Mathematical Modeling and Stability Analysis of Conflict among three Nations

A Thesis Submitted to the Department of Mathematics, Jimma University in Partial Fulfillment for the Requirements of the Degree of Masters of Science (MSc.) in Mathematics.

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DECLARATION

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ABSTRACT

The main objective of this research is to formulate a mathematical model that describes the conflict among three nations and to investigate its stability by taking into account their defense and restraint coefficient. In order to achieve the stated objective analytical method was employed. The result of the research indicated that the model was analyzed in terms of local stability condition by considering four different cases using Routh-Hurwitz stability criteria. Moreover, numerical examples were provided in order to verify the applicability of the research in all cases.

Key Words and Phrases: *Model formulation, Model analysis, Local stability condition and Routh-Hurwitz stability criteria.*

CHAPTER ONE

INTRODUCTION

1.1. Background of the Study

In this section terminologies and concepts such as mathematical model, stability, and stability in the linear systems used for the development of the research are introduced.

1.1.1. Mathematical Model

A mathematical model is a description of a system using mathematical concept and language. The process of developing a mathematical model is termed as mathematical modeling. Mathematical models are used in the natural sciences (such as physics, biology, earth science, meteorology), engineering disciplines (such as computer science, artificial intelligence), medicine as well as in the social sciences (such as economics, psychology, sociology, political science) (Sandip, 2014). Physicists, engineers, statisticians, operations research analysts, and economists use mathematical models most extensively. So an application to be solved is clearly outlined one or more differential equations are derived as a model (M .Braun, 1983).

Mathematical model is a powerful tool for understanding historical, practical and biologically observed phenomena which cannot be understood by verbal reasoning alone (Alder, 2001). Throughout history, there has been constant debate on cause of war. Over two thousand years ago (Thucydides, 1998) claimed that" armaments cause the war". The increase of armaments that is intended in each nation to produce consciousness strength lead to war (Sir Edward Grey, 2016). The armaments were only the symptoms of conflict ambitions and ideals of those nationalist forces which created war (Amery, 1953). Several mathematicians have tried to model and make analysis of the model to have a contribution in resolving conflicts among nations in the course of history. In modeling the situation, several simplifying assumptions such as reducing the number of variables to be included in a model, focusing on addressing specific issues of stability or focusing on the main causes for unstability among nations were taken into account.

“It is not an attempt to make a scientific description or statement on politics or to predict the date on which the next war will breakout. It is a description of what people would do if they didn’t stop to think about conflict or war” (Richardson, 1957).

The objectives of the researchers were not to forecast when the next war is going to explode among nations, rather it is to inform people what they should do in order to reduce tensions that may arise between nations and control the causes that may lead to disagreements and affect security of sustainable peace among nations. The effect of war on the economy of a country entering into conflict with other countries was also one of the variables used in modeling of the researchers.

(Gardner, 1995) presents two brief analysis of the formation of international alliances. The first is limited to a study of how defense costs should be distributed among three nation-states interested in defending themselves from a common external threat. The basis for this distribution is the length of the boundaries of the nation-states in direct contact with the enemy's territories. It is implicitly assumed that the nation-states will benefit equally from the protection provided by the alliance and the capabilities of the nation-states to contribute the resources needed for their collective defense are not considered.

In second analysis, the author studied the condition that appeared in around Bosnia (Poundstone, 1993), and shows that no alliance of any two of the three warring factions (Serbs, Croats and Muslins) could have led to a sustainable peace. (Powell, 1999) makes an extensive and systematic study of alliances by applying game theoretic technique. He analyzes the interactions of the three nation-states, two of them involved in direct confrontation that could lead to war, and a third that decide whether to take one side or the other. So that conceptual modification of the arm race model involves the assumption that the strategies available to the nation-states specify the size of the defense budgets.

When it is assumed that the size of defense budgets is the instrument that each defending nation-state can attack by the other, it is possible to include a characteristic of arm races not considered in the prisoners Dilemma model (Hamburger, 1979). Specifically, the defending nation-states do not simply maintain their military capabilities at a fixed level. Instead, they tend to increase as much as they can in an effort to gain the upper hand in their contest. This process can continue for as long as the resources of the nation-states involved permitting or until an enforceable agreement is reached among them to limit their arm building.

1.1.2. The Concept of Stability

Definition 1: A linear system of ordinary differential equation is the form

$$\frac{dx}{dt} = Ax, x \in R^n \ ,$$

where the constant coefficient matrix A is nxn square matrix and

$$\frac{dx}{dt} = \begin{pmatrix} \frac{dx_i}{dt} \\ . \\ . \\ . \\ \frac{dx_n}{dt} \end{pmatrix} \ ,$$

where i = 1, 2, 3… n.

Definition 2: (Merkin, 1997). An equilibrium point is a point x_o such that $\frac{dx}{dt}(x_o) = 0.$

Definition 3: (James, 2007). The equilibrium point $x_o = 0$ is

a) Stable if for each $\varepsilon > 0$, there is $\delta > 0$ such that $\|x(0)\| < \sigma \rightarrow \|x(t)\| < \varepsilon$, for all $t \geq 0$,

b) Unstable if it is not stable.

c) Asymptotically stable if it is stable and δ can be chosen such that

$$\|x(t_o)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0 \ .$$

Definition 4: (Lyapunov, 1992). An autonomous system of ordinary differential equation is one that has the form $y' = f(y)$. We say that y_o is a critical point of the system, if it is a constant solution of the system, namely if $f(y_o)$.

Definition 5: (Khalil, 1996). A critical point is said to be stable, if every solution which is initially close to it remains close to it for all times. It is said to be asymptotically stable, if it is stable and every solution which is initially close to it converges as $t \to \infty$.

1.1.3. Routh-Hurwitz Stability Criteria

1. Hurwitz polynomial: A polynomial with real coefficients and all its roots have negative real parts.
2. It determines if all the roots of a polynomial lie in the open LHP (left half-plane) or equivalently have negative real parts.
3. It also determines the number of roots of characteristic polynomial with positive real parts is equal to the number of changes in sign of the first column of Routh array for the given characteristic polynomial. Consider a polynomial

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + ... + a_1 s + a_0,$$

where $a_n, a_{n-1}, \ldots, a_0$ are real coefficients with $a_0 \neq 0$ and n is positive integer.

Table 1.1: Routh Array

a_n	a_{n-2}	a_{n-4}	$\cdots$
a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
b_1	b_2	$\cdots$	$\cdots$
c_1	c_2	$\cdots$	$\cdots$
. . .	. . .	. . .	. . .
m_1	$\cdots$	$\cdots$	$\cdots$

Where $b_1, b_2, c_1, ...$ are determined from the given polynomial as follows;

$$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$

$$b_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

$$c_1 = \frac{b_1 a_{n-3} - a_{n-1} b_2}{b_1} \text{ , etc.}$$

1.1.4. Stability of Linear Systems

An equilibrium point is stable if all solutions starting at nearby points stay nearby; otherwise, it is unstable. It is asymptotically stable if all solutions starting at nearby points not only stay nearby, but also tends to the equilibrium as time approaches infinity. For a linear invariant system $\frac{dx}{dt} = Ax$, the stability of equilibrium point $\frac{dx}{dt} = 0$ can be completely characterized by location of eigenvalue of matrix A. This is expressed as follows;

1. If the Eigenvalues of the Jacobian matrix all have real parts less than zero then, the linear system is stable.
2. If at least one of the Eigenvalues of Jacobian matrix has real part greater than zero then, the linear system is unstable.
3. Otherwise there is no conclusion (then we have a borderline case between stability and unstability of linear system).

Stability theory plays a central role in the system of engineering, special in the field of control systems and automation, with regard to both dynamics and control. Stability of a dynamical system is a fundamental requirement for its practical value, particularly in most real world applications (Merkin, 1997).

1.2. Statement of the Problem

Consider two neighboring countries c_1 and c_2 such that the expenditures on arms by these two countries in a given monetary unit. A simple mathematical model was constructed by assuming the notion of mutual fear, that is, the more one country spends on arms; it encourages the other one to increase its expenditure on arms. Thus, assume that each country spends on arms at a rate which is directly proportional to the existing expenditure of the other nation. This model was first developed by (Richardson, 1957)

Let $x(t)$ and $y(t)$ be the war potential or armaments of nation c_1 and c_2 respectively then, the rate of change of $x(t)$ depends on the war readiness of $y(t)$. In the most simplistic model let us represent these term by ky where k is a positive constant. With the similar argument for the rate of change of $y(t)$. We have

$$\begin{aligned} \frac{dx}{dt} &= ky \\ \frac{dy}{dt} &= lx \end{aligned} , \tag{1}$$

where l has the same property as k. In modifying the model, other than the mutual fear, we also assume that the excessive expenditure on the arms puts the country's economy in the compromising position and hence, the rate of change of one country's expenditure (the cost of armaments) have a restraining effect on $\frac{dx}{dt}$. Let us represent this by $-\alpha x$ where α is positive constant. A similar analysis holds for $\frac{dy}{dt}$. Moreover, the rate of change of $x(t)$ depends on the grievance that nation c_1 feels towards nation c_2 (Presence of continues complaint between two nations). Let us represent these terms by g and h (non-negative constants), for c_1 and c_2 respectively. Hence $x(t)$ and $y(t)$ is a solution for the linear system of differential equations;

$$\begin{aligned} \frac{dx}{dt} &= ky - \alpha x + g \\ \frac{dy}{dt} &= lx - \beta y + h \end{aligned} , \tag{2}$$

where k, α, l and β are positive constants, g and h non-negative constants.

The system of equation (2) would be analyzed by considering several important stability cases like:

1. Suppose g and h to be both zero,
2. Assume x, y, g and h are all made zero simultaneously and,
3. Let x and y to be both zero for which $g > 0$ and $h > 0$ etc.

Stability analysis in each of these cases leads to developing different stability conditions for the nations' peace condition.

In this research the model described by equation (2) is extended to a mathematical model describing conflict among three nations and investigation on the stability of the model for different causes that move a country to arm, such as differences in defense and restraint coefficients were made. Moreover, numerical examples of a particular case would be provided in order to verify the accuracy of the analysis of the model.

1.3. Objective of the study

1.3.1. General objective of the study

The general objective of this study is to formulate mathematical model that describes conflict among three nations and investigate stability of the model.

1.3.2. Specific objectives of the study

The specific objectives of the study would be to:

1. formulate mathematical model that describe conflict among three nations that have defense and restraint coefficients.
2. analyze local stability of the mathematical model of conflict among three nations by using Routh-Hurwitz stability criteria.

1.4. Significance of the study

The output of this research can be used to open a discussion among three nations to stabilize their region or reduce the causes of conflict that may lead to war and apply the stability conditions to guarantee sustainable peace. Moreover, it can be used as a stepping stone for further researchers solving problems related to conflict among nations.

1.5. Delimitation of the study

This study was delimited to discussing the stability analysis of formulated mathematical model of conflict among three nations taking into account few variables that cause grievance or conflict among nations.

CHAPTER TWO

LITERATURE REVIEW

2.1. History and Definition of Differential Equation

The subject of differential equation originated in the study of calculus by Isaac Newton (1642-1716) and Gottfried Wilhelm Leibniz (1646-1716) independently in the seventeenth century. Differential equation is a mathematical equation that relates some functions with its derivatives. In the applications, the functions usually represent physical quantities, the derivatives represent their rates of change, and the equation defines relationship between the two. Because such relations are extremely common, differential equations play a prominent role in many disciplines including engineering, physics, economics, biology and other social science.

2.2. History of Mathematical Model of Conflict among Nations

The main objective of two nation-states in a direct confrontation is to protect themselves against the possibility of destruction or domination by the other. As mentioned by (Reynolds, 1994) the ability of nation-states to defend themselves and survive depends to a large extent on their economic capabilities. So the nation-state is involved to feel more secure if it acquires weapons, even if this is done purely from defensive reasons. On the other hand, since weapons can be used as much for defense as for attack. The other nation-states can never be sure of the intension of the first. For, this reason, it feels obliged to produce or purchase weapons to prepare itself to defend its interest.

(Mitchell, 1985) indicates that the investigations available of its causes, initiation, process and consequences in its economic, political, social and military aspects reflect the great complexity of the phenomena. The decision of whether to attack first or only respond when attacked is studied in the first type. (Poundstone, 1993) provides that this analysis was particularly relevant during the cold war, in view of the progressive development of nuclear weapons and their delivery. (Taylor, 1995) indicates that particularly at the time when USA had the monopoly on these weapons and systems, a first strike against the USSR was considered, at least by several distinguished and influential personalities, to be the most recommendable policy. (Maoz, 1985) extends these analyses with a model that studies international relationships beginning at the pre-conflict stages, followed by the initiation of a war, its management and ending with a study of conflict termination. For instance, by (Davis, 1970) and (Gintis, 2000) usually assumed that what one of the nation-states gains, the other loses and vice-verse. For this reason, they form part of what are called zero sum games.

A substantially more sophisticated analysis of war is presented by (Varoufakis, 1991) who studies the Peloponnesian war between Athens and Sparta. As (Brams, 1975) the Battle of Bismarck Sea that took place during War World II and involved Japanese and American naval and air forces is frequently used example. (Mishal, 1990) have a substantially more elaborate the Israel-Palestinian conflict. This would make it possible to consider that in the two nation-states, different population groups are the real actors in the conflict. Another additional aspect not taken into consideration in the framework being outlined is that all other nation-states in the Arab World are much more than simply spectators at the interactions between Israelis and Palestinians. Finally, at least in a part in view of the economic significance of the Arab World, China, European Union, Japan, Russia and USA could consider actors more or less directly involved in the conflict.

CHAPTER THREE

METHODOLOGY

3.1. Study area and Period

This research was conducted at Jimma University in the department of Mathematics from September 2016 to October 2017 G.C.

3.2. Study Design

In order to achieve the objective of the research analytical method was employed.

3.3. Source of Information

The sources of information for this research were data obtained from different reference books, internet and different published research articles.

3.4. Mathematical Procedures

This research is based on the following procedures

1. Constructing mathematical model that describe conflict among three nations system,
2. Determining characteristic polynomial of the formulated mathematical model of conflict among three nations system,
3. Demonstrating local stability analysis of the formulated mathematical model of conflict among three nations system by Routh-Hurwitz criteria,
4. Verifying the formulated mathematical model of conflict among three nations system using numerical example.

CHAPTER FOUR

RESULT AND DISCUSSION

4.1. Mathematical Model Formulation

Our model is based on the work of (Richardson, 1957). We consider two neighboring countries c_1 and c_2. Let $x(t)$ and $y(t)$ be the war potential or armaments of nation c_1 and c_2 respectively. The rate of change of $x(t)$ depends on the war readiness of $y(t)$. In a most simplistic model let us represent these term by ky where k is a positive constant. In similar argument for rate of change of $y(t)$. We have

$$\begin{aligned}\frac{dx}{dt} &= ky \\ \frac{dy}{dt} &= lx\end{aligned} \quad , \tag{4.1}$$

where l has the same property as k. In modifying the model, other than the mutual fear, we also assume that the excessive expenditure on the arms puts the country's economy in the compromising position and hence, the rate of change of one country's expenditure (the cost of armaments) has a restraining effect on $\frac{dx}{dt}$. Let us represent this by $-\alpha x$ where α is positive constant. A similar analysis holds for $\frac{dy}{dt}$. Moreover, the rate of change of $x(t)$ depends on the grievance that nation c_1 feels towards nation c_2 (Presence of continues complaint between two nations). Let us represent these terms by g and h (non-negative constants), for c_1 and c_2 respectively. Hence $x(t)$ and $y(t)$ is a solution for the linear system of differential equation;

$$\begin{aligned}\frac{dx}{dt} &= ky - \alpha x + g \\ \frac{dy}{dt} &= lx - \beta y + h\end{aligned} \tag{4.2}$$

where k, α, l and β are positive constants, g and h non-negative constants.

The system of equation (4.2) has several important implications.

I) Suppose g and h are both zero. Then the system reduces to

$$\frac{dx}{dt} = ky - \alpha x$$
$$\frac{dy}{dt} = lx - \beta y \qquad (4.3)$$

Clearly $(0,0)$ is the only steady state solution, provided that $\alpha\beta - lk \neq 0$. The characteristic equation given by

$$\begin{vmatrix} -\alpha-\lambda & k \\ l & -\beta-\lambda \end{vmatrix}$$

$$\lambda^2 - (-\alpha-\beta)\lambda + \alpha\beta - lk = 0.$$

Hence, the system is stable if $\alpha\beta - lk > 0$ or $\alpha\beta > lk$ by Routh-Hurwitz stability criteria. This implies if the product of the rate of depreciation $(\alpha\beta)$ on the expenditure of arms of both countries c_1 and c_2 is greater than the product of expenditure (lk) on arms of both countries, the system will be stable and countries will spend an allocated amount of money on arms. So the economy of the country is not compromised. Note that in this case since $(0,0)$ is the only equilibrium solution.

II) If x, y, g and h are all made zero simultaneously, then $x(t)$ and $y(t)$ will remain zero for the rest of the time. This ideal condition is permanent peace by disarmament and satisfaction.

III) Mutual disarmament without satisfaction does not lead to permanent peace. Indeed, suppose x and y vanishes at some time t_o.

$$\frac{dx}{dt} = g$$
$$\frac{dy}{dt} = h \ , \qquad (4.4)$$

then x and y will not remain zero, if $g > 0$ and $h > 0$. Instead both nations will rearm.

Now assume that each country spends on arm at the rate which is directly proportional to the existing expenditure of other nation.

$$\begin{aligned} \frac{dx}{dt} &= ky \\ \frac{dy}{dt} &= lx \end{aligned} ,$$

and has the following solution given by

$$\begin{aligned} x &= A_1 e^{\sqrt{kl}\,t} + A_2 e^{-\sqrt{kl}t} \\ y &= B_1 e^{\sqrt{kl}t} + B_2 e^{-\sqrt{kl}t} \end{aligned}$$

Thus, $x, y \to \infty$ as $t \to \infty$, if A_1 and B_1 are positive and we conclude that both the countries c_1 and c_2 spend more and more money on arms with increasing time and no limits on expenditure. This infinity can be interpreted as war.

Let us consider equation (4.2) again

$$\begin{aligned} \frac{dx}{dt} &= ky - \alpha x + g \\ \frac{dy}{dt} &= lx - \beta y + h \end{aligned}$$

Equating this differential equation to zero, we have

$$\begin{aligned} ky - \alpha x + g &= 0 \\ lx - \beta y + h &= 0 \end{aligned} ,$$

has unique solution steady state solution is given by

$$x = \frac{\beta g + kh}{\alpha\beta - kl},\ y = \frac{gl + \alpha h}{\alpha\beta - kl},$$

provided that $\alpha\beta - lk \neq 0$.

We are interested in determining whether this equilibrium solution is stable or unstable.

To this end, we write equation above in the form of $\dot{W} = Aw + f$, where

$$w(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},\ f = \begin{pmatrix} g \\ h \end{pmatrix}\ and\ A = \begin{pmatrix} -\alpha & k \\ l & -\beta \end{pmatrix}$$

The equilibrium solution is $w = w_o = \begin{pmatrix} x \\ y \end{pmatrix}$, where $Aw_o + f = 0$.

Setting $z = w - w_o$, we obtain that

$$\dot{Z} = \dot{W} = Aw + f = A(z + w_o) + f$$
$$= Az + Aw_0 + f = Az.$$

So the equilibrium solution $w(t) = w_0$ of $Aw + f$ is stable, if and only if, $z = 0$ is a stable solution of $\dot{Z} = Az$.

The present model is based on the following;

Consider the case of three nations where governments move to arm based on the magnitude of other nation's armaments. Now based on the above discussion made for two nations, we can extend the mathematical model to take into account for three nations and obtain the following system of equations:

$$\begin{aligned} \frac{dx}{dt} &= k_1 y + l_1 z - \alpha_1 x + g_1 \\ \frac{dy}{dt} &= k_2 x + l_2 z - \alpha_2 y + g_2 \\ \frac{dz}{dt} &= k_3 x + l_3 y - \alpha_3 z + g_3, \end{aligned} \qquad (4.5)$$

where k_1, k_2, k_3 and l_1, l_2, l_3 are defense coefficients of nations c_1, c_2, c_3 respectively and α_1 , α_2 and α_3 are restraint coefficients of nations c_1, c_2 and c_3 respectively. Moreover, $k = l$ i.e. $k_1 = l_1$, $k_2 = l_2$, $k_3 = l_3$. Since k and l are defense coefficients of one nation with respect to other two nations respectively. Similarly, g_1, g_2 and g_3 are non-negative constant.

4.2. Stability Analysis of Formulated Mathematical Model

The formulated mathematical model given by equation (4.5) has different cases of stability condition analyzed as follows;

1. The case where the three nations have the same defense coefficient k and restraint coefficient α for α and k non-negative real number.

$$\frac{dx}{dt} = ky + kz - \alpha x + g_1$$
$$\frac{dy}{dt} = kx + kz - \alpha y + g_2 \qquad (4.6)$$
$$\frac{dz}{dt} = kx + ky - \alpha z + g_3$$

The above differential equation (4.6) would be written as:

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} = \begin{pmatrix} -\alpha & k & k \\ k & -\alpha & k \\ k & k & -\alpha \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}$$

in matrix form, we can write as

$$\frac{dm}{dt} = Aw + f, \text{ where}$$

$$\frac{dm}{dt} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix}, \; A = \begin{pmatrix} -\alpha & k & k \\ k & -\alpha & k \\ k & k & -\alpha \end{pmatrix}, \; w = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } f = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix},$$

From the coefficient matrix we determine characteristic polynomial

$$\left| A - \lambda I \right| = 0$$

$$\begin{vmatrix} -\alpha - \lambda & k & k \\ k & -\alpha - \lambda & k \\ k & k & -\alpha - \lambda \end{vmatrix} = 0$$

$$-(\alpha + \lambda)\left[(\alpha + \lambda)^2 - k^2\right] - k\left[-\alpha k - k\lambda - k^2\right] + k\left[k^2 + \alpha k + k\lambda\right] = 0$$

$$-(\alpha + \lambda)^3 + \alpha k^2 + k^2\lambda + \alpha k^2 + k^2\lambda + k^3 + k^3 + \alpha k^2 + k^2\lambda = 0$$

$$-[\lambda^3+3\alpha\lambda^2+3\alpha^2\lambda+\alpha^3]+3\alpha k^2+3k^2\lambda+2k^3=0$$

$$-\lambda^3-3\alpha\lambda^2-3\alpha^2\lambda+3\alpha k^2+3k^2\lambda+2k^3-\alpha^3=0$$
$$-\lambda^3-3\alpha\lambda^2+\left(3k^2-3\alpha^2\right)\lambda+3\alpha k^2+2k^3-\alpha^3=0.$$

Hence, this is the required characteristic polynomial of the model given by equation (4.6). Now to determine an eigenvalues:

$$\left(\lambda+\alpha+k\right)[-\lambda^2+\left(-2\alpha+k\right)\lambda-\alpha^2+\alpha k+2k^2]=0$$
$$\left(\lambda+\alpha+k\right)\left(\lambda+\alpha+k\right)\left(\lambda+\alpha-2k\right)=0$$
$$\left(\lambda+\alpha+k\right)^2\left(\lambda+\alpha-2k\right)=0$$

Therefore, the eigenvalue are

$$\lambda=-\alpha-k$$
$$\lambda=-\alpha+2k$$

The system is said to be stable;

$$\text{if } -\alpha-k<0 \text{ and } 2k-\alpha<0.$$

$$k<\frac{\alpha}{2} \text{ and k > -}\alpha,$$

But, by taking an intersection of all inequality, we have the following result.

$$\left(k<\frac{\alpha}{2}\right)\cap\left(k>-\alpha\right)\cap\left(k>0\right)\cap\left(\alpha>0\right)$$

$$\left(-\alpha<k<\frac{\alpha}{2}\right)\cap\left(0,\infty\right)$$

$$0<k<\frac{\alpha}{2}$$

$$k<\frac{\alpha}{2} \tag{4.7}$$

As a result, in the case where the three nations have the same defense coefficient k and restraint coefficient α. The nations stabilize their regions for a long period of time provided that condition in an inequality (4.7) is satisfied. That is the defense coefficient of the countries must be kept less than half of the restraint coefficient of the countries.

2. The case where the first two nations have the same defense coefficient k and restraint coefficient α.

$$\begin{aligned}\frac{dx}{dt} &= k_1 y + k_1 z - \alpha_1 x + g_1 \\ \frac{dy}{dt} &= k_1 x + k_1 z - \alpha_1 y + g_2 \\ \frac{dz}{dt} &= k_2 x + k_2 y - \alpha_2 z + g_3\end{aligned} \tag{4.8}$$

By rearranging differential equation given by equation (4.8) we obtain:

$$\begin{aligned}\frac{dx}{dt} &= -\alpha_1 x + k_1 y + k_1 z \\ \frac{dy}{dt} &= k_1 x - \alpha_1 y + k_1 z \\ \frac{dz}{dt} &= k_2 x + k_2 y - \alpha_2 z\end{aligned}$$

From this we determine characteristic polynomial using the determinant of the matrix i.e.

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -\alpha_1 - \lambda & k_1 & k_1 \\ k_1 & -\alpha_1 - \lambda & k_1 \\ k_2 & k_2 & -\alpha_2 - \lambda \end{vmatrix} = 0$$

$$(-\alpha_1 - \lambda)\begin{vmatrix} -\alpha_1 - \lambda & k_1 \\ k_2 & -\alpha_2 - \lambda \end{vmatrix} - k_1 \begin{vmatrix} k_1 & k_1 \\ k_2 & -\alpha_2 - \lambda \end{vmatrix} + k_1 \begin{vmatrix} k_1 & -\alpha_1 - \lambda \\ k_2 & k_2 \end{vmatrix} = 0$$

$$(-\alpha_1 - \lambda)[\alpha_1\alpha_2 + \alpha_1\lambda + \alpha_2\lambda + \lambda^2 - k_1 k_2] - k_1[-\alpha_2 k_1 - k_1\lambda - k_1 k_2] + k_1[k_1 k_2 + \alpha_1 k_2 + k_2\lambda] = 0$$

$$\left(-\alpha_1-\lambda\right)[\lambda^2+\left(\alpha_1+\alpha_2\right)\lambda+\alpha_1\alpha_2-k_1k_2]+\alpha_2k_1^2+k_1^2\lambda+k_1^2k_2+k_1^2k_2+\alpha_1k_1k_2+k_1k_2\lambda=0$$

$$-\lambda^3+\left(-2\alpha_1-\alpha_2\right)\lambda^2+\left(k_1^2-2\alpha_1\alpha_2-\alpha_1^2+2k_1k_2\right)\lambda+\alpha_2k_1^2-\alpha_1^2\alpha_2+2\alpha_1k_1k_2+2k_1^2k_2=0$$

$$-\lambda^3+\left(-2\alpha_1-\alpha_2\right)\lambda^2+\left(k_1^2-\alpha_1^2-2\alpha_1\alpha_2+2k_1k_2\right)\lambda+\alpha_2k_1^2-\alpha_1^2\alpha_2+2k_1^2k_2+2\alpha_1k_1k_2=0.$$

Therefore, this gives characteristic polynomial of degree three and we apply Routh-Hurwitz stability criteria to determine the stability of the system.

Table 4.1: Routh array for the case where the first two nations have the same defense coefficient k and restraint coefficient α.

-1	$k_1^2-\alpha_1^2-2\alpha_1\alpha_2+2k_1k_2$	0
$-2\alpha_1-\alpha_2$	$\alpha_2k_1^2-\alpha_1^2\alpha_2+2k_1^2k_2+2\alpha_1k_1k_2$	0
$-\dfrac{\left[\begin{array}{l}2k_1^2\left(k_2-\alpha_1\right)+2\alpha_1^2\left(\alpha_1+2\alpha_2\right)\\+2\alpha_1\alpha_2^2-2k_1k_2\left(\alpha_1+\alpha_2\right)\end{array}\right]}{2\alpha_1+\alpha_2}$	0	0
$\alpha_2k_1^2-\alpha_1^2\alpha_2+2k_1^2k_2+2\alpha_1k_1k_2$	0	0

Where, $a_3=-1$

$$a_1=k_1^2-\alpha_1^2-2\alpha_1\alpha_2+2k_1k_2$$

$$a_2=-2\alpha_1-\alpha_2$$

$$a_0=\alpha_2k_1^2-\alpha_1^2\alpha_2+2k_1^2k_2+2\alpha_1k_1k_2$$

$$b_1=\frac{\left(-2\alpha_1-\alpha_2\right)\left(k_1^2-\alpha_1^2-2\alpha_1\alpha_2+2k_1k_2\right)-\left(-1\right)\left(\alpha_2k_1^2-\alpha_1^2\alpha_2+2k_1^2k_2+2\alpha_1k_1k_2\right)}{-2\alpha_1-\alpha_2},$$

$$for,\ 2\alpha_1 + \alpha_2 \neq 0$$

$$b_1 = \frac{-[2k_1^{\ 2}\left(k_2 - \alpha_1\right) + 2\alpha_1^{\ 2}\left(\alpha_1 + 2\alpha_2\right) + 2\alpha_1\alpha_2^{\ 2} - 2k_1k_2\left(\alpha_1 + \alpha_2\right)]}{2\alpha_1 + \alpha_2},$$

$$for,\ 2\alpha_1 + \alpha_2 \neq 0$$

$$b_2 = \frac{\left(-2\alpha_1 - \alpha_2\right)\left(0\right) - \left(-1\right)\left(0\right)}{-2\alpha_1 - \alpha_2} = 0$$

$$for,\ 2\alpha_1 + \alpha_2 \neq 0$$

$$c_1 = \frac{b_1 a_0 - b_2 a_2}{b_1} = a_0,$$

$$\text{Since,}\quad b_2 = 0$$

$$c_1 = a_0 = \alpha_2 k_1^{\ 2} - \alpha_1^{\ 2}\alpha_2 + 2k_1^{\ 2}k_2 + 2\alpha_1 k_1 k_2$$

To establish stability conditions we apply Routh-Hurwitz stability criteria described below.

$$\mathrm{a}_3 = -1 < 0$$

$$\mathrm{a}_2 < 0$$

$$\alpha_2 + 2\alpha_1 > 0$$

$$b_1 = \frac{-[2k_1^{\ 2}\left(k_2 - \alpha_1\right) + 2\alpha_1^{\ 2}\left(\alpha_1 + 2\alpha_2\right) + 2\alpha_1\alpha_2^{\ 2} - 2k_1k_2\left(\alpha_1 + \alpha_2\right)]}{2\alpha_1 + \alpha_2} < 0$$

$$2k_1^{\ 2}\left(k_2 - \alpha_1\right) + 2\alpha_1^{\ 2}\left(\alpha_1 + 2\alpha_2\right) + 2\alpha_1\alpha_2^{\ 2} - 2k_1k_2\left(\alpha_1 + \alpha_2\right) > 0$$

$$c_1 = \alpha_2 k_1^{\ 2} - \alpha_1^{\ 2}\alpha_2 + 2k_1^{\ 2}k_2 + 2\alpha_1 k_1 k_2 < 0$$

$$\alpha_2 k_1^{\ 2} - \alpha_1^{\ 2}\alpha_2 + 2k_1^{\ 2}k_2 + 2\alpha_1 k_1 k_2 < 0$$

$$\alpha_2\left(k_1^2-\alpha_1^2\right)+2k_1^2k_2+2\alpha_1k_1k_2<0$$
$$\alpha_2\left(k_1-\alpha_1\right)\left(k_1+\alpha_1\right)+2k_1k_2\left(k_1+\alpha_1\right)<0$$
$$[\alpha_2\left(k_1-\alpha_1\right)+2k_1k_2][k_1+\alpha_1]<0$$

Now considering the case, $\alpha_2\left(k_1-\alpha_1\right)+2k_1k_2<0$ or $\mathrm{k}_1+\alpha_1>0$

$$\alpha_2k_1-\alpha_1\alpha_2+2k_1k_2<0 \text{ or } \mathrm{k}_1>-\alpha_1$$

Since, $\alpha_1>0$ implies $k_1>0$, we have

$$k_1\left(\alpha_2+2k_2\right)<\alpha_1\alpha_2$$
$$k_1<\frac{\alpha_1\alpha_2}{\alpha_2+2k_2},$$

where $\alpha_2+2k_2\neq 0$

$$k_1<\frac{\alpha_1\alpha_2}{\alpha_2+2k_2} \tag{4.9}$$

Hence, by applying Routh-Hurwitz stability criteria the three nations stabilize their region under this circumstance for a long period of time for which the value of the defense coefficients k_1, k_2 and restraint coefficients α_1, α_2 of the nations was satisfied an inequality (4.9). Otherwise, the three nations lead to conflict among themselves.

3. The case where each of the three nations has different defense coefficient k and restraint coefficient α.

$$\begin{aligned}\frac{dx}{dt}&=k_1y+k_1z-\alpha_1x+g_1\\\frac{dy}{dt}&=k_2x+k_2z-\alpha_2y+g_2\\\frac{dz}{dt}&=k_3x+k_3y-\alpha_3z+g_3\end{aligned} \tag{4.10}$$

By rearranging differential equation given by equation (4.10) we obtain:

$$\frac{dx}{dt} = -\alpha_1 x + k_1 y + k_1 z$$
$$\frac{dy}{dt} = k_2 x - \alpha_2 y + k_2 z$$
$$\frac{dz}{dt} = k_3 x + k_3 y - \alpha_3 z$$

In the same way, we get the characteristic equation by using

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -\alpha_1 - \lambda & k_1 & k_1 \\ k_2 & -\alpha_2 - \lambda & k_2 \\ k_3 & k_3 & -\alpha_3 - \lambda \end{vmatrix} = 0$$

$$(-\alpha_1 - \lambda)\begin{vmatrix} -\alpha_2 - \lambda & k_2 \\ k_3 & -\alpha_3 - \lambda \end{vmatrix} - k_1 \begin{vmatrix} k_2 & k_2 \\ k_3 & -\alpha_3 - \lambda \end{vmatrix} + k_1 \begin{vmatrix} k_2 & -\alpha_2 - \lambda \\ k_3 & k_3 \end{vmatrix} = 0$$

$$(-\alpha_1 - \lambda)[(-\alpha_2 - \lambda)(-\alpha_3 - \lambda) - k_2 k_3] - k_1[-\alpha_3 k_2 - k_2 \lambda - k_2 k_3] + k_1[k_2 k_3 + \alpha_2 k_3 + k_3 \lambda] = 0$$

$$\begin{aligned}&(-\alpha_1 - \lambda)[\alpha_2 \alpha_3 + \alpha_2 \lambda + \alpha_3 \lambda + \lambda^2 - k_2 k_3] + \alpha_3 k_1 k_2 + k_1 k_2 \lambda + k_1 k_2 k_3 \\ &\quad + k_1 k_2 k_3 + \alpha_2 k_1 k_3 + k_1 k_3 \lambda = 0\end{aligned}$$

$$\begin{aligned}&-\lambda^3 + (-\alpha_1 - \alpha_2 - \alpha_3)\lambda^2 + (k_1 k_2 + k_2 k_3 + k_1 k_3 - \alpha_1 \alpha_2 - \alpha_1 \alpha_3 - \alpha_2 \alpha_3)\lambda + \alpha_1 k_2 k_3 \\ &\quad + \alpha_2 k_1 k_3 + \alpha_3 k_1 k_2 + 2 k_1 k_2 k_3 - \alpha_1 \alpha_2 \alpha_3 = 0.\end{aligned}$$

Consequently, to discuss the stability conditions among the three nations systems we apply Routh-Hurwitz stability criteria on the characteristic polynomial obtained above.

Table 4.2: Routh array for the case where each of the three nations has different defense coefficient k and restraint coefficient α.

-1	$k_1k_2+k_2k_3+k_1k_3$ $-\alpha_1\alpha_2-\alpha_1\alpha_3-\alpha_2\alpha_3$	0
$-\alpha_1-\alpha_2-\alpha_3$	$[\alpha_1k_2k_3+\alpha_2k_1k_3+\alpha_3k_1k_2+$ $2k_1k_2k_3-\alpha_1\alpha_2\alpha_3]$	0
$-\dfrac{\begin{bmatrix}\alpha_1^2(\alpha_2+\alpha_3)+\alpha_2^2(\alpha_1+\alpha_3)+\alpha_3^2(\alpha_1+\alpha_2)\\-\alpha_1k_1(k_2+k_3)-\alpha_2k_2(k_1+k_3)\\-\alpha_3k_3(k_1+k_2)+2k_1k_2k_3+2\alpha_1\alpha_2\alpha_3\end{bmatrix}}{\alpha_1+\alpha_2+\alpha_3}$	0	0
$\alpha_1k_2k_3+\alpha_2k_1k_3+\alpha_3k_1k_2+2k_1k_2k_3-\alpha_1\alpha_2\alpha_3$	0	0

Where, $a_3=-1$

$$a_1=k_1k_2+k_2k_3+k_1k_3-\alpha_1\alpha_2-\alpha_1\alpha_3-\alpha_2\alpha_3$$

$$a_2=-\alpha_1-\alpha_2-\alpha_3$$

$$a_0=\alpha_1k_2k_3+\alpha_2k_1k_3+\alpha_3k_1k_2+2k_1k_2k_3-\alpha_1\alpha_2\alpha_3$$

$$b_1=-\frac{\begin{bmatrix}\alpha_1^2(\alpha_2+\alpha_3)+\alpha_2^2(\alpha_1+\alpha_3)+\alpha_3^2(\alpha_1+\alpha_2)\\-\alpha_1k_1(k_2+k_3)-\alpha_2k_2(k_1+k_3)\\-\alpha_3k_3(k_1+k_2)+2k_1k_2k_3+2\alpha_1\alpha_2\alpha_3\end{bmatrix}}{\alpha_1+\alpha_2+\alpha_3}$$

$$for,\ \alpha_1+\alpha_2+\alpha_3\neq 0$$

$$c_1=a_0=\alpha_1k_2k_3+\alpha_2k_1k_3+\alpha_3k_1k_2+2k_1k_2k_3-\alpha_1\alpha_2\alpha_3$$

Now, the stability condition is given as;

$$a_3 = -1 < 0$$

$$\mathrm{a}_2 < 0.$$
$$-\alpha_1 - \alpha_2 - \alpha_3 < 0$$
$$\alpha_1 + \alpha_2 + \alpha_3 > 0$$

$$b_1 = -\frac{\begin{bmatrix} \alpha_1^{\,2}(\alpha_2+\alpha_3)+\alpha_2^{\,2}(\alpha_1+\alpha_3)+\alpha_3^{\,2}(\alpha_1+\alpha_2) \\ -\alpha_1 k_1(k_2+k_3)-\alpha_2 k_2(k_1+k_3) \\ -\alpha_3 k_3(k_1+k_2)+2k_1k_2k_3+2\alpha_1\alpha_2\alpha_3 \end{bmatrix}}{\alpha_1+\alpha_2+\alpha_3} < 0$$

$$\frac{\begin{bmatrix} \alpha_1^{\,2}(\alpha_2+\alpha_3)+\alpha_2^{\,2}(\alpha_1+\alpha_3)+\alpha_3^{\,2}(\alpha_1+\alpha_2) \\ -\alpha_1 k_1(k_2+k_3)-\alpha_2 k_2(k_1+k_3) \\ -\alpha_3 k_3(k_1+k_2)+2k_1k_2k_3+2\alpha_1\alpha_2\alpha_3 \end{bmatrix}}{\alpha_1+\alpha_2+\alpha_3} > 0$$

$$\text{Since, } b_2 = 0$$

$$c_1 = a_0 = \alpha_1 k_2 k_3 + \alpha_2 k_1 k_3 + \alpha_3 k_1 k_2 + 2k_1 k_2 k_3 - \alpha_1\alpha_2\alpha_3$$

$$c_1 = \alpha_1 k_2 k_3 + \alpha_2 k_1 k_3 + \alpha_3 k_1 k_2 + 2k_1 k_2 k_3 - \alpha_1\alpha_2\alpha_3 < 0$$

$$\alpha_1 k_2 k_3 + \alpha_2 k_1 k_3 + \alpha_3 k_1 k_2 + 2k_1 k_2 k_3 - \alpha_1\alpha_2\alpha_3 < 0$$

$$\alpha_1 k_2 k_3 + \alpha_2 k_1 k_3 + \alpha_3 k_1 k_2 + 2k_1 k_2 k_3 < \alpha_1\alpha_2\alpha_3$$

$$\alpha_1 k_2 k_3 + \alpha_2 k_1 k_3 + 2k_1 k_2 k_3 < \alpha_1\alpha_2\alpha_3 - \alpha_3 k_1 k_2$$

$$k_3(\alpha_1 k_2 + \alpha_2 k_1 + 2k_1 k_2) < \alpha_3(\alpha_1\alpha_2 - k_1 k_2)$$

$$k_3 < \frac{\alpha_3(\alpha_1\alpha_2 - k_1 k_2)}{\alpha_1 k_2 + \alpha_2 k_1 + 2k_1 k_2} \qquad (4.11)$$

Thus, the system is stable for which the defense coefficients and restraint coefficients in the inequality (4.11) satisfied and $\alpha_1\alpha_2 > k_1k_2$.

4. The case where the first two nations have same defense coefficients and all the three countries have the same restraint coefficients but, the third country has no defense coefficient.

Suppose the Z nation is a Pacifist nation, while X and Y are Pugnacious nations. Then,

$$\begin{aligned}\frac{dx}{dt} &= ky + kz - \alpha x + g_1\\ \frac{dy}{dt} &= kx + kz - \alpha y + g_2\\ \frac{dz}{dt} &= 0x + 0y - \alpha z + g_3\end{aligned} \tag{4.12}$$

The above differential equation given by equation (4.12) would be written as a matrix form of;

$$\frac{dm}{dt} = Aw + f \quad \text{i.e.}$$

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} = \begin{pmatrix} -\alpha & k & k \\ k & -\alpha & k \\ 0 & 0 & -\alpha \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix},$$

$$\frac{dm}{dt} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix},\ A = \begin{pmatrix} -\alpha & k & k \\ k & -\alpha & k \\ 0 & 0 & -\alpha \end{pmatrix},\ w = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } f = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}$$

Where, A is a coefficient Jacobian matrix.

w is variable in x, y and z.

f is a constant column matrix.

To determine the characteristic equation we consider

$$|A-\lambda I|=0$$

$$\begin{vmatrix} -\alpha-\lambda & k & k \\ k & -\alpha-\lambda & k \\ 0 & 0 & -\alpha-\lambda \end{vmatrix}=0$$

By expanding the determinant of this matrix along the third row and third column the following characteristic equation was obtained.

$$(-\alpha-\lambda)[(-\alpha-\lambda)(-\alpha-\lambda)-k^2]=0$$

$$(-\alpha-\lambda)[(-\alpha-\lambda)^2-k^2]=0$$

$$(-\alpha-\lambda)^3-(-\alpha-\lambda)k^2=0$$

$$-(\alpha+\lambda)^3+(\alpha+\lambda)k^2=0$$

$$-\lambda^3-3\alpha\lambda^2+(k^2-3\alpha^2)\lambda-\alpha^3+\alpha k^2=0$$

This is the characteristic equation of (4.12). Now let us apply Routh-Hurwitz stability criteria in order to determine the stability of system.

Table 4.3: Routh array for the case where the first two nations have the same defense coefficients and all the three nations have the same restraint coefficients but, the third nation has no defense coefficient.

-1	$k^2-3\alpha^2$	0
-3α	$\alpha k^2-\alpha^3$	0
$\dfrac{2\alpha k^2-8\alpha^3}{3\alpha}$	0	0
$\alpha k^2-\alpha^3$	0	0

$$\text{Where,}\quad a_3 = -1$$

$$a_1 = k^2 - 3\alpha^2$$

$$a_2 = -3\alpha$$

$$a_0 = \alpha k^2 - \alpha^3$$

$$b_1 = \frac{2\alpha k^2 - 8\alpha^3}{3\alpha}$$

$$for,\quad \alpha \neq 0$$

$$c_1 = \alpha k^2 - \alpha^3$$

Now, we investigate stability as follows;

$$\mathrm{a}_3 < 0$$
$$-1 < 0$$

$$a_2 < 0,$$
$$-3\alpha < 0$$
$$\alpha > 0$$
$$b_1 < 0$$
$$\Rightarrow \mathrm{k} < 2\alpha$$

$$c_1 = \alpha k^2 - \alpha^3 < 0$$
$$\Rightarrow \mathrm{c}_1 < 0$$
$$\Rightarrow \alpha k^2 - \alpha^3 < 0$$

$$k^2 - \alpha^2 < 0$$

$$(k-\alpha)(k+\alpha) < 0$$

Let us consider the case when $k-\alpha < 0$ and $k+\alpha > 0$.

Which implies $k < \alpha$ and $k > -\alpha$. Now by taking intersection of the following inequality we have

$$(k < \alpha) \cap (k > -\alpha) \cap (k > 0) \cap (\alpha > 0)$$

$$k < \alpha \tag{4.13}$$

Hence, for all possible value of the defense coefficient k and the restraint coefficient α related by this stability condition given by inequality (4.13) lead to guaranteed the stability among three nations for their common benefit permanently; otherwise, it increased the causes that lead to unstable among the nations which created tension among themselves.

4.3. Numerical Examples

In order to verify the formulated mathematical model of conflict among three nations system the following numerical examples are used for each cases separately.

Example 1: In case of equation (4.6), let us take some value for $g_1 = g_2 = g_3 = 1, \alpha = 4$ and $k = 1$. Since the stability condition of inequality (4.7) i.e. $k < \frac{\alpha}{2}$ and was satisfied. Then, equation (4. 6)

$$\begin{aligned} \frac{dx}{dt} &= y + z - 4x + 1 \\ \frac{dy}{dt} &= x + z - 4y + 1 \\ \frac{dz}{dt} &= x + y - 4z + 1 \end{aligned} \tag{4.14}$$

By rearranging equation (4.14) and equating to zero. We have

$$\begin{aligned} -4x + y + z &= -1 \\ \mathrm{x} - 4y + z &= -1 \\ x + y - 4z &= -1 \end{aligned}$$

$$A=\begin{pmatrix}-4 & 1 & 1\\ 1 & -4 & 1\\ 1 & 1 & -4\end{pmatrix},\ \text{w}=\begin{pmatrix}x\\ y\\ z\end{pmatrix}\ \text{and}\ \text{f}=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

Then, to determine the characteristic polynomial of the coefficient matrix, we have

$$|A-\lambda I|=0$$

$$\begin{vmatrix}-4-\lambda & 1 & 1\\ 1 & -4-\lambda & 1\\ 1 & 1 & -4-\lambda\end{vmatrix}=0$$

$$(-4-\lambda)\left[(-\lambda-4)^2-1\right]-(1)\left(-4-\lambda-1\right)+(1)(1+\lambda+4)=0$$

$$(-4-\lambda)\left[(\lambda^2+8\lambda+15\right]+\lambda+5+\lambda+5=0$$

$$-\lambda^3-12\lambda^2-45\lambda-50=0.$$

$$\left(\lambda+2\right)\left(\lambda+5\right)^2=0$$

$$\lambda_1=-2\ \text{and}\ \lambda_2\text{=-5.}$$

Since, all eigenvalue of the matrix A are all negative then, the system is stable.

Example 2: In case of equation (4.8), let us take some value for $g_1=4$, $g_2=5$, $g_3=6$, $\alpha_1=8$, $\alpha_2=2$, $k_1=3$, $k_2=1$. So that stability condition of inequality (4.9) was satisfied. Then, equation (4. 8) becomes

$$\begin{aligned}\frac{dx}{dt}&=3y+3z-8x+4\\ \frac{dy}{dt}&=3x+3z-8y+5\\ \frac{dz}{dt}&=x+y-2z+6\end{aligned}\tag{4.15}$$

By rearranging equation (4.15) and equating zero. We have

$$-8x+3y+3z=-4$$
$$3x-8y+3z=-5$$
$$x+y-2z=-6$$

$$A=\begin{pmatrix}-8 & 3 & 3\\ 3 & -8 & 3\\ 1 & 1 & -2\end{pmatrix},\ w=\begin{pmatrix}x\\ y\\ z\end{pmatrix}\ \text{and}\ f=\begin{pmatrix}4\\ 5\\ 6\end{pmatrix}$$

Then, to determine the characteristic polynomial of the coefficient matrix, we have

$$|A-\lambda I|=0$$
$$\begin{vmatrix}-8-\lambda & 3 & 3\\ 3 & -8-\lambda & 3\\ 1 & 1 & -2-\lambda\end{vmatrix}=0$$

$$(-8-\lambda)\left[(-\lambda-8)(-\lambda-2)-3\right]-(3)(-3\lambda-6-3)+(3)(3+\lambda+8)=0$$

$$(-\lambda-8)\left[(\lambda^2+10\lambda+13\right]+9\lambda+27+3\lambda+33=0$$

$$-\lambda^3-18\lambda^2-81\lambda-44=0.$$

$$(\lambda+11)(-\lambda^2-7\lambda-4)=0$$

Hence, λ_1= -11, $\lambda_2=-6.3733$ *and* $\lambda_3=-0.6277$

Since, all eigenvalue of the matrix A are all negative then, the three nations stabilize their region under the given stability condition of Routh-Hurwitz.

Example 3: In case of equation (4.10), let us give some constants for $g_1=6$, $g_2=7$, $g_3=8$

$\alpha_1=11$, $\alpha_2=1$, $\alpha_3=8$, $k_1=4$, $k_2=1$, $k_3=2$. As a result of the stability condition of inequality (4.11) was holds true. The equation (4. 10) becomes

$$\frac{dx}{dt} = 4y + 4z - 11x + 6$$
$$\frac{dy}{dt} = x + z - y + 7 \quad (4.16)$$
$$\frac{dz}{dt} = 2x + 2y - 8z + 8$$

From equation (4.16), we have

$$-11x + 4y + 4z = -6$$
$$\text{x} - y + z = -7$$
$$2x + 2y - 8z = -8$$

Then, to determine the characteristic polynomial of the coefficient matrix A, we have

$$|A - \lambda I| = 0$$
$$\begin{vmatrix} -11-\lambda & 4 & 4 \\ 1 & -1-\lambda & 1 \\ 2 & 2 & -8-\lambda \end{vmatrix} = 0$$

$$(-\lambda-11)\left[(-\lambda-1)(-\lambda-8)-2\right]-(4)(-\lambda-8-2)+(4)(2+2\lambda+2)=0$$

$$(-\lambda-11)\left[(\lambda^2+9\lambda+8-2\right]-(4)(-\lambda-10)+(4)(2\lambda+4)=0$$

$$(-\lambda-11)\left[(\lambda^2+9\lambda+6\right]+4\lambda+40+8\lambda+16=0$$

$$\Rightarrow -\lambda^3 - 20\lambda^2 - 93\lambda - 10 = 0.$$

Hence, λ_1=-12.7896, $\lambda_2 = -7.1003$ *and* $\lambda_3 = -0.1101$

As results of all eigenvalue of the matrix are all negative then, the system is stable.

Example 4: In case of equation (4.12) it follows that taking value for $g_1 = 1$, $g_2 = 3$, $g_3 = 5$

$\alpha = 4$, $k = 2$. For which conditions of inequality (4.13) hold. Then, equation (4. 12) becomes

$$\frac{dx}{dt} = 2y + 2z - 4x + 1$$
$$\frac{dy}{dt} = 2x + 2z - 4y + 3 \qquad (4.17)$$
$$\frac{dz}{dt} = 0x + 0y - 4z + 5$$

From equation (4.17), we have

$$-4x + 2y + 2z = -1$$
$$2x - 4y + 2z = -3$$
$$0x + 0y - 4z = -5$$

$$A = \begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 0 & 0 & -4 \end{pmatrix}, \text{ w=} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and f=} \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

Then, to determine the characteristic polynomial of the coefficient matrix A, we have

$$|A - \lambda I| = 0$$
$$\begin{vmatrix} -4-\lambda & 2 & 2 \\ 2 & -4-\lambda & 2 \\ 0 & 0 & -4-\lambda \end{vmatrix} = 0$$

$$(-\lambda - 4)\left[(-\lambda - 4)(-\lambda - 4)\right] - (2)(-2\lambda - 8) = 0$$

$$(-\lambda - 4)\left[(\lambda^2 + 8\lambda + 16\right] + 4\lambda + 16 = 0$$

$$-\lambda^3 - 12\lambda^2 - 44\lambda - 48 = 0.$$

$$(\lambda + 2)(\lambda + 4)(\lambda + 6) = 0$$

Therefore, λ_1= -2, $\lambda_2 = -4$ *and* $\lambda_3 = -6$

As all eigenvalue of the matrix are all negative then, the system is stable.

CHAPTER FIVE

CONCLUSION AND FUTURE SCOPE

5.1. Conclusion

In this research, mathematical model that describe conflict among three nations was formulated. The model incorporates defense coefficient and restraint coefficient. Mathematical manipulation such as local stability condition by Routh-Hurwitz criteria using four different cases such as when the three nations have the same defense and restraint coefficient and the case where first two nations have the same defense coefficient and restraint coefficient were analyzed. In addition to this the case where each of the three nations has different defense coefficient and restraint coefficient and the case where the first two nations have same defense coefficient and all the three countries have the same restraint coefficient but, the third country has no defense coefficient were carried out. Therefore, for each four cases of stability condition carried out the numerical examples were used to verify the applicability of formulated mathematical model.

5.2. Scope for Future Work

In this research stability analysis of conflict among three nations using Routh-Hurwitz criteria condition was discussed. Based on this, the upcoming post graduate student and other researchers who are interested in this area can use this result as a stepping stone and make further investigation on the stability analysis among four or more nations. Furthermore, modified mathematical model of the conflict among three nations are another further investigation.

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