Jimma University

College of Natural Sciences

Department of Mathematics

Harmonic Equations and the Mean Value Property

By

Kemer Abibaker

A Thesis Submitted to Department of Mathematics, College of Natural Sciences, Jimma University, in Partial Fulfillment of the Requirements for the Master of Science Degree in Mathematics (Differential Equations)

Jimma, Ethiopia

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Jimma, Ethiopia

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Abstract

The Harmonic equation is one of the most widely used partial differential equations for modeling science and Engineering problems. The purpose of this study was to present the solution and various properties of Harmonic equations in rectangular and spherical coordinate systems. Dirichlet condition and Neumann condition were discussed. The theories of spherical harmonics were also discussed as it arises from the solution of the Dirichlet problems.

Keywords: Harmonic Equation, Dirichlet condition and Neumann condition, Analytic solution.

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CHAPTER ONE

1. INTRODUCTION

1.1 Background of the Study

Many problems of physical interest are described by partial differential equations with appropriate initial or boundary conditions. These problems are usually formulated as initial value problems, boundary-value problems, or initial boundary-value problems (Hillen et al.)[10].

The Harmonic equation is arguably the most important differential Equation in all of applied mathematics, mathematical physics and Engineering. It arises in an astonishing variety of mathematical and physical systems, ranging through fluid mechanics, electromagnetism, potential theory, solid mechanics, heat conduction, geometry, probability, number theory and so on (COLIN.HARPHAM ,)[7].

A regular solution of the Harmonic equation is called a harmonic function. The first boundary value problem for the Harmonic equation is often referred to as the Dirichlet problem, and the second boundary value problem as the Neumann problem (Andrei .D et al.)[3].

Many problems of mathematical physics and electromagnetic are related to the Laplacian differential operator. Among them, it is worth mentioning those relevant to the Harmonic and Helmholtz equations. However, most of the mentioned differential problems can be solved in explicit way only in canonical domains with special symmetries, such as intervals, cylinders or spheres (N. Lebedev) [16]. However, it is clear that conformal mapping techniques cannot be used in the three dimensional case where approaches based on suitable spatial discretization procedures, such as finite-difference or finite-element methods, are usually adopted.

Legendre's function Introduced in 1784 by the French mathematician A. M. Legendre (1752-1833).The Legendre functions are important in problems involving spherical coordinate system. Due to their Orthogonality properties. They are also useful in numerical analysis. Also known as spherical harmonics or zonal harmonics (Attar.E) [4].

In this study Legendre functions are solution to Legendre's differential equation:

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_n(x)\right]+n(n+1)P_n(x)=0 \qquad (1.1)$$

It occurs when solving Harmonic equation in the spherical coordinates. An analytic solution for fluxes at interior points for the two dimensions Harmonic equation is given by (Yoon and S.D. Heister) [17]. Finally the Poisson integral is presented as solution of Harmonic equation by (C.R.Wiley et al) [6].

Product method is a well-established technique for solving ordinary differential equations (A. Jeffrey) [2]. This method adaptable to almost all linear homogeneous partial differential equations with constant coefficients in canonical form, and exhibits the power of the superposition principle to construct the general solution of such equations. The product method is usually applied to solve higher order partial differential equations (A. Jeffrey) [2]. A lot of applications of the product method can be found in literatures.

A fundamental solution of Harmonic equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{1.2}$$

Is given by (apart from a multiplicative factor of 4π)

$$U(x, x_o) = |x - x_o| \quad \text{where} \quad x = (x, y, z) \neq x_o = (x_o, y_o, z_o) \tag{1.3}$$

And $|x - x_o|$ denotes the Euclidean distance between x and x_o.Examples of such applications include electrostatics, magneto statics, quantum direct and exchange Coulomb interactions, Newtonian gravity, potential flow and steady state heat transfer. In (Morse and Feshbach) [15]. See also (Hobson)[11] and reference there in provide a list of such expansions for various coordinate systems but the formulas for several coordinate systems are missing. It is the goal of this study to provide these expansions in rectangular and spherical coordinate system.

In mathematics, the Dirichlet (or first-type) boundary condition is a type of boundary condition named after Johann Peter Gustav Lejeune Dirichlet (1805–1859), when imposed on an ordinary or a partial differential equation (Cheng. A) [5].It specifies the values a solution needs to take on the boundary of the domain. The question of finding solutions to such equations is known as the Dirichlet. Another also the Neumann (or second-type) boundary condition is a type of boundary condition named after Carl Neumann, when imposed on an ordinary or a partial differential equation (Cheng. A) [5]. It specifies the values that the derivatives of solutions are take on the boundary of the domain.

In three-dimensional space, the simplest product cases are problems formulated on rectangular, cylindrical or spherical domains. Since the first two are straightforward extensions of their two-dimensional counterparts, we were only discussing spherically product solutions in any detail. The simplest domain to which the product method applies is a rectangular box.

A first aim of this work is to provide and compile document on the solution of Harmonic equation in 3D. In order to make it easier to understand, some necessary mathematical fundamentals are presented at the beginning, including the basic knowledge about Definition, fundamental solution and property of harmonic function. In the next part concentrations are focused on the detailed procedures to solve

Dirichlet and Neumann problem with different boundary conditions in rectangular and spherical coordinate system. The results are applied, as examples

1.2. Statement of the Problem

Harmonic equation is the most important Partial differential equations. The solutions of Harmonic equation are the harmonic function , which are important in many fields of science, notably the fields of electromagnetism, astronomy, and fluid dynamics, because they can be used to accurately describe the behavior of electric, gravitational, and fluid potentials(Arfken.et.al)[9]. The Harmonic equation is very important in applications. It appears in physical phenomena (Asmara.N.H) [1].

The purpose of this study was to provide and compile a document on the solution of Harmonic equation and mean value property with regard to three dimensional in rectangular and spherical coordinates systems. This title was selected, since it is more applicable in partial differential equation. But the derivation of Harmonic equation in rectangular coordinates and application in spherical coordinates lacks clarity and are not well organized in various books. Therefore, this study was providing detail information by answering the following questions:

- How to find the solution of Harmonic equation in rectangular coordinate **s**ystems?
- How to find the solution of Harmonic equation in spherical coordinate **s**ystems?
- How to solve boundary value problems that involve the Harmonic equation in spherical coordinate system?
- What are the solutions of Neumann problem for the Harmonic equation in spherical coordinate system?
- How to prove the uniqueness of solution of Harmonic equations in a bounded domain?
- What are the general solutions of the interior Dirichlet problem for the Harmonic equation in spherical coordinates system?
- What are the general solutions of the exterior Dirichlet problem for the Harmonic equation in spherical coordinates system?

1.3. Objective

1.3.1. General Objective

The general objective of this research was to investigate solutions of Harmonic equations in spherical coordinate system and study properties of solution.

1.3.2. Specific Objectives

The followings are specific objectives of the study:

- To find the solution of Harmonic equation in rectangular coordinate **s**ystems
- To find the solution of Harmonic equation in spherical coordinate **s**ystems.
- To solve boundary value problems that involves the Harmonic equation in spherical coordinate system.

- To find the solutions of Neumann problem for the Harmonic equation in spherical coordinate system?
- To prove the uniqueness of solution of Harmonic equations in a bounded domain.
- To find the general solutions of interior Dirichlet problem for the Harmonic equation in spherical coordinates system.
- To find the general solutions of exterior Dirichlet problem for the Harmonic equation in spherical coordinates system.

1.4. Significance of the Study

We addressed the above mentioned objectives associated with Harmonic equations in spherical coordinates system have following important points as a significant contribution of the study.

- It will help students to develop skills of solving problems by using product method,
- It will be used as reference materials for students, teachers and others who work on this area.
- It provides techniques of transforming partial differential equation from Cartesian to spherical coordinates.
- It may support student/researchers dealing on Harmonic equations to simply understand the concept.

1.5. Delimitation of the Study

The fundamental partial differential equations that govern the equilibrium mechanics of multidimensional media are the Harmonic equation. The Harmonic equation is arguably the most important differential equation in all of applied mathematics. It arises in surprising variety of mathematical and physical systems, ranging through fluid mechanics, electromagnetism, potential theory, solid mechanics, heat conduction, geometry, probability, number theory, and so on. Though Harmonic equation is vast topic and has many applications in the real world. This study is delimited to focus only on solutions of three dimensional Harmonic equations in the spherical coordinate system and studying their properties.

CHAPTER TWO

2. METHODOLOGY

2.1 .Study Site and Period

The study was conducted from January 2012 –June 2013 G.C.in Jimma University, Mathematics Department.

2.2 Study Design

This study was the document review design on harmonic equation in spherical coordinate system and mean value property. Secondary data were collected from the relevant sources of information to achieve each specific objectives of the study.

2.3 .Sources of Information

To conduct this study secondary data was used. Hence the sources of these data were Different mathematics reference books, Internet and published research.

2.4. Procedures of Study

In order to achieve the above mentioned objectives document data analysis approaches together with the following methods were followed:

- To transform the Harmonic equation from three dimensional Cartesian coordinates $u(x, y, z)$ to the spherical coordinate system $U(r,\theta,\phi)$,the transformation were used $X=\sin\theta\cos\phi$,$y=\sin\theta\sin\phi$, $z=r\cos\theta$,where $x^2+y^2+z^2=r^2$,$\tan\phi=\frac{y}{x}$
- To derive the three dimensional equation
$$\Delta u = u_{xx} + u_{yy} + u_{zz} = 0 \ .$$
Gauss's theorem has been used.
- To find the solution of harmonic equation in rectangular coordinate **s**ystems and solve boundary value problems that involve the harmonic equation in spherical coordinate system product method was used.
- To find the general solutions of the interior Dirichlet problem for the harmonic equation in spherical coordinate system product method was used.
- The maximum principle was used to show uniqueness of solution initial boundary value problem/boundary value problems.

2.5. Instrumentation and Administration

Since this study was a document review study, it uses purely secondary data. For this study the materials like paper, flash disk, RW-CD, pen are needed until the completion of the study. Also the available secondary sources of information were collected by the two assistant document collectors from the resource center regularly by the supervision of the researcher.

2.6 .Limitation of the Study

It is clear that any researcher would not be free from some limitations. Some of the problems that the researcher faced in this study are the following:

- Shortage of time given to carry out the study.
- Class activities parallel to research.
- Lack of some reference materials and limited hours of library.

2.7. Ethical Consideration

In order to conduct this research, the researcher made communication with the concerned office of Jimma University to get an official cooperation letter. Moreover, the researchers keep the rules and regulations and have good approaches during data collection period.

CHAPTER THREE

3. RESULT AND DISCUSSION

3.1. Harmonic Equation in Rectangular Coordinate System

Definition 3.1.1. Let Ω denote an open set in R^3. A real valued function *u*(*x, y, z*) on Ω with continuous second partial derivatives are said to be *harmonic* if and only if it satisfies Laplacian.

$$\Delta u = \nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0 \qquad \text{on} \quad \Omega.$$

The prototypical equilibrium system is the three-dimensional Harmonic equation

$$\boldsymbol{\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0} \quad . \tag{3.1}$$

Definition 3.1.2.A function u $\in \mathbf{C}^2$ (Ω) which satisfies the Harmonic equation is called a *harmonic function.* The inhomogeneous Harmonic equation where

$$\Delta u = \nabla^2 u = \boldsymbol{f} \tag{3.2}$$

Is known as the Poisson equation where f is a given function. The Harmonic equation is very important in applications. It appears in physical phenomena.

Examples of harmonic functions are

1. $u = x^2 + y^2 - 2z^2, \Omega = R^3$
2. $u = \frac{1}{r}$, $\Omega = R^3 - (0, 0,0)$; Where $r = \sqrt{x^2 + y^2 + z^2}$

Definition 3.1.3.The function $u(x) = \begin{cases} \frac{-1}{2\pi}\log r + c & (n = 2) \\ \frac{1}{n(n-2)\sigma(n)|x|^{n-2}} & (n \geq 3) \end{cases}$

is defined for $x \in R^n, x \neq 0,$ is fundamental solution of Harmonic equation.

3.1.1 Derivation of Laplace equation

In this study we first discuss the model of temperature distribution in a body in space, which leads to the so-called the Laplace equation. We then solve this model under the initial and boundary Conditions by product method.

Physical Assumptions

i. The specific heat σ and the density $\rho$of the material of the body are constants. No heat is produced or disappears in the body.
ii. Experiments show that in a body the heat flows in the direction of decreasing temperature and the rate of flow is proportional to the gradient of the temperature, that is, the velocity v of the heat flow in the body is of the form

$$v = -K grad u \tag{3.2}$$

Where $u(x, y, z, t)$ is the temperature at a point(x, y, z) and instant t.

iii. The thermal conductivity K is a constant, as is the case for homogeneous material and non-extreme temperatures.

Under these assumptions we now derive the model of heat flow in a body as follows. Let D be any region in the body bounded by a smooth surface S with outer unit normal vector n. Then v .n is the component of v in the direction of n. Hence $|v.n\Delta S|$ is the amount of heat leaving D if v .n > 0 at some point P, hence $v.n\Delta S > 0$ (or entering D if v. n < 0 at point P, hence $v.n\Delta S < 0$) per unit time at some point P of S through a small portion of S of area ΔS . Hence the total amount of heat that flows across S entering D per unit time is

$$-\iint v.ndS \tag{3.3}$$

Using Gauss's theorem, we now convert the surface integral into a volume integral over the region D. By (3.3) we have

$$-\iint v.ndS = \mathrm{K}\iint gradu.ndS = K\iiint div(gradu)dxdydz$$
$$= k\iiint \nabla^2 udxdydz \tag{3.4}$$

Where $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ is The Laplacian of u. On the other hand, the total amount of heat in D is

$$H(t) = \iiint \sigma\rho udxdydz \tag{3.5}$$

Hence the time rate of increase of the total amount of heat H in region D is

$$\frac{\partial H}{\partial t} = \iiint \sigma\rho\frac{\partial u}{\partial t}dxdydz \tag{3.6}$$

Since (3.8) represents the amount of heat cross surface S entering D per unit time and no heat is produced or disappears in the body, we obtain

$$\iiint \delta\rho\frac{\alpha u}{\alpha t}dxdydz = k\nabla^2\iiint \nabla^2 udxdydz \tag{3.7}$$

Dividing by $\delta\rho$ and writing $c^2 = \frac{K}{\delta\rho}$

$$\iiint \frac{\alpha u}{\alpha t}dxdydz = c^2\iiint \nabla^2 udxdydz \tag{3.8}$$

$$\iiint \frac{\alpha u}{\alpha t}\left(\frac{\alpha u}{\alpha t} - c^2\nabla^2 u\right)dxdydz = 0 \tag{3.9}$$

Suppose that the integrand is continuous. Since the above integral holds for any region D in the body, the integrand is zero everywhere, namely,

$$\frac{\alpha u}{\alpha t} = c^2\nabla^2 u \tag{3.10}$$

To be heat equation. It gives the temperature u(x, y, z, t) in a body of homogeneous material in space. The constant $c^2 = \frac{K}{\delta\rho}$ is the thermal diffusivity; K is the thermal conductivity, δ the specific heat, and ρ the density of the material of the body. For the steady state (that is, time independent) problems,
$\frac{\alpha u}{\alpha t} = 0$. Therefore, equation (3.10) reduced to

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0. \quad (3.11)$$

This is called Harmonic equation.

Definition 3.1.4. (Superposition principle**)**

Superposition principle is of fundamental importance in the study of partial differential equations. This principle is used extensively in solving linear partial differential equations by the method of separation of variables. If u_1, u_2 ...$u_{N,}$ are solutions of a linear partial differential equation, where N is a positive integer, and if$C_{1,}$. $C_1, C_2 \dots C_N$ are arbitrary real constants, then

$$C_1 u_1 + C_2 u_2 + \cdots C_N u_N \quad (3.12)$$

Also a solution. See more in detail (MYint-U.T.)[14].

Definition3.1.5. A Sturm-Liouville problem is called regular if and only if p(a) > 0, p(b) > 0,q is continuous on [a, b],and the constants a_1,b_1,a_2, and b_2 are nonnegative. A Sturm-Liouville problem that is not regular is called singular.

3.1.2. Boundary Value Problem

Practical problems involving Harmonic equations are boundary value problems in a region T in space with boundary surface S. Such a problem is called

1. First boundary value problem or Dirichlet problem if u is prescribed on S
2. Second boundary value problem or Neumann problem if the normal derivative $u_n = \frac{\partial u}{\partial n}$ is prescribed on S.
3. Third or mixed boundary value problem or Robin problem if the normal derivative $u_n = \frac{\partial u}{\partial n}$ is prescribed on portion of S and u_n on the remaining portion of S.

In this section we discuss only Dirichlet and Neumann problem. Let us consider first Dirichlet problem.

3.1.3. Dirichlet Problem

To find a harmonic function u which is regular (continues and defined) in a domain D and which coincides with a given continuous function ϕ on the boundary Γ of D. The problem of finding the solution of a second-order elliptic equation which is regular in the domain is also known as the Dirichlet or first boundary value problem. The solution u of the Dirichlet problem for a domain D with a sufficiently smooth boundary can be represented by the integral formula

$$u(x) = \int_{\tau} \phi(x_o) \frac{\partial G(x, x')}{\partial n_o} d\delta \tag{3.13}$$

Where $\frac{\partial G(x, x')}{\partial n_o}$ is the derivative in the direction of the interior normal at the point $x_0 \in \Gamma$ of the Green's function, which is characterized by the following property:

1. $G(x, x_o) = {S_n}^{-1} r^{2-n} + y(x, x_o)$ if $n \geq 3$ where $r = |x - x_o|$ is the distance between the point x and x_o. Also S_n is the surface area of the unit sphere in $R^n$and , $y(x, x_o)$ is harmonic function which is regular in D both with respect to the coordinate x andx_o .
2. $G(x, x_o) = 0, x_0 \in D, x \in \Gamma$.

For the sphere, the half-space and certain other most simple domains the Green function is constructed explicitly, and formula (3.14) yields an effective solution of the Dirichlet problem. The formulas thus obtained for the sphere and the half-space are known as the Poisson formulas (poison integral formula) more in detail see reference (Evans. L. C) [8].

In this section we demonstrate how boundary value problems for Laplace's equation can be solved by separation of variables in the case of cubes in three dimensions.

Let us first consider the following Dirichlet problem:

- Finding the general solution of Harmonic equation in three-dimensional box

D = $\{0 < x < \pi, 0 < y < \pi, 0 < z < \pi\}$

Satisfying Boundary conditions

$$u(\pi, y, z) = u(x, \pi, z) = u(x, y, 0) = 0, for\ 0 < x < \pi, 0 < y < \pi \tag{3.14}$$

Initial conditions$u(x, y, c) = f(x, y)$ (3.15)

<u>Solution</u>

First finding the partial differential equation subject to the problem which is given by

$$\nabla^2 \boldsymbol{u} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \boldsymbol{0} \tag{3.16}$$

Suppose that the solution has the form

$$U(x, y, z) = X(x)\, Y(y)\, Z(z) \quad (3.17)$$

Substituting this in (3.6), we obtain

$$X''YZ + XY''Z + XYZ'' = 0. \quad (3.18)$$

Now, dividing both sides of (3.18)by XY Z yields

$$\frac{x''}{x} + \frac{y''}{y} + \frac{z''}{z} = 0$$

$$\frac{x''}{x} + \frac{y''}{y} = -\frac{z''}{z} = \eta \quad (3.19)$$

Since the right side depends only on z and the left side is independent of z, thus both terms must be equal to a constant. Thus, we have

$$\frac{x''}{x} + \frac{y''}{y} = -\frac{z''}{z} = \eta$$

$$\frac{x''}{x} + \frac{y''}{y} - \eta = 0\,, \frac{z''}{z} + \eta = 0$$

$$\frac{x''}{x} = \eta - \frac{y''}{y} \quad (3.20)$$

Here also x and y are independent variables. Therefore, (3.9) is true only if both sides must have a constant let μ (say) we get

$$\frac{x''}{x} = \eta - \frac{y''}{y} = \mu$$

Hence; we obtain the three ordinary differential equations

$$\begin{cases} X'' - \mu X = 0 & (3.21) \\ Y'' - Yv = 0 \;\; where\; v = \eta - \mu & (3.22) \\ Z'' + \eta Z = 0 \;\; Where\; \eta = v + \mu & (3.23) \end{cases}$$

Considering different cases for constant v, η *and* μ

Case I let $\mu = 0$

i. From (3.21) we have $X'' - \mu X = 0$,substituting $\mu = 0$

We get t $X'' = 0$

Up on integrating both sides two time we get X(x) = $ax + b$

ii. let v=0 from (3.22) we have $y'' - \mu y = 0$,Substituting $v = 0$ we get $Y'' = 0$

Up on integrating both sides two times, we get Y(y) = $cy + d$

Now $\eta = 0$ from (3.23) we have $Z'' + \eta Z = 0$, substituting $\eta = 0$,

We get $Z'' = 0$

Up on integrating both sides two time we get Z (z) = $ez + f$

Hence, combining the above result, we get

$$U(x, y, z) = X(x)\, Y(y)\, Z(z) = (ax + b)(cy + d)(ez + f)$$

Applying boundary condition: U (0, y, z) =U $(\pi, y, z) = 0$ means that

$$X(0) = X(\pi) = 0$$

$$X(0) = a.o + b = 0 \Rightarrow b = 0$$

$$X(x) = ax \Rightarrow X(\pi) = a\pi = 0 \;\; \Rightarrow a = 0$$

Therefore, X(x) =0 ⇒ $U(x, y, z) = 0$

Hence the solution is trivial, because of product solution.

Case II. Let $\mu = w^2 > 0$ and v=$k^2 > 0$ from this we get

$$p^2 = w^2 + k^2 > 0, \text{ where } \eta = p^2$$

i From (3.21) we have $X'' - \mu X = X'' - w^2 X = 0$.

Let X=e^t, $X' = me^{mx}$, $x^{\wedge}'' = m^{\wedge}2\ e^{\wedge}mx$, Substituting $X'' = m^2 e^{mx}$ into the above equation, we get

$$m^2 e^{mx} + e^{mx} w^2 = 0$$

$$\Rightarrow e^{mx}(m^2 + w^2) = 0, \text{ Since } e^{mx} \neq 0 \; for\ all\ x$$

$$\Rightarrow m^2 - w^2 = 0 \quad \text{or } m = \pm w$$

Thus, $X = ae^{wx} + be^{-wx}$

Applying boundary condition, we obtain

U (0, y, z) =U $(\pi, y, z) = 0$, Means that

$$X(0) = X(\pi) = 0$$

$$X(0) = ae^0 + be^0 = a + b \Rightarrow a = -b$$

$$X(x) = ae^{wx} - ae^{-wx}$$

$$\Rightarrow X(\pi) = a(e^{w\pi} - e^{-w\pi}) = 0 \ ,$$

$$a(e^{w\pi} - e^{-w\pi}) \neq 0$$

$$\Rightarrow a = 0 = b$$

Therefore, X(x) = 0⇒ $U(x, y, z) = 0$, hence also the solution is trivial.

ii. From (3.22) we have $Y'' + \mu Y = Y'' + k^2 Y = 0$

Now, let Y=e^{my},$Y' = me^{my}$, $Y'' = m^2 e^{my}$,

Substituting this into the above equation, we get

$$m^2 e^{my} - e^{my} k^2 = 0$$

$$\Rightarrow e^{my}(m^2 - k^2) = 0, \text{ since } e^{my} \neq 0 \text{ for all y}$$

$$\Rightarrow m^2 - k^2 = 0 \text{ or } m = \pm k$$

Thus, $X = ce^{ky} + de^{-ky}$

iii. From (3.23) we have $Z'' + \eta Z = Z'' + p^2 Z = 0$.

Let Z=e^{mz},$Z' = me^{mz}$, $Y'' = m^2 e^{my}$

Substituting $Z'' = m^2 e^{mz}\ into\ the\ above\ equation\ \ , we\ get$

$$m^2 e^{mz} + e^{mz} p^2 = 0, \text{ since } e^{mz} \neq 0 \; for\ all\ z$$

$$\Rightarrow m^2 + p^2 = 0 \text{ or } \quad m = \pm pi$$

Thus, $Z = \text{e} \cos pz + \text{f} \sin pz$, where $p^2 = w^2 + k^2$,

$$Hence\ \ , U(x, y, z) = (a\ e^{wx} + b\ e^{-wx})(c\ e^{ky} + d\ e^{-ky})(\text{e} \cos pz + \text{f} \sin pz)$$

Case 3 let $\mu = -w^2 < 0$

i. From (3.21) we have . $X'' - \mu X = X'' + w^2 X = 0$

Now, let X=e^{mx},$X' = me^{mx}$, $X'' = m^2 e^{mx}$,

Substituting, this into the above equation , $we\ get$ $m^2 e^{mx} + e^{mx} w^2 = 0$

$$\Rightarrow e^{mx}(m^2 + w^2) = 0, \text{ Since } e^{mx} \neq 0 \; for\ all\ x$$

$$\Rightarrow m^2 + w^2 = 0 \text{ or } m = \pm wi$$

Thus, $X = \text{a} \cos wx + \text{b} \sin wx$ (3.24)

Applying boundary condition:

U (0, y, z) =U $(\pi, y, z) = 0$

$$\text{Means that } X(0) = X(\pi) = 0$$

$$X(0) = \mathrm{a}\cos 0 + \mathrm{b}\sin 0 = a = 0$$

Now, if $b = 0$,the solution is again trivial. In order to get non trivial solution

Let $b \neq 0 \Rightarrow \sin w\pi\, b = 0 \Rightarrow w\pi = n\pi \Rightarrow w = n$, for n=0, 1, 2...

Therefore, X(x) =bsin $nx \Rightarrow X_n(x) = b_n \sin nx$,

ii From (3.22) we have $Y'' + \mu X = Y'' + k^2 Y = 0$.

Let Y= e^{my},$Y' = me^{my}$, $Y'' = m^2 e^{my}$

Substituting this into the above equation, we get

$\Rightarrow e^{my}(m^2 + w^2) = 0$, Since $e^{my} \neq 0$ *for all y* ,

$\Rightarrow m^2 + w^2 = 0 \Rightarrow m = \pm wi$, *thus* , $Y = \mathrm{c}\cos ky + \mathrm{d}\sin ky$

Applying boundary condition, we obtain

$$\mathrm{U}(\mathrm{x}, 0, \mathrm{z}) = \mathrm{U}\,(x.\pi, z) = 0$$

$$\text{Means that } Y(0) = Y(\pi) = 0$$

$$Y(0) = \mathrm{c}\cos 0 + \mathrm{d}\sin 0 = c = 0$$

Now, $if d = 0$ our solutions are trivial .In order to get non trivial solution

Let $d \neq 0 \Rightarrow \sin k\pi = 0$

$\Rightarrow k\pi = m\pi \Rightarrow k = m$, for m=0, 1, 2,. . .

Therefore, Y(y) =bsin $my \Rightarrow Y_m(y) = d_m \sin my$ (3.25)

iii. From (3.23) we have

$$Z'' - \eta Z = Z'' - p^2 Z = 0 \ .$$

Let Z=e^{mz}, $Z' = me^{mz}$, $Z'' = m^2 e^{mz}$

Substituting this into the above equation, we get

$$m^2 e^{mz} + e^{mz} w^2 = 0$$

Since $e^{mz} \neq 0$ for all z , we obtain from above that $m^2 - w^2 = 0$

or $m = \pm w$. Thus, $Z = e\, e^{pz} + f e^{-pz}$, where $p^2 = w^2 + k^2$

Hence , $U(x, y, z) = (\mathrm{a}\cos wx + \mathrm{b}\sin wx)(\mathrm{c}\cos ky + \mathrm{d}\sin ky)(\, e\, e^{pz} + f\, e^{-pz})$

$$U_n(x, y, z) = X_n Y_m Z_{nm} = (b_n \sin nx)(d_m \sin my)\,)(\, E_{nm} e^{pz} + F_{nm} e^{-pz})$$

$= \sin nx \sin my)\,)(\, D_{nm} e^{pz} + C_{nm} e^{-pz})$

Where $D_{nm} = b_n d_m$ *and* $C_{nm} = b_n d_m F_{nm}$

Applying superposition principles, we obtain

$U(x, y, z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sin nx \sin my\, (\, D_{nm} e^{pz} + C_{nm} e^{-pz})$ (3.26)

Applying boundary condition, we get

$$U(x,y,\pi) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \sin nx \sin my)(D_{nm}e^{p\pi} + C_{nm}e^{-p\pi}) = 0$$

$$\Rightarrow \sin nx \sin my \neq 0, (D_{nm}e^{p\pi} + C_{nm}e^{-p\pi}) = 0$$

$$\Rightarrow D_{nm}e^{p\pi} = -C_{nm}e^{-p\pi} \text{ or } D_{nm} = \frac{=-C_{nm}e^{-p\pi}}{e^{p\pi}}$$

Substituting this into (3.26) we get

$$U(x,y,z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \sin nx \sin my) \left(\frac{-C_{nm}e^{-p\pi}}{e^{p\pi}} e^{pz} + C_{nm}e^{-pz}\right)$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{nm} \sin nx \sin my) \left(e^{-pz} - \frac{e^{-p\pi}}{e^{p\pi}} e^{pz}\right)$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{nm} \sin nx \sin my)\left(e^{p(\pi-z)} - e^{-p(\pi-z)} e^{pz}\right)\frac{2}{2e^{p\pi}}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{nm} \sin nx \sin my) \sinh\sqrt{n^2+m^2}(\pi-z)\frac{2}{e^{p\pi}}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} G_{nm} \sin nx \sin my) \sinh\sqrt{n^2+m^2}(\pi-z) \text{ .where } G_{nm} = \frac{2}{e^{p\pi}}C_{nm}$$

$$U(x,y,0) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} G_{nm} \sin nx \sin my \sinh\sqrt{n^2+m^2}(\pi) = f(x, y)$$

$$where\ G_{nm} \sinh\sqrt{n^2+m^2}(\pi) = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} f(x,y) \sin nx \sin my \ dx\, dy$$

The coefficient double flourier series becomes

$$G_{nm} = \frac{4}{\sinh\sqrt{n^2+m^2}(\pi)\,\pi^2}\int_0^{\pi}\int_0^{\pi} f(x,y) \sin nx \sin my \ dx\, dy$$

Therefore, the general solution of the Dirichlet problem for a cube may be written in the form:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{4\sinh\sqrt{n^2+m^2}(\pi-z)}{\sinh\sqrt{n^2+m^2}(\pi)\,\pi^2} \sin nx \sin my \int_0^{\pi}\int_0^{\pi} f(x,y)\sin nx \sin my \ dx\, dy$$

The strategy of solving boundary value problems for Harmonic equation in a cube is summarized as in the following scheme:

- Look for product solutions
- Solve the Eigen value problem for the component with two homogeneous boundary conditions
- Find the other component using the obtained Eigen values, and the third homogeneous boundary condition
- Form the series solution, and find the coefficients from the inhomogeneous boundary condition.

In this manner we can do any Dirichlet problem on a rectangular parallelepiped in the form of an infinite series.

Theorem 3.1.1 (The maximum principle for the Harmonic equation)

Let D be an open, bounded set in R^3, let ∂D denote its boundary, and let $\overline{D}$ be the union of D and ∂D. If u: $\overline{D}$ →IR is a continuous function that satisfies the Harmonic equation in D, then u attains its maximum and minimum values on ∂D.

Proof

Assume that u has its maximum value M at a point (x_o, Y_o, Z_o) in D, and
Let m< M be its maximum value on ∂D. Then, if d is the diameter of a sphere containing D, Define v: $\overline{D}$ →IR

$v(x, y, z)$ = u(x, y, z) $+\frac{(M-m)}{d^2}[(x-x_o)^2+(y-y_o)^2+(z-z_o)^2]$

If (x, y, z) is in ∂D, we have

$$v(\mathrm{x,y,z}) < u(\mathrm{x,y,z}) + \frac{(M-m)}{2} \leq \mathrm{m} + \frac{(M-m)}{2} < M$$

Since v (x_o, Y_o, Z_o) = u (x_o, Y_o, Z_o) =M.

V attains its maximum value at some point in D and this implies that u_{xx}, u_{yy} *and* u_{zz} are less than zero or equal to zero at that point.

This is imposable because

$v_{xx}+v_{yy}+v_{zz}=u_{xx}+u_{yy}+u_{zz}+\frac{3(M-m)}{d^2}=\frac{3(M-m)}{d^2}$

And is contradiction show that u attains its maximum ∂D.the same type of arguments show that –u attains its minimum value on D and hence u attains its minimum value on ∂D.

Theorem 3.1.2. If Dirichlet problem for a bounded region has a solution, and then it is unique.

Proof if $u_1 and\ u_2$ are two solutions of interior Dirichlet problem, then

Δu_1=0 in IR, u_1 =f on ∂R

$\Delta u_2=0$ in IR u_2=f on ∂R and let u=u_1-u_2 .

$\Rightarrow\Delta u=\Delta u_1-\Delta u_2$=0 in IR

$u=u_1-u_2$=0 in IR,

$\Rightarrow f-f$=0 on the boundary.

Therefore, $\Delta \mathrm{u}=0$ on IR .and $\Delta u=0$ on the boundary

The solution of Dirichlet problem is unique.

3.1.4. Neumann condition

Theorem 3.1.3 .If Neumann problem in the bounded region has the solution, then It is unique or differs from one another by a constant only.

Proof

Let u_1 *and* u_2 be two distict solutions Neumann problem, then we have

Δu_1=0 in IR,$\frac{\partial u_1}{\partial n} = f$ on ∂R

$\Delta u_2 = 0$ in IR $\frac{\partial u_2}{\partial n}$ =f on ∂R and let u=$u_1 - u_2$

$\Rightarrow \Delta u = \Delta u_1 - \Delta u_2$=0 in IR

$u = u_1 - u_2$=0 in IR, then

$\frac{\partial u}{\partial n} = \frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n}$=0 on the boundary.

Hence u is constant. Therefore, the solution of the Neumann is not unique.

Thus, the solution of a certain Neumann can be different from one another by a constant only.

3.1.4. Properties of Harmonic Functions

Solution of Laplace equation is called harmonic function which possesses number of interesting properties and they are presented in the following theorem:

Theorem 3.1.4.1. If harmonic function vanishes everywhere on the boundary, then its identical everywhere.

Proof

If u is harmonic function, then ∇^2u $= 0$ in IR and also u=0 on ∂R.

We shall show that u=0 in$\bar{R} = IR \cup \partial$R. We know that the first green identity

$$\iiint (\Delta u)^2 \text{d u}=\iint u \frac{\partial u}{\partial n} dS - \iiint u(\Delta u)^2 \text{du}.$$

Since $(\Delta u)^2$ is positive.it follows that the integral will be satisfied only if Δu=0, this implies that u is constant in IR. Since u continuous in $\overline{\text{IR}}$ and u is zero on ∂R.

It follows that u=0 in IR.

Theorem 3.1.4.2. If v is harmonic function in IR and $\frac{\partial u}{\partial n} = 0$ on∂R, and then v is constant in $\bar{R}$

Proof

Using first Green's identity and the data of the theorem, we arrive at

$$\iiint (\Delta v)^2 \text{Dv} = 0.$$

Implying Δv =0 that is v is constant in IR.

Since the value of v is not known on the boundary ∂R while $\frac{\partial u}{\partial n} = 0$, its implies that v is constant on ∂R and hence $\overline{\mathbb{R}}$

Symmetry properties

A function f on R^n is radial if it is rotation invariant ($f0T = T$ for all rotation T). Therefore, f is radially precisely when it is constant on every sphere about the origin or when f(x) depends only on $|x|$.

Harmoncity

Harmonic functions that arise in physics are determined by their singularity and boundary conditions (such as Dirichlet boundary conditions or Neumann boundary condition). On regions without boundaries, adding the real or imaginary part of any entire function will produce a harmonic function with the same singularity, so in this case the harmonic function is not determined by its singularities; however, we can make the solution unique in physical situations by requiring that the solution goes to 0 as you go to infinity. In this case, uniqueness follows by mean value property.

The mean value property

If B(x, r) is a ball with center x and radius r which is completely contained in the open set $\Omega \subset R^n$, then the value u(x) of a harmonic function u: $\Omega \rightarrow R$ at the center of the ball is given by the average value of u on the surface of the ball, this average value is also equal to the average value of u in the interior of the ball.

Theorem3.1.6 (Mean value theorem)

Let $\Omega \subset R^n$. If u$\in C^2(\Omega)$ is harmonic, then

$$u(x) = \int_{\partial B(x,r)} u(y)dS(y) = \int_{B(x,r)} u(y)dy \text{ , For every ball } B(x,r) \subset \Omega$$

<u>Proof</u>

Assume that u$\in C^2(\Omega)$ is harmonic.

$$\phi\ (r) = \int_{\partial B(x,r)} u(y)dS(y) \text{. For r=0, define } \phi\ (r) = u(x).$$

Notice that if u is smooth function then

$\lim_{r \to 0+} \phi(r) = u(x)$ .And therefore ϕ is continuous function.

Thus, we can show that $\phi'(r) = 0$

Thus, we conclude that ϕ is a constant function and therefore,

$$U\ (x) = \int_{\partial B(x,r)} u(y)dS(y)$$

Now, we prove that $\phi'(r) = 0$ as follows

i. Making a change of variables we have

$$\phi(\mathbf{r}) = \int_{\partial B(x,r)} u(y)dS(y) = \int_{\partial B(x,r)} u(x+rz)dS(z)$$

Therefore, $\phi'(\mathbf{r})= \int_{\partial B(0,1)} \Delta u(x+rz)\cdot z dS(z) = \int_{\partial B(x,r)} \Delta u(y)\cdot\left(\frac{y-x}{r}\right) dS(y)$

$$= \int_{\partial B(x,r)} \frac{\partial u}{\partial v}(y) dS(y) = \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(x,r)} \frac{\partial u}{\partial v}(y) dS(y)$$

$$= \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(x,r)} \nabla\cdot(\nabla u) dy \quad \text{(by divergence theorem)}$$

$$\frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(x,r)} \Delta u dy = \mathbf{0}$$

Using the fact that u is harmonic. Therefore we have proven that the first part of the theorem.

ii. It remains to prove that

$$\text{U(x)} = \int_{B(x,r)} u(y) dy.$$

We do as follow using the first result.

$$\int_{B(x,r)} u(y) dy. = \int_0^r \left(\int_{\partial B(x,s)} u(y) dS(y) \right) ds = \int_0^r n\alpha(n) s^{n-1} \left(\int_{\partial B(x,s)} u(y) dS(y) \right) ds$$

$$= \int_0^r n\alpha(n) s^n u(x) ds = n\partial(n) u(x) \int_0^r S^{n-1} dS = \alpha(n) u(x) S^{n}\Big|_{s=0}^{s=r} = \alpha(n) u(x) r^n$$

Therefore, $\int_{B(x,r)} u(y) dy = \alpha(n) u(x) r^n$ which implies

$$\text{U(x)} = \frac{1}{\alpha(n) r^n} \int_{B(x,r)} u(y) dy = \int_{B(x,r)} u(y) dy$$

3.1.5. Legendre’s Equation and Legendre's Function

Legendre's Equation arises naturally when solving some PDE in Spherical Coordinate systems. Usually it forms part of the Sturm-Liouville problem which requires it to have bounded Eigen functions over a fixed domain (peter Oliver) [12].

Definition3.1.6 (Legendre's Equation**)**

The Legendre's Equations is families of differential equations differ μ by the parameter in the following form:

$$(1-x^2)y'' - 2xy' + \mu y = 0. \qquad (3.27)$$

$$\text{Or} \quad \frac{d}{dx}(1-x^2)y + \mu y = 0. \qquad (3.28$$

The Legendre's equation is a linear two second order Ordinary Differential Equation (ODE),at x = ± 1 are two singular points of the ODE.A solution near the ordinary point x = 0 is a power series

$$y = \sum_{m=-\infty}^{\infty} a_m x^m,\ a_m = 0 \ \text{ for all m<0.}$$

The radius of convergence for the power series is the distance from the center of the series to the nearest singular point, i.e. R = 1.Substitue the power series into the ODE, we obtain the recurrence relation

$$a_{m+2} = \frac{m(m+1)-\mu}{(m+1)(m+2)},\ \text{m=0, 1, 2 …} \qquad (3.29)$$

The recurrence relation gives two series solutions known as the Legendre's functions, where one is an odd function and the other one is an even function.

By using convergence tests, we can show that the two series are convergence for

$|x| < 1$

However, the series are generally not convergent at x = ± 1 except if μ_n= n (n+1)

Definition3.1.7. (Associated Legendre's Equation)

The Associated Legendre's Equation is defined as

$$(1-x^2)y'' - 2xy' + \left(n(n+1) - \frac{m^2}{1-x^2}\right)y=0, -1 < x < 1 \qquad (3.30)$$

$$\text{Or } \frac{d}{dx}\left((1-x^2)y'\right) + \left(n(n+1) - \frac{m^2}{1-x^2}\right)y=0, -1 < x < 1 \qquad (3.31)$$

The case m = 0 is known as the ordinary Legendre differential equation; the case of non-zero m is known as Legendre's equation. One of the difficulties in this equation is that the coefficient $(1-x^2)$ is zero at the ends of the interval $[-1, 1]$. Equations like this are called singular differential equations and are often solved by the method of Frobenius. The only bounded solutions of Legendre's equation occur when n = 0, 1, 2, . . And these solutions are polynomials

$Pn(x), -1 \leq x \leq 1$ (Legendre polynomials)

$n = 0, P0(x)$ =1

$n = 1, P0(x)$=x

$n = 2, P2(x) = \frac{1}{2}(3x^2 - 3)$

$n = 3, P3(x) = \frac{1}{2}(5x^2 - 3x)$

.

.

.

.

$$n = n, Pn(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n] \text{ (Rodriguez's formula)} \qquad (3.32)$$

For the case of μ_n= n (n+1).

The Legendre's equation of order n, for n=0, 1, 2, …

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0, -\mathbf{1} < x < 1$$

Has Solution which is called the Legendre Polynomial of degree n, $P_n(x)$. The other solution is a series solution known as the Legendre function of the second kind, $Qn(x)$. $Qn(x)$ Converges on the -1 < x < 1 but unbounded in $\mathbf{-1 \leq x \leq 1.}$

Since $P_n(x)$ and $Q_n(x)$ are linearly independent, thus the general solution to the ODE is

$$Y=c_1 P_n(x)+c_2 Q_n(x) \qquad (3.33)$$

3.2. Harmonic Equation in Spherical Coordinate System

An important problem is to find the potential inside or outside a sphere when the potential is given on the boundary.

3.2.1. Transformations of three dimensional Harmonic Equations in Spherical Coordinates System.

Spherical coordinates {r,θ, ϕ} are related to Cartesian coordinate {x, y, z} as follows In the spherical coordinate the Harmonic equation.

$$\nabla^2 u = U_{xx} + U_{yy} + U_{zz} = 0. \tag{3.34}$$

Has the following relation:

$$X = r\cos\phi\sin\theta\ ,\ r \geq 0\ ,\ y = r\sin\phi\sin\theta \qquad -\pi \leq \phi \leq \pi\ ,$$

$$z = r\ \cos\theta \qquad 0 \leq \theta \leq \pi$$

Where $r = \sqrt{x^2+y^2+z^2} = \sqrt{\rho^2+z^2}$ and $\rho = \sqrt{x^2+y^2}$, X=$\rho\cos\phi$, z=rcos θ

$$\rho_{xx} = \frac{y^2}{\rho^3}, \rho_{yy} = \frac{x^2}{\rho^3}, \phi_{xx} = \frac{2xy}{\rho^4}, \phi_{yy} = \frac{-2xy}{\rho^4}, Y = \rho\sin\phi\ ,\ \rho = r\sin\theta,$$

Using the polar form of the Laplacian with x and y we obtain the following result

Let $\rho = r\sin\theta \Rightarrow x = \rho\cos\phi$, y= $\rho\sin\phi$, where $\rho^2 = x^2 + y^2$

$$\rho_y = \frac{y}{r}, \rho_x = \frac{x}{\rho}, \phi_x = \frac{-y}{\rho^2}, \phi_y = \frac{x}{\rho^2}$$

$$\rho_{xx} = \frac{y^2}{\rho^3}, \rho_{yy} = \frac{x^2}{\rho^3}, \qquad \phi_{xx} = \frac{2xy}{\rho^4}, \phi_{yy} = \frac{-2xy}{\rho^4}$$

Now, by using chain rule in two dimension with x and y, we can find the following differentiation

$$u_x = u_\rho\rho_x + u_\phi\phi_x = u_\rho\frac{x}{\rho} + u_\phi\left(\frac{-\sin\phi}{\rho}\right)$$

$$u_{xx} = \left(u_{\rho\rho}r_x + u_{r\theta}\phi_x\right)\rho_x + \left(u_{\phi\phi}\phi_x + u_{\rho\phi}\rho_x\right)\phi_x$$

$$u_{xx} = u_{\rho\rho}(\rho_x)^2 + u_{\phi\phi}(\phi_x)^2 + 2u_{r\phi}\phi_x r_x + u_r r_{xx} + u_\phi\phi_{xx} \tag{3.35}$$

Similarly $u_y = u_\rho\rho_y + u_\theta\theta_y = u_\rho\frac{y}{\rho} + u_\theta\left(\frac{\cos\phi}{\rho}\right)$

$$u_{yy} = \left(u_{\rho\rho}\rho_y + u_{\rho\phi}\phi_y\right)\rho_y + \left(u_{\phi\phi}\phi_y + u_{\rho\phi}\rho_y\right)\phi_y$$

$$= u_{\rho\rho}\left(\rho_y\right)^2 + u_{\phi\phi}\left(\phi_y\right)^2 + 2u_{\rho\phi}\phi_y\rho_y + u_\rho\rho_{yy} + u_\phi\phi_{yy} \tag{3.36}$$

Adding (3.35)) and (3.36) we get

$$u_{xx} + u_{yy} = u_{\rho\rho}(\rho_x)^2 + u_{\phi\phi}(\phi_x)^2 + 2u_{r\phi}\phi_x r_x + u_r r_{xx} + u_\phi\phi_{xx} u_{\rho\rho}\left(\rho_y\right)^2$$

$$+ u_{\phi\phi}\left(\phi_y\right)^2 + 2u_{\rho\phi}\phi_y\rho_y + u_\rho\rho_{yy} + u_\phi\phi_{yy}$$

$$= u_{\rho\rho}(\rho_x)^2 + u_{\phi\phi}(\phi_x)^2 + 2u_{\rho\phi}\phi_x\rho_x + u_\rho r_{xx} + u_\phi\phi_{xx} + u_{\rho\rho}\left(\rho_y\right)^2 u_{\phi\phi}\left(\phi_y\right)^2 +$$

$$+ 2u_{\rho\phi}\phi_y\rho_y + u_\rho\rho_{yy} + u_\phi\phi_{yy}$$

$$= u_{\rho\rho}\left((\rho_x)^2 + \left(\rho_y\right)^2\right) + u_{\phi\phi}\left((\phi_x)^2 + \left(\phi_y\right)^2\right) + u_\rho\left(\rho_{xx} + \rho_{yy}\right) +$$

$$+ u_\phi\left(\phi_{xx} + \phi_{yy}\right) + 2u_{\rho\phi}\left(\phi_y\rho_y + \theta_y\rho_y\right)$$

$$=u_{\rho\rho}\left(\frac{x^2}{\rho^2}+\frac{y^2}{\rho^2}\right)+u_{\phi\phi}\left(\frac{x^2}{\rho^4}+\frac{y^2}{\rho^4}\right)+u_{\rho}\left(\frac{y^2}{r^3}+\frac{x^2}{r^3}\right)+2u_{\rho\phi}\left(\frac{-yx}{\rho^3}+\frac{yx}{\rho^3}\right)$$

$$=u_{\rho\rho}+\frac{1}{\rho^2}u_{\phi\phi}+\frac{1}{\rho}u_{\rho}$$

$$u_{xx}+u_{yy}=u_{\rho\rho}+\frac{1}{\rho^2}u_{\phi\phi}+\frac{1}{\rho}u_{\rho}=0 \qquad (3.37)$$

Again using the polar form of the Laplacian with z and ρ (in place of x and y) we can obtain another result.

Let z=$r\cos\theta$, ρ = r$\sin\theta$, tans θ=$\frac{\rho}{z}$ $\Rightarrow \theta=\tan^{-1}\frac{\rho}{z}, r_z=\frac{z}{r}$, $r_{zz}=\frac{\rho^2}{r^3}$,

$r_{\rho}=\frac{\rho}{r}$, $r_{\rho\rho}=\frac{z^2}{r^3}$, $\theta_z=\frac{-\rho}{r}$, $\theta_{\rho}=\frac{z}{r}$,$\theta_{zz}=\frac{z^2}{r^4}$,$\theta_{\rho\rho}=\frac{z^2}{r^4}$

$$u_z=u_r r_z+u_{\theta}\theta_z=u_r\frac{z}{r}+u_{\theta}\left(\frac{-\sin\theta}{r}\right)$$

$$u_{zz}=(u_{rr}r_z+u_{r\theta}\theta_z)r_z+(u_{\theta\vartheta}\theta_z+u_{r\theta}r_z)\theta_z$$

$$=u_{rr}(r_z)^2+u_{\theta\theta}(\theta_z)^2+2u_{r\theta}\theta_z r_z+u_r r_{zz}+u_{\theta}\theta_{zz} \qquad (3.38)$$

Similarly $u_{\rho}=u_r r_{\rho}+u_{\theta}\theta_{\rho}=u_r\frac{\rho}{r}+u_{\theta}\left(\frac{\cos\theta}{r}\right)$

$$u_{\rho\rho}=\left(u_{rr}r_{\rho}+u_{r\theta}\theta_{\rho}\right)r_{\rho}+\left(u_{\theta\vartheta}\theta_{\rho}+u_{r\theta}r_{\rho}\right)\theta_{\rho}$$

$$=u_{rr}\left(r_{\rho}\right)^2+u_{\theta\theta}\left(\theta_{\rho}\right)^2+2u_{r\theta}\theta_{\rho}r_{\rho}+u_r r_{\rho\rho}+u_{\theta}\theta_{\rho\rho} \qquad (3.39)$$

Adding (3.38) and (3.39) we get

$$u_{zz}+u_{\rho\rho}=u_{rr}(r_z)^2+u_{\theta\theta}(\theta_z)^2+2u_{r\theta}\theta_z r_z+u_r r_{zz}+u_{\theta}\theta_{zz}+u_{rr}\left(r_{\rho}\right)^2+u_{\theta\theta}\left(\theta_{\rho}\right)^2$$
$$+2u_{r\theta}\theta_{\rho}r_{\rho}+u_r r_{\rho\rho}+u_{\theta}\theta_{\rho\rho}$$

$$=u_{rr}\left((r_z)^2+\left(r_{\rho}\right)^2\right)+u_{\theta\theta}\left((\theta_z)^2+\left(\theta_{\rho}\right)^2\right)$$
$$+u_r\left(r_{zz}+r_{\rho\rho}\right)+u_{\theta}\left(\theta_{zz}+\theta_{\rho\rho}\right)+2u_{r\theta}\left(\theta_y r_y+\theta_y r_y\right)$$

$$=u_{rr}\left(\frac{z^2}{r^2}+\frac{y^2}{r^2}\right)+u_{\theta\theta}\left(\frac{z^2}{r^4}+\frac{y^2}{r^4}\right)+u_r\left(\frac{\rho^2}{r^3}+\frac{z^2}{r^3}\right)+2u_{r\theta}\left(\frac{-\rho z}{r^4}+\frac{\rho z}{r^4}\right)$$

$$u_{zz}+u_{\rho\rho}=u_{rr}+\frac{1}{r^2}u_{\theta\theta}+\frac{1}{r}u_r \qquad (3.40)$$

Now adding (3.37) and (3.40) we get

$$u_{xx}+u_{yy}+u_{zz}+u_{\rho\rho}=u_{rr}+\frac{1}{r^2}u_{\theta\theta}+\frac{1}{r}u_r+u_{\rho\rho}+\frac{1}{\rho}u_{\phi\phi}+\frac{1}{\rho}u_{\rho}=0$$

$$u_{xx}+u_{zz}+u_{yy}=u_{rr}+\frac{1}{r^2}u_{\theta\theta}+\frac{1}{r}u_r+\frac{1}{\rho^2}u_{\phi\phi}+\frac{1}{\rho}u_{\rho}=0$$

$$=u_{rr}+\frac{1}{r^2}u_{\theta\theta}+\frac{1}{r}u_r+\frac{1}{r^2 sin^2\theta}u_{\phi\phi}+\frac{1}{r\sin\theta}u_{\rho}=0 \qquad (3.41)$$

Here it remains to express $\frac{\partial u}{\partial\rho}$ in spherical coordinates. From the relation

$\theta=\tan^{-1}\left(\frac{\rho}{z}\right)$ We get $\frac{\partial\theta}{\partial\rho}=\frac{z}{r^2}$

Differentiating partially $\rho = r\sin\theta$ with respect to ρ we get

$$1=\frac{\partial r}{\partial\rho}\sin\theta + r\cos\theta\frac{\partial\theta}{\partial\rho}=\frac{\partial r}{\partial\rho}\sin\theta+cos^2\theta$$

Hence $\frac{\partial r}{\partial\rho} = \frac{1-cos^2\theta}{\sin\theta} = \sin\theta$

We know that ϕ and ρ are polar coordinates in the x y-plane.

Hence, $\frac{\partial\phi}{\partial\rho} = 0$

$$\frac{\partial u}{\partial\rho} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial\rho} + \frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial\rho} + \frac{\partial u}{\partial\phi}\frac{\partial\phi}{\partial\rho} = \frac{\partial u}{\partial r}\frac{\rho}{r} + \frac{\partial u}{\partial\theta}\frac{\cos\theta}{r} = \sin\theta\, u_r + \frac{\partial u}{\partial\theta}\frac{\cos\theta}{r}$$

Substituting this into (3.29) we get

$$= u_{rr} + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{r}u_r + \frac{1}{r^2sin^2\theta}u_{\phi\phi} + \frac{1}{r\sin\theta}\left(\sin\theta\, u_r + \frac{\partial u}{\partial\theta}\frac{\cos\theta}{r}\right) = 0$$

$$u_{rr} + \frac{1}{r^2}u_{\theta\theta} + \frac{2}{r}u_r + \frac{1}{r^2sin^2\theta}u_{\phi\phi} + \frac{1\cos\theta}{r^2\sin\theta}u_\theta = 0$$

$$u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left[u_{\theta\theta} + \cot\theta u_\theta + \frac{1}{sin^2\theta}u_{\phi\phi}\right]=0 \qquad (3.42)$$

Therefore, Equation (3 .42) is called the three dimensions of Harmonic equation in the spherical coordinate system.

3.2.2. Solutions of Harmonic Equation in Spherical Coordinate System

The solutions to the Harmonic equation are called harmonic functions. Here we review the explicit forms of the equation in spherical coordinates, illustrating their separation and integration leading to the respective spherical harmonics harmonic (Arfken and Weber) [9]. In fact that, the Harmonic equation in the familiar spherical coordinates(r,θ,ϕ) has the forms:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial\theta^2} + \cot\theta\frac{\partial u}{\partial\theta} + \frac{1}{sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}\right) = 0 \qquad (3.43)$$

To solve this equation, applying Product method as

$$u(\mathrm{r},\theta,\phi) = \mathrm{R(r)Y}(\theta,\phi) \qquad (3.44)$$

Substituting (3.44) into (3.43), we get

$$R''Y + \frac{2}{r}\frac{\partial u}{\partial r}R'Y + \frac{1}{r^2}\left(R\frac{\partial^2 Y}{\partial\theta^2} + \cot\theta\frac{\partial Y}{\partial\theta}R + \frac{1}{sin^2\theta}R\frac{\partial^2 Y}{\partial\phi^2}\right) = 0 \qquad (3.45)$$

Multiplying both sides of equation (3.45) through by $\frac{r^2}{RY}$ and then placing all the terms involving r on one side yields

$$\frac{r^2R''}{R} + \frac{2rR'}{R} = -\frac{1}{Y}\left(\frac{\partial^2 Y}{\partial\theta^2} + \cot\theta\frac{\partial Y}{\partial\theta} + \frac{1}{sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right) \qquad (3.46)$$

Since we have found that a function of r alone is equal to a function of only θ and ϕ. Since R and Y are independent variables. Therefore, equation (3.46) is true only if both sides of each must have separation constant let μ (say).

$$\frac{r^2R''}{R} + \frac{2rR'}{R} = -\left(\frac{\partial^2 Y}{Y\partial\theta^2} + \cot\theta\frac{\partial Y}{Y\partial\theta} + \frac{1}{Ysin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right) = \mu$$

$$\Rightarrow \frac{r^2R''}{R} + \frac{2rR'}{R} - \mu = -\left(\frac{\partial^2 Y}{Y\partial\theta^2} + \cot\theta\frac{\partial Y}{Y\partial\theta} + \frac{1}{Ysin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right) = \mu = -\frac{\Delta Y}{Y} = \mu$$

$$\Rightarrow r^2R'' + 2rR' - R\mu = 0 \text{ (Euler equation)} \qquad (3.47)$$

Where μ is the separation constant and

$$-\frac{\Delta Y}{Y} = -\frac{1}{Y}\left(\frac{\partial^2 Y}{\partial\theta^2} + \cot\theta\frac{\partial Y}{\partial\theta} + \frac{1}{sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right) = \mu \qquad (3.48)$$

As a consequence, the radial component R(r) satisfies the ordinary differential equation.

Assume that the solution of the angular components in (3.36) has the form

$$\Delta Y + Y\mu = \left(\frac{\partial^2 Y}{\partial\theta^2} + \cot\theta\frac{\partial Y}{\partial\theta} + \frac{1}{sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right) + Y\mu = 0 \qquad (3.49)$$

This second order partial differential equation can be regarded as the Eigen value equation for the spherical Laplacian operator $\Delta Y$and is known as the spherical Helmholtz equation.

To solve (3.49), we assume a further separation of angular variables,

$$Y(\theta,\phi) = S(\theta)Q(\phi) = 0 \qquad (3.50)$$

Substituting equation (3.50) into (3.49) we get

$$\left(S''Q + S'Q\cot\theta + SQ''\frac{1}{sin^2\theta}\right) + SQ\mu = 0 \qquad (3.51)$$

Multiplying both sides of (3.39) by $\frac{sin^2\theta}{SQ}$ we obtain

$$sin^2\theta\frac{S''}{S} + sin^2\theta\cot\theta\frac{S'}{S} + \frac{Q''}{Q} + \mu = 0$$

$$= sin^2\theta\frac{S''}{S} + \sin\theta\cos\theta\frac{S'}{S} + \frac{Q''}{Q} + \mu = 0$$

$$sin^2\theta\frac{S''}{S} + \sin\theta\cos\theta\frac{S'}{S} + \mu = -\frac{Q''}{Q} \qquad (3.52)$$

The left hand side depends only on the zenith coordinate θ while the right hand side depends only on the azimuthal coordinate ϕ (z axis only). Since these two angles are independent, the only way this could hold is when the two sides are equal to common separation constant, denoted by v

$$sin^2\theta\frac{S''}{S} + \sin\theta\cos\theta\frac{S'}{S} + \mu = -\frac{Q''}{Q} = v$$

$$sin^2\theta S'' + \sin\theta\cos\theta S' + sin^2\theta S\mu = -\frac{Q''}{Q} = v$$

From this equation, we get two ordinary differential equation

$$\begin{cases} sin^2\theta S'' + \sin\theta\cos\theta S' + sin^2\theta S\mu - Sv = 0 & (3.53) \\ Q'' + Qv = 0 & (3.54) \end{cases}$$

Let us see first considering equation (3. 54) satisfies the following

$$Q'' + vQ = 0$$

$$Q(0) = Q(2\pi)$$

$$Q'(0) = Q'(2\pi)$$

Now we consider different cases for a constant v in equation (3.54)

Case I: Let v=o

Since v=0, so that from equation (3.54) we have $Q'' + Q0 = 0$

$$\Rightarrow Q'' = 0,$$

Up on integrating both sides two times we get

$$\Rightarrow Q(\phi) = A\phi + B \quad \text{, since the solution linear}$$

Therefore, this equation satisfies the regular Sturm-Liouville problem

$$Q'' + vQ = 0$$
$$Q(0) = Q(2\pi)$$
$$Q'(0) = Q'(2\pi)$$
$$Q(0) = Q(2\pi)$$
$$\Rightarrow A0 + B = A2\pi + B$$
$$\Rightarrow A = 0 \text{ and}$$

and also $Q'(0) = Q'(2\pi) = A = 0$. Hence B is arbitrary. So with Eigen values and corresponding Eigen functions given by

$$Q_0(\phi) = B_0 \Rightarrow v = o \qquad (3.55)$$

Case II: Let V$=-m^2 < 0$

Again also substitute this into equation (3.42) we get

$$Q'' + Qv = Q'' - m^2 Q = 0$$

Let Q$=e^{\alpha x}$,$Q' = \alpha e^{\alpha x}, Q'' = \alpha^2 e^{\alpha x}$, now substituting this values into the above equation, we get $\alpha^2 e^{\alpha x} - m^2 e^{\alpha x} = 0$

$$\Rightarrow e^{\alpha x}(\alpha^2 - m^2) = 0 \; since \; e^{\alpha x} \neq 0$$
$$\Rightarrow \alpha^2 - m^2 = or \; \alpha = \pm m$$

So that $Q(\phi) = c_1 e^{\phi m} + c_2 e^{-\phi m}$

Applying regular Sturm Liouville problem

$$Q(0) = Q(2\pi) \Rightarrow c_1 e^{\phi 0} + c_2 e^{\phi 0} = c_1 e^{2\pi m} + c_2 e^{-2\pi m}$$
$$\Rightarrow c_1 + c_2 = c_1 e^{2\pi m} + c_2 e^{-2\pi m}$$
$$c_1(1 - e^{2\pi m}) + c_2(1 - e^{-2\pi m}) = 0$$

Again, using

$$Q'(0) = Q'(2\pi) \Rightarrow we\ have\ \ c_1 e^{\phi 0} + mc_2 e^{\phi 0} = mc_1 e^{2\pi m} + mc_2 e^{-2\pi m}$$
$$mc_1(1 - e^{2\pi m}) + mc_2(1 - e^{-2\pi m}) = 0$$

Since $1 - e^{2\pi m} \neq 0$ and $1 - e^{-2\pi m} \neq 0$, where m>0, we get

$$c_1 = c_2 = 0$$

Thus, the solution is trivial.

Case III let V$=m^2 > 0$

Again also substitute this into equation (3.54) we get

$$Q'' + Qv = Q'' + m^2 Q = 0$$

Let Q$=e^{\alpha x}$, $Q' = \alpha e^{\alpha x}, Q'' = \alpha^2 e^{\alpha x}$ now substituting these values into the above equation, we get

$$\alpha^2 e^{\alpha x} + m^2 e^{\alpha x} = 0 \Rightarrow e^{\alpha x}(\alpha^2 + m^2) = 0, since \; e^{\alpha x} \neq 0$$
$$\Rightarrow \alpha^2 + m^2 = 0 \Rightarrow \alpha = \pm mi \;.$$

Therefore, $Q(\phi) = c_1 \cos \phi m + c_2 \sin m\phi$

Here Q is periodic function because of the azimuthal angle ϕ increases from $-\pi$ to π, so Q (ϕ) must be a periodic function with period 2π

Thus, Q (ϕ) satisfies Q (ϕ) $= Q(\phi + 2\pi)$

$$c_1 \cos \phi m + c_2 \sin \phi m == c_1 \cos(m\phi + 2\phi\pi) + c_2 \sin(\phi m + 2\phi\pi)$$

$$c_1(\cos m\phi + \cos(m\phi + 2\phi\pi)) = c_2(\sin \phi m + \sin(m\phi + 2\phi\pi))$$

But $\cos \phi m + \cos(m\phi + 2\phi\pi)$= $\cos m\phi + \cos \phi m \cos 2\phi\pi - \sin m\phi \sin 2\phi\pi$.

$\cos \phi m - \cos \phi m\,(cos^2\phi\pi - sin^2\phi\pi) - 2 \sin \phi\pi \sin \phi m \cos \phi\pi$.

$\cos \phi m - \cos \phi m\,(1 - 2sin^2\phi\pi) - 2 \sin \phi\pi \sin \phi m \cos \phi\pi$=

$$\cos \phi m - \cos \phi m + 2sin^2\phi\pi - 2 \sin \phi\pi \sin \phi m \cos \phi\pi$$

$$2\sin \phi\pi\,(\sin \phi\pi + \sin \phi m \cos \phi\pi) \qquad (3.56)$$

Similarly $(\sin(m\phi + 2\phi\pi) - \sin \phi\, m)$

$$= -\sin \phi m + \sin \phi m \cos 2\phi\pi + \sin 2\phi\pi \cos \phi m$$

$$= -\sin \phi m + \sin \phi m\,(cos^2\phi\pi - sin^2\phi\pi) + 2 \sin \phi\pi \sin \phi m \cos \phi\pi$$

$$= -\sin \phi m + \sin \phi m\,(1 - 2sin^2\phi\pi) + 2 \sin \phi\pi \sin \phi m \cos \phi\pi$$

$$=-\sin \phi m + \sin \phi m - 2 \sin \phi m sin^2\phi\pi + 2 \sin \phi\pi \sin \phi m \cos \phi\pi$$

$$=2 \sin \phi\pi\,(\sin \phi m \cos \phi\pi - \sin \phi\pi \sin \phi m) \qquad (3.57)$$

Substituting the result of (3.56) and (3.57) in the original equation we get

$$2\, c_1 \sin \phi\pi\,(\sin \phi\pi + \sin \phi m \cos \phi\pi) =$$

$$2\, c_2 \sin \phi\pi\,(\sin \phi m \cos \phi\pi - \sin \phi\pi \sin \phi m)$$

$$2\, c_1 \sin \phi\pi\,(\sin \phi\pi + \sin \phi m \cos \phi\pi)$$

$$= 2\, c_2 \sin \phi\pi\,(\sin \phi m \cos \phi\pi - \sin \phi\pi \sin \phi m)$$

$$2\sin \phi\pi\,\big(c_1(\sin \phi\pi + \sin \phi m \cos \phi\pi) + c_2(\sin \phi m \cos \phi\pi - \sin \phi\pi \sin \phi m)\big) = 0$$

$$\Rightarrow\big(c_1(\sin \phi\pi + \sin \phi m \cos \phi\pi) - c_2(\sin \phi m \cos \phi\pi - \sin \phi\pi \sin \phi m)\big) \neq 0$$

$$\Rightarrow 2 \sin \phi\pi = 0$$

$$\Rightarrow \phi\pi = m\pi \Rightarrow \phi = m,\ m=0, 1, 2\ldots$$

$$Q_m(\phi) = C_m \cos \phi\theta + D_m \sin \theta \qquad (3.58)$$

With this information, we endeavor to solve the zenith equation.

Now substituting the value of v=m^2 into equation (3.43) we obtain

$$sin^2\theta S'' + \sin \theta \cos \theta\, S' + sin^2\theta S\mu - m^2 S = 0 \qquad (3.59)$$

This is not easy, and constructing analytic formulas for its solutions requires some effort. The motivation behind the following steps will not be immediately apparent to the reader, since they are the result of a long, detailed study of this important differential equation by mathematicians over the last 200 years. This equation is complicated, because it involves trigonometric functions.

As an initial simplification, we can eliminate the trigonometric functions. To this end, We invoke the change of variables

Let x=cos θ with p(cos θ)=p(x)=S(θ) for $0 < \theta < \pi$ with $\sqrt{1-x^2} \leq \sin\theta \leq 1$

Differentiating the function p with respect to θ where x=cos θ as follow:

According to chain rule

$$\frac{dp}{d\theta} = \frac{dp}{dx}\frac{dx}{d\theta} = -\sin\theta\frac{dp}{dx} = -\sqrt{1-x^2}\frac{dp}{dx}$$

$$\frac{d^2p}{d\theta^2} = \frac{d}{d\theta}\left(-\sin\theta\frac{dp}{dx}\right) = -\cos\theta\frac{dp}{dx} + \sin^2\theta\frac{d^2p}{dx^2}$$

$$= \sin^2\theta\frac{dp}{dx} - \cos\theta\frac{dp}{dx} = (1-x^2)\frac{d^2p}{dx^2} - x\frac{dp}{dx}$$

To complete our solution to the Harmonic equation on the spherical coordinates, We still now substituting these expressions into (3.59), we get

$$(1-x^2)^2\frac{d^2p}{dx^2} - 2x(1-x^2)\frac{dp}{dx} + p(1-x^2)\mu - m^2p = 0. \qquad (3.60)$$

Now also dividing both sides by $1-x^2$,

We get the generalized Legendre Equation

$$(1-x^2)\frac{d^2p}{dx^2} - 2x\frac{dp}{dx} + (\mu - \frac{m^2}{1-x^2})p \quad =0 \qquad (3.61)$$

We can obtain the following solution from (3.60)

$$T(X) = E_{nm}p_n^m(\cos\theta) + F_{nm}Q_n^m(\cos\theta) \qquad (3.62)$$

To complete our solution to the Laplace equation on the spherical coordinates, we still need to analyze the ordinary differential equation (3.47) for the radial component

In view of our analysis of the spherical Helmholtz equation, the original separation constant is $\mu = n(n+1)$ for some non-negative integer n ≥ 0, and so the radial equation takes the

$$\frac{d}{dr}\left(r^2\frac{d}{dr}R\right) - n(n+1)R = r^2R'' + 2Rr - n(n+1) = 0$$

Let $R(r) = r^\alpha, R' = \alpha r^{\alpha-1}$,$\alpha(\alpha-1)r^{\alpha-2}$,

Substituting this into the above equation, we obtain

$$r^2\alpha(\alpha-1)r^{\alpha-2} + 2r\alpha(r^{\alpha-2} - n(n+1)r^\alpha = 0$$

$$\Rightarrow \alpha^2r^\alpha - \alpha r^\alpha + 2\alpha r^\alpha - n(n+1)r^\alpha = 0$$

$$= r^\alpha\left(\alpha^2 + \alpha - n(n+1)\right) = 0.$$

Since $r^\alpha \neq 0$ .thus, $\alpha^2 + \alpha - n(n+1)) = \alpha^2 + \alpha - n^2 - n) = 0$

$$\Rightarrow (\alpha - n)(\alpha + n + 1) = 0$$

$$\Rightarrow \alpha = n \text{ or } \alpha = -n-1$$

Thus, the general solution of this equation becomes

$$\Rightarrow R(r) = Ar^n + Br^{-(n+1)} \qquad (3.63)$$

Combining the result of (3.58), (3.62) and (3.63) we get the general solution

$$U(r,\theta,\phi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\begin{bmatrix}\left(A_n r^n + B_n r^{-(n+l)}\right)(c_{nm}\cos\phi\theta + c_{nm}\sin\theta\,) + \\ +\left(E_{nm}p_n^m(\cos\theta) + F_{nm}Q_n^m(\cos\theta)\right)\end{bmatrix} \quad (3.64)$$

Hence, $F_{nm}\, Q_n^m(cos\theta)\ \rightarrow 0$, because of unbounded at $\theta = 0, \pi$.

From the above equation (3.63) we can obtain the general solution of interior Dirichlet problem as follow

$$U(r,\theta,\phi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}[A_n r^n(c_{nm}\cos\phi\theta + c_{nm}\sin\theta\,)E_{nm}p_n^m(\cos\theta)] \quad (3.65)$$

Again also we can obtain the general solution of exterior Dirichlet problem as follow

$$U(r,\theta,\phi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[B_n r^{-(n+l)}(c_{nm}\cos\phi\theta + c_{nm}\sin\theta\,)E_{nm}p_n^m(\cos\theta)\right] \quad (3.66)$$

Therefore, the general solution for a boundary-value problem in spherical coordinates can be written as

$$U(r,\theta,\phi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[A_n r^n + B_n r^{-(n+l)}\right]y_n^m(\theta,\phi) \quad (3.67)$$

Where $Y_n^m(\theta,\phi)$ is the spherical harmonic function which is defined as

$Y_n^m(\theta,\phi)=\sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)}}\,p_n^m(\cos\theta))e^{im\theta}$ and also Orthogonally is defined as$\langle Y_n^m|Y_{n'}^{m\prime}\rangle=\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}Y_n^{m^*}Y_{n'}^{m\prime}\sin\theta\, d\theta d\phi=\sigma_{nn}\sigma_{mm\prime}$ (Asmara)[1].

3.2.3. Boundary Value Problems with Azimuthal Symmetry

In this case we consider physical situations with complete rotational symmetry about the *z*-axis (azimuthal symmetry or axial symmetry). This means that the general solution is independent of ϕ that is the following equation

$$U(r,\theta)=\sum_{n=0}^{\infty}\left[A_n r^n + B_n r^{-(n+l)}\right]p_n(\cos\theta) \quad (3.68)$$

The coefficients A_n and B_n can be determined by the boundary conditions.

Hence for interior Dirichlet problem equation (3.68) is reduced to

$$U(r,\theta)=\sum_{n=0}^{\infty}A_n r^n p_n(\cos\theta) \quad (3.69)$$

But exterior Dirichlet problems of equation (3.68) is reduced to

$$U(r,\theta)=\sum_{n=0}^{\infty}B_n r^{-(n+l)}p_n(\cos\theta) \quad (3.70)$$

Examples

3.2.4 .1. Find the general solution of $U(r, \theta, \phi)$ of the interior Dirichlet problem for the Harmonic equation with the boundary condition

$$U(a,\theta,\phi)=\sin(3\theta)\cos\phi \quad (3.71)$$

Solutions:

Now, we observe that from the given information, the general solution has the following form from equation (3.64)

$$u(r,\theta,\phi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}r^n(A_{nm}\cos\phi + B_{nm}\sin m\phi)\,p_n^m(\cos\theta) \quad (3.72)$$

Applying boundary conditions, we obtain

$$u(a,\theta,\phi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}a^n(A_{nm}\cos\phi + B_{nm}\sin m\phi)\,p_n^m(\cos\theta)=\sin 3\theta\,\cos\phi.$$

But, $\sin 3\theta = \sin\theta + 2\theta = \sin\theta\cos 2\theta + \sin 2\theta\cos\theta = \sin\theta(\cos^2\theta - \sin^2\theta) + 2\sin\theta\cos^2\theta$

$= \sin\theta\left(4\cos^2\theta - 1\right)$, So that for m=1, we have

$\sin\theta\left(4\cos^2\theta-1\right)\cos\phi=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}a^n(A_{nm}\cos\phi+B_{nm}\sin m\phi)\,p_n^m(\cos\theta)$

$$=a^n(A_{n1}\cos\phi+B_{n1}\sin\phi)p_n^1(\cos\theta)$$

$$\Rightarrow \sin 3\theta\cos\phi=a^n(A_{n1}\cos\phi+B_{n1}\sin\phi)p_n^1(\cos\theta)$$

Where $\sin 3\theta=\sin\theta\left(4\cos^2\theta-1\right)$, $p_n^1(\cos\theta)=\sin\theta\,\dfrac{dp_n(\cos\theta)}{d(\cos\theta)}$,

$p_1(x)=x$, $p_3(x)=\frac{1}{2}\left(5x^3-3x\right)$.

Therefore, $\left(4\cos^2\theta-1\right)\sin\theta = \sin\theta\,[a.A_{11}.1+a^3.A_{31.}\,\frac{1}{2}\left(15\cos^2\theta-3\right)$

Which gives $A_{11}=\dfrac{-1}{5a}$, $A_{31}=\dfrac{8}{15a^3}, A_{n1}=0$, n=2, 4, 5,…

Hence, we have certain values of A_n and p_n. Thus, we conclude that the solution of our problem has the form

$$u(r,\theta,\phi)=\left(\frac{-1}{5}\right)\frac{r}{a}P_1^1(\cos\theta)\cos\phi+\frac{8}{15}\left(\frac{r}{a}\right)^3P_3^1(\cos\theta)\cos\phi$$

3.2.4.2. Find the general solution of Dirichlet problem inside the sphere with the boundary function is given by

$$u(a,\theta)=f(\theta)=\begin{cases}100 \text{ if } 0<\theta<\pi\\ 0 \quad \text{if} \frac{\pi}{2}<\theta<\pi\end{cases} \tag{3.73}$$

Since f is independent of ϕ. Thus, from equation (3.68) we have

$$u(r,\theta)=\sum_{m=0}^{\infty}A_n r^n p(\cos\theta) \tag{3.74}$$

Applying boundary conditions, we obtain

$$u(a,\theta)=\sum_{m=0}^{\infty}A_n a^n p(\cos\theta) \tag{3.75}$$

From this equation we have $A_n=\frac{2n+1}{2a^n}\int_0^{\pi}1oo\,p_n(\cos\theta)\sin\theta d\theta$.

But we know that

$x=\cos\theta \Rightarrow dx=-\sin\theta d\theta$. so that

$$A_n=50(2n+1)\int_0^1 p_n(x)dx.$$

At this point, we can use the explicit formulas for the $p_n's$ to compute as many coefficients as we wish.

For example since $p_0(x)=1$, we get, $A_0=50\int_0^1 p_0(x)dx=\ 50, p_1(x)=x$,

We get $A_1=50(3)\int_0^1 xdx.=75, p_3(x)=\frac{1}{2}(2x-3$,…..

Continuing in this manner by appealing to the explicit formulas of the $p_n's$.

We arrive at $A_0=50, A_1=75, A_3=\frac{-175}{4}, A_4=0$

Hence , the temperature inside the sphere is

$$u(r,\theta) = 50 + 75r\, p_1(cos\theta) - \frac{174}{4} r^3 p_1(cos\theta) + \cdots$$

In the following table we have used the first three non-zero terms of the series solution to approximate the temperature inside the sphere at the indicated points

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	
$u(\frac{1}{4},\theta)$	68.1	63.4	50	32	
$u(\frac{1}{2},\theta)$	82	77.5	50	18	
$u(\frac{3}{4},\theta)$	87.8	93	50	12.2	

Table 1 approximation of the temperature inside the ball

From this table we conclude that as r→ 1 , $u \to 100$ .But at $\theta \to \pi$, u *is decreasing*

3.2.4.3.Find a function *u*, harmonic inside the spherical layer $R_1 < \rho < R_2$ and such that

$$u\Big|_{r=R_1} = p_2^{(1)}(\cos\theta)\cos\phi ,\; u\Big|_{r=R_2} = p_5^{(3)}(\cos\theta)\sin\phi$$

Solution:

The mathematical formulation of the problem is

$$\begin{cases} \Delta u = o \\ u(R_1,\theta,\phi) = p_2^{(1)}(\cos\theta)\cos\phi \\ u(R_2,\theta,\phi) = p_2^{(1)}(\cos\theta)\cos\phi \end{cases} \tag{3.76}$$

The solution of this problem is written in the form

$$U(r,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \begin{bmatrix} \left(A_{nm}r^n + \frac{B_{nm}}{r^{-(n+l)}}\right)\cos\phi + \\ + \left(C_{nm}r^n + \frac{D_{nm}}{r^{(n+l)}}\right)\sin\phi \end{bmatrix} p_n^m(\cos\theta) \tag{3. 77}$$

Applying boundary condition at R_1

$$U(R_1,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \begin{bmatrix} \left(A_{nm}{R_1}^n + \frac{B_{nm}}{{R_1}^{-(n+l)}}\right)\cos\phi + \\ + \left(C_{nm}{R_1}^n + \frac{D_{nm}}{{R_1}^{-(n+l)}}\right) \end{bmatrix} p_n^m(\cos\theta)$$

$$= p_2^1(\cos\theta)\cos\phi$$

$$= \left(\left(A_{21}{R_1}^2 + \frac{B_{21}}{{R_1}^3}\right)\cos\phi + \left(C_{21}{R_1}^2 + \frac{D_{21}}{{R_1}^3}\right)\right)p_2^1(\cos\theta) +$$

$$+\left(\left(A_{21}{R_2}^2 + \frac{B_{21}}{{R_2}^3}\right)\cos\phi + \left(C_{21}{R_2}^2 + \frac{D_{21}}{{R_2}^3}\right)\right)p_2^1(\cos\theta) \tag{3.78}$$

Comparing both sides (3.78) with the same components, we obtain

$$A_{21}R_1 + \frac{B_{21}}{{R_1}^3} = 0 \tag{3.79}$$

$$C_{21}{R_1}^1+\frac{D_{21}}{{R_1}^{-3}}=1 \qquad (3.80)$$

$$A_{21}{R_2}^2+\frac{B_{21}}{{R_2}^3}=0 \qquad (3.81)$$

$$C_{21}{R_2}^2+\frac{D_{21}}{{R_2}^3}=0 \qquad (3.82)$$

Again also applying boundary condition at R_2

$$U(R_2,\theta,\phi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\begin{bmatrix}\left(A_{nm}{R_2}^n+\frac{B_{nm}}{{R_2}^{(n+l)}}\right)\cos\phi+\\ \left(+C_{nm}{R_2}^n+\frac{D_{nm}}{{R_2}^{(n+l)}}\right)\sin\phi\end{bmatrix}p_n^m(\cos\theta)$$

$=p_5^3(\cos\theta)\sin\phi$

$$=\left(\left(A_{53}{R_1}^5+\frac{B_{53}}{{R_1}^6}\right)\cos\phi+\left(C_{53}{R_1}^5+\frac{D_{53}}{{R_1}^6}\right)\sin\phi\right)p_5^3(\cos\theta)+$$

$$+\left(\left(A_{53}{R_2}^5+\frac{B_{53}}{{R_1}^6}\right)\cos\phi+\left(C_{53}{R_2}^5+\frac{D_{53}}{{R_2}^6}\right)\sin\phi\right)p_5^3(\cos\theta) \qquad (3.83)$$

Comparing both sides of (3.83) with the same components we get

$$A_{53}{R_1}^5+\frac{B_{53}}{{R_1}^6}=0 \qquad (3.84)$$

$$C_{53}{R_1}^5+\frac{D_{53}}{{R_1}^6} \qquad (3.85)$$

$$A_{53}{R_2}^5+\frac{B_{53}}{{R_1}^6} \qquad (3.86)$$

$$C_{53}{R_2}^5+\frac{D_{53}}{{R_2}^6} \qquad (3.87)$$

Where the numbersA_{nm},B_{nm},$C_{nm}$and $D_{nm}$are subject to determination. The boundary conditions yield the following systems of equations for the coefficients of the expansion:

1) $\begin{cases}A_{21}{R_1}^2+\frac{B_{21}}{{R_1}^3}=0\\ C_{21}{R_1}^2+\frac{D_{21}}{{R_1}^3}=1\\ C_{21}{R_2}^2+\frac{D_{21}}{{R_2}^3}=0\\ A_{21}{R_2}^2+\frac{B_{21}}{{R_2}^3}=0\end{cases}$ 2) $\begin{cases}A_{53}{R_1}^5+\frac{B_{53}}{{R_1}^6}=1\\ C_{53}{R_1}^5+\frac{D_{53}}{{R_1}^6}=0\\ A_{53}{R_2}^5+\frac{B_{53}}{{R_2}^3}=0\\ C_{53}{R_2}^5+\frac{D_{53}}{{R_2}^6}=0\end{cases}$

All the remaining coefficients are equal to zero. Now, solving the above systems simultaneously

From (1) we have

$$A_{21}{R_1}^2+\frac{B_{21}}{{R_1}^3}=0 \quad\Rightarrow A_{21}{R_1}^5+B_{21}=0 \qquad (3.88)$$

$$A_{21}{R_2}^2+\frac{B_{21}}{{R_2}^3}=0\Rightarrow \qquad A_{21}{R_2}^5+B_{21}=0 \qquad (3.89)$$

Subtracting (3.89) from (3.88) we get

$$A_{21}\left(r_1^5-r_2^5\right)=0\Rightarrow A_{21}=0 \text{ .since } R_1\neq 0 \text{ and } R_2\neq o$$

$$\Rightarrow B_{21}=0$$

From (1) we have also

$$C_{21}R_1^2 + \frac{D_{21}}{R_1^3} = 1 \Rightarrow C_{21}R_1^5 + D_{21} = R_1^3 \quad (3.90)$$

$$C_{21}R_2^2 + \frac{D_{21}}{R_2^3} = 0 \Rightarrow C_{21}R_2^5 + D_{21} = 0 \quad (3.91)$$

Subtracting (3.90) from. 3.9 1) we get

$$C_{21}\left(R_2^5 - R_1^5\right) = R_1^3 \Rightarrow C_{21} = \frac{-R_1^3}{R_2^5 - R_1^5} \quad (3.92)$$

Substituting (3.92) into (3.90) we get

$$D_{21} = \frac{-R_2^5 R_1^3}{R_2^5 - R_1^5} \quad (3.93)$$

From (2) we have

$$A_{53}R_1^5 + \frac{B_{53}}{R_1^6} = 1 \Rightarrow A_{53}R_1^{11} + B_{53} = R_1^6 \quad (3.94)$$

$$A_{53}R_2^5 + \frac{B_{53}}{R_2^6} = 1 \Rightarrow A_{53}R_2^{11} + B_{53} = 0 \quad (3.95)$$

Subtracting (3.94) from (3.95) we get

$$A_{53}\left(R_2^{11} - R_1^{11}\right) = -R_1^6 \Rightarrow A_{53} = \frac{-R_1^6}{R_2^{11} - R_1^{11}} \quad (3.96)$$

Substituting (3.94) into (3.96)

$$B_{53} = \frac{R_2^{11} R_1^6}{R_2^{11} - R_1^{11}} \quad (3.97)$$

$$C_{53}R_1^5 + \frac{D_{53}}{R_1^6} = 0 \Rightarrow C_{53}R_1^{11} + D_{53} = 0 \quad (3.98)$$

$$C_{53}R_2^5 + \frac{D_{53}}{R_2^6} = 0 \Rightarrow C_{53}R_2^{11} + D_{53} = 0 \quad (3.99)$$

Subtracting (3.98) from (3.99) we get $C_{53}\left(r_1^{11} - r_2^{11}\right) = 0 \Rightarrow C_{53} = 0$

Since $R_1 \neq 0$ and $R_2 \neq o$

$$\Rightarrow D_{53} = 0$$

Therefore, the harmonic function sought has the form

$$U(r,\theta,\phi) = \left(C_{21}{R_1}^2 + \frac{D_{21}}{{R_1}^3}\right)p_2^1(\cos\theta)\sin\phi + \left(A_{53}{R_1}^5 + \frac{B_{53}}{{R_1}^6}\right)\sin\phi +$$

$$\left(A_{53}{R_2}^3 + \frac{B_{53}}{{R_2}^6}\right)\cos\phi + \left(C_{53}{R_2}^5 + \frac{D_{53}}{{R_2}^6}\right)p_5^3(\cos\theta)\cos 3\phi$$

3.2.4.4.Find a function u, harmonic inside the spherical layer $a \leq r \leq b$ with boundary conditions such that

$$v(a,\theta) = \begin{cases} v_o, 0 \leq \theta \leq \frac{\pi}{2} \\ 0, \frac{\pi}{2} \leq \theta \leq \pi \end{cases} \quad (3.100)$$

$$v(b,\theta) = \begin{cases} 0, 0 \leq \theta \leq \frac{\pi}{2} \\ v_o, \frac{\pi}{2} \leq \theta \leq \pi \end{cases} \quad (3.101)$$

Solution:

From the geometry of the problem shown in above figure, the solution is assumed to be axially symmetric. Thus the general form of the solution is

$$v(r,\theta) = \sum_{n=0}^{\infty} \left[A_n r^n + B_n r^{-(n+l)}\right] p_n(\cos\theta) \quad (3.102)$$

In the region $a \leq r \leq b$ we have (note that the origin r=0 and $r = \infty$ are both excluded in this problem)

Applying boundary condition

At r=a: $v(a,\theta) = \sum_{n=0}^{\infty} \left[A_n a^n + B_n a^{-(n+l)}\right] p_n(\cos\theta) = \begin{cases} v_o, 0 \leq \theta \leq \frac{\pi}{2} \\ 0, \frac{\pi}{2} \leq \theta \leq \pi \end{cases}$ (3.103)

At r=b: $v(r,\theta) = \sum_{n=0}^{\infty} \left[A_n b^n + B_n b^{-(n+l)}\right] p_n(\cos\theta) = \begin{cases} 0, 0 \leq \theta \leq \frac{\pi}{2} \\ v_o, \frac{\pi}{2} \leq \theta \leq \pi \end{cases}$ (3.104)

Now, to determine the coefficients A_n and B_n

Multiply the above expression on both side by $p_n(\cos\theta)$ and integrate over $\int_{-1}^{1} d(\cos\theta)$ to find m[th] term, thus, at r=a: we have

$$\sum_{n=0}^{\infty} \left[A_n a^n + B_n a^{-(n+l)}\right] \int_{-1}^{1} p_n(\cos\theta) p_m(\cos\theta) d(\cos\theta)$$

$$= \int_{0}^{1} v_o P_m \cos\theta) d(\cos\theta) \quad (3.105)$$

At r=b: $\sum_{n=0}^{\infty} \left[A_n b^n + B_n b^{-(n+l)}\right] \int_{-1}^{1} p_n(\cos\theta) p_m(\cos\theta) d(\cos\theta)$

$$= \int_{-1}^{0} v_o p_m \cos\theta) d(\cos\theta) \quad (3.106)$$

Using orthogonally of the Legendre polynomials we can find the following expression

But $\int_{-1}^{1} p_n(\cos\theta) p_m(\cos\theta) d(\cos\theta) = \frac{2}{2m+1} \delta_{nm} \begin{cases} \delta_{nm} = 1\ for\ n = m \\ \delta_{nm} = 0, n \neq m \end{cases}$

$$\text{And} \int_0^1 p_n \cos\theta) d(\cos\theta) = \begin{cases} 1 = \ for\ n = 0 \\ \left(\frac{-1}{2}\right)^{\frac{n-1}{2}} \frac{(n-2)!}{\left(\frac{n+1}{2}\right)!} \quad for\ n = 1,3,\ldots \\ o\ for\ n = 2,4,\ldots \end{cases}$$

$$\text{At r=a: } \left[A_m a^n + B_m a^{-(n+l)}\right] \frac{2}{2m+1} = \left(\frac{-1}{2}\right)^{\frac{m-1}{2}} \frac{(m-2)!}{\left(\frac{m+1}{2}\right)!} v_0$$

For m>0 and m is an odd integer only

$$\text{At r=b: } \left[A_m b^m + B_m b^{-(m+l)}\right] \frac{2}{2m+1} = \left(\frac{-1}{2}\right)^{\frac{m-1}{2}} \frac{(m-2)!}{\left(\frac{m+1}{2}\right)!} v_0$$

For m>0 and m is an odd integer only

For m=0, $A_m = 0$ and $B_m = 0$ for m=2, 4…

$$\text{At r=a: } A_o + \frac{B_o}{a} = \frac{1}{2} v_o \tag{3.107}$$

$$\text{At r=b: } A_o + \frac{B_o}{b} = \frac{1}{2} v_o \tag{3.108}$$

Subtracting (3.108) from (3.107) we get $B_0(\frac{1}{a} - \frac{1}{b}) = 0 \Rightarrow B_o = 0$ and $A_o = \frac{1}{2} v_o$

For m=1

$$\text{At r=a: } A_1 a + \frac{B_1}{a^2} = \frac{3}{4} v_o \tag{3.109}$$

$$\text{At r=a: } A_1 b + \frac{B_1}{b^2} = -\frac{3}{4} v_o \tag{3.110}$$

Multiplying both sides of equation (3.109) by a^2 and Multiplying both sides of equation (3.110) by b^2 , we obtain

$$a^3 A_1 + B_1 = \frac{3}{4} a^3 v_o \tag{3.111}$$

$$\text{And } b^3 A_1 + B_1 = \frac{3}{4} b^3 v_o \tag{3.112}$$

Adding (3.111) and (3.112) we get

$$A_1 = \frac{3}{4}\left(\frac{a^2 + b^2}{a^3 - b^3}\right) v_0 \quad \text{And} \quad B_1 = -\frac{3}{4} a^2 b^2 \left(\frac{a^2 + b^2}{a^3 - b^3}\right) v_o$$

For m=3

$$\text{At r=a: } A_3 a^7 + B_3 = \frac{-7}{16} v_o a^4 \tag{3.113}$$

$$\text{At r=b: } A_3 b^7 + B_3 = \frac{7}{16} v_o b^4 \tag{(3.114}$$

Adding equation (3.113) and (3.114) we get

$$A_3 = \frac{-7}{16}\left(\frac{a^4 + b^4}{a^7 - b^7}\right) v_0 \quad \text{And} \quad B_3 = \frac{7}{16} a^4 b^4 \left(\frac{a^3 + b^3}{a^7 - b^7}\right) v_0$$

In general

$$A_n = (-1)^{n+1} \frac{2n+1}{2^{n+1}} v_o \left(\frac{a^{n+1} + b^{n+1}}{a^{2n+1} - b^{2n+1}} \right). \text{ For } n= 1,3,\ldots.. \text{ Odd integer}$$

And $B_n = (-1)^{n!!} \frac{2n+1}{2^{n+1}} v_o \left(\frac{a^n + b^n}{a^{2n+1} - b^{2n+1}} \right) a^{n+1} b^{n+1}$ for n= 1, 3,… Odd integer

But $A_o = \frac{1}{2} v_o, B_o = 0 \quad A_n = B_n = 0$ for n= 2, 4 …

Solution for $v(r,\theta)$ in a region $a \le r \le b$

$$v(r,\theta) = \sum_{n=0}^{\infty} \left[A_n b^n + B_n b^{-(n+l)} \right] p_n(\cos\theta)$$

$$v(r,\theta) = \frac{1}{2} v_o + \sum_{n=odd}^{\infty} \left((-1)^{(n+1)!!} \frac{(2n+1)}{2^{n+1}} v_o \left(\frac{a^n + b^n}{a^{2n+1} - b^{2n+1}} \right) \right) r^n +$$

$$(-1)^{n!!} \frac{(2n+1) a^{n+1} b^{n+1}}{2^{n+1}} v_o \left(\frac{a^n + b^n}{a^{2n+1} - b^{2n+1}} \right) \frac{1}{r^{n+1}} p_n(\cos\theta)$$

$$v(r,\theta) = \frac{1}{2} v_o + + \frac{3}{4} v_o \left[\left(\frac{a^2 + b^2}{a^3 - b^3} \right) r - a^2 b^2 \left(\frac{a+b}{a^3 - b^3} \right) \frac{1}{r^2} \right] p_1(\cos\theta) -$$

$$+ \frac{7}{16} v_o \left[\left(\frac{a^4 + b^4}{a^3 - b^3} \right) r^3 - a^4 b^4 \left(\frac{a^3 + b^3}{a^7 - b^7} \right) \frac{1}{r^4} \right] p_3(\cos\theta) + \ldots\ldots$$

Note that if $b \to \infty$ then $v(r,\theta) = \frac{1}{2} v_o$ is the potential outside the two hemispheres held at potential v_o and 0.

If $a \to 0$ then

$$v(r,\theta) = \frac{1}{2} v_o + \frac{3}{4} v_o \left(\frac{r}{b} \right)^2 p_1(\cos\theta) - \frac{7}{16} v_o \left(\frac{r}{b} \right)^3 p_3(\cos\theta) + \ldots$$

The potential inside the two hemispheres held at potential v_o and 0.

3.2.4.5. Find the general solution of Harmonic equation in three-dimensional box $D = \{0 < a < x < a, 0 < y < b, 0 < z < c\}$

Satisfying Boundary conditions on the six sides as

$$u(0,y,z) = f_1(y,z), u(a,y,z) = f_1(y,z), 0 < y < b, 0 < z < c$$
$$u(x,0,z) = g_1(x,z), u(x,b,z) = g_2(x,z), 0 < x < a, 0 < z < c$$
$$u(x,y,0) = h_1(x,y), u(x,y,c) = h_2(x,y), 0 < x < a, 0 < y < b.$$

Solution:

The solution of this problem can be obtained by summing the solutions of six problems .This problem can be solved by separating it into six problems, each of which has one non homogeneous boundary condition and the rest zero. Thus, determining each solution as in the preceding problem in section (3.1.2) and then adding the all results, we get the general solution.

1. Consider for u_1 with boundary conditions

u_1 (o, y, z) =0 X=sin αx
u_1 (x, 0, z) = 0 ⇒ Y = sin βy
u_1 (x, y, o) = 0 Z=sinh $\sqrt{\alpha^2+\beta^2}z$
u_1 (a, y, z) =0 ⇒ $\alpha_n = \frac{n\pi}{a}$=sin αx
u_1 (x, b, z) = 0 ⇒ $\beta_m = \frac{m\pi}{b}$

Notice that only the sine functions are used and also only the hyperbolic sine. The reason for the latter is that the potential must vanish at z = 0. This condition rules out the use of the hyperbolic cosine which is not zero at zero argument. The cosine could be used but is not needed as the sine functions with arguments introduced above are complete on the appropriate intervals. The coefficients in the sum for the solution are determined by looking at the potential on the top face of the box:

u_{1nm} (x, y, z) = $X_m Y_m Z_{nm}$

u_1 (X, y, z) =$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{nm} \sin nx \sin my) \sinh\sqrt{{\alpha_n}^2+{\beta_m}^2}z$

Initial condition u(x, y, c) = f (x, y), 0 ≤ x ≤ a, 0 ≤ y ≤ b

u_1(x, y, c) = f (x, y) = $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{nm} \sin nx \sin my) \sinh\sqrt{{\alpha_n}^2+{\beta_m}^2}c$

$$where\ A_{nm} == \frac{4}{ab \sinh\sqrt{{\alpha_n}^2+{\beta_m}^2}(c)} \int_0^a \int_0^b f_1(x,y) \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b}\, dx\, dy$$

is called the double flourier series coefficient . Similarly we can find the values of $u_2, u_3, u_4 u_5$ & u_6

Therefore, we have the general solutions of the form:

U=$u_1+u_2+u_3+u_4+u_5+u_6 =$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{nm} \sin n\frac{n\pi x}{a} \sin\frac{m\pi y}{b} \sinh\sqrt{{\alpha_{1n}}^2+{\beta_{1m}}^2}\, z$$
$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} B_{nm} \sin n\frac{n\pi x}{a} \sin\frac{m\pi y}{b} \sinh\sqrt{{\alpha_{2n}}^2+{\beta_{2m}}^2}\,(c-z)$$
$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{nm} \sin n\frac{n\pi y}{b} \sin\frac{m\pi z}{c} \sinh\sqrt{{\alpha_{3n}}^2+{\beta_{3m}}^2}\, x$$
$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} D_{nm} \sin n\frac{n\pi y}{b} \sin\frac{m\pi z}{c} \sinh\sqrt{{\alpha_{4n}}^2+{\beta_{4m}}^2}\,(a-x)$$
$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} E_{nm} \sin n\frac{n\pi x}{a} \sin\frac{m\pi z}{c} \sinh\sqrt{{\alpha_{5n}}^2+{\beta_{5m}}^2}$$
$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} F_{nm} \sin\frac{n\pi x}{a} \sin\frac{m\pi z}{c} \sinh\sqrt{{\alpha_{6n}}^2+{\beta_{6m}}^2}\,(b-y)$$

Where $A_{nm}, B_{nm}, C_{nm}, D_{\partial nm}, E_{nm}$ & F_{nm} are called the double Fourier coefficients.

CHAPTER FOUR

4. CONCLUSION AND FUTURE SCOPE

4.1. Conclusions

In this study, we have carried out the analytic analysis of three dimensional Harmonic equations in rectangular and spherical coordinate systems. The basic mathematical tools as well as various properties of harmonic functions are discussed. Further, the Dirichlet and Neumann boundary value problems are solved using variable separable method in rectangular and spherical coordinate systems. The method of variable separable is simple, elegant and it also provides greater insight into the problem concerned.

4.2. Future Scopes

The method presented in this study is also used to solve some other instances of Harmonic equations such as: Electrostatics, steady fluid flow, the Brownian motion and the like. Two of among the methods not based on product method that can be used to find the solution of Harmonic equation problems are Double Laplace transform and image method. Therefore, the subsequent researcher can also use these two methods to provide additional information on the solution of Harmonic equation.

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