MODIFIED REDUCED DIFFERENTIAL TRANSFORM METHOD FOR SOLVING HEAT-LIKE EQUATION

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS, COLLEGE OF NATURAL SCIENCE, JIMMA UNIVERSITY IN PARTIAL FULFILLMENT OF THE REQUIREMENT FOR THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS.

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OCTOBER, 2017

JIMMA, ETHIOPIA

Declaration

I, undersigned declare that, this research entitled "**Modified Reduced Differential Transform Method for solving heat-like equation**" is my own original work and it has not been submitted for the award of any academic degree or the like in this or in any other university or other institution of learning and that all the sources I have used or quoted have been indicated and acknowledged.

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Acknowledgment

First of all I would like to express my gratitude to our almighty **God** who helped me from the start up to the completion of this study. Next I would like to express my deepest gratitude to my advisor Dr. Yesuf Obsie for his unreserved guidance, encouragement, and valuable suggestions and providing me related materials during the development of this thesis.

Finally I would you thank Jimma University for the financial support via the college of natural science so that this research manuscript become reality.

Abstract

In this study, Modified Reduced Differential Transform Method (MRDTM) is used to find approximate solutions of one dimensional heat-like equation subject to initial condition. The MRDTM basically involves the combination of Laplace pade Resummation method and RDTM. First theories, their proofs and basic properties of these procedures are given. Three test examples are given to demonstrate the validity and usefulness of the under consideration method. The result shows that this method is efficient simply applicable and has powerful effect on problems of partial differential equations.

Key words: Modified Reduced Differential Transform Method, Laplace pade Resummation Method, one dimensional heat-like equation

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Acronym

ADM - Adomian Decomposition Method

DTM - Differential Transforms Method

HAM - Homotopy Analysis Method

HPM - Homotopy Perturbation Method

LPRDTM - Laplace Pade Reduced Differential Transform Method

MRDTM - Modified Reduced Differential Transform Method

RDTM - Reduced Differential Transform Method

VIM - Variational Iteration Method

IVPs - Initial Value Problems

CHAPTER ONE

Introduction

1.1 Background of the study

Most physical phenomena described by functions which depend on two or more independent variables can be modeled by partial differential equations (Molabahrami and Khani, 2009).Heat-like equations are second-order partial differential equations of parabolic type which can be found in a wide variety of engineering and scientific applications.

In recent years, numerous works have focused on the development of more advanced and efficient methods for solving heat-like equations such as the Adomian Decomposition Method (Wazwaz, 2007), Variational Iteration Method (VIM) (Abbasbandy and Darvishi, 2005), Differential Transform Method(Khatereh *et al*., 2012), and others . The differential transform method (DTM) was first introduced by (Zhou, 1986) to solve initial value problems (IVPs) associated with electrical circuit analysis. This method is very effective and powerful for solving various kinds of differential equations as two point boundary value problems (Debnath, 2014).However, the Reduced Differential Transform Method (RDTM) is an iterative procedure for obtaining Taylor series solution of differential equation and it will be employed in a straightforward manner without any need of linearization or smallness assumptions. Moreover, using RDTM, the solution to initial value problems can be expressed as an infinite power series. Later, taking advantage of the Resummation methods capabilities the domain of convergence of such power series can be extended leading in some cases to the exact solution. But the solution obtained from Reduced Differential Transform Method have limited region of convergence (not much closer to the exact), even if we take a large number of terms (Brahim *et al.,* 2014). So solutions to one dimensional heat-like equation are first obtained in convergent series form using RDTM. To improve the solution, we apply Laplace transform on the result obtained from RDTM truncated series to increase the degree of the truncated series by one. Then we convert the transformed series into a meromorphic function by forming its Pade approximant. Finally, we take the inverse Laplace transform of the Pade approximant to obtain the analytical solution.

Therefore, in this work, relative to those described methods above, a new version of the modification referred to as Modified Reduced Differential Transform Method (MRDTM) is used to obtain the better approximate solution of initial value problem of one dimensional heat-like equations of the form:

$$\frac{\partial u(x,t)}{\partial t} - c^2 \frac{\partial^2 u(x,t)}{\partial x^2} = g(x,t)$$

Subject to initial condition $u(x,0) = f(x)$

where $u(x,t)$ is unknown analytic and continuously differentiable function, $g(x.t)$ is known analytic function and c^2 is variable coefficient.

1.2 Statement of the problem

It is described in the literature that the solution to an initial value problem using reduced differential transform method is expressed as an infinite power series. The solution obtained in this way has limited region of convergence, even if we take a large number of terms. So as a way out of this problem we apply the Laplace pade Resummation on the truncated series obtained from RDTM to get better approximate solution to one dimensional heat-like equation. Therefore, this study focused on:

a. Finding the approximate solution of initial value problem of one dimensional heat-like equation by using Laplace pade Resummation method.

b. Demonstrate the validity of the proposed method to the solution of one dimensional heat-like equation subject to the initial condition by using examples.

1.3. Objectives of the study

1.3.1. General objective

The general objective of this study is to find the solution of one dimensional heat-like equation subject to the initial condition by using Modified Reduced Differential Transform Method.

1.3.2 Specific objective

The specific objectives of this study are to:

A. Apply Modified Reduced Differential Transform Method to obtain approximate solution of initial value problems of one dimensional heat-like equation.

B. Demonstrate the validity of the proposed method by using examples.

1.4 Significance of the study

In this study we used the Modified Reduced Differential Transform Method to find approximate solution to one dimensional heat-like equation subject to initial condition. The study is expected to have the following importance:

a. It develops techniques of solving initial value problem of Partial Differential Equation by using MRDTM
b. It develops the skill of conducting mathematical research.
c. The results obtained in this study may contribute to research activities in this area.

1.5 Delimitation of the study

The study is delimited to find approximate solution of one dimensional homogeneous and non-homogeneous heat-like equation subject to initial condition. The Laplace pade Resummation Method is used as a basic tool to find the solution

CHAPTER TWO

Literature review

Mathematical approaches to partial differential equations are divided into two methods called analytical method which strives to find exact formulae for the dependent variables as a function of independent variable and numerical methods which result in approximate value of dependent variables (Saravanan and Magesh, 2013). But there are also mathematical approaches which can be neither of the two methods. These are semi-analytical and semi-numerical method. Several analytical methods were developed for solving partial differential equations, such us the Homotopy Perturbation Method (Turgut and Deniz, 2008), Differential Transform Method (Zhou, 1986), and Reduced Differential Transform Method (Keskin and Galip, 2010), and others.

As the science history in last decades indicated, the nonlinear problems are one of the most important phenomena in mathematics, physics and engineering and efficiency of these problems show us the importance of obtaining exact or approximate solution which still needs better methods (Reza *et al*., 2013). Researchers used new methods for this requirement such As: Homotopy Analysis Method (HAM) (Molabahrami and Khani, 2009) , Variation Iteration Method (VIM) (Inc, 2007), etc. One of the well-known models of these problems is heat-like equations (Cannon, 1984).This model has essential role in various fields of science and engineering which is investigated widely by many researchers.

Recently researchers have applied different method successfully to obtain analytic solution. For example: The Homotopy perturbation method(HPM) was applied to both non-linear and linear fractional differential equation and it was showed that HPM is an alternative analytical method for fractional differential equations(Ablowitz and Clarkson, 1990).The perturbation methods, like other nonlinear analytical techniques, have their own limitations. At first, almost all perturbation methods are based on the assumption that a small parameter must exist in the equation. This so-called small parameter assumption greatly restricts applications of perturbation techniques. As is well known, an overwhelming majority of nonlinear problems have no small parameters at all. Secondly, the determination of small parameters seems to be a special art requiring special techniques. An appropriate choice of small parameters leads to the ideal results, but an unsuitable choice may create serious problems. Furthermore, the approximate solutions solved by perturbation methods are valid, in most cases, only for the small values of the parameters. It is obvious that all these limitations come from the small parameter assumption (Tauseef and Aslam Noor, 2009). Another improved approach for solving initial value problem for partial differential equation, known as reduced differential transform method (RDTM) has recently been used by (Ayaz, 2004), and developed the reduced differential transform method for the fractional differential equations and gives the exact solution for both the linear and nonlinear differential equation. The method of separation of variables is also applicable to a large number of classical linear homogeneous equations. The choice of the coordinate system in general depends on the shape of the boundary condition (Sankara, 2014). The variable coefficient in the partial differential equation led to the variable coefficient in the ordinary differential equation which results from the separation of variable. Obtaining the general solution of this ordinary differential equation can then be formidable task in its own right. But, even if these general solutions can be obtained they may involve functions which will lead to further difficulties when one attempts to apply certain of supplementary conditions (Shepley, 1984). Therefore, this study concentrates on finding the solution of one dimensional heat-like equation using the MRDTM.

CHAPTER THREE

Methodology

3.1 Study Area and period

This study is conducted to find approximate solution of one dimensional heat-like equation subject to initial conditions from January to September 2017G.C in the department of Mathematics, Jimma University.

3.2 Study design

The researcher used semi-analytical design to conduct the study.

3.3 Source of information

The information that is used to conduct this study is collected from secondary source such as reference books, research papers, and internet

3.4 Mathematical procedure of the study

In order to achieve the objective of the study, the following procedures are undertaken.

1. Laplace transform will be applied to the power series solution obtained from RDTM to improve the solution (to increase the degree of the truncated series by 1).
2. The variable $\boldsymbol{s}$ is replaced by $\frac{1}{t}$ to simplify the expression in step 1.
3. Convert the transformed series (in step 2) into a meromorphic function by forming its pade approximation of order $[\frac{M}{N}]$, where M and N are arbitrary chosen constants from the set of positive integers. But they should be of smaller values than the order of the power series. In this step the pade approximate extends the domain of the truncate series solution to obtain better accuracy and convergence.
4. Then t is replaced by $\frac{1}{s}$ to take the expression (in step 3) back to the original variable.
5. Finally the inverse Laplace transform of the equation in step 4 will be taken to obtain better approximate solution of the problem under consideration.

3.5 Ethical consideration

Ethical consideration was taken care by getting the consent of official concerned body from Jimma University.

CHAPTER FOUR

Results and Discussions

4.1Preliminaries

4.1.1 The basic definitions and theorems of reduced differential transform method are introduced as follows (.,Vazquezz-leal,H., and Sarmiento-Reyes, A.,).

Definition 4.1(Brahim *et el*., 2014). If a function $u(x,t)$ is analytic and continuously differentiable with respect to time and space x in the studied domain, then

$$\mathrm{U_K(x)} = \frac{1}{\mathrm{k!}}\left[\frac{\partial^{\mathrm{k}}\mathrm{u(x,t)}}{\partial \mathrm{x^k}}\right]_{t=0}, \quad k = 0,1,2 \ldots. \tag{1}$$

Definition4.2 (Brahim *et el*., 2014).The differential inverse function of $\{U_k(x)\}_{k=0}^{n}$is defined by:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)\, t^k \tag{2}$$

where $\mathrm{U_K(x)}$ is t-dimensional spectrum function.

Then combining equation (1) and (2), we obtain

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial x^k}\right]_{t=0} t^k \tag{3}$$

From the above definition (4.1) and (4.2) the concept of RDTM is obtained from the power series expansion.

Theorem 4.1(Brahim *et el*., 2014). If $f(x,t) = \frac{\partial^n u(x,t),}{\partial x^n}$ then $F_k(x) = \frac{\partial^n U_K(x)}{\partial x^n}$ (4)

Proof: Let $F_k(x)$ and $U_k(x)$ are the t-dimensional spectrum functions (transformed function) of $f(x,t)$ and $u(x,t)$ respectively.

We want show that: $F_k(x) = \frac{\partial^n U_k(x)}{\partial x^n}$

Applying reduced differential operator (RDT) on both sides of equation (4)

$$RDT[f(x,t)] = RDT\left[\frac{\partial^n u(x,t)}{\partial x^n}\right]$$

Using definition (4.1) $RDT[f(x,t)] = \frac{1}{k!}\left[\frac{\partial^k f(x,t)}{\partial t^k}\right]_{t=0}$ and

$$RDT\left[\frac{\partial^n u(x,t)}{\partial x^n}\right] = \frac{\partial^n}{\partial x^n}\frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial t^k}\right]_{t=0} \quad (5)$$

Substitute (5) into (4), we have

$$\frac{1}{k!}\left[\frac{\partial^k f(x,t)}{\partial t^k}\right] = \frac{\partial^n}{\partial x^n}\frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial t^k}\right] \quad (6)$$

And also by definition 4.1, we have

$$F_K(x) = \frac{1}{k!}\left[\frac{\partial^k f(x,t)}{\partial t^k}\right]_{t=0} \text{ and } U_k(x) = \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial t^k}\right]_{t=0} \quad (7)$$

Substitute (7) in to (6), we obtain

$$F(x) = \frac{\partial^n U_K(x)}{\partial x^n}$$

Therefore, the theorem is proved.

Meromorphic function: A simple definition states that the meromorphic function is a function $f(z)$ of the form $\frac{g(z)}{h(z)}$, where $g(z)$ and $h(z)$ are entire functions with $h(z) \neq 0$ (Krantz, 1999).

4.1.2 Laplace transform

Definition4.3 :(Shepley, L.R. 1984)

Let f be real valued function of the real variable t, defined for $t < 0$. Let s be a variable that we shall assume to be real. The Laplace transform of f is F defined by the integral:

$$F(s) = \int_0^\infty e^{-st} f(t)dt \quad (8)$$

For all value of s for which this integral exist. The function F defined by integral (8) is called the Laplace transform of the function f . The Laplace transform of f denoted by $L\,(f)$ and $F(s)$ by $L\,[f\,(t)]$.

4.1.2.1 Properties of Laplace transform `

Theorem 4.2: Linearity (Shepley, L.R. 1984).

If $c_1$and $c_2$are any two constants and $F_1(s)$and $F_2(s)$ are the Laplace transform of $f_1(t)$ and $f_2(t)$ respectively, then $L[c_1 f_1(t) + c_2 f_2(t)] = c_1 L[f_1(t)] + c_2\, L[f_2(t)]$

$$= c_1 F_1(s) + c_2 F_2(s) \qquad (9)$$

Proof

From definition 4.3, we have:$L[c_1 f_1(t) + c_2 f_2(t)] \ = \ \int_0^\infty e^{-st}\, [c_1 f_1 + c_2 f_2(t)]dt$

$$= \int_0^\infty e^{-st}\, c_1 f_1(t)dt + \int_0^\infty e^{-st}\, c_2 f_2(t)dt$$

$$= c_1 \int_0^\infty e^{-st}\, f_1(t)dt + c_2 \int_0^\infty e^{-st}\, f_2(t)dt$$

$$= c_1 L[f_1(t)] + c_2\, L[f_2(t)]$$

$$= c_1 f_1(s) + c_2 f_2(s)$$

$$= L[c_1 f_1(t) + c_2 f_2(t)]$$

$$= c_1 F_1(s) + c_2 F_2 S)$$

Hence $L[c_1 f_1(t) + c_2 f_2(t)] = c_1 L[f_1(t)] + c_2\, L[f_2(t)] \ = c_1 F_1(s) + c_2 F_2(s)$

Theorem 4.3: Shifting property (Shepley, L.R. 1984).

If a function multiplied by e^{at}, then transform of the result is obtained by replacing s by $(s - a)$ in the transform of the original function.

That is if $L[f(t)] = F(s)$ then $L[e^{at} f(t)] = \ F(s - a\ )$ (10)

Proof

From definition4.3, we have$L\ [e^{at} f\ (t)] = \ \int_0^\infty e^{-st} e^{at} f(t)dt$

$$= \int_0^\infty e^{-(s-a)t} f(t)dt$$

$$= F(s - a)$$

Hence $L[e^{at} f(t)] = \ F(s - a]$

4.1.2.2 The inverse Laplace transforms (Shefley, L.R. 1984).

Definition 4.4: Given a function $F(s)$, the inverse Laplace transform of $F(s)$, denoted by $L^{-1}\{F(s)\}$,is that function f whose Laplace transform is F .That is $f(t) = L^{-1}\{F(s)\}$or $L\{f(t)\} = F(s)$

Theorem 4.4(Shefley, L.R. 1984). If $L[f(t)] = F(s)$, then $L^{-1}[F(s)e^{-at}] = L^{-1}[F(s)$ (11)

Proof

From definition (4.4), $[Lf(t)] = F(s)$ and $L^{-1}[F(s)] = f(t)$

By theorem (4.3) $L\,[e^{-at}f\,(t)] = F(s+a)$ and $L^{-1}[F(s+a)] = e^{-at}f(t)$

Thus $L^{-1}[F(s+a)] = e^{-at}L^{-1}[F(s)]$

4.1.3 Pade approximate

Definition 4.5 :(Brezenski, C.1996)

Given a function f and two integers $M \geq 1$ and $N \geq 1$, the Pade approximant of order $[\frac{M}{N}]$ is the rational function:

$$R(x) = \frac{\sum_{j=0}^{M} a_j x^j}{1+\sum_{k=1}^{N} b_k x^k} = \frac{a_0+a_1x+a_2x^2+\cdots+a_Mx^M}{1+b_1x+b_2x^2+\cdots+b_Nx^N} \quad (12)$$

This agrees with $f(x)$ to the highest possible order, which amount to

$$f(0) = R(0)$$

$$f'(0) = R'(0)$$

$$f''(0) = R''(0)$$

$$\cdots$$

$$f^{M+N}(0) = R^{M+N}(0)$$

The Pade approximant is unique for given M and N, that is the coefficients $a_0,\ a_1,\ a_2,\ \ldots,\ a_M$ and $b_1,\ b_2,\ b_3\ ,\ldots,\ b_N$ can be uniquely determined. It is for reasons of uniqueness that the zero-th order term at the denominator of $R(x)$ was chosen to be 1, otherwise the numerator and denominator of R(x) would have been unique only up to multiplication by a constant. For a fixed value of $N + M$, the error is smallest when $M = N$ or when $M = N + 1$ .The Pade approximant often gives better approximation of the function than truncating its Taylor series and it may still work where the Taylor series does not converge. For this reason Pade approximants are used extensively in computer calculations.

Definition 4.6: (Forenberg, B. 1998)

A Taylor expansion can often be accelerated quite dramatically (or turned from divergent to convergent) by being rearranged into a ratio of two such expansion.

A pade approximation $P_N^M(x) = \frac{\sum_{n=0}^{M} a_n x^n}{\sum_{n=0}^{N} b_n x^n}$

(Normalized $b_0 = 1$) generalizes the Taylor expansion with equally many degrees of freedom.

$$T_{M+N}(x) = \sum_{n=0}^{M+N} c_n x^n \qquad (13)$$

The two equation being the same in case N=0. The pade coefficients are normally best found from a Taylor expansion: $c_0 + c_1 x + c_2 x^2 + \ldots = \frac{a_o + a_1 x + a_2 x^2 + \cdots}{1 + b_1 x + b_2 x^2 + \cdots}$ multiplying up the denominator gives the following equivalent set of coefficient relations.

$$a_o = c_o$$

$$a_1 = c_1 + c_o b_1 \qquad (14)$$

$$a_2 = c_2 + c_1 b_1 + c_o b_2$$

$$a_3 = c_3 + c_2 b_1 + c_1 b_2 + c_o b_3$$

$$\cdots$$

$$a_M = c_M + c_{M-1} b_1 + \ldots\ldots + c_o b_M$$

and

$$c_M q_1 + \cdots + c_{M-N+1} q_1 + c_{M+1} = 0$$

$$c_{M+1} q_1 + \cdots + c_{M-N+2} q_M + c_{M+2} = 0$$

$$\cdots$$

$$c_{M+N-1} q_1 + \cdots + c_M q_N + c_{M+N} = 0,$$

with the c_i given, each new line introduces two new unknowns,a_i andb_i. The system would appear to be severely underdetermined. However, if we specify the degree of the numerator to be M and of the denominator to be N, and of the truncated Taylor expansion to be M+N, there will be just as many equations as unknowns (ignoring all terms that are $0(X^{M+N+1})$.We can then solve for all the unknown coefficients, as the following example shows:

Example :(Forenberg , B. 1998). Given $T_5(x)$, determine $P_2^3(x)$.

In this case $M = 3, N = 2,\ M + N = 5$, the system (14) becomes 'cut off' as follow.

$$a_o = c_0$$

$$a_1 = c_1 + c_0 b_1$$

$$a_2 = c_2 + c_1 b_1 + c_0 b_2$$

$$a_3 = c_3 + c_2 b_1 + c_1 b_2 + c_0 b_3$$

No more a′s available ⇒

0	$c_3 + c_2 b_1 + c_1 b_2 + c_0 b_3$	No more b′s available
0	$c_4 + c_3 b_1 + c_2 b_2 + c_1 b_3$	
0	$c_5 + c_4 b_1 + c_3 b_2 + c_2 b_3$	
Past limit $0(x^{(3+2+1)})$		

The bottom three equations can be solved for b_1, b_2, b_3 after which the top three explicitly givea_1, a_2, a_3.The same idea carries through for any values M and N.

A key usage of pade approximation is to extract the information from power series expansion with only a few known terms. Transformation to pade form usually accelerates convergence, and often allows good approximations to be found even outside a power series expansion's radius of convergence.

4.2 Main results

Based on the above theorem and definition, the main result of this study will be presented as follows. First we consider the following one dimensional heat-like equation written in an operator form.

$$Lu(x,t) - c^2R\,u(x,t) = g(x,t) \quad (15)$$

Subject to initial condition $u(x,0) = f(x)$ (16)

Where $L = \frac{\partial}{\partial t}$, R is linear differential operator, $u(x,t)$ is unknown analytic and continuously differentiable function, $g\,(x.t)$ is an inhomogeneous term and c^2 is a variable coefficient.

Here we consider two cases of (15), the homogeneous and non-homogenous.

<u>Case</u>I: Let equation (15) is homogeneous, i.e. $Lu(x,t) - c^2R\,u(x,t) = 0$

$$Lu(x,t) = c^2R\,u(x,t) \quad (17)$$

Applying RDTM on both sides of (17) and (16), we have

$$RDT\,(Lu(x,t)) = RDT\,(c^2R\,u(x,t))$$

$$(k+1)U_{K+1}(x) = c^2RU_k(x) \quad (18)$$

$$u(x,0) = f(x) \quad (19)$$

Substitute (19) into (18), we have

$$U_1(x) = \frac{1}{1!}R(c^2\,f(x)) = \frac{1}{1!}a_1h(x$$

$$U_2(x) = \frac{1}{2!}R^2(c^4f(x)) = \frac{1}{2!}a_2h(x)$$

$$U_3(x) = \frac{1}{3!}R^3(c^6f(x)) = \frac{1}{3!}a_3h(x) \quad (20)$$

$$U_4(x) = \frac{1}{4!}R^4\big(c^8f(x)\big) = \frac{1}{4!}a_4h(x)$$

$$\cdots$$

By definition (4.1) $U_k(x) = \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial x^k}\right] \;\; = \frac{1}{k!}a_n h(x)$

Again by definition differential(4.2), the inverse transform of the set of values $\{\mathrm{U_K(x)}\}_{k=0}^{n}$ gives approximate solution as, $\overline{U}_n(x,t) \cong \sum_{k=0}^{n} \mathrm{U_K(x)} t^k$, (21)

where n is the approximation order of the solution.

From equation (20) and (21), we have $u(x,t) \cong \sum_{k=0}^{n} \mathrm{U_K(x)} t^k$

$u(x,t) = U_o t^o + U_1 t^1 + U_2 t^2 + U_3 t^3 + U_4 t^4 + \cdots + U_n t^n.$

$$u(x,t) = f(x) + a_1 h(x)t + \frac{a_2}{2!}h(x)t^2 + \frac{a_3}{3!}h(x)t^3 + \frac{a_4}{4!}h(x)t^4 + \cdots + \frac{a_n}{n!}h(x)t^n$$

$$u(x,t) = f(x) + h(x)(a_1 t + \frac{a_2}{2}t^2 + \frac{a_3}{6}t^3 + \frac{a_4}{24}t^4 + \cdots + \frac{a_n}{n!}t^n) \tag{22}$$

Hence (22) is the approximate solution of (17) that obtained from RDTM truncated series. The exact solution is $u(x,t) = \lim_{n\to\infty} \overline{U}_n(x,t)$.

But it is described in the literature the solutions obtained from the RDTM truncated series may have limited regions of convergence, even if we take a large number of terms. Therefore, we proposed to apply the Pade approximation technique to this truncated series to increase the convergence region (the steps are mentioned in chapter three). First Laplace transform is applied to (22). Then, s is substituted by $\frac{1}{t}$ and the Pade approximant is applied to the transformed series. Finally, t is substituted by $\frac{1}{s}$ and the inverse Laplace transform is applied to the resulting expression to obtain the solution.

Apply Laplace transform to $(a_1 t + \frac{a_2}{2}t^2 + \frac{a_3}{6}t^3 + \frac{a_4}{24}t^4 + \cdots + \frac{a_n}{n!}t^n)$, we get

$$L[u(x,t) = L[a_1 t + \frac{a_2}{2}t^2 + \frac{a_3}{6}t^3 + \frac{a_4}{24}t^4 + \cdots + \frac{a_n}{n!}t^n$$

$L[u(x,t)] = (\frac{a_1}{s^2} + \frac{a_2}{s^3} + \frac{a_3}{s^4} + \frac{a_4}{s^5} + \cdots + \frac{a_n}{s^{n+1}})$, using equation (8) (23)

From step 2, replacing s by $\frac{1}{t}$, we get

$$L[u(x,t)] = (\, a_1 t^2 + a_2 t^3 + a_3 t^4 + a_4 t^5 + \cdots + a_n t^{n+1}) \tag{24}$$

Based on the definition4.6, all of $[\frac{M}{N}]$ pade approximation of (24) with $N \geq 1$ and $M \geq 1$, where N is the degree of the denominator and M is the degree of the numerator gives:

$$\left[\frac{M}{N}\right](a_1t^2 + a_2t^3 + a_3t^4 + a_4t^5 + \cdots + a_nt^{n+1}) = \frac{p_o+p_1\, t+p_2t^2+p_3t^3+\ ...+\ p_Mt^M}{q_0+q_1t+q_2t^2+\ ...\ +q_Nt^N} \qquad (25)$$

where q_0=1, and the numerator and denominator polynomials have no common factors (unique).

Now multiplying both sides of equation (25) by $q_0 + q_1t + q_2t^2 + \ ... + q_Nt^N$ gives the following equivalence relation.

$$p_o = 0$$

$$p_1 = 0$$

$$p_2 = a_1$$

$$p_{3=}a_2 + a_1q_1$$

$$p_4 = a_3 + a_2q_1 + a_1q_2$$

$$\cdot\ \cdot\ \cdot$$

$$p_M = a_{M-1} + a_{M-2}q_1 + \cdots + a_1q_N$$

and

$$0 = a_4 + a_3q_1 + a_2q_3 + \cdots + a_1q_N$$

$$0 = a_5 + a_4q_1 + a_3q_1 + \cdots + a_2q_N$$

$$\cdot\ \cdot\ \cdot$$

$$0 = a_{M+N-1} + a_{M+N-2}q_1 + \ \dots\dots\ + \ a_Nq_N$$

The unknown coefficients $p_{0\,,}p_1, p_3\ \dots.\ p_M$ and $q_1, q_2, q_3\ \dots\dots\ q_N$ of the right hand side of equation (25) can be determined from the condition that the first$(N + M + 1)$terms vanish in the Taylor series expansion.(i.e. Ignoring all terms that are $0(X^{(M+N+1)})$. So now let $b_0, b_1, b_2\ \dots\dots.\ b_M$ be the value of $p_{0\,,}p_1, p_{3\,\dots,}\ p_M$ and $d_1, d_2, d_3, \dots, d_N$ be the value of $q_1, q_2, q_3, \dots,\ q_N$ respectively (26)

Using definition 4.5, let the degree of the numerator is greater than the degree of the denominator by one (i.e.$M = N + 1$).

Substitute (26) into (25) we obtain:

$$\left[\frac{M}{N}\right](a_1t^2 + a_2t^3 + a_3t^4 + a_4t^5 + \cdots + a_nt^{n+1}) = \frac{b_o + b_1t + b_2t^2 + b_3t^3 + \ \ldots\ b_Mt^M}{1 + d_1t + d_2t^2 + \ldots + d_Nt^N}$$

$$\left[\frac{M}{N}\right](a_1t^2 + a_2t^3 + a_3t^4 + a_4t^5 + \cdots + a_nt^{n+1}) = \frac{b_2t^2 + b_3t^3 + \ \ldots\ b_Mt^M}{1 + d_1t + d_2t^2 + \ldots + d_Nt^N} \quad (27)$$

In step 4, replacing t by $\frac{1}{s}$ in the right hand side of (27), we obtain

$$\left[\frac{M}{N}\right](a_1t^2 + a_2t^3 + a_3t^4 + a_4t^5 + \cdots + a_nt^{n+1}) = \frac{b_2s^{M-2} + b_3s^{M-3} + \ \ldots\ b_M}{s^{M-N}[s^N + d_1s^{N-1} + d_2S^{N-2} + \ldots + d_N]}$$

Since $M = N + 1$

$$= \frac{b_2s^{M-2} + b_3s^{M-3} + \ \ldots\ b_M}{s[s^N + d_1s^{N-1} + d_2S^{N-2} + \ldots + d_N]}$$

$$\left[\frac{M}{N}\right](a_1t^2 + a_2t^3 + a_3t^4 + a_4t^5 + \cdots + a_nt^{n+1}) = \frac{b_2s^{M-2} + b_3s^{M-3} + \ \ldots\ b_M}{s[s^N + s^{N-1} + d_1s^{N-2} + d_2S^{N-3} + \ldots + d_N]} \quad (28)$$

Decompose the right side of equation (28) into partial fraction reduction, we get

$$\left[\frac{M}{N}\right](a_1t^2 + a_2t^3a_3t^4 + a_4t^5 + .. + a_nt^{n+1}) = \frac{A_1}{s} + \frac{A_2}{s+j_0} + \frac{A_3}{s+j_1} + \frac{A_4}{s+j_2} + \cdots + \frac{A_{N+1}}{s+j_{N-1}} \quad (29)$$

Where $A_1, A_2, A_3, \ldots.\ A_{N+1}$ and $j_0, j_1, j_3, \ldots\ j_{N-1}$ are arbitrary constants.

Finally, applying the inverse Laplace transform to Pade approximants (29), we obtain

$$L^{-1}\left[\frac{A_1}{s} + \frac{A_2}{s + j_0} + \frac{A_3}{s + j_1} + \frac{A_4}{s + j_2} + .. + \frac{A_{N+1}}{s+j_{N-1}}\right] = (A_1 + A_2e^{-j_0t} + A_3e^{-j_1t} + .. + A_{N+1}e^{j_{N-1}t})$$

Therefore, the approximate solution of (15) is given by:

$$u(x,t) = f(x) + h(x)(A_1 + A_2e^{-j_0t} + A_3e^{-j_1t} + \cdots + A_{N+1}e^{j_{N-1}t})$$

<u>Case </u>II: let equation (15) is non-homogeneous

$$Lu(x,t) - c^2Ru(x,t) = g(x,t) \quad (30)$$

Subject to the initial condition $u(x,0) = f(x)$, (31) where $L = \frac{\partial}{\partial t}$, R is linear differential operator, $u(x,t)$ is unknown analytic and continuously differentiable function, $g\ (x.t)$ is an non-homogeneous term and c^2 is a variable coefficient.

Applying both side RDTM in (30) & (31), we have

$$RDT\ (Lu(x,t))\ =\ RDT\ (c^2 R\, u(x,t))\ +\ RDT\ (g(x,t)$$

$$(k+1)U_{k+1}(x) = c^2 R U_K(x) + G_k(x) \tag{32}$$

$$U_O(x) = f(x), \tag{33}$$

Substitute (33) into (32), we obtain the following $U_k(x)$values successively:

$$U_1(x) = \frac{1}{1!}[c^2 Rf(x) + G_0(x)] = a_1 m(x)$$

$$U_2(x) = \frac{1}{2!}c^4 R^2 f(x) + \frac{1}{2!}c^2 R G_0(x) + \frac{1}{2!}G_1(x) \tag{34}$$

$$= \frac{1}{2!}R^2\left[c^4 f(x) + c^2 \frac{1}{R}G_0(x) + \frac{1}{R^2}G_1(x)\right] = a_2 m(x)$$

$$U_3(x) = \frac{1}{3!}R^3\left[c^6 f(x) + c^4 \frac{1}{R}G_0(x) + c^2 \frac{1}{R^2}G_1(x) + \frac{1}{R^3}G_2(x)\right] = a_3 m(x)$$

$$\cdot\ \cdot\ \cdot\ \cdot$$

$$U_k\ (x) = \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial x^k}\right]\ = a_n m(x),$$

where $m(x)$ is analytic function.

The differential inverse transform of set of values $\{\mathrm{U_k(x)}\}_{k=0}^{n}$ gives the approximation solution as, $\overline{U}_n(x,t) = \sum_{k=0}^{n} \mathrm{U_K(x)} t^k$ (35)

where n is approximation order of the solution.

From equation (34) and (35), we have

$$u(x,t) \cong \sum_{k=0}^{n} \mathrm{U_K(x)} t^k$$

$$= U_o t^o + U_1 t^1 + U_2 t^2 + U_3 t^3 + U_4 t^4 + \cdots U_n t^n$$

$$= f(x) + a_1 m(x)\ t + \frac{a_2}{2}m(x)t^2 + \frac{a_3}{6}m(x)t^3 + \frac{a_4}{24}m(x)t^4 + \cdots + \frac{a_n}{n!}m(x)t^n$$

$$= f(x) + m(x)(a_1 t + \frac{a_2}{2} t^2 + \frac{a_3}{6} t^3 + \frac{a_4}{24} t^4 + \cdots + \frac{a_n}{n!} t^n) \quad (36)$$

Hence (36) is the approximate solution of (30) that obtained from RDTM truncated series. The exact solution is $u(x,t) = \lim_{n\to\infty} \overline{U}_n(x,t)$

By the same reason given in **case I**, we have to follow the next steps.

Applying Laplace transform to$(a_1 t + \frac{a_2}{2} t^2 + \frac{a_3}{6} t^3 + \frac{a_4}{24} t^4 + \cdots + \frac{a_n}{n!} t^n)$, we get

$$L[u(x,t)] = L[a_1 t + \frac{a_2}{2} t^2 + \frac{a_3}{6} t^3 + \frac{a_4}{24} t^4 + \cdots + \frac{a_n}{n!} t^n]$$

$L[u(x,t)] = (\frac{a_1}{s^2} + \frac{a_2}{s^3} + \frac{a_3}{s^4} + \frac{a_4}{s^5} + \cdots + \frac{a_n}{s^{n+1}})$ using equation (8) (37)

In step 2, replacing s by $\frac{1}{t}$, we get

$$L[u(x,t)] = (a_1 t^2 + a_2 t^3 + a_3 t^4 + a_4 t^5 + \cdots + a_n t^{n+1}) \quad (38)$$

Based on definition4.6, all of $[\frac{M}{N}]$ pade approximation of (38) with $N \geq 1 \; and \; M \geq 1$, where N is the degree of the denominator and M is the degree of the numerator gives:

$$\left[\frac{M}{N}\right] (a_1 t^2 + a_2 t^3 + a_3 t^4 + a_4 t^5 + \cdots + a_n t^{n+1}) = \frac{p_o + p_1 t + p_2 t^2 + p_3 t^3 + \ldots + p_M t^M}{q_0 + q_1 t + q_2 t^2 + \ldots + q_N t^N} \quad (39)$$

where$q_0 = 1$, and the numerator and denominator polynomials have no common factors.

Now multiplying both sides of equations (39) by $1 + q_1 t + q_2 t^2 + \ldots + q_N t^N$ gives the following equivalence relation:

$$p_o = 0$$

$$p_1 = 0$$

$$p_2 = a_1 + q_1$$

$$p_3 = a_2 + a_1 q_1 + q_2$$

$$p_4 = a_3 + a_2 q_1 + a_1 q_2 + q_3$$

$$p_5 = a_4 + a_3 q_1 + a_2 q_2 + a_1 q_3 + q_4$$

. . .

$$p_M = a_{M-1} + a_{M-2}q_1 + \cdots + a_1 q_N$$

and

$$0 = a_4 + a_3 q_1 + a_2 q_3 + \cdots + a_1 q_N$$

$$0 = a_5 + a_4 q_1 + a_3 q_1 + \cdots + a_2 q_N$$

. . .

$$0 = a_{M+N-1} + a_{M+N-2}q_1 + \ \dots\dots + \ a_N q_N$$

Now let $b_0, b_1, b_2 \dots\dots . b_M$ be the value of$p_{0\,,}p_1, p_{3\,\dots,}\, p_M$ and $d_1, d_2, d_3, \dots, d_N$be the value of $q_1, q_2, q_3, \dots,\ q_N$ respectively. (40)

As explained in homogenous case for the constants p's and q's and then using definition4.5 for the degree of the numerator and denominator and further substituting (40) into (39), we obtain

$$\left[\frac{M}{N}\right](a_1 t^2 + a_2 t^3 + a_3 t^4 + a_4 t^5 +..+ a_n t^{n+1}) = \frac{b_2 t^2 + b_3 t^3 + \ \dots\ b_M t^M}{1 + d_1 t + d_2 t^2 + \dots + d_N t^N} \tag{41}$$

In step 4, replacing t by $\frac{1}{s}$ in the right hand side of (41), we have

$$\left[\frac{M}{N}\right](a_1 t^2 + a_2 t^3 + a_3 t^4 + a_4 t^5 + \cdots + a_n t^{n+1}) = \frac{b_2 s^{M-2} + b_3 s^{M-3} + \ \dots\ b_M}{s^{M-N}[s^N + d_1 s^{N-1} + d_2 S^{N-2} + \dots + d_N]}$$

$$= \frac{b_2 s^{M-2} + b_3 s^{M-3} + \ \dots\ b_M}{s[s^N + s^{N-1} + d_1 s^{N-2} + d_2 S^{N-3} + \dots + d_N]}$$

$$[\frac{M}{N}]\,(a_1 t^2 + a_2 t^3 + a_3 t^4 + a_4 t^5 +..+ a_n t^{n+1}) = \frac{b_2 s^{M-2} + b_3 s^{M-3} + \ \dots + b_M}{s(s^N + s^{N-1} + d_1 s^{N-2} + d_2 S^{N-3} + \dots + d_N)} \tag{42}$$

Decompose the right side of (42) into partial fraction reduction, we get

$$\left[\frac{M}{N}\right](a_1 t^2 + a_2 t^3 a_3 t^4 + a_4 t^5 +..+ a_n t^{n+1}) = \frac{h_1}{s} + \frac{h_2}{s + j_0} + \frac{h_3}{s + j_1} + \frac{h_4}{s + j_2} + \cdots + \frac{h_{N+1}}{s + j_{N-1}} \tag{43}$$

Where $h_1, h_2, h_3, \dots.\ \ h_{N+1}$ and $j_0, j_1\, , j_3, \dots\ \ j_{N-1}$ are arbitrary constants

Finally, applying the inverse Laplace transform to Pade approximants (43), we obtain

$$L^{-1}\left[\frac{h_1}{s} + \frac{h_2}{s + j_0} + \frac{h_3}{s + j_1} + \frac{h_4}{s + j_2} +..+ \frac{h_{N+1}}{sj +_{N-1}}\right] = h_1 + h_2 e^{-j_0 t} + h_3 e^{-j_1 t} +..+ h_{N+1} e^{j_{N-1} t})$$

Therefore, the better approximate solution of (30) is given by:

$$u(x,t) = f(x) + m(x)(h_1 + h_2 e^{-J_0 t} + h_3 e^{-J_1 t} + \cdots + h_{N+1} e^{J_{N-1} t}).$$

Remark: Here we observe that the solution obtained in case I & case II are in a convergent function form than the solution obtained from RDTM. Hence the approximate solution obtained after the application of Laplace Pade Resummation Method is better than the solution obtained from RDTM.

Test problems

In this section, we will demonstrate the effectiveness and accuracy of the MRDTM described in the previous sections on one dimensional heat-like equation subject to initial condition using the following three examples.

Example 1: Now consider the one dimensional initial value problem which describes the homogeneous heat-like equation.

$$\frac{\partial u(x,t)}{\partial t} = \frac{x^2}{2} \frac{\partial^2 u(x,t)}{\partial x^2} \tag{44}$$

With the initial condition $u(x,0) = x^2$ (45)

Taking the reduced differential transform of (44) and (45), we have

$$(k+1) U_{K+1}(x) = \frac{x^2}{2} \frac{\partial^2 U_K(x)}{\partial x^2} \tag{46}$$

$$U_0(x) = x^2 \delta(k) \tag{47}$$

Where $\delta(k) = \begin{cases} 1, if\ k = 0 \\ 0, if\ k \neq 0 \end{cases}$

Now, substitute (47) into (46), we obtain the following $U_k(x)$ values successively:

$$U_1(x) = x^2,\ U_2(x) = \frac{x^2}{2}, U_3(x) = \frac{x^2}{6}, U_4(x) = \frac{x^2}{24}, U_5(x) = \frac{x^2}{120},$$

$$U_6(x) = \frac{x^2}{720}, \ldots, U_k(x) = \frac{x^2}{k!} \tag{48}$$

Then using (21) and (48), we get approximate solution of (44).

Let the order of approximation (n) =6, then we have

$$u(x,t) \cong \sum_{K=0}^{6} U_K\,(x)t^k$$

$$= U_0t^0 + U_1t^1 + U_2t^2 + U_3t^3 + U_4t^4 + U_5t^5 + u_6t^6$$

$$= x^2 + x^2t + \frac{1}{2}x^2t^2 + \frac{1}{6}x^2t^3 + \frac{1}{24}x^2t^4 + \frac{1}{120}x^2t^5 + \frac{1}{720}x^2t^6$$

$$= x^2(1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{1}{120}t^5 + \frac{1}{720}t^6) \quad (49)$$

Hence (49) is the approximate solution of (44) that obtained from RDTM truncated series. By the same reason given in case I above, we have to follow the next steps.

We apply Laplace transforms to $(1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{1}{120}t^5 + \frac{1}{720}t^6)$, we get

$L\,[u(x,t)] = (\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} + \frac{1}{s^4} + \frac{1}{s^5} + \frac{1}{s^6} + \frac{1}{s^7})$ Using equation (8)

For simplicity, replacing s by $\frac{1}{t}$, we get

$$L\,[u(x,t)] = (\,t + t^2 + t^3 + t^4 + t^5 + t^6 + t^7\,) \quad (50)$$

All of the $[\frac{M}{N}]$ pade approximate of (50) with $N \geq 1, M \geq 1$ and $M + N \leq 7$ where the degree of the numerator is 4 and the degree of the denominator is 3 gives:

$$[\,\frac{N}{M}\,](\,t + t^2 + t^3 + t^4 + t^5 + t^6 + t^7\,) = \frac{a_0 + a_1t + a_2t^2 + a_3t^3 a_4t^4}{1 + b_1t + b_2t^2 + b_3t^3} \quad (51)$$

Using (14) and (51), we have

$$Co = 0,\ c_1 = 1,\ c_2 = 1, c_3 = 1,\ c_4 = 1,\ c_5 = 1,\ c_6 = 1,\ c_7 = 1$$

$$a_0 = c_0 = 0$$

$$a_1 = c_1 + c_0b_1 = 1$$

$$a_2 = c_0 + c_1b_1 + c_0b_2 = b_1$$

$$a_3 = c_3 + c_2b_1 + c_1b_2 + c_0b_3 = 1 + b_1 + b_2$$

$$a_4 = c_4 + c_3b_1 + c_2b_2 + c_1b_3 + c_0b_4 = 1 + b_1 + b_2 + b_3$$

$$0 = c_5 + c_4b_1 + c_3b_2 + c_2b_3 + c_1b_4 + c_0b_5 = 1 + b_1 + b_2 + b_3$$

$$0 = c_6 + c_5b_1 + c_4b_2 + c_3b_3 + c_2b_4 + c_1b_5 + c_0b_6 = 1 + b_1 + b_2 + b_3$$
$$0 = c_7 + c_6b_1 + c_5b_2 + c_4b_3 + c_3b_4 + c_2b_5 + c_1b_6 + c_0b_7 = 1 + b_1 + b_2 + b_3$$

Solving this, we have:

$a_1 = 1, a_2 = a_3 = a_4 = b_2 = b_3 = 0$ and $b_1 = -1$ (52)

Substitute (52) into (51), we obtain:

$$\left[\frac{N}{M}\right](t + t^2 + t^3 + t^4 + t^5 + t^6 + t^7) = \frac{t}{1-t} \quad (53)$$

In step 4, replacing t by $\frac{1}{s}$ in the right hand side of (53), we get:

$$\left[\frac{N}{M}\right](t + t^2 + t^3 + t^4 + t^5 + t^6 + t^7) = \frac{1}{s-1} \quad (54)$$

Finally, applying the inverse Laplace transform to the right hand side of Pade approximants (54), we obtain: $[\frac{N}{M}](t + t^2 + t^3 + t^4 + t^5 + t^6 + t^7) = e^t$

Therefore, the approximate solution of (44) is given by $u(x,t) = x^2e^t$ which in this case is an exact solution (Gupta, V. G., and Gupta, S., 2011).

Table1. Comparison between the solutions obtained from RDTM and MRDTM of one dimensional homogenous heat-like equation $\frac{\partial u(x,t)}{\partial t} = \frac{x^2}{2}\frac{\partial^2 u(x,t)}{\partial x^2}$

t	x	RDTM	MRDTM	Exact solution $u(x,t) = x^2e^t$ (Gupta & Gupta, 2011)
1	1	2.71805…	2.71828…	$e = 2.71828 \ldots$
	2	10.87222…	10.87312…	$4e = 10.87312 \ldots$
	3	24.4625…	24.4645….	$9e = 24.4645 \ldots.$
	4	43.4888…	43.4925…	$16e = 43.4925 \ldots$
	5	67.9513…	67.9570…	$25e = 67.9570 \ldots$
2	1	7.35555..	7.3890…	$e^2 = 7.3890 \ldots$
	2	29.4222…	29.5556…	$4e^2 = 29.5556 \ldots$
	3	66.2000…	66.5015…	$9e^2 = 66.5015 \ldots$
	4	117.6888…	118.2248…	$16e^2 = 118.2248 \ldots$
	5	183.8888…	184.7264…	$25e^2 = 184.7264 \ldots$
3	1	19.4125…	20.0855…	$e^3 = 20.0855 \ldots$
	2	77.6500…	80.34214…	$4e^3 = 80.34214 \ldots$
	3	174.7125..	180.7698…	$9e^3 = 180.7698 \ldots$
	4	310.6000…	321.3685…	$16e^3 = 321.3685 \ldots$
	5	485.3125…	502.1384…	$25e^3 = 502.1384 \ldots$

From table 1, we observe that the solution obtained from MRDTM is better than the solution obtained from RDTM and fits with the exact solution.

Example2: Consider the one dimensional initial value problem which describes the non-homogeneous heat like equation.

$$\frac{\partial u(x,t)}{\partial t} = \frac{x^2}{2}\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{x^2}{2} \tag{55}$$

With the initial condition $u(x,0) = \frac{x^2}{2}$ (56)

Taking the reduced differential transform of (55) and (56), we have

$$(k+1)U_{K+1}(x) = \frac{x^2}{2}\frac{\partial^2 U_K(x)}{\partial x^2} + \frac{1}{2}x^2\delta(k), \text{ Where } \delta(k)\begin{cases}1, if\ k=0\\0, if\ k\neq 0\end{cases} \tag{57}$$

$$U_0(x) = \frac{x^2}{2}\delta(k) \tag{58}$$

Plunging (58) into (57), we have:

$$U_1(x) = x^2, U_2(x) = \frac{1}{2}x^2, U_3(x) = \frac{1}{6}x^2, U_4(x) = \frac{1}{24}x^2$$

$$U_5(x) = \frac{1}{120}x^2\ , U_6(x) = \frac{1}{720}x^2, \ldots, U_k(x) = \frac{x^2}{k!} \tag{59}$$

Then using (35) and (59), we get an approximate solution of (55)

Let the order of approximation (n) =6

$$u(x,t) \cong \sum_{K=0}^{6} U_K\,(x) t^k$$
$$= U_0 t^0 + U_1 t^1 + U_2 t^2 + U_3 t^3 + U_4 t^4 + U_5 t^5 + u_6 t^6$$

$$= \frac{1}{2}x^2 + x^2 t + \frac{1}{2}x^2 t^2 + \frac{1}{6}x^2 t^3 + \frac{1}{24}x^2 t^4 + \frac{1}{120}x^2 t^5 + \frac{1}{720}x^2 t^6$$
$$= x^2\left(\frac{1}{2} + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{1}{120}t^5 + \frac{1}{720}t^6\right) \quad (60)$$

Hence (60) is the approximate solution of (55) that obtained from (RDTM) truncated series. By the same reason given in case I above, we have to follow the next steps.

We apply Laplace transforms to$\left(\frac{1}{2} + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{1}{120}t^5 + \frac{1}{720}t^6\right)$, we get

$L\,[u(x,t)] = \left(\frac{1}{2s} + \frac{1}{s^2} + \frac{1}{s^3} + \frac{1}{s^4} + \frac{1}{s^5} + \frac{1}{s^6} + \frac{1}{s^7}\right)$ Using equation (8)

For simplicity, replacing s by $\frac{1}{t}$, we get

$$L\,[u(x,t)] = \left(\frac{1}{2}t + t^2 + t^3 + t^4 + t^5 + t^6 + t^7\right) \quad (61)$$

All of the $[\frac{M}{N}]$pade approximation of (61) with $N \geq 1$ and $M \geq 1$ and $M + N \leq 7$ where the degree of the numerator is 4 and the degree of the denominator is 3gives:

$$[\frac{N}{M}]\left(\frac{1}{2}t + t^2 + t^3 + t^4 + t^5 + t^6 + t^7\right) = \frac{a_0 + a_1 t + a_2 t^2 + a_3 t^3 a_4 t^4}{1 + b_1 t + b_2 t^2 + b_3 t^3} \quad (62)$$

Using (14) and (62), we have:

$$Co = 0,\ c_1 = \frac{1}{2},\quad c_2 = 1, c_3 = 1,\quad c_4 = 1,\quad c_5 = 1,\quad c_6 = 1,\quad c_7 = 1$$
$$a_0 = c_0 = 0$$
$$a_1 = c_1 + c_0 b_1 = \frac{1}{2}$$
$$a_2 = c_2 + c_1 b_1 + c_0 b_2 = 1 + \frac{1}{2}b_1$$
$$a_3 = c_3 + c_2 b_1 + c_1 b_2 + c_0 b_3 = 1 + b_1 + \frac{1}{2}b_2$$
$$a_4 = c_4 + c_3 b_1 + c_2 b_2 + c_1 b_3 + c_0 b_4 = 1 + b_1 + b_2 + \frac{1}{2}b_3$$
$$0 = c_5 + c_4 b_1 + c_3 b_2 + c_2 b_3 + c_1 b_4 + c_0 b_5 = 1 + b_1 + b_2 + b_3$$
$$0 = c_6 + c_5 b_1 + c_4 b_2 + c_3 b_3 + c_2 b_4 + c_1 b_5 + c_0 b_6 = 1 + b_1 + b_2 + b_3$$
$$0 = c_7 + c_6 b_1 + c_5 b_2 + c_4 b_3 + c_3 b_4 + c_2 b_5 + c_1 b_6 c_0 b_7 = 1 + b_1 + b_2 + b_3$$

Solving this we have $a_1 = \frac{1}{2}, a_2 = \frac{1}{2}\ , b_1 = -1, a_0 = a_3 = a_4 = b_2 = b_3 = 0$ (63)

Substitute (63) into (62), we obtain:

$$[\frac{N}{M}]\ (\frac{1}{2}t + t^2 +\ t^3\ + t^4\ + t^5 + t^6\ + t^7)\ =\ \frac{\frac{1}{2}(\ t+t^2)}{1-t} \quad (64)$$

In step2, replacing t by $\frac{1}{s}$ in the right hand side of (64), we get:

$$[\ \frac{N}{M}\ ]\ (t + t^2 +\ t^3 + t^4\ + t^5 + t^6\ + t^7)\ =\ \frac{s+1}{2s(s-1)} \quad (65)$$

Decompose the right side of (65) in to partial fraction reduction, we obtain

$$[\ \frac{N}{M}\ ]\ (t + t^2 +\ t^3\ + t^4\ + t^5 + t^6\ + t^7)\ = \frac{1}{2}(\frac{2}{s-1} - \frac{1}{s}) \quad (66)$$

Finally, applying the inverse Laplace transform to the right hand side of Pade approximant (66), we obtain: $[\ \frac{N}{M}\ ]\ (t + t^2 +\ t^3\ + t^4 + t^5 + t^6\ + t^7)\ = e^t - \frac{1}{2}$

Therefore, the approximate solution of (55) is given by $u(x,t) = x^2(e^t - \frac{1}{2})$ (67)

Substituting (67) into (55), we conclude that the approximate solution coincides with the exact one (Ahmed, 2014)

Table2. Compression between the solutions obtained from RDTM and MRDTM of one dimensional non-homogenous heat-like equation $\frac{\partial u(x,t)}{\partial t} = \frac{x^2}{2}\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{x^2}{2}$

t	x	RDTM	MRDTM	Exact solution $u(x,t) = x^2(e^t - 0.5)$ (Ahmed, 2014)
1	1	2.21805...	2.21828...	$e - 0.5 = 2.71828$ ...
	2	8.8722....	8.8731...	$4(e - 0.5) = 8.8731$ ...
	3	19.9625...	19.9645....	$9(e - 0.5) = 19.9645$ ....
	4	35.4888...	35.4924...	$16(e - 0.5) = 35.4924$ ...
	5	55.4512...	55.4570...	$25(e - 0.5) = 55.4570$ ...
2	1	6.8555..	6.8890...	$e^2 - 0.5 = 6.8890$ ...
	2	27.4220...	27.5562...	$4(e^2 - 0.5) = 27.5562$ ...
	3	61.6995...	62.0001...	$9(e^2 - 0.5) = 62.0001$ ... ...
	4	109.6880...	110.2248...	$16(e^2 - 0.5) = 110.2248$ ...
	5	171.3875...	172.2264...	$25(e^2 - 0.5) = 172.2264$ ...
3	1	18.9125...	19.5855...	$e^3 - 0.5 = 19.5855$ ....
	2	75.6500...	78.3421...	$4(e^3 - 0.5) = 78.3421$ ...
	3	170.2125..	176.2698...	$9(e^3 - 0.5) = 176.2698$ ...
	4	302.6000...	313.3685...	$16(e^3 - 0.5) = 313.3685$ ...
	5	472.8125...	489.6384...	$25(e^3 - 0.5) = 489.6384$ ...

From table2, we observe that the solution obtained from MRDTM is better than the solution obtained from RDTM and fits with the exact solution.

Example3: Consider another one dimensional initial value problem which describes the non-homogeneous heat like equation.

$$\frac{\partial u(x,t)}{\partial t} = \frac{x^2}{6}\frac{\partial^2 u(x,t)}{\partial x^2} + x^3 \quad (68)$$

With the initial condition $u(x,0) = 0$ (69)

Taking the reduced differential transform of (68) and (69), we have

$$(k+1)U_{K+1}(x) = \frac{x^2}{6}\frac{\partial^2 U_K(x)}{\partial x^2} + x^3\delta(k) \quad (70)$$

$$U_0(x) = 0 \quad (71)$$

Substitute (71) into (70), we have:

$$U_1(x) = x^3, U_2(x) = \frac{1}{2}x^3, U_3(x) = \frac{1}{6}x^3, U_4(x) = \frac{1}{24}x^3$$

$$U_5(x) = \frac{1}{120}x^3\ ,\ U_6(x) = \frac{1}{720}x^3, \dots, U_k(x) = \frac{x^3}{k!} \quad (72)$$

Then using (35) and (72), we get approximation solution of (68)

Let the order of approximation (n) =5

$$u(x,t) \cong \sum_{K=0}^{5} U_K\,(x)t^k$$

$$= U_0t^0 + U_1t^1 + U_2t^2 + U_3t^3 + U_4t^4 + U_5t^5$$

$$= 0 + x^3t + \frac{1}{2}x^3t^2 + \frac{1}{6}x^3t^3 + \frac{1}{24}x^3t^4 + \frac{1}{120}x^3t^5$$

$$= x^3(t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{1}{120}t^5) \quad (73)$$

Hence (73) is the approximate solution of (68) that obtained from RDTM truncated series. By the same reason given in case I above, we have to follow the next steps.

Applying Laplace transform to $(t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{1}{120}t^5)$, we get:

$L\,[u(x,t)] = (\frac{1}{s^2} + \frac{1}{s^3} + \frac{1}{s^4} + \frac{1}{s^5} + \frac{1}{s^6}\ )$ Using equation (8)

In step 2, replacing s by $\frac{1}{t}$, we get

$$L\,[u(x,t)] = (t^2 + t^3 + t^4 + t^5 + t^6) \quad (74)$$

All of the $[\frac{M}{N}]$pade approximation of (74) with $N \geq 1$ and $M \geq 1$ and $M + N \leq 6$ where the degree of the numerator and the degree of the denominator are equal (i.e., M=N=3) gives:

$$[\ \frac{N}{M}\ ](t^2 + t^3 + t^4 + t^5 + t^6) \ = \frac{a_0+a_1t+a_2t^2+a_3t^3}{1+b_1t+b_2t^2+b_3t^3} \tag{75}$$

Using (14) and (75), we have

$$Co \ = \ 0, \quad c_1 \ = 0, \quad c_2 \ = \ 1, c_3 \ = \ 1, \quad c_4 \ = 1, \quad c_5 = 1, \quad c_6 \ = \ 1,$$

$$a_0 \ = \ \mathrm{c}_0 \ = 0$$

$$a_1 = \ \mathrm{c}_1 + \ \mathrm{c}_0\mathrm{b}_1 = 0$$

$$a_2 = \ \mathrm{c}_2 + \ \mathrm{c}_1\mathrm{b}_1 \ + \ \mathrm{c}_0\mathrm{b}_2 = 1$$

$$a_3 = \mathrm{c}_3 + \ \mathrm{c}_2\mathrm{b}_1 + \ \mathrm{c}_1\mathrm{b}_2 + \ \mathrm{c}_0\mathrm{b}_3 = 1 + \mathrm{b}_1$$

$$0 = \ \mathrm{c}_4 + \ \mathrm{c}_3\mathrm{b}_1 + \ \mathrm{c}_2\mathrm{b}_2 + \ \mathrm{c}_1\mathrm{b}_3 + \ \mathrm{c}_0\mathrm{b}_4 = 1 + \ \mathrm{b}_1 + \ \mathrm{b}_2$$

$$0 = \ \mathrm{c}_5 + \ \mathrm{c}_4\mathrm{b}_1 + \ \mathrm{c}_3\mathrm{b}_2 + \ \mathrm{c}_2\mathrm{b}_3 + \ \mathrm{c}_1\mathrm{b}_4 + \ \mathrm{c}_0\mathrm{b}_5 = 1 + \ \mathrm{b}_1 + \ \mathrm{b}_2 + \ \mathrm{b}_3$$

$$0 \ = \ c_6 + \ c_5b_1 + \ c_4b_2 \ + \ c_3b_3 + \ c_2b_4 + \ c_1b_5 + \ c_0b_6 = 1 + \ b_1 + \ b_2 \ + \ b_3$$

Solving this we have $a_2 = 1\ , b_1 = -1, a_1 = a_0 = a_1 = a_3 = b_2 = b_3 = 0$ (76)

Substitute (76) into (75), we obtain: $[\frac{N}{M}]\ (t + t^2 + \ t^3 \ + t^4 \ + t^5 + t^6) \ = \ \frac{t^2}{1-t}$ (77)

In step 4, replacing t by $\frac{1}{s}$ in the right hand side of (77), we get

$$[\frac{N}{M}]\ (t + t^2 + \ t^3 + t^4 \ + t^5 + t^6\ ) \ = \ \frac{1}{s(s-1)} \tag{78}$$

Decompose the right hand side of (78) in to partial fraction reduction, we obtain

$$[\frac{N}{M}]\ (t + t^2 + \ t^3 \ + t^4 \ + t^5 + t^6 \ + t^7) \ = \frac{1}{s-1} - \frac{1}{s} \tag{79}$$

Finally, applying the inverse Laplace transform to the right hand side of Pade approximant (79), we obtain: $\left[\frac{N}{M}\right](t + t^2 + \ t^3 \ + t^4 + t^5 + t^6) = e^t - 1$

Therefore, the approximate solution of (67) is given by: $u(x, t) = x^3(e^t - 1)$ (80)

Substituting (80) into (68), we conclude that the approximate solution coincides with the exact one (Ahmed, 2014)

Table3. Compression between the solutions obtained from RDTM and MRDTM of one dimensional non-homogenous heat-like equation $\frac{\partial u(x,t)}{\partial t} = \frac{x^2}{6}\frac{\partial^2 u(x,t)}{\partial x^2} + \mathrm{x}^3$

t	x	RDTM	MRDTM	Exact solution $u(x,t) = x^3(e^t - 1)$ (Ahmed, 2014)
1	1	1.7166…	1.71828…	$e - 1 = 1.71828 \ldots$
	2	13.7333…	13.7462…	$8(e - 1) = 13.7462 \ldots$
	3	46.3500..	46.3936…	$27(e - 1) = 46.3936 \ldots.$
	4	109.8666..	109.9700..	$64(e - 1) = 109.9700 \ldots$
	5	214.5833…	214.7852…	$125(e - 1) = 214.7852 \ldots$
2	1	6.2666…	6.3890…	$e^2 - 1 = 6.3890 \ldots$
	2	50.1333…	51.1124…	$8(e^2 - 1) = 51.1124 \ldots$
	3	169.20000…	172.5045…	$27(e^2 - 1) = 172.5045 \ldots.$
	4	401.0666…	408.8995…	$64(e^2 - 1) = 408.8995 \ldots$
	5	770.8000..	798.6320…	$125(e^2 - 1) = 798.6320 \ldots$
3	1	17.4000…	19.0855…	$e^3 - 1 = 19.0855 \ldots$
	2	139.2000..	152.6842…	$8(e^3 - 1) = 152.6842 \ldots$
	3	469.8000…	515.3094…	$27(e^3 - 1) = 515.3094 \ldots$
	4	1113.6000..	1221.4743..	$64(e^3 - 1) = 1221.4743 \ldots$
	5	2175.0000..	2385.6921..	$125(e^3 - 1) = 2385.6921 \ldots$

From table3, we observe that the solution obtained from MRDTM is better than the Solution obtained from RDTM and fits with the exact solution.

CHAPTER FIVE

Conclusion and Future scope

5.1 Conclusion

In this study, Modified Reduced Differential Transform Method has been applied to construct approximated analytical solution of one dimensional heat-like equation with initial condition. The MRDTM is the combination of Laplace pade Resummation method and RDTM. Few examples are demonstrated to show the validity and effectiveness of the method, also the solutions obtained by using RDTM and MRDTM are compared. Results show that the MRDTM gives better approximate solution of the indicated model in a wider domain of convergence than the result obtained using RDTM, that means the technique, that we call LPRDTM greatly improves RDTM truncated series solution in convergence rate and often leads to the exact solution. Finally, we point out that LPRDTM is very powerful and easily applicable mathematical tool for PDEs.

5.2 Future scope

In this study, Modified Reduced Differential Transform Method (MRDTM) has been applied to find better approximate solutions of one dimensional heat-like equation with initial condition. Hence further research may be performed to solve two or more dimensional heat-like models.

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