Higher order Stable Central Difference with Richardson Extrapolation Method for Second Order Self-Adjoint Singularly Perturbed Boundary Value Problems

A Thesis Submitted to the Department of Mathematics, Jimma University in Partial Fulfillment for the Requirements of the Degree of Masters of Sciences in Mathematics

(Numerical Analysis)

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DECLARATION

I hereby declare that the work which is being presented in this thesis entitled **"Higher order Stable Central Difference Method with Richardson Extrapolation for Second Order Self-Adjoint Singularly Perturbed Boundary Value Problems"** in partial fulfillment of the requirement for the degree of Masters of Science in Mathematics, submitted to Jimma University, department of Mathematics is my original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

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Table of Contents

List of Tables

List of Figures

ABSTRACT

In this thesis, higher order stable central difference scheme with Richardson extrapolation method have been presented for solving second order self-adjoint singularly perturbed boundary value problems. First, the derivatives of the differential equation are transformed into finite difference approximations that make linear system of algebraic equations in the form of a three-term recurrence relation. Secondly, applying Richardson extrapolation method and then solve by Thomas algorithm. Thirdly, investigate the consistency and stability that guarantees convergence of the proposed method very well. Then, the applicability of the proposed method is validated by implementing it with two model examples and the present method is compared with other methods reported in the literature and exact solution.

Finally, maximum absolute error for each model example was shown both by tables and graphs with different perturbation parameters and mesh sizes which shows the betterment of the present method.

CHAPTER ONE

INTRODUCTION

1.1. Background of the Study

Any differential equation obtained from a given differential equation and having the property that its solution is an integrating factor of the other is known as adjoint differential equation. If the coefficients $a_0(x)$, $a_1(x)$ and $a_2(x)$ in the differential equation of the form:

$$a_0(x)y''(x)+a_1(x)y'(x)+a_2(x)y(x)=0$$

are continuous and $a_0(x)\neq 0$ with the given domain, the obtained differential equation can be transformed into the equivalent self-adjoint equation of $\left(a(x)y'(x)\right)'+b(x)y(x)=0$ for the functions $a(x)=e^{\int\frac{a_2(x)}{a_0(x)}dx}$ and $b(x)=\dfrac{a_2(x)}{a_0(x)}a(x)$.

Singularly perturbed differential equation is a differential equation whose highest order derivative is multiplied by a small positive parameter. A self-adjoint differential equation, whose highest order derivative is multiplied by a small positive parameter, ε $(0<\varepsilon<<1)$, which has the form: $-\varepsilon\left(a(x)y'(x)\right)'+b(x)y(x)=g(x)$ is called second order self-adjoint singular perturbation problem. A singular perturbation problem is a problem containing a small positive parameter that cannot be approximated by setting the parameter value to zero.

In singularly perturbed differential problem, small positive parameter affecting the highest order derivative(s) of the differential equation which gives rise to large gradients in the solution over narrow regions of the domain. The presence of a small perturbation parameter in the differential equation typically leads to boundary layers in the solution, which makes the convergence analysis very difficult (Suayip and Niyazi, 2013). As Miller *et. al*., (1996), Boundary layer is a region of the independent variable over which the dependent variable changes rapidly.

Singularly perturbed second order two-point boundary value problem occur very frequently in fluid motion, chemical reactor theory, elasticity, diffusion in polymer, reaction- diffusion equation, control of chaotic system and so on (Kadalbajoo and Kumar, 2008). If the order of singularly perturbed differential equations of the reduced problem is reduced by one then the problem called as convection-diffusion type and if the order is reduced by two it is called reaction-diffusion type. Hence, Second order singularly perturbed self-adjoint ordinary differential equations are types of reaction-diffusion problem.

Due to the importance of these problems in real life situations, the need to develop numerical methods for approximating its solution is advantageous. But, numerically solving the singularly perturbed differential equations depends upon the small positive parameters. The solution varies rapidly in some parts of the domain and varies slowly in some other parts of the domain because of the existence of boundary layer.

The solution of second order self-adjoint singularly perturbed two point boundary value problem exhibits one or two layers. For solving this problem having two layers, the existing numerical methods give good results when the mesh size h is smaller than the perturbation parameter ε (*i.e.,* $h \le \varepsilon$). But it is expensive and time consuming process (Fasika*et al*., 2017). If we take $h \ge \varepsilon$, the existing numerical methods produce oscillatory solution and pollute the solution in the entire interval, because of boundary layer behavior. As a result, developing numerical methods for solving self-adjoint singular perturbation problems yield consideration of the researches.

Recently, different scholars like Fasika *et al*., (2016), Fasika *et al*., (2017) and Feyisa and Gemechis, (2017) have developed a higher(fourth, sixth, eighth and tenth) order compact finite difference method to solve singularly perturbed reaction diffusion problems. These authors' developed higher order compact finite difference methods, by considering the condition for the coefficients of diffusion and reaction terms are constant only. Thus, even if their methods produce more accurate numerical solution, it is restricted to treat the problems with constant coefficients of diffusion and reaction term. Also, other scholar's, Terefe *et. al*., (2016) and Yitbarek *et .al* (2017) have presented fourth and sixth order stable central difference method for solving self–adjoint singularly perturbed two –

point boundary value problem. As well in all of these currently developed numerical methods, the perturbation parameter ε is comparable with the mesh size h of the domain.

The purpose of this study is to formulate fourth order stable central finite difference scheme with Richardson extrapolation method which give more accurate results for solving second order self-adjoint singularly perturbed two point boundary value problems.

1.2. Objectives of the Study

1.2.1. General Objective

The general objective of this study is to formulate higher order stable central difference method with Richardson extrapolation for solving second order self-adjoint singularly perturbed two point boundary value problems.

1.2.2. Specific Objectives

The specific objectives of this study are:

1. To formulate forth order stable central difference method for solving second order self- adjoint singularly perturbed boundary value problem.
2. To apply Richardson extrapolation on the formulated fourth order stable central finite difference method for obtaining the sixth order convergent scheme.
3. To establish the stability and convergence of the proposed method.

1.3. Significance of the Study

The outcomes of this study may have the following importance:

- Provide some background information for other researchers who work on this area.
- Introduce the application of numerical methods in different field of studies.
- Help the graduate students to acquire research skills and scientific procedures.

1.4. Delimitation of the Study

This study delimited to solve second order self-adjoint singularly perturbed boundary value problem of the form:

$$-\varepsilon\left(a(x)y'(x)\right)' + b(x)y(x) = g(x), \quad 0 < x < 1 \tag{1.1}$$

with the boundary conditions:

$$y(0) = \alpha \quad \text{and} \quad y(1) = \beta \tag{1.2}$$

where ε is a parameter that satisfies $0 < \varepsilon << 1$, α, β are given constants and the functions $a(x)$, $b(x)$ and $g(x)$ are assumed to be sufficiently continuous differentiable functions.

CHAPTER TWO

LITERATURE REVIEW

2.1. Finite Difference Method

Most problems cannot be solved analytically, henceforth finding good approximation solutions using numerical methods will be very useful. From different classification of numerical methods like: finite difference, spectral method, finite element method, finite volume method, and so on; finite difference method seems to be the simplest approach for the numerical solution of linear differential equations, (Roos*et. al.*, 2008).Finite difference methods are one of the most widely used numerical methods to solve differential equations. It proceeds by replacing the derivatives appearing in the differential equations by finite difference approximations. The replacement or transformation of differential equation into finite difference approximations and incorporating the boundary conditions in the difference equations gives a large algebraic system of equations to be solved by different possible iterative techniques(Jain *et. al.*, 2007). Hence, the solution obtained by solving finite difference equations indicates that the solution of differential equation at the grid points or discrete solution rather than continuous solution, so that finite difference methods are called discretization methods.

In numerical analysis, Richardson extrapolation is a sequence acceleration method, used to improve the rate of convergence of a sequence. The basic idea behind extrapolation is that whenever the leading term in the error for an approximation formula is known, we can combine two approximations obtained from that formula using different values of the parameter mesh size h to obtain a higher-order approximation and the technique is known as Richardson extrapolation.

2.3. Recent development

In year 2003, Kadalbajoo and Patidar presented, 'Spline approximation method for solving self-adjoint singular perturbation problems on non-uniform grids'. In this article, a numerical method based on cubic spline with adaptive grid was given for the self-adjoint singularly perturbed two point boundary value problems of the form:

$$Ly \equiv -\varepsilon\left(a(x)y'(x)\right)' + b(x)y(x) = g(x),\ 0 < x < 1$$

subject to the boundary conditions:

$$y(0) = \alpha \quad \text{and} \quad y(1) = \beta .$$

where α, β are given constants and ε is a small positive parameter. Further, the coefficients of diffusion term $a(x)$ and the coefficient of reaction term $b(x)$ are smooth functions and satisfy the condition $a(x) \geq a > 0,\ \ a'(x) \geq 0$ and $b(x) \geq b > 0$. The scheme derived in this method is second order accurate and model numerical examples are given to support the predicted theory.

Kailash (2005) offered, 'High order fitted operator numerical method for self-adjoint singular perturbation problems'. Here, authors consider self-adjoint singularly perturbed two-point boundary value problems in conservation form. Reducing the original problem into the normal form and then using the theory of inverse monotone matrices, a fitted operator finite difference method is derived via the standard Numerov's method. The scheme thus derived is higher order accurate for moderate values of the perturbation parameter ε whereas for very small values of this parameter the method is ε -uniformly convergent with order two.

Jean *et. al*.,(2006), proposed uniformly convergent non-standard finite difference methods for self-adjoint singular perturbation problems. Author's design non-standard finite difference schemes for self-adjoint singularly perturbed two-point boundary value problems. Essential physical properties (e.g., dissipativity) of the solutions of such problems are captured in the schemes by an appropriate ere normalization of the denominator of the discrete derivative. The schemes are analyzed for ε -uniform convergence.

Kadalbajoo and Kumar (2010) proposed ‘Variable mesh finite difference method for self-adjoint singularly perturbed two-point boundary value problems’. In this article, a numerical method based on finite difference method with variable mesh is given for solving self-adjoint singularly perturbed two-point boundary value problems. To obtain parameter- uniform convergence, a variable mesh is constructed, which is dense in the boundary region and coarse in the outer region. The uniform convergence analysis of the method discussed. The original problem is reduced to its normal form and the reduced problem solved by finite difference method taking variable mesh.

Aruna and Kanth (2012) suggested, ‘A spline based computational simulations for solving self-adjoint singularly perturbed two-point boundary value problems’. They proposed a spline based computational simulations for solving self-adjoint singularly perturbed two-point boundary value problems. The original problem is reduced to its normal form and the reduced boundary value problem is treated by using difference approximations via cubic splines in tension. The convergence of the method was analyzed.

Khuri and Sayfy (2014), proposed “A patching approach for Self-adjoint singularly perturbed second-order two-point boundary value problems. In this article, the basic aim was to introduce and describe a patching approach based on a novel combination of the variational iterative method and adaptive cubic spline collocation scheme for the solution of a class of self-adjoint singularly perturbed second-order two-point boundary value problems that model various engineering problems. The domain of the problem was decomposed into two subintervals: the variational iterative method is implemented in the vicinity of the boundary layer while in the outer region the resulting problem is tackled by applying an adaptive cubic spline collocation scheme, which comprises the use of mapping/transformation redistribution functions or constructed grading functions.

Thus, it is necessary to improve the accuracy with higher order of convergence for solving second order self-adjoint singularly perturbed boundary value problems which involves variable coefficient of reaction and diffusion terms. Furthermore, for self-adjoint singularly perturbed boundary value problems with two boundary layers, it is essential to develop numerical method which produces more accurate numerical solution.

CHAPTER THREE

METHODOLOGY

3.1. Study Area and Period

The study was conducted at Jimma University under the department of Mathematics from October 2018 to June 2019.

3.2. Study Design

This study employed mixed-design (documentary review design and numerical experimentation design) on second order self-adjoint singularly perturbed boundary value problem.

3.3. Source of Information

The relevant sources of information for this study are books, published articles and related studies from internet and the experimental results obtained by writing MATLAB code for the present numerical methods.

3.4. Mathematical Procedures

In order to achieve the stated objectives, the study followed the following procedures:

1. Defining the problem,
2. Discretizing the solution domain,
3. Replacing the derivatives in the differential equation by finite difference approximations that gives the algebraic system of equations which can be solved by Thomas algorithm
4. Apply Richardson extrapolation method.
5. Establishing the stability and consistency of the proposed scheme,
6. Writing MATLAB code for the proposed scheme.
7. Validate, using numerical examples and results.

CHAPTER FOUR

DESCRIPTION OF THE METHOD, RESULTS AND DISCUSSION

4.1 Description of the Method

Consider the singularly perturbed self-adjoint boundary value problem of the form:

$$-\varepsilon\left(a(x)y'(x)\right)' + b(x)y(x) = g(x), \qquad x \in \Omega := (0,1) \tag{4.1}$$

subject to the boundary conditions :

$$y(0) = \alpha \text{ and } y(1) = \beta \tag{4.2}$$

where ε is a perturbation parameter that satisfies $0 < \varepsilon << 1$, α, β are given constants and the functions $a(x) \neq 0$, $b(x) \neq 0$ and $g(x)$ are assumed to be sufficiently continuous differentiable functions.

By product rule of differentiation, Eq. (4.1) can be re-written as:

$$-\varepsilon y''(x) + p(x)y'(x) + q(x)y(x) = f(x) \tag{4.3}$$

where $p(x) = \dfrac{-\varepsilon a'(x)}{a(x)}$, $\qquad q(x) = \dfrac{b(x)}{a(x)}$ and $\quad f(x) = \dfrac{g(x)}{a(x)}$

In order to develop the finite difference method for the problem in Eq. (4.3) the interval $[0,1]$ is divided into N equal sub-intervals. For this, we introduce set of grid points $x_i = x_0 + ih$, for $i = 0,1,2,...,N$, where $h = \dfrac{1}{N}$. For convenience, let us denote $p(x_i) = p_i$ $q(x_i) = q_i$, $y(x_i) = y_i$, $y'(x_i) = y_i',...,$ $y^{(n)}(x_i) = y_i^{(n)}$,

Assume that $y(x)$ has continuous higher order derivatives on $[0,1]$, using Taylor series expansion. To find a description of fourth order stable central difference scheme, we use Taylor's series expansion in order to get central difference formula for y_i'' and y_i' as:

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!} y_i'' + \frac{h^3}{3!} y_i''' + \frac{h^4}{4!} y_i^{(4)} + \frac{h^5}{5!} y_i^{(5)} + \frac{h^6}{6!} y_i^{(6)} + \ldots \tag{4.4}$$

$$y_{i-1} = y_i - hy_i' + \frac{h^2}{2!} y_i'' - \frac{h^3}{3!} y_i''' + \frac{h^4}{4!} y_i^{(4)} - \frac{h^5}{5!} y_i^{(5)} + \frac{h^6}{6!} y_i^{(6)} + \ldots \tag{4.5}$$

From Eq. (4.4) and (5.5) we have:-

$$\left.\begin{aligned} y_i' &= \frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6} y_i''' - \frac{h^4}{120} y_i^{(5)} + \tau_1 \\ y_i'' &= \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{h^2}{12} y_i^{(4)} + \tau_2 \end{aligned}\right\} \tag{4.6}$$

where $\tau_1 = \frac{-h^6}{7!} y_i^{(7)} + O(h^8)$ and $\tau_2 = \frac{-h^4}{360} y_i^{(6)} + O(h^6)$

Substituting Eq. (4.6) into the discrete form of Eq. (4.3) gives:

$$q_i y_i + \frac{p_i}{2h}(y_{i+1} - y_{i-1}) - \frac{\varepsilon}{h^2}(y_{i+1} - 2y_i + y_{i-1}) - \frac{p_i h^2}{6} y_i''' + \frac{\varepsilon h^2}{12} y_i^{(4)} - \frac{p_i h^4}{120} y_i^{(5)} + \tau_0 = f_i \tag{4.7}$$

where $\tau_0 = p_i \tau_1 - \varepsilon \tau_2$

Differentiating Eq. (4.3) successively and considering at discretized mesh point:

$$y_i''' = \frac{1}{\varepsilon}\left(p_i y_i'' + (p_i' + q_i) y_i' + q_i' y_i - f_i' \right) \tag{4.8}$$

$$y_i^{(4)} = \frac{1}{\varepsilon}\left(p_i y_i''' + (2p_i' + q_i) y_i'' + (p_i'' + 2q_i') y_i' + q_i'' y_i - f_i'' \right) \tag{4.9}$$

$$y_i^{(5)} = \frac{1}{\varepsilon}\left(p_i y_i^{(4)} + (3p_i' + q_i) y_i''' + (3p_i'' + 3q_i') y_i'' + (p_i''' + 3q'') y_i' + qi''' y_i + f_i''' \right) \tag{4.10}$$

Using Eq. (4.10), the term which contains $y_i^{(5)}$ from Eq. (4.7) becomes:

$$\frac{-p_i h^4}{120} y_i^{(5)} = -\frac{p_i^2 h^4}{120\varepsilon} y_i^{(4)} - \frac{p_i h^{(4)}}{120\varepsilon}(3p_i' + q_i) y_i''' - \frac{p_i h^{(4)}}{120\varepsilon}(3p_i'' + 3q_i') y_i''$$
$$\frac{-p_i h^4}{120\varepsilon}(p_i''' + 3q_i'') y_i' - \frac{p_i q_i''' h^4}{120\varepsilon} y_i + \frac{p_i h^4}{120\varepsilon} f_i''' \quad (4.11)$$

Also, from Eqs. (4.4) and (4.5) we have the central finite difference approximation:

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} + \tau_3 \text{ and } y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + \tau_4 \quad (4.12)$$

where $\tau_3 = \frac{-h^2}{6} y_i''' + O(h^4)$ and $\tau_4 = \frac{-h^2}{12} y_i^{(4)} + O(h^4)$

Substituting Eq. (4.12), into Eq. (4.11) gives:

$$\frac{-p_i h^4}{120} y_i^{(5)} = -\frac{p_i^2 h^4}{120\varepsilon} y_i^{(4)} - \frac{p_i h^4}{120\varepsilon}(3p_i' + q_i) y_i'''$$
$$-\left(\frac{p_i h^2}{120\varepsilon}(3p_i'' + 3q_i')\right)(y_{i+1} - 2y_i + y_{i-1}) \quad (4.13)$$
$$-\left(\frac{p_i h^3}{120\varepsilon}(p_i''' + 3q_i'')\right)(y_{i+1} - y_{i-1}) - \frac{p_i q_i''' h^4}{120\varepsilon} y_i + \frac{p_i h^4}{120\varepsilon} f_i''' + \tau_5$$

where $\tau_5 = -\frac{p_i h^4}{120\varepsilon}\left(p_i''' + 3q_i''\right)\tau_3 - \frac{p_i h^4}{120\varepsilon}(3p_i'' + 3q')\tau_4$

Substituting Eq. (4.13) into Eq. (4.7) and by using the approximation in Eq. (4.12) we get:

$$\left(q_i - \frac{p_i q_i''' h^4}{120\varepsilon}\right) y_i + \left(\frac{p_i}{2h} - \frac{p_i h^3}{240\varepsilon}\left(p_i''' + 3q_i''\right)\right)(y_{i+1} - y_{i-1})$$
$$-\left(\frac{\varepsilon}{h^2} + \frac{p_i^2}{120\varepsilon}(3p_i'' + 3q_i')\right)\left(y_{i+1} - 2y_i + y_{i-1}\right) - \left(\frac{p_i h^2}{6} + \frac{p_i h^4}{120\varepsilon}(3p_i' + q_i)\right) y''' \quad (4.14)$$
$$+\left(\frac{\varepsilon h^2}{12} - \frac{p_i^2 h^2}{120\varepsilon}\right) y_i^4 + \tau_6 = f_i - \frac{p_i h^4}{120\varepsilon} f_i'''$$

where $\tau_6 = \tau_0 + \tau_5$

Again, using Eq. (4.9), the term which contain $y_i^{(4)}$ from Eq.(4.14) becomes:

$$\left(\frac{\varepsilon h^2}{12} - \frac{p_i^2 h^4}{120\varepsilon}\right) y_i^4 = \left(\frac{p_i h^2}{12} - \frac{p_i^3 h^4}{120\varepsilon^2}\right) y_i''' + \left(\frac{q_i'' h^2}{12} - \frac{p_i^2 q_i'' h^4}{120\varepsilon^2}\right) y_i + \tau_7$$
$$+\left(\left(\frac{1}{12} - \frac{p_i^2 h^2}{120\varepsilon^2}\right)(2p_i' + qi)\right)(y_{i+1} - 2y_i + y_{i-1}) \tag{4.15}$$
$$+\left(\left(\frac{h}{24} - \frac{p_i^2 h^3}{240\varepsilon^2}\right)\left(p_i'' + 2q_i'\right)\right)\left(y_{i+1} - y_{i-1}\right) - \left(\frac{h^2}{12} - \frac{p_i^2 h^4}{120\varepsilon^2}\right) f_i''$$

where $\tau_7 = \left(\frac{h^2}{12} - \frac{p_i^2 h^4}{120\varepsilon^2}\right)\left(2p_i' + q_i\right)\tau_4 + \left(\frac{h^2}{12} - \frac{p_i^2 h^4}{120\varepsilon^2}\right)\left(p_i'' + 2q_i'\right)\tau_3$

Substituting Eq.(15) into Eq.(14) gives:-

$$\left(q_i - \frac{p_i q_i'' h^4}{120\varepsilon} + \frac{q_i'' h^2}{12} - \frac{p_i^2 q_i'' h^4}{120\varepsilon^2}\right) y_i$$
$$+\left(\frac{p_i}{2h} - \frac{p_i h^3}{240\varepsilon}\left(p_i''' + 3q_i''\right) + \left(\frac{h}{24} - \frac{p_i^2 h^3}{240\varepsilon^2}\right)\right)\left(p_i'' + 2q_i'\right)(y_{i+1} - y_{i-1})$$
$$-\left(\frac{\varepsilon}{h^2} + \frac{p_i h^2}{120\varepsilon}(3p_i'' + 3q_i') - \left(\frac{1}{12} - \frac{p_i^2 h^2}{120\varepsilon^2}\right)\right)\left(2p_i' + q_i\right)\left(y_{i+1} - 2y_i + y_{i-1}\right) \tag{4.16}$$
$$+\left(\frac{p_i h^2}{12} + \frac{p_i h^4}{120\varepsilon}(3p_i' + qi) - \frac{p_i h^2}{6} - \frac{p_i^3 h^4}{120\varepsilon^2}\right) y''' + \tau_8$$
$$= f_i + \left(\frac{h^2}{12} + \frac{p_i^2 h^4}{120\varepsilon^2}\right) f_i'' - \frac{p_i h^4}{120\varepsilon} f_i'''$$

where $\tau_8 = \tau_6 + \tau_7$

For simplicity, let us denote:

$$A_i = q_i - \frac{p_i q_i'' h^4}{120\varepsilon} + \frac{q_i'' h^2}{12} - \frac{p_i^2 q_i'' h^4}{120\varepsilon^2},\ B_i = \frac{p_i}{2h} - \frac{p_i h^3}{240\varepsilon}\left(p_i''' + 3q_i''\right) + \left(\frac{h}{24} - \frac{p_i^2 h^3}{240\varepsilon^2}\right)\left(p_i'' + 2q_i'\right)$$

$$C_i = \frac{\varepsilon}{h^2} + \frac{p_i h^2}{120\varepsilon}(3p_i'' + 3q_i') - \left(\frac{1}{12} - \frac{p_i^2 h^2}{120\varepsilon^2}\right)\left(2p_i' + q_i\right)$$

$$D_i = \frac{p_i h^2}{12} + \frac{p_i h^4}{120\varepsilon}(3p_i' + q_i) - \frac{p_i h^2}{6} - \frac{p_i^3 h^4}{120\varepsilon^2} \text{ and } Hh_i = f_i + \left(\frac{h^2}{12} + \frac{p_i^2 h^4}{120\varepsilon^2}\right) f_i'' - \frac{p_i h^4}{120\varepsilon} f'''$$

Then Eq. (4.16) re-written as:-

$$A_i y_i + B_i(y_{i+1} - y_{i-1}) - C_i(y_{i+1} - 2y_i + y_{i-1}) + D_i y_i''' + \tau_8 = Hh_i \tag{4.17}$$

Also, from Eq. (4.17), the term that contains y_i''' becomes:

$$D_i y_i''' = \frac{p_i D_i}{\varepsilon} y_i'' + \frac{D_i}{\varepsilon}(p_i' + q_i) y_i' + \frac{D_i q_i'}{\varepsilon} y_i - \frac{D_i}{\varepsilon} f_i'$$

Hence, considering Eq. (4.12) for y_i' and y_i'' we get:

$$D_i y_i''' = \frac{p_i D_i}{\varepsilon}(y_{i+1} - 2y_i + y_{i-1}) + \frac{D_i}{2h\varepsilon}(p_i' + q_i)\left(y_{i+1} - y_{i-1}\right) + \frac{D_i q_i'}{\varepsilon} y_i - \frac{D_i}{\varepsilon} f_i' + \tau_9 \tag{4.18}$$

where $\tau_9 = \frac{p_i D_i}{\varepsilon}\tau_4 + \frac{D_i}{\varepsilon}\tau_3$

Putting Eq. (4.18) into Eq. (4.17), gives:-

$$\begin{aligned} &\left(A_i + \frac{D_i q_i}{\varepsilon}\right) y_i + \left(B_i + \frac{D_i}{2h\varepsilon}(p_i' + q_i)\right)(y_{i+1} - y_{i-1}) \\ &-\left(C_i - \frac{p_i D_i}{\varepsilon h^2}\right)(y_{i+1} - 2y_i + y_{i-1}) + \tau_{10} = Hh(i) + \frac{D_i}{\varepsilon} f_i' \end{aligned} \tag{4.19}$$

where $\tau_{10} = \tau_8 + \tau_9$ which can be written in terms of each local truncation error

This can be written in three term recurrence relation as:-

$$-E_i y_{i-1} + F_i y_i - G_i y_{i+1} = H_i \tag{4.20}$$

where $E_i = C_i - \frac{p_i D_i}{\varepsilon h^2} + B_i + \frac{D_i}{2h\varepsilon}(p_i' + q_i),\ F_i = A_i + \frac{D_i q_i'}{\varepsilon} + 2\left(C_i - \frac{P_i D_i}{\varepsilon h^2}\right)$

$G_i = C_i - \frac{P_i D_i}{\varepsilon h^2} - B_i - \frac{D_i}{2h\varepsilon}(p_i' + q_i)$ and $H_i = Hh(i) + \frac{D_i}{\varepsilon} f_i'$

4.2. Richardson extrapolation

The basic idea behind extrapolation is that whenever the leading term in the error for an approximation formula is known, we can combine two approximations obtained from that formula using different values of the parameter mesh sizes h and $0.5h$ to obtain a higher-order approximation and the technique is known as Richardson extrapolation. This procedure is a convergence acceleration technique which consists of considering a linear combination of two computed approximations of a solution (on two nested meshes). The linear combination turns out to be a better approximation.

Since from the beginning at Eq. (4.6), we know that the truncation error of the formulated method is $O(h^4)$. Hence, we have

$$\left|y(x_i) - Y_N\right| \le C\left(h^4\right) \tag{4.21}$$

where $y(x_i)$ and Y_N are exact and approximate solutions respectively, C is constant independent of mesh sizes h.

Let Ω^{2N} be the mesh obtained by bisecting each mesh interval in Ω^N and denote the approximation of the solution on Ω^{2N} by Y_{2N}. Consider Eq. (4.21) works for any $h \neq 0$, which implies:

$$y(x_i) - Y_N \le C\left(h^4\right) + R^N, \quad x_i \in \Omega^N \tag{4.22}$$

So that, it works for any $\frac{h}{2} \neq 0$ yields:

$$y(x_i) - Y_{2N} \leq C\left(\left(\frac{h}{2}\right)^4\right) + R^{2N}, \qquad x_i \in \Omega^{2N} \tag{4.23}$$

where the remainders, R^N and R^{2N} are $O(h^6)$. A combination of inequalities in Eqs. (4.22) and (4.23) leads to $15y(x_i) - \left(16Y_{2N} - Y_N\right) = O(h^6)$, which suggests that

$$\left(Y_N\right)^{ext} = \frac{1}{15}\left(16Y_{2N} - Y_N\right) \tag{4.24}$$

is also an approximation of $y(x_i)$. Using this approximation to evaluate the truncation error, we obtain:

$$\left|y(x_i) - \left(Y_N\right)^{ext}\right| \leq Ch^6 \tag{4.25}$$

Now, using these two different solutions which are obtained by the same scheme given by Eq. (4.20), we get another third solution in terms of the two by Eq. (4.25). This is Richardson extrapolation method for the fourth order finite difference scheme only to accelerate the rate of convergence to sixth order.

4.3. Consistency of the method

When a finite difference method is used to solve a differential equation, it is important to know how accurate the resulting approximate solution is compared to the true solution. Local truncation errors refer to the differences between the original differential equation and its finite difference approximations at grid points. Local truncation errors measure how well a finite difference discretization approximates the differential equation (Zhilin*et al.*, 2018). In our case, the last truncation error in Eq. (4.19) is:

$$\tau_{10} = \tau_8 + \tau_9$$

But, form Eqs. (4.16) and (4.18), we have $\tau_8 = \tau_6 + \tau_7$ and $\tau_9 = \frac{p_i D_i}{\varepsilon}\tau_4 + \frac{D_i}{\varepsilon}\tau_3$, So that:

$$\tau_{10} = \tau_6 + \tau_7 + \frac{p_i D_i}{\varepsilon}\tau_4 + \frac{D_i}{\varepsilon}\tau_3$$

$$\tau_{10} = \tau_0 + \tau_5 + \tau_7 + \frac{p_i D_i}{\varepsilon}\tau_4 + \frac{D_i}{\varepsilon}\tau_3, \quad \text{Because of Eq.(4.14)}$$

Also, from Eqs. (4.13) and (4.15), we have:

$$\tau_5 = -\frac{p_i h^4}{120\varepsilon}\left(p_i''' + 3q_i''\right)\tau_3 - \frac{p_i h^4}{120\varepsilon}(3p_i'' + 3q')\tau_4$$

$$\tau_7 = \left(\frac{h^2}{12} - \frac{p_i^2 h^4}{120\varepsilon^2}\right)\left(2p_i' + q_i\right)\tau_4 + \left(\frac{h^2}{12} - \frac{p_i^2 h^4}{120\varepsilon^2}\right)\left(p_i'' + 2q_i'\right)\tau_3$$

Hence, the truncation errors re-written as:

$$\tau_{10} = \tau_0 - \frac{p_i h^4}{120\varepsilon}\left(p_i''' + 3q_i''\right)\tau_3 - \frac{p_i h^4}{120\varepsilon}(3p_i'' + 3q')\tau_4 + \frac{p_i D_i}{\varepsilon}\tau_4 + \frac{D_i}{\varepsilon}\tau_3$$
$$+\left(\frac{h^2}{12} - \frac{p_i^2 h^4}{120\varepsilon^2}\right)\left(2p_i' + q_i\right)\tau_4 + \left(\frac{h^2}{12} - \frac{p_i^2 h^4}{120\varepsilon^2}\right)\left(p_i'' + 2q_i'\right)\tau_3$$

$$\begin{aligned}\tau_{10} = \tau_0 &+ \left(\frac{D_i}{\varepsilon} + \left(\frac{h^2}{12} - \frac{p_i^2 h^4}{120\varepsilon^2}\right)\left(p_i'' + 2q_i'\right) - \frac{p_i h^4}{120\varepsilon}\left(p_i''' + 3q_i''\right)\right)\tau_3 \\ &+ \left(\frac{p_i D_i}{\varepsilon} + \left(\frac{h^2}{12} - \frac{p_i^2 h^4}{120\varepsilon^2}\right)\left(2p_i' + q_i\right) - \frac{p_i h^4}{120\varepsilon}(3p_i'' + 3q')\right)\tau_4\end{aligned} \quad (4.26)$$

Again, from Eqs. (4.6), (4.7) and (4.12), we have the value of:

$$\tau_0 = p_i\tau_1 - \varepsilon\tau_2, \qquad \tau_1 = \frac{-h^6}{7!}y_i^{(7)},\ \tau_2 = \frac{-h^4}{360}y_i^{(6)},\ \ \tau_3 = \frac{-h^2}{6}y_i'''' \text{ and } \tau_4 = -\frac{h^2}{12}y_i^{(4)}$$

which also re-written as:

$$\left.\begin{aligned}&\tau_0 = \varepsilon\frac{h^4}{360}y_i^{(6)} - p_i\frac{h^6}{5040}y_i^{(7)} \\ &\tau_3 = \frac{-h^2}{6}y_i''' \quad \text{and} \quad \tau_4 = -\frac{h^2}{12}y_i^{(4)}\end{aligned}\right\} \quad (4.27)$$

Substituting Eq. (4.27) into Eq. (4.26), yields:

$$\begin{aligned}\tau_{10} = \varepsilon\frac{h^4}{360}y_i^{(6)} - p_i\frac{h^6}{5040}y_i^{(7)} \\ -\frac{h^2}{6}y_i'''\left(\frac{D_i}{\varepsilon} + \left(\frac{h^2}{12} - \frac{p_i^2 h^4}{120\varepsilon^2}\right)\left(p_i'' + 2q_i'\right) - \frac{p_i h^4}{120\varepsilon}\left(p_i''' + 3q_i''\right)\right) \\ -\frac{h^2}{12}y_i^{(4)}\left(\frac{p_i D_i}{\varepsilon} + \left(\frac{h^2}{12} - \frac{p_i^2 h^4}{120\varepsilon^2}\right)\left(2p_i' + q_i\right) - \frac{p_i h^4}{120\varepsilon}(3p_i'' + 3q')\right)\end{aligned} \quad (4.28)$$

From Eq. (4.16), we have:

$$\begin{aligned} D_i &= -\frac{p_i h^2}{12} - \frac{p_i h^4}{120\varepsilon}(3p_i' + qi) - \frac{p_i^3 h^4}{120\varepsilon^2} \\ &= -h^2\left(\frac{p_i}{12} + \frac{p_i h^2}{120\varepsilon}(3p_i' + qi) + \frac{p_i^3 h^2}{120\varepsilon^2}\right) \end{aligned} \tag{4.29}$$

Considering Eq. (4.29) into Eq. (4.28), we get:

$$\begin{aligned} \tau_{10} = {} & \frac{h^4}{6} y_i'''\left(\frac{p_i}{12\varepsilon} + \frac{p_i h^2}{120\varepsilon^2}(3p_i' + qi) + \frac{p_i^3 h^2}{120\varepsilon^3} + \left(\frac{1}{12} - \frac{p_i^2 h^2}{120\varepsilon^2}\right)\left(p_i'' + 2q_i'\right) - \frac{p_i h^2}{120\varepsilon}\left(p_i''' + 3q_i''\right)\right) \\ & + \frac{h^4}{12} y_i^{(4)}\left(\frac{p_i^2}{12\varepsilon} + \frac{p_i^2 h^2}{120\varepsilon^2}(3p_i' + qi) + \frac{p_i^4 h^2}{120\varepsilon^3} + \left(\frac{1}{12} - \frac{p_i^2 h^2}{120\varepsilon^2}\right)\left(2p_i' + q_i\right) - \frac{p_i h^2}{120\varepsilon}(3p_i'' + 3q')\right) \\ & + \varepsilon \frac{h^4}{360} y_i^{(6)} - p_i \frac{h^6}{5040} y_i^{(7)} \end{aligned}$$

This can be rearranged as:

$$\begin{aligned} \tau_{10} = h^4 \Bigg[& \frac{1}{6} y_i'''\left(\frac{p_i}{12\varepsilon} + \frac{p_i h^2}{120\varepsilon^2}(3p_i' + qi) + \frac{p_i^3 h^2}{120\varepsilon^3} + \left(\frac{1}{12} - \frac{p_i^2 h^2}{120\varepsilon^2}\right)\left(p_i'' + 2q_i'\right) - \frac{p_i h^2}{120\varepsilon}\left(p_i''' + 3q_i''\right)\right) \\ & + \frac{1}{12} y_i^{(4)}\left(\frac{p_i^2}{12\varepsilon} + \frac{p_i^2 h^2}{120\varepsilon^2}(3p_i' + qi) + \frac{p_i^4 h^2}{120\varepsilon^3} + \left(\frac{1}{12} - \frac{p_i^2 h^2}{120\varepsilon^2}\right)\left(2p_i' + q_i\right) - \frac{p_i h^2}{120\varepsilon}(3p_i'' + 3q')\right) \\ & + \frac{\varepsilon}{360} y_i^{(6)} - \frac{p_i h^2}{5040} y_i^{(7)} \Bigg] \end{aligned}$$

This can be written as:

$$|TE| \le Ch^4 \tag{4.30}$$

where $TE = \tau_{10}$ and

$$\begin{aligned} C = \Bigg| & \frac{1}{6} y_i'''\left(\frac{p_i}{12\varepsilon} + \frac{p_i h^2}{120\varepsilon^2}(3p_i' + qi) + \frac{p_i^3 h^2}{120\varepsilon^3} + \left(\frac{1}{12} - \frac{p_i^2 h^2}{120\varepsilon^2}\right)\left(p_i'' + 2q_i'\right) - \frac{p_i h^2}{120\varepsilon}\left(p_i''' + 3q_i''\right)\right) \\ & + \frac{1}{12} y_i^{(4)}\left(\frac{p_i^2}{12\varepsilon} + \frac{p_i^2 h^2}{120\varepsilon^2}(3p_i' + qi) + \frac{p_i^4 h^2}{120\varepsilon^3} + \left(\frac{1}{12} - \frac{p_i^2 h^2}{120\varepsilon^2}\right)\left(2p_i' + q_i\right) - \frac{p_i h^2}{120\varepsilon}(3p_i'' + 3q')\right) \\ & + \frac{\varepsilon}{360} y_i^{(6)} - \frac{p_i h^2}{5040} y_i^{(7)} \Bigg| \end{aligned}$$

In general, the local truncation error is then defined as $TE = y - Y_N$, where y is the exact solution and Y_n is the approximate solution. If $|TE| \le Ch^p,\ p > 0$, then we say that the numerical method is $p-th$ order accurate. Hence, from Eq. (4.30) the value of $p = 4$. Thus, the developed scheme without applying Richardson extrapolation is 4th order accurate or order of convergence is $O(h^4)$.

Definition 4.1: A finite difference scheme is called consistent if the limit of truncation error *TE* is equal to zero as the mesh size *h* goes to zero (Zhilin*et al.*, 2018).

Now, by this definition the consistency of the proposed method which is given in Eq. (4.20) with the local truncation error in Eqs. (4.25) and (4.30) satisfied as:

$$\lim_{h\to 0} TE = \lim_{h\to 0} Ch^4 = \lim_{h\to 0} Ch^6 = 0$$

Thus, the proposed method is consistent.

4.4. Stability of the method

Consider the developed scheme in Eq. (4.20) which is given by:

$$-E_i y_{i-1} + F_i y_i - G_i y_{i+1} = H_i$$

But, the coefficients E_i, F_i and G_i given in terms of A_i, B_i, C_i and D_i with its values stated in Eq. (4.17). If we multiply both sides of Eq. (4.17) by h^2 and consider the limit as $h \to 0$, we get:

$$A_i = B_i = D_i = 0 \text{ and } C_i = \varepsilon \qquad (4.31)$$

Using the values in Eq. (4.31), the coefficients E_i, F_i and G_i in Eq. (4.20) becomes:

$$E_i = G_i = \varepsilon \ \text{ and } F_i = 2\varepsilon \qquad (4.32)$$

Now, after rearranging both Eqs. (4.31) and (4.32), Consider the developed scheme in Eq. (4.20), which can be written as:

$$MY = H \tag{4.33}$$

where the matrices:

$$M = \begin{bmatrix} F_1 & -G_1 & 0 & \cdots & \cdots & 0 \\ -E_2 & F_2 & -G_2 & 0 & \vdots & \vdots \\ 0 & -E_3 & F_3 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & -G_{N-2} & 0 \\ \vdots & \vdots & 0 & -E_{N-2} & F_{N-2} & -G_{N-2} \\ 0 & \cdots & 0 & 0 & -E_{N-1} & F_{N-1} \end{bmatrix} = \begin{bmatrix} 2\varepsilon & -\varepsilon & 0 & \cdots & \cdots & 0 \\ -\varepsilon & 2\varepsilon & -\varepsilon & 0 & \vdots & \vdots \\ 0 & -\varepsilon & 2\varepsilon & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & -\varepsilon & 0 \\ \vdots & \vdots & 0 & -\varepsilon & 2\varepsilon & -\varepsilon \\ 0 & \cdots & 0 & 0 & -\varepsilon & 2\varepsilon \end{bmatrix}$$

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ y_{N-2} \\ y_{N-1} \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} h^2 H_1 + \varepsilon y_0 \\ h^2 H_2 \\ \vdots \\ \vdots \\ h^2 H_{N-2} \\ h^2 H_{N-1} + \varepsilon y_N \end{bmatrix}$$

Here, M is a tri-diagonal matrix. M is irreducible if its co-diagonals contain non-zero elements only. The co-diagonal contains E_i, G_i. It is easily seen that, for sufficiently small h (*i.e.* $h \to 0$), $E_i \neq 0$ and $G_i \neq 0$, $\forall\, i = 1, 2, \cdots, N-1$.

Hence, M is irreducible. Again one can observe that, $|E_i| > 0$ and $|G_i| > 0$ and in each row of M, the sum of the two off-diagonal elements less than or equal to the modulus of the diagonal element (*i.e*. $|F_i| \geq |E_i| + |G_i|$). This proves the diagonal dominant of M. Under these conditions the Thomas algorithm is stable for sufficiently small *h*, (Kadalbajoo and Reddy, 1986).

As proved by Smith (1985), the eigenvalues of a tri-diagonal matrix $(N-1)\times(N-1)$ of matrix M are:

$$\begin{aligned} \lambda_s &= F_i - 2\sqrt{E_i G_i}\cos\frac{s\pi}{N}, \quad s = 1, 2, \ldots, N-1 \\ &= 2\varepsilon - 2\sqrt{\varepsilon^2}\cos\frac{s\pi}{N} = 2\varepsilon - 2\varepsilon\cos\frac{s\pi}{N} = 2\varepsilon\left(1 - \cos\frac{s\pi}{N}\right) \end{aligned} \tag{4.34}$$

Also, from trigonometric identity, we have: $1-\cos\frac{s\pi}{N}=2\sin^2\frac{s\pi}{2N}$.

Hence, the eigenvalues of matrix M re-written as:

$$\lambda_s = 2\varepsilon\left(2\sin^2\frac{s\pi}{2N}\right)=4\varepsilon\sin^2\frac{s\pi}{2N}\leq 4\varepsilon \tag{4.35}$$

Definition4.2: A finite difference method for the BVPs is **stable** if M is invertible and

$$\left\|M^{-1}\right\|\leq C,\quad \forall\ 0<h<h_0 \tag{4.36}$$

where C and h_0 are two constants that are independent of h,(Zhilin*et al*., 2018).

Since, matrix M is symmetric also its inverse matrix M^{-1} is symmetric and the eigenvalues $M^{-1}$is given by $\frac{1}{\lambda_s}$. Thus, by the definition 4.2, we have:

$$\left\|M^{-1}\right\|=\frac{1}{\lambda_s}=\frac{1}{4\varepsilon}\leq C$$

where C is independent of h. Hence, the developed scheme in Eq. (4.20) is *stable*.

A consistent and stable finite difference method is **convergent** by Lax's equivalence theorem (Smith 1985). Hence, as we have shown above the proposed method is satisfying the criteria for both consistency and stability which are equivalents to convergence of the method.

4.5. Numerical Examples and Results

In order to test the validity of the proposed method and to demonstrate their convergence computationally, we have taken two model examples of singularly perturbed self-adjoint second order two point boundary value problems with exact solutions. The maximum absolute errors (AE) at the nodal points are given by:

$$|AE|=\max_{1\leq i\leq N-1}|y(x_i)-(Y_N)^{ext}|$$

And the rate of convergence (R) can be calculated by the formula:

$$R=\frac{\log\left(Y_N\right)^{ext}-\log\left(Y_{2N}\right)^{ext}}{\log 2}$$

where $y(x_i)$ and $(Y_N)^{ext}$ are exact solution and numerical solution respectively, at the nodal point x_i .And for the rate of convergence Y_N and Y_{2N} are the numerical solutions obtained by the mesh size h and $\frac{h}{2}$ respectively.

Example 4.1: Consider the singularly perturbed self-adjoint problem:

$$-\varepsilon\left((1+x^2)y'(x)\right)' + (1+x-x^2)y(x) = f(x), \quad 0 < x < 1$$

subject to the boundary conditions $y(0) = y(1) = 0$, where $f(x)$ is chosen such that the exact solution is given by: $y(x) = 1 + (x-1)e^{\frac{-x}{\sqrt{\varepsilon}}} - xe^{\frac{(1-x)}{\sqrt{\varepsilon}}}$

Table 4.1: Comparison of maximum absolute errors for Example 4.1

ε	$N=8$	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$
Present Method						
2^{-3}	8.4257e-08	1.3480e-09	2.1109e-11	3.3035e-13	1.2490e-14	7.4385e-15
2^{-5}	8.5202e-07	1.4078e-08	2.2312e-10	3.5109e-12	5.4401e-14	2.3620e-14
2^{-8}	6.6238e-05	1.6978e-06	2.8399e-08	4.5168e-10	7.0878e-12	1.0836e-13
2^{-12}	4.5141e-03	4.1790e-04	2.1462e-05	9.9671e-07	1.8913e-08	3.0955e-10
Terefe*et. al*., (2016)						
2^{-3}	1.44e-03	3.64E-04	9.06E-05	2.26E-05	5.66E-06	1.41E-06
2^{-5}	3.71E-03	8.57E-04	2.08E-04	5.09E-05	1.26E-05	3.16E-06
2^{-8}	4.63E-03	1.65E-03	2.33E-04	6.09E-05	1.73E-05	4.37E-06
2^{-12}	6.76E-02	3.76E-02	7.40E-03	6.17E-04	3.54E-05	4.74E-06

Table 4.2: Comparison between with and without Richardson methods of maximum absolute errors for Example 4.1

ε	$N=8$	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$
With Richardson extrapolation						
2^{-3}	8.4257e-08	1.3480e-09	2.1109e-11	3.3035e-13	1.2490e-14	7.4385e-15
2^{-5}	8.5202e-07	1.4078e-08	2.2312e-10	3.5109e-12	5.4401e-14	2.3620e-14
2^{-8}	6.6238e-05	1.6978e-06	2.8399e-08	4.5168e-10	7.0878e-12	1.0836e-13
2^{-12}	4.5141e-03	4.1790e-04	2.1462e-05	9.9671e-07	1.8913e-08	3.0955e-10
Without Richardson extrapolation						
2^{-3}	1.9663e-05	1.2739e-06	8.1142e-08	5.0815e-09	3.1775e-10	1.9855e-11
2^{-5}	1.4465e-04	9.7270e-06	6.1899e-07	3.8861e-08	2.4324e-09	1.5212e-10
2^{-8}	7.1344e-03	5.8790e-04	3.6784e-05	2.3006e-06	1.4381e-07	9.0025e-09
2^{-12}	7.6315e-02	3.7522e-02	7.8643e-03	6.9834e-04	4.4580e-05	2.8040e-06

Table 4.3: Comparison of rate of convergence for Example 4.1

ε	$N=8$	$N=16$	$N=32$	$N=64$
With Richardson extrapolation				
2^{-5}	5.9194	5.9795	5.9898	6.0121
2^{-8}	5.2859	5.9017	5.9744	5.9938
2^{-12}	3.4332	4.2833	4.4285	5.7197
Without Richardson extrapolation				
2^{-5}	4.0747	4.0277	4.0061	4.0011
2^{-8}	3.6016	3.9980	3.9989	3.9997
2^{-12}	1.0472	2.2541	3.4933	3.9694

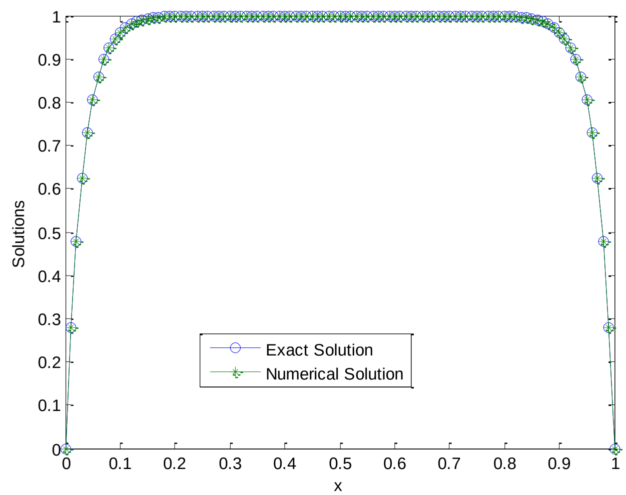

Figure 4.1: The behavior of exact and numerical solution for Example 4.1 at $\varepsilon = 10^{-3}$ and $N = 100$

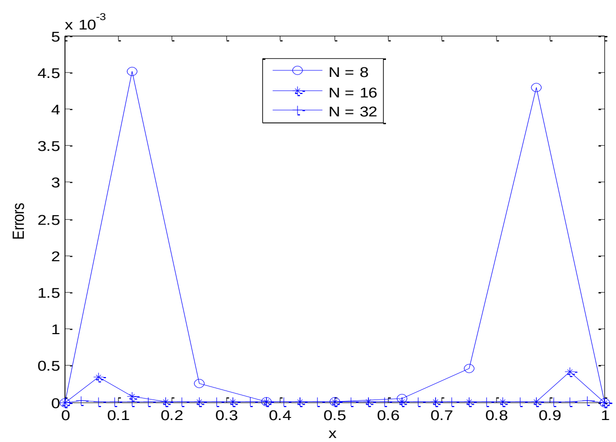

Figure 4.2: Point wise absolute errors for Example 4.1 at $\varepsilon = 2^{-12}$ with different mesh size h.

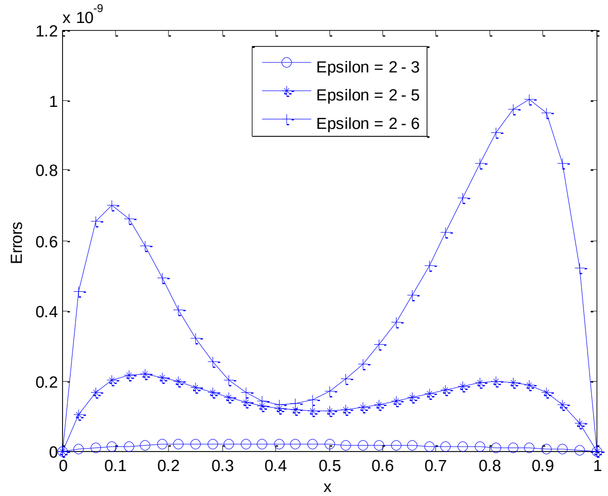

Figure 4.3: Point wise absolute errors for Example 4.1 at $N = 32$ and different perturbation parameters.

Example 4.2: Consider the following self-adjoint singular perturbation problem:

$$-\varepsilon y''(x) + \frac{4}{(x+1)^4}\left((x+1)\sqrt{\varepsilon}\right) y(x) = f(x), \qquad 0 < x < 1$$

with boundary conditions $y(0) = 2$ and $y(1) = -1$, where *f(x)* is chosen such that the exact solution is given by: $y(x) = -\cos\left(\frac{4\pi x}{x+1}\right) + \frac{3\left(\exp\left(\frac{-2x}{\sqrt{\varepsilon}(x+1)}\right) - \exp\left(\frac{-1}{\sqrt{\varepsilon}}\right)\right)}{1 - \exp\left(\frac{-1}{\sqrt{\varepsilon}}\right)}$

Table 4.4: Comparison of maximum absolute errors for Example 4.2

N	$\varepsilon = \left(\frac{1}{N}\right)^{0.25}$	$\varepsilon = \left(\frac{1}{N}\right)^{0.5}$	$\varepsilon = \left(\frac{1}{N}\right)^{0.75}$	$\varepsilon = \left(\frac{1}{N}\right)^{1.0}$
Present Method				
16	1.3049e-06	1.2226e-06	1.2374e-06	1.6007e-06
32	1.7972e-08	1.7143e-08	2.1783e-08	5.6182e-08
64	2.6954e-10	2.7457e-10	5.3161e-10	5.7507e-09
128	4.1296e-12	4.7784e-12	2.1287e-11	5.4866e-10
256	6.0396e-14	2.8866e-13	1.4388e-12	5.0873e-11
Yitbarik*et. al*., (2017)				
16	2.9718e-04	4.9658e-04	8.9268e-04	1.7181e-03
32	2.0905e-05	4.1607e-05	9.0798e-05	2.3653e-04
64	1.4884e-06	3.4999e-06	9.8228e-06	3.9036e-05
128	1.0650e-07	3.0026e-07	1.1659e-06	7.4775e-06
256	7.6403e-09	2.6424e-08	1.5321e-07	1.5612e-06

Table 4.5: Rate of convergence for Example 4.2

N	$\varepsilon=\left(\frac{1}{N}\right)^{0.25}$	$\varepsilon=\left(\frac{1}{N}\right)^{0.5}$	$\varepsilon=\left(\frac{1}{N}\right)^{0.75}$	$\varepsilon=\left(\frac{1}{N}\right)^{1.0}$
16	6.1820	6.1562	5.8280	4.8325
32	6.0591	5.9643	5.3567	3.2883
64	6.0284	5.8445	4.6423	3.3898
128	6.0954	4.0491	3.8870	3.4309

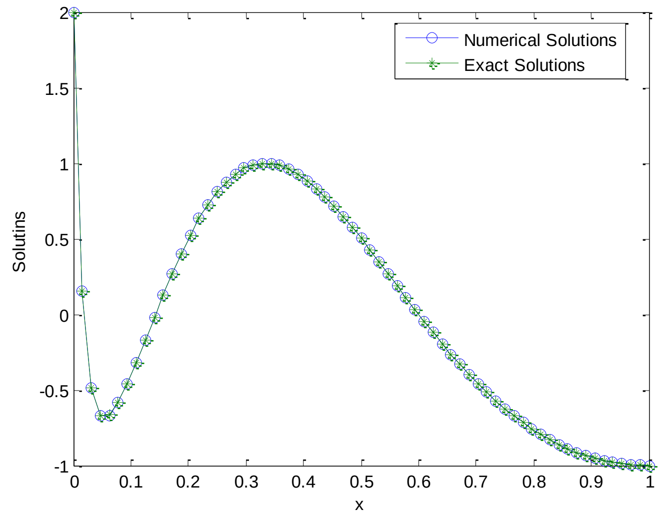

Figure 4.4: The physical behavior of solution for Example 4.2 at $N=64$ and $\varepsilon=10^{-3}$

4.6. Discussion

In this study, the higher order stable central difference with Richardson extrapolation method is presented for solving singularly perturbed self-adjoint second order boundary value problems. Hence, the Richardson extrapolation method accelerates fourth order into sixth order convergent; the developed method is higher order stable central difference method. The given domain of the problem is discretized, the derivatives of the differential equation are replaced by stable central finite difference approximations and the scheme is obtained in form of tri-diagonal algebraic system. Since, the developed method is fourth order as shown in Eq. (4.30),by applying Richardson extrapolation method its order increased to order six (see Eq. (4.25)). Then, the system is solved by using Thomas algorithm.

The stability and consistency analysis of the obtained scheme have been investigated very well to insure the convergence of the method. To validate the applicability of the proposed method, model examples are considered and numerical results displayed for different values of mesh size h and perturbation parameter ε . The numerical results are presented in Tables 4.1-4.5 and Figures4.1-4.4. The numerical results obtained by the present method have been compared with the numerical results presented by more recent authors like, Terefe *et. al*., (2016) and Yitbarik *et. al*., (2017); and it is observed that the present method gives more accurate results than some findings reported in literatures. Moreover, the absolute error decreases rapidly as number of meshes N increases which show the convergence of the proposed method.

CHAPTER FIVE

CONCLUSION AND SCOPE OF THE FUTURE WORK

5.1. Conclusion

In this thesis the higher order stable central difference scheme with Richardson extrapolation method is developed for solving second order self-adjoint singularly perturbed boundary value problems. The stability and convergence analysis is investigated and shows that the present method is of sixth order convergent. As the formulated scheme is validated by numerical model examples and results, one can realize that the maximum absolute error decreases as mesh size h decreases, which in turn shows the convergence of the computed solution and the rate of convergence is conformed to the theoretical results. Furthermore, the result of the present method is compared with previous findings and shows that, it is more accurate than some existing numerical methods reported in the literature and approximates the exact solution very well. Generally, the present method is consistent, stable, and gives more accurate numerical solution for solving second order self-adjoint singularly perturbed boundary value problems.

5.2. Scope for Future Work

In this study, higher order stable central difference method with Richardson extrapolation numerical method has been presented for solving second order self-adjoint singularly perturbed boundary value problems. The scheme proposed in this study can also be extended to sixth or more order stable central difference method with Richardson extrapolation numerical method for solving second order self-adjoint singularly perturbed boundary value problems.

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