)^{-1},\ \mathrm{c}_2=0\ \&\ \mathrm{c}_1\neq 0,\mathrm{E}\neq 0\ \&\ \mathrm{W}=0$$

$$\mathrm{V}_{5,b_2}(\eta)=\beta_1\left(\frac{B}{2}+\frac{1}{\eta}\right)^{-1},\mathrm{c}_2\neq 0\ \&\ \mathrm{c}_1=0,\mathrm{E}\neq 0\ \&\ \mathrm{W}=0$$

c) When $\alpha_1=0$ we found that:

Kind 1

$$\left\{\begin{array}{l}\alpha_0=-\dfrac{(1+n)(-BEk^2q+An(Anr+E\lambda_1))}{2AEknp}\\ \beta=\dfrac{Ekq(n+1)}{Anp},\quad \alpha_1=0\\ \omega=\lambda_1\qquad \text{(see }\lambda_1\text{ in appendix I)}\end{array}\right\}$$

Kind 2

$$\begin{cases} \alpha_0 = -\dfrac{(1+n)(-BEk^2q + An(Anr + E\lambda_2))}{2AEknp} \\ \beta = \dfrac{Ekq(n+1)}{Anp}, \quad \alpha_1 = 0 \\ \omega = \lambda_2 \quad \text{(see } \lambda_2 \text{ in appendix I)} \end{cases}$$

Kind 3

$$\begin{cases} \alpha_0 = -\dfrac{(1+n)(-BEk^2q + An(Anr + E\lambda_3))}{2AEknp} \\ \beta = \dfrac{Ekq(n+1)}{Anp}, \quad \alpha_1 = 0 \\ \omega = \lambda_3 \quad \text{(see } \lambda_3 \text{ in appendix I)} \end{cases}$$

REMARK: $E \neq 0$ **in Kind1-Kind3**

In this case, using eq. (9-13) we found solutions of eq. (19) as follows:

Using **Kind 1**

$$\begin{cases} \alpha_0 = \alpha_0 = -\dfrac{(1+n)(-BEk^2q + An(Anr + E\lambda_1))}{2AEknp} \\ \beta = \dfrac{Ekq(n+1)}{Anp}, \quad \alpha_1 = 0 \\ \omega = \lambda_1 \quad \text{(see } \lambda_1 \text{ in appendix I)} \end{cases}$$

Case1: When $B \neq 0$ and $\Omega = B + 4EW > 0$, then

$$F=\left(\frac{G'}{G}\right)=\frac{B}{2W}+\frac{\sqrt{\Omega}}{2W}\frac{c_1\sinh\left(\frac{\sqrt{\Omega}}{2A}\eta\right)+c_2\cosh\left(\frac{\sqrt{\Omega}}{2A}\eta\right)}{c_1\cos\left(\frac{\sqrt{\Omega}}{2A}\eta\right)+c_2\sinh\left(\frac{\sqrt{\Omega}}{2A}\eta\right)}$$

$$V1(\eta)=\alpha_0+\beta_1\left(\frac{G'}{G}\right)^{-1}$$

$$V_1(\eta)=\alpha_0+\beta_1\left(\frac{B}{2W}+\frac{\sqrt{\Omega}}{2W}\frac{c_1\sinh\left(\frac{\sqrt{\Omega}\eta}{2A}\right)+c_2\cosh\left(\frac{\sqrt{\Omega}\eta}{2A}\right)}{c_1\cosh\left(\frac{\sqrt{\Omega}\eta}{2A}\right)+c_2\sinh\left(\frac{\sqrt{\Omega}\eta}{2A}\right)}\right)^{-1}$$

Case 2: When B $\neq$ 0 and Ω = B+ 4EW > 0, then

$$F=\left(\frac{G'}{G}\right)=\frac{B}{2W}+\frac{\sqrt{-\Omega}}{2W}\frac{-c_1\sin\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)+c_2\cos\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)}{c_1\cos\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)+c_2\sin\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)}$$

$$V(\eta)=\alpha_0+\beta_1\left(\frac{G'}{G}\right)^{-1}$$

$$V_2(\eta)=\alpha_0+\beta_1\left(\frac{B}{2W}+\frac{\sqrt{-\Omega}}{2W}\frac{-c_1\sin\left(\frac{\sqrt{-\Omega}\eta}{2A}\right)+c_2\cos\left(\frac{\sqrt{-\Omega}\eta}{2A}\right)}{c_1\cos\left(\frac{\sqrt{-\Omega}\eta}{2A}\right)+c_2\sin\left(\frac{\sqrt{-\Omega}\eta}{2A}\right)}\right)^{-1}$$

where $\eta=kx+\omega t$, $\omega=\lambda_1$ in(w(η)-w2(η))

Case 3: When B = 0 and Δ = EW > 0, then

$$F=\left(\frac{G'}{G}\right)=\frac{\sqrt{\Delta}}{W}\frac{c_1\sinh\left(\frac{\sqrt{\Delta}}{A}\right)+c_2\cosh\left(\frac{\sqrt{\Delta}}{A}\right)}{c_1\cosh\left(\frac{\sqrt{\Delta}}{A}\right)+c_2\sinh\left(\frac{\sqrt{\Delta}}{A}\right)}$$

$$V(\eta)=\alpha_0+\beta_1\left(\frac{G'}{G}\right)^{-1}$$

$$V_3(\eta)=\alpha_0+\beta_1\left(\frac{\sqrt{\Delta}}{W}\frac{c_1\sinh\left(\frac{\sqrt{\Delta}\eta}{A}\right)+c_2\cosh\left(\frac{\sqrt{\Delta}\eta}{A}\right)}{c_1\cosh\left(\frac{\sqrt{\Delta}\eta}{A}\right)+c_2\sinh\left(\frac{\sqrt{\Delta}\eta}{A}\right)}\right)^{-1}$$

Similarly, Case 4: When B = 0 and $\Delta = EW < 0$, then

$$F=\left(\frac{G'}{G}\right)=\frac{\sqrt{-\Delta}}{W}\frac{-c_1\sin\left(\frac{\sqrt{-\Delta}}{A}\eta\right)+c_2\cos\left(\frac{\sqrt{-\Delta}}{A}\eta\right)}{c_1\cos\left(\frac{\sqrt{-\Delta}}{A}\eta\right)+c_2\sin\left(\frac{\sqrt{-\Delta}}{A}\eta\right)}$$

$$V(\eta)=\alpha_0+\beta_1\left(\frac{G'}{G}\right)^{-1}$$

$$V_4(\eta)=\alpha_0+\beta_1\left(\frac{\sqrt{-\Omega}}{W}\frac{-c_1\sin\left(\frac{\sqrt{-\Omega}\eta}{A}\right)+c_2\cos\left(\frac{\sqrt{-\Omega}\eta}{A}\right)}{c_1\cos\left(\frac{\sqrt{-\Omega}\eta}{A}\right)+c_2\sin\left(\frac{\sqrt{-\Omega}\eta}{A}\right)}\right)^{-1}$$

where $\eta=kx+\omega t$, $\omega=\lambda_1, B=0$ in α_0 & λ_1 for (w3 (η) & w4 (η))

Case 5: When B $\neq$ 0 and $\Omega = EW=0$, then

$$F=\left(\frac{G'}{G}\right)=\frac{B}{2}+\frac{c_2}{c_1+c_2\eta}$$

$$V(\eta)=\alpha_0+\beta_1\left(\frac{G'}{G}\right)^{-1}$$

$$V_5(\eta)=\alpha_0+\beta_1\left(\frac{B}{2}+\frac{c_2}{c_1+c_2\eta}\right)^{-1}$$

Where $\eta=kx+\omega t,\ \omega=\lambda_1,\ W=0\ in\lambda_1$

Some exact traveling wave solutions for **Kind 1**

By setting $c_2\neq 0$ & $c_1=0$ and $c_2=0$ & $c_1\neq 0$ we found respectively:

$$V_{1,1'}(\eta)=\alpha_0+\beta_1\left(\frac{B}{2W}+\frac{\sqrt{\Omega}}{2W}\coth\left(\frac{\sqrt{\Omega}\eta}{2A}\right)\right)^{-1}$$

$$V_{1,2'}(\eta)=\alpha_0+\beta_1\left(\frac{B}{2W}+\frac{\sqrt{\Omega}}{2W}\tanh\left(\frac{\sqrt{\Omega}\eta}{2A}\right)\right)^{-1}$$

$$V_{2,1'}(\eta)=\alpha_0+\beta_1\left(\frac{B}{2W}+\frac{\sqrt{-\Omega}}{2W}\cot\frac{\sqrt{-\Omega}\eta}{2A}\right)^{-1}$$

$$V_{2,2'}(\eta)=\alpha_0+\beta_1\left(\frac{B}{2W}-\frac{\sqrt{-\Omega}}{2W}\tan\frac{\sqrt{-\Omega}\eta}{2A}\right)^{-1}$$

$$\mathrm{V}_{3,1'}(\eta)=\alpha_0+\beta_1\left(\frac{\sqrt{\Delta}}{W}\coth\left(\frac{\sqrt{\Delta}\eta}{A}\right)\right)^{-1}$$

$$\mathrm{V}_{3,2'}(\eta)=\alpha_0+\beta_1\left(\frac{\sqrt{\Delta}}{W}\tanh\left(\frac{\sqrt{\Delta}\eta}{A}\right)\right)^{-1}$$

$$V_{4,1'}(\eta)=\alpha_0+\beta_1\left(\frac{\sqrt{-\Omega}}{W}\cot\left(\frac{\sqrt{-\Omega}\eta}{A}\right)\right)^{-1}$$

$$V_{4,2'}(\eta)=\alpha_0+\beta_1\left(-\frac{\sqrt{-\Omega}}{W}\tan\left(\frac{\sqrt{-\Omega}\eta}{A}\right)\right)^{-1}$$

$$V_{5,1'}(\eta)=\alpha_0+\beta_1\left(\frac{B}{2}+\frac{1}{\eta}\right)^{-1}$$

$$V_{5,2'}(\eta)=\alpha_0+\beta_1\left(\frac{B}{2}\right)^{-1}$$

Using **Kind 2:**

$$\left\{\begin{array}{l}\alpha_0=\alpha_0=-\dfrac{(1+n)(-BEk^2q+An(Anr+E\lambda_2))}{2AEknp}\\ \beta=\dfrac{Ekq(n+1)}{Anp},\quad \alpha_1=0\\ \omega=\lambda_2\qquad \text{(see } \lambda_2 \text{ in appendix II)}\end{array}\right\}$$

Again by eq. (9)-(13) and for simplicity substituting

λ_2 by λ_1 in **Kind 1** we found the following exact traveling wave solutions

Now, by setting $c_2\neq 0$ & $c_1=0$ *and* $c_2=0$ & $c_1\neq 0$ *we had respectively*:

$$V_{1,1}(\eta)=\alpha_0+\beta_1\left(\frac{B}{2W}+\frac{\sqrt{\Omega}}{2W}\coth\left(\frac{\sqrt{\Omega}\eta}{2A}\right)\right)^{-1}$$

$$V_{1,2}(\eta)=\alpha_0+\beta_1\left(\frac{B}{2W}+\frac{\sqrt{\Omega}}{2W}\tanh\left(\frac{\sqrt{\Omega}\eta}{2A}\right)\right)^{-1}$$

$$V_{2,1}(\eta)=\alpha_0+\beta_1\left(\frac{B}{2W}+\frac{\sqrt{-\Omega}}{2W}\cot\frac{\sqrt{-\Omega}\eta}{2A}\right)^{-1}$$

$$V_{2,2}(\eta)=\alpha_0+\beta_1\left(\frac{B}{2W}-\frac{\sqrt{-\Omega}}{2W}\tan\frac{\sqrt{-\Omega}\eta}{2A}\right)^{-1}$$

where $\eta=kx+\omega t$,$\omega=\lambda_2$,

$$V_{3,1}(\eta)=\alpha_0+\beta_1\left(\frac{\sqrt{\Delta}}{W}\coth\left(\frac{\sqrt{\Delta}\eta}{A}\right)\right)^{-1}$$

$$V_{3,2}(\eta)=\alpha_0+\beta_1\left(\frac{\sqrt{\Delta}}{W}\tanh\left(\frac{\sqrt{\Delta}\eta}{A}\right)\right)^{-1}$$

$$\mathrm{V}_{4,1}(\eta)=\alpha_0+\beta_1\left(\frac{\sqrt{-\Omega}}{W}\cot\left(\frac{\sqrt{-\Omega}\eta}{A}\right)\right)^{-1}$$

$$\mathrm{V}_{4,2}(\eta)=\alpha_0+\beta_1\left(-\frac{\sqrt{-\Omega}}{W}\tan\left(\frac{\sqrt{-\Omega}\eta}{A}\right)\right)^{-1}$$

Where η=kx+ωt, $\omega=\lambda_2$, B=0 in α_0 & λ_2 in $(w_{3,1}(\eta)-w_{4,2}(\eta))$

$$\mathrm{V}_{5,1}(\eta)=\alpha_0+\beta_1\left(\frac{B}{2}+\frac{1}{\eta}\right)^{-1}$$

$$\mathrm{V}_{5,2}(\eta)=\alpha_0+\beta_1\left(\frac{B}{2}\right)^{-1}$$

Where η=kx+ωt, $\omega=\lambda_2$, W=0 in λ_2

Using **Kind 3:**

$$\left\{\begin{array}{l}\alpha_0 = -\dfrac{(1+n)(-BEk^2q + An(Anr + E\lambda_3))}{2AEknp} \\ \beta = \dfrac{Ekq(n+1)}{Anp}, \quad \alpha_1 = 0 \\ \omega = \lambda_3 \quad \text{(see } \lambda_3 \text{ in appendix I)} \\ \end{array}\right\}$$

Again by eq. (9)-(13) and for simplicity substituting

λ_3 by λ_1 in **Kind 1** we found the following exact traveling wave solutions

by setting $c_2 \neq 0$ & $c_1 = 0$ *and* $c_2 = 0$ & $c_1 \neq 0$ *we had respectively*:

$$V_{1_{1'}}(\eta) = \alpha_0 + \beta_1\left(\frac{B}{2W} + \frac{\sqrt{\Omega}}{2W}\coth\left(\frac{\sqrt{\Omega}\eta}{2A}\right)\right)^{-1}$$

$$V_{1_{2'}}(\eta) = \alpha_0 + \beta_1\left(\frac{B}{2W} + \frac{\sqrt{\Omega}}{2W}\tanh\left(\frac{\sqrt{\Omega}\eta}{2A}\right)\right)^{-1}$$

$$V_{2_{1'}}(\eta) = \alpha_0 + \beta_1\left(\frac{B}{2W} + \frac{\sqrt{-\Omega}}{2W}\cot\frac{\sqrt{-\Omega}\eta}{2A}\right)^{-1}$$

$$V_{2_{2'}}(\eta) = \alpha_0 + \beta_1\left(\frac{B}{2W} - \frac{\sqrt{-\Omega}}{2W}\tan\frac{\sqrt{-\Omega}\eta}{2A}\right)^{-1}$$

$$V_{3_{1'}}(\eta) = \alpha_0 + \beta_1\left(\frac{\sqrt{\Delta}}{W}\coth\left(\frac{\sqrt{\Delta}\eta}{A}\right)\right)^{-1}$$

$$V_{3_{2'}}(\eta) = \alpha_0 + \beta_1\left(\frac{\sqrt{\Delta}}{W}\tanh\left(\frac{\sqrt{\Delta}\eta}{A}\right)\right)^{-1}$$

$$V_{4_{1'}}(\eta)=\alpha_0+\beta_1\left(\frac{\sqrt{-\Omega}}{W}\cot\left(\frac{\sqrt{-\Omega}\eta}{A}\right)^{-1}\right) \qquad V_{5_{1'}}(\eta)=\alpha_0+\beta_1\left(\frac{B}{2}+\frac{1}{\eta}\right)^{-1}$$

$$V_{4_{2'}}(\eta)=\alpha_0+\beta_1\left(-\frac{\sqrt{-\Omega}}{W}\tan\left(\frac{\sqrt{-\Omega}\eta}{A}\right)\right)^{-1} \qquad V_{5_{2'}}(\eta)=\alpha_0+\beta_1\left(\frac{B}{2}\right)^{-1}$$

4.1.4 More General Solution OF The generalized Burger-Fisher's equation as compared to the solution obtained in the existing literature [3]

Solving the algebraic equation for α_0, α_1, and β_1 we found

Case1: When $B \neq 0$ and $\Omega = B+4EW>0$, then

$$F=\left(\frac{G'}{G}\right)=\frac{B}{2W}+\frac{\sqrt{\Omega}}{2W}\frac{c_1\sinh\left(\frac{\sqrt{\Omega}}{2A}\eta\right)+c_2\cosh\left(\frac{\sqrt{\Omega}}{2A}\eta\right)}{c_1\cos\left(\frac{\sqrt{\Omega}}{2A}\eta\right)+c_2\sinh\left(\frac{\sqrt{\Omega}}{2A}\eta\right)}$$

$$V(\eta)=\alpha_0+\alpha_1\left(\frac{G'}{G}\right)+\beta_1\left(\frac{G'}{G}\right)^{-1}$$

$$V_1(\eta)=\alpha_0+\alpha_1\left(\frac{B}{2W}+\frac{\sqrt{\Omega}}{2W}\frac{c_1\sinh\left(\frac{\sqrt{\Omega}}{2A}\eta\right)+c_2\cosh\left(\frac{\sqrt{\Omega}}{2A}\eta\right)}{c_1\cos\left(\frac{\sqrt{\Omega}}{2A}\eta\right)+c_2\sinh\left(\frac{\sqrt{\Omega}}{2A}\eta\right)}\right)+$$
$$\beta_1\left(\frac{B}{2W}+\frac{\sqrt{\Omega}}{2W}\frac{c_1\sinh\left(\frac{\sqrt{\Omega}}{2A}\eta\right)+c_2\cosh\left(\frac{\sqrt{\Omega}}{2A}\eta\right)}{c_1\cos\left(\frac{\sqrt{\Omega}}{2A}\eta\right)+c_2\sinh\left(\frac{\sqrt{\Omega}}{2A}\eta\right)}\right)^{-1}$$

Case 2: When $B \neq 0$ and $\Omega = B+4EW<0$, then

$$F=\left(\frac{G'}{G}\right)=\frac{B}{2W}+\frac{\sqrt{-\Omega}}{2W}\frac{-c_1\sin\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)+c_2\cos\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)}{c_1\cos\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)+c_2\sin\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)}$$

$$V(\eta)=\alpha_0+\alpha_1\left(\frac{G'}{G}\right)+\beta_1\left(\frac{G'}{G}\right)^{-1}$$

$$V_2(\eta)=\alpha_0+\alpha_1\left(\frac{B}{2W}+\frac{\sqrt{-\Omega}}{2W}\frac{-c_1\sin\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)+c_2\cos\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)}{c_1\cos\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)+c_2\sin\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)}\right)+$$

$$\beta_1\left(\frac{B}{2W}+\frac{\sqrt{-\Omega}}{2W}\frac{-c_1\sin\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)+c_2\cos\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)}{c_1\cos\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)+c_2\sin\left(\frac{-\sqrt{\Omega}}{2A}\eta\right)}\right)^{-1}$$

Case 3: When B = 0 and $\Delta = E\mathrm{W} > 0$, then

$$F=\left(\frac{G'}{G}\right)=\frac{\sqrt{\Delta}}{W}\frac{c_1\sinh\left(\frac{\sqrt{\Delta}}{A}\right)+c_2\cosh\left(\frac{\sqrt{\Delta}}{A}\right)}{c_1\cosh\left(\frac{\sqrt{\Delta}}{A}\right)+c_2\sinh\left(\frac{\sqrt{\Delta}}{A}\right)}$$

$$V(\eta)=\alpha_0+\alpha_1\left(\frac{G'}{G}\right)+\beta_1\left(\frac{G'}{G}\right)^{-1}$$

$$V_3(\eta) = \alpha_0 + \alpha_1 \left(\frac{\sqrt{\Omega}}{W} \frac{c_1 \sinh\left(\frac{\sqrt{\Omega}}{A}\eta\right) + c_2 \cosh\left(\frac{\sqrt{\Omega}}{A}\eta\right)}{c_1 \cos\left(\frac{\sqrt{\Omega}}{A}\eta\right) + c_2 \sinh\left(\frac{\sqrt{\Omega}}{A}\eta\right)} \right) +$$

$$\beta_1 \left(\frac{\sqrt{\Omega}}{W} \frac{c_1 \sinh\left(\frac{\sqrt{\Omega}}{A}\eta\right) + c_2 \cosh\left(\frac{\sqrt{\Omega}}{A}\eta\right)}{c_1 \cos\left(\frac{\sqrt{\Omega}}{A}\eta\right) + c_2 \sinh\left(\frac{\sqrt{\Omega}}{A}\eta\right)} \right)^{-1}$$

Where, $\eta = kx - \frac{k(qr(n+1)^2 + p^2)}{p(n+1)} t$, k is arbitrary constant, $i^2 = -1$

Similarly, Case 4: When B = 0 and $\Omega = EW < 0$, then

$$F = \left(\frac{G'}{G}\right) = \frac{\sqrt{-\Omega}}{W} \frac{-c_1 \sin\left(\frac{\sqrt{-\Omega}}{A}\eta\right) + c_2 \cos\left(\frac{\sqrt{-\Omega}}{A}\eta\right)}{c_1 \cos\left(\frac{\sqrt{-\Omega}}{A}\eta\right) + c_2 \sin\left(\frac{\sqrt{-\Omega}}{A}\eta\right)}$$

$$V(\eta) = \alpha_0 + \alpha_1 \left(\frac{G'}{G}\right) + \beta_1 \left(\frac{G'}{G}\right)^{-1}$$

$$V_4(\eta)=\alpha_0+\alpha_1\left(\frac{\sqrt{-\Omega}}{W}\frac{-c_1\sin\left(\frac{\sqrt{-\Omega}}{A}\eta\right)+c_2\cos\left(\frac{\sqrt{-\Omega}}{A}\eta\right)}{c_1\cos\left(\frac{\sqrt{-\Omega}}{A}\eta\right)+c_2\sin\left(\frac{\sqrt{-\Omega}}{A}\eta\right)}\right)+$$

$$\beta_1\left(\frac{\sqrt{-\Omega}}{W}\frac{-c_1\sin\left(\frac{\sqrt{-\Omega}}{A}\eta\right)+c_2\cos\left(\frac{\sqrt{-\Omega}}{A}\eta\right)}{c_1\cos\left(\frac{\sqrt{-\Omega}}{A}\eta\right)+c_2\sin\left(\frac{\sqrt{-\Omega}}{A}\eta\right)}\right)^{-1}$$

Case 5: When B $\neq$ 0 and Ω = 4EW=0, then

$$F=\left(\frac{G'}{G}\right)=\frac{B}{2}+\frac{c_2}{c_1+c_2\eta}$$

$$V(\eta)=\alpha_0+\alpha_1\left(\frac{G'}{G}\right)+\beta_1\left(\frac{G'}{G}\right)^{-1}$$

$$V_5(\eta)=\alpha_0+\alpha_1\left(\frac{B}{2}+\frac{c_2}{c_1+c_2\eta}\right)+\beta_1\left(\frac{B}{2}+\frac{c_2}{c_1+c_2\eta}\right)^{-1}$$

4.1.5 Comparison of the solutions obtained by Abdollah Boharfani and Reza Abazari [3] and our solutions.

Table4.1. Comparison of the solutions of Abdollah Boharfani and Reza Abazari [3] and our solutions.

Solutions by Abdollah Boharfani and Reza Abazari[3]	Our Solutions
i). eq. (32) $u_{1,1}\left[\left(\frac{qk}{2p}\sqrt{\frac{p^2}{q^2k^2}}\tanh\left(\frac{n}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\xi\right)\right)+\frac{1}{2}\right]^{\frac{1}{n}}$	i). eq.(29) $V^{\frac{1}{n}}{}_{1_1}(\eta)=\left[\left(\frac{qk}{2p}\sqrt{\frac{p^2}{q^2k^2}}\tanh\left(\frac{n}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)\right)+\frac{1}{2}\right]^{\frac{1}{n}}$
ii). eq.(33) $u_{1,2}=\left[\left(\frac{qk}{2p}\sqrt{\frac{p^2}{q^2k^2}}\coth\left(\frac{n}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\xi\right)\right)+\frac{1}{2}\right]^{\frac{1}{n}}$	ii). eq.(30) $V^{\frac{1}{n}}{}_{2_1}=\left[\left(\frac{qk}{2p}\sqrt{\frac{p^2}{q^2k^2}}\coth\left(\frac{n}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)\right)+\frac{1}{2}\right]^{\frac{1}{n}}$
iii). eq.(34) $u_{2,1}=\left[\left(\frac{-qki}{2p}\sqrt{\frac{p^2}{q^2k^2}}\tan\left(\frac{ni}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\xi\right)\right)+\frac{1}{2}\right]^{\frac{1}{n}}$	iii).eq.(32) $V^{\frac{1}{n}}{}_{1_2}(\eta)=\left[\left(\frac{-qki}{2p}\sqrt{\frac{p^2}{q^2k^2}}\tan\left(\frac{ni}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)\right)+\frac{1}{2}\right]^{\frac{1}{n}}$
iv). eq.(35) $u_{2,2}=\left[\left(\frac{qki}{2p}\sqrt{\frac{p^2}{q^2k^2}}\cot\left(\frac{ni}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\xi\right)\right)+\frac{1}{2}\right]^{\frac{1}{n}}$	iv). eq.(33) $V^{\frac{1}{n}}{}_{2_2}(\eta)=\left[\left(\frac{qki}{2p}\sqrt{\frac{p^2}{q^2k^2}}\cot\left(\frac{ni}{2(n+1)}\sqrt{\frac{p^2}{q^2k^2}}\eta\right)\right)+\frac{1}{2}\right]^{\frac{1}{n}}$

Where $\xi = \eta = kx - \frac{k(-Ar(n+1)^2+p^2)}{p(n+1)}t,$

Beside the above solutions given in the table we found some new exact traveling wave solutions i.e. the nth root of $V_{5_{1''}}(\eta), V_{5_{2''}}(\eta)$,

$$V_{1,1}(\eta),V_{1,2}(\eta),V_{2,1}(\eta),V_{2,2}(\eta),V_{3,1}(\eta),V_{3,2}(\eta),V_{4,1}(\eta),V_{4,2}(\eta),V_{5,1}(\eta),V_{5,2}(\eta),$$

$$V_{1,1'}(\eta),V_{1,2'}(\eta),V_{2,1'}(\eta),V_{2,2'}(\eta),V_{3,1'}(\eta),V_{3,2'}(\eta),V_{4,1'}(\eta),V_{4,2'}(\eta),V_{5,1'}(\eta),V_{5,2'}(\eta),$$

$$V_{1,a}(\eta),V_{1,b}(\eta),V_{2,a}(\eta),V_{2,b}(\eta),V_{3,a}(\eta),V_{3,b}(\eta),V_{4,a}(\eta),V_{4,b}(\eta),V_{5,a_1},V_{5,a_2}(\eta),V_{5,b_1}(\eta),V_{5,b_2}(\eta)$$

$$V_{1,a'}(\eta),V_{1,b'}(\eta),V_{2,a'}(\eta),V_{2,b'}(\eta),V_{3,a'}(\eta),V_{3,b'}(\eta),V_{4,a'}(\eta),V_{4,b'}(\eta),V_{5,a'}(\eta),V_{5,b'}(\eta)$$

$$V_{1,_1}(\eta),V_{1,_2}(\eta),V_{2,_1}(\eta),V_{2,_2}(\eta),V_{3,_1}(\eta),V_{3,_2}(\eta),V_{4,_1}(\eta),V_{4,_2}(\eta),V_{5,_1}(\eta),V_{5,_2}(\eta),$$

$$V_{1_{1'}}(\eta),V_{1_{2'}}(\eta),V_{2_{1'}}(\eta),V_{2_{2'}}(\eta),V_{3_{1'}}(\eta),V_{3_{2'}}(\eta),V_{4_{1'}}(\eta),V_{4_{2'}}(\eta),V_{5_{1'}}(\eta),V_{5_{2'}}(\eta),$$

which were not found in the existing literature (Abdollah Boharfani and Reza Abazari [3])).

5 Conclusion and Future Scope

5.1 Conclusion

In this study, the improved $\left(\frac{G'}{G}\right)$-expansion method was applied to find exact (a particular) solution of a generalized Burger-Fisher equation. As a result exact traveling wave solutions were obtained in terms of hyperbolic, trigonometric and rational functions with free parameters. Some of the solutions were new, which were not in the work of Boharfani and Reza Abazari [3]. This study outlined that the improved $\left(\frac{G'}{G}\right)$- expansion method is well suited to be used by Mathematica 7 Software to find exact traveling wave solutions of Burger - Fisher's equation.

Thus, we can conclude that the improved $\left(\frac{G'}{G}\right)$-expansion method can be used to find exact traveling wave solutions of nonlinear partial differential equations (which are nonlinear evolution equations) which can arise in physics, engineering science, and other related areas.

5.2 Future Scope

The improved $\left(\frac{G'}{G}\right)-$expansion method is a simple and efficient method to find exact traveling wave solutions of NLEE. So an individual who interested in this area can use this method to find some new exact traveling wave solutions of some NLEE.

Appendix I

$$\lambda_1 = \frac{p(-k(-1+n)p(B(1+n)^2 - 4A^2E^2knp^2) - 3An(1+n)^3 r)}{\begin{array}{l}(2A^2E^2kn(1+n)^2 qr(2k(-1+n)p(An(E+Bn)p - 3BEk(1+n)q) \\ +3An(1+n)(An(-1+2n)p - 3Ek(1+n)q)r))\end{array}}$$

$$\begin{aligned}
\lambda_2 = {}& (3A^4Ek^2n^4(1+n)(B+4E+3(B+4E)n+4(B+E)n^2)pqr^2 + \\
& 3A^6n^7(1+n)pr^3 + A^5kn^5r^2(n(B+4E+Bn)p^2 + 9E(1+n)^2(1+2n)qr) - \\
& 3A^2E^2k^4n^2(1+n)pq^2r(B(4E+8En(2+n) + \\
& B(-3+n(8+n(21+11n))))+4E(2+n)(1+n+n^2)W) + \\
& 3BE^3k^6(1+n)pq^3(B^2(-4+n(1+3n)(4+3n))+4En(4+n(3+n))W) - \\
& AE^2k^5nq^2(B^2(Bn(-4+n(1+3n)(4+3n))p^2 - 3E(1+n)^2((2-6n)p^2 + 3(1+2n(2+n))qr)) + \\
& 4E(2E(1+n)^3 + Bn^2(4+n(3+n)))p^2W) + 2A^3Ek^3n^3qr(B^2n^2(2+n)p^2 + BE(2(1+n(1+n)(2+n))p^2 - \\
& 9(1+n)^2(1+n(3+n))qr) + 2Ep^2(2E(1+n)^2 + nW)))/(A^2E^2n^2(2k(-1+n)p(An(E+Bn)p - \\
& 3BEk(1+n)q) - 3An(1+n)(A(1-2n)np + 3Ek(1+n)q)r))
\end{aligned}$$

$$\begin{aligned}
\lambda_3 = {}& -(p(12A^6E^2k^2n^5(1+n)(-1+3n)p^3r^2 + 3A^4kn^3(1+n)pr(8E^3k^3(B-2(B+2E)n)p^2q + \\
& n(1+n)^2(B+4E+Bn)r) + A^5n^4r(9n(1+n)^4r^2 + 8E^3k^3p^2((-2+4n)p^2 - 9(1+n)^2qr)) - \\
& 3BE^2k^5(1+n)^3pq^2(B^2(-1+n)+4E(1+n)W) + AE^2k^4n(1+n)^2q(B^2(4(-1+n)p^2 + \\
& 9(1+n)^2qr) + 8(E+(B+E)n)p^2W) - 6A^2Ek^3n(1+n)pq(-2BE^3(B+8E)k^3(-1+n)p^2q + \\
& B(B+2E)n(1+n)^2r - 2E(1+n)(-4E^3k^3p^2q + n(1+n)r)W) + 2A^3Ek^2n^2(B(8E^3k^3(1+n-2n^2)p^4q + \\
& 2(1+n)^2p^2(n+18E^3k^3q^2)r - 9n(1+n)^4qr^2) + 4Ep^2(n(1+n)^2r + \\
& 2E^3k^3q(-2(-1+n)p^2 + 9(1+n)^2qr) + 4E^2k^3n^2p^2qW)))) / \\
& (2A^2E^3kn(1+n)^2q(2k(-1+n)p(An(E+Bn)p - 3BEk(1+n)q) - 3An(1+n)(A(1-2n)np + \\
& 3Ek(1+n)q)r)(Anr(Ek(2Anp - 3Bk(1+n)q) + \\
& 3A^2n^2(1+n)r) + 2E^2k^3(-1+n)pqW))
\end{aligned}$$

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