NEW FIXED POINT RESULTS FOR RATIONAL TYPE CONTRACTIONS IN PARTIALLY ORDERED b-METRIC SPACES

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILLMENT FOR THE REQUIREMENTS OF THE DEGREE OF MASTERS OF SCIENCE IN MATHEMATICS

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Declaration

I, the undersigned declare that, this research paper entitled ”New Fixed Point Results for Rational Type Contractions in Partially Ordered *b*-Metric Spaces” is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged.

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Abstract

In this study, we established fixed point results for rational type contractions in partially ordered b-metric spaces by extending the work of (Reza Arab and Kolsoum Zare, 2016). In this research work we followed analytical design and secondary source of data such as articles and different books related to the study area were used. The study procedure we followed was that of the standard techniques used by (Reza Arab and Kolsoum Zare, 2016). Our results extend and generalize comparable results in the Literature. We also provided examples in support of our result.

Contents

Chapter 1

Introduction

1.1 Background of the study

Let X be a non-empty set and $T : X \to X$ be mapping we call T is a self- map of X. An element $x \in X$ is said to be a fixed point of T if $Tx = x$.
In modern mathematics, one of the most important tools is the theory of fixed point. The usefulness of this theory is confined to pure and applied mathematics. In addition to this, it is a very popular area of interaction between analysis and topology and also to examine the quantitative problems involving certain mappings and space structures required in various areas such as: economics, chemistry, biology, computer science, engineering, and others. In order to gain more information about importance of fixed point, any one can refer (Banach, 1922; Nieto and Lopez, 2005; Czerwik, 1993) etc. In 1922 the Polish mathematician Stefan Banach was the author who gave the first most useful result of metric fixed point theory, which is known as Banach contraction principle.
The known Banach contraction principle (Banach, 1922) states that if (X,d) is a complete metric space and a self-map $T : X \to X$ is a contraction, that is , there is a real number k such that $0 \leq k < 1$,

$$d(Tx,Ty) \leq kd(x,y)$$

for all x,y in X.
Then T has a unique fixed point in X. Besides this, the contraction mapping is continuous. But many authors studied and established fixed point theorems in which the map under consideration need not be continuous.
Kannan (1968) studied a fixed point theorem for a self- map $T : X \to X$ which need not be continuous but continuous at a fixed point and satisfying the contractive condition

$$d(Tx,Ty) \leq \alpha[d(x,Tx)+d(y,Ty)],$$

for all $x,y \in X$ where $0 \leq \alpha < 1/2$. In addition to Kannans work, many authors invested their time to obtain fixed point theorems for contractive conditions in which the given map need not be continuous on the given metric space one of them is, (Chttereja, 1972) gave the dual of Kannan fixed point theorem satisfying the contractive condition:

$$d(Tx,Ty) \leq \beta[d(x,Ty)+d(y,Tx)],$$

for all $x,y \in X$ where $0 \leq \beta < 1/2$.
Banach contractions, Kannan mappings and Chatterjea mappings are independent, (Rohades, 1977). Generalizing the mappings of Banach, Kannan and Chatterjea, (Zamfirescu, 1972) proved the following fixed point theorem.

Theorem 1.1.1 *Let (X,d) be a complete metric space and $T: X \to X$ be self- map for which there exist the real numbers α, β and γ satisfying $0 \leq \alpha < 1$, $0 \leq \beta < 1/2$ and $0 \leq \gamma < 1/2$ such that for each pair at least one of the following holds:*
$Z_1)$ $d(Tx,Ty) \leq \alpha d(x,y)$
$Z_2)$ $d(Tx,Ty) \leq \beta[d(x,Tx)+d(y,Ty)]$
$Z_3)$ $d(Tx,Ty) \leq \gamma[d(x,Ty)+d(y,Tx)]$
Then T is a Picard Operator.

Definition 1.1.1 *A partially ordered set is a set X and a binary relation $\preceq$ denoted by $(X,\preccurlyeq)$ such that for all a,b,c in X satisfying:*
$a)$ $a \preceq a$ (reflexivity)
$b)$ If $a \preceq b$ and $b \preccurlyeq a \Rightarrow a = b$ (anti-symmetry)
$c)$ If $a \preceq b$ and $b \preceq c$, then $a \preceq c$ (transitivity).

The idea of b-metric space was introduced by (Czerwik, 1993). From that time onwards, several papers have been published on the fixed point theory of various classes of single-valued and multi-valued operators in b-metric spaces (see, Singh and Prasad, 2008; Singh and Prasad, 2009; Boriceanu, 2009; Abbas, Chemeda and razani, 2016).

Definition 1.1.2 *(Czerwik, 1993) Let X be a non-empty set, $s \geq 1$ be a given real number. A function $d \colon X \times X \to [0,\infty)$ is called b-metric if and only if for all $x,y,z \in X$,*

a) $d(x,y) = 0 \Leftrightarrow x = y$ (positive definiteness)

b) $d(x,y) = d(y,x)$ (symmetry of distance)

c) $d(x,y) \leq s[d(x,z) + d(z,y)]$ (b- triangular inequality).

Then the triple (X,d,s) is called a *b*-metric space with a parameter *s*. Clearly, a standard metric space is also a *b*-metric space, but the converse is not always true.

Definition 1.1.3 *Let (X,d,s) be a b-metric space and a sequence $\{x_n\}$ in X we say that*

a)$\{x_n\}$ b-converges to $x \in X$ if $d(x_n,x) \to 0$ as $n \to \infty$.

b) $\{x_n\}$ is a b-Cauchy sequence if $d(x_m,x_n) \to 0$ as $m,n \to \infty$.

c) (X,d,s) is b-complete if every b-Cauchy sequence in X is $b-$convergent to $x \in (X,d,s)$.

Each *b*-convergent sequence in a *b*-metric space has a unique limit and it is also a *b*-Cauchy sequence. Moreover, a *b*-metric is not necessarily continuous.

Lemma 1.1.2 *(Aghajani and Roshan, 2014) Let (X,d,s) be a b-metric space with $s \geq 1$ and suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x and y respectively. Then we write*

$$\frac{1}{s^2} d(x,y) \leq \liminf_{n\to\infty} d(x_n,y_n) \leq \limsup_{n\to\infty} d(x_n,y_n) \leq s^2 d(x_n,y_n).$$

In particular, if $x = y$ then we have $\lim_{n\to\infty} d(x_n,y_n) = 0$. Moreover, for each $z \in X$,

$$\frac{1}{s} d(x,z) \leq \liminf_{n\to\infty} d(x_n,z) \leq \limsup_{n\to\infty} d(x_n,z) \leq s d(x_n,z).$$

Cabrera *et al.* (2013) extended the result of (Jaggi, 1975) and established a fixed point result in partially ordered metric spaces. Recently, (Chandok *et al.*, 2014) proved the following theorem.

Theorem 1.1.3 *Let $(X,\preccurlyeq)$ be partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Suppose that T is a continuous self- mapping on X and T is monotone non- decreasing mapping*

$$\begin{aligned} d(Tx,Ty) &\leq \alpha\frac{d(y,Ty)d(x,Tx)}{d(x,y)}+\beta d(x,y)+\gamma[d(x,Tx)+d(y,Ty)] \\ &\quad +\delta[d(y,Tx)+d(x,Ty)], \end{aligned}$$

and for all $x,y\in X$ with $x\succ y, x\neq y$ and for some $\alpha,\beta,\gamma,\delta\in[0,1)$ with $\alpha+\beta+\gamma+\delta<1$. If there exists $x_0\in X$ such that $x_0\preccurlyeq Tx_0$, then T has a fixed point.

Theorem 1.1.4 *Let $(X,\preccurlyeq,d)$ is a partially ordered complete b-metric space. Let $T:X\to X$ be a continuous and nondecreasing self mapping. Suppose there exist mappings $a_i\colon X\times X\to[0,1)$ such that for all $x,y\in X$ and $i=1,2,3,\ldots,7$,*

$$a_i(Tx,Ty)\leq a_i(x,y).$$

Also, for all $x,y\in X$ with $x\preccurlyeq y$,

$$\begin{aligned} d(Tx,Ty) &\leq a_1(x,y)d(x,y)+a_2(x,y)[d(x,Tx)+d(y,Ty)]+a_3(x,y)\frac{d(y,Tx)+d(x,Ty)}{s} \\ &\quad +a_4(x,y)d(y,Ty)\varphi(d(x,y),d(x,Tx))+a_5(x,y)d(y,Tx)\varphi(d(x,y),d(x,Ty)) \\ &\quad +a_6(x,y)d(x,y)\varphi(d(x,y),d(x,Tx))+d(y,Tx))+a_7(x,y)d(y,Tx), \end{aligned}$$

where $\varphi\colon\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ is a function such that $\varphi(t,t)=1$ for all $t\in\mathbb{R}^+=[0,\infty)$ and

$$\sup_{x,y\in X}\{\,a_1(x,y)+a_2(x,y)+a_3(x,y)+a_4(x,y)+a_6(x,y)\}\leq\frac{1}{s+1}.$$

If there exists $x_0\in X$ such that $x_0\preccurlyeq Tx_0$, then T has a fixed point.
Inspired and motivated by the works of (Reza Arab and Kolsoum Zare, 2016) we have established some fixed point results satisfying contractive condition of rational type in the setting of partially ordered complete b-metric spaces and illustrative examples which support the result of the study were also provided.

1.2 Statements of the problem

This study concentrated on establishing the existence of new fixed point result for mappings satisfying rational type contractions in partially ordered b-metric spaces.

1.3 Objectives of the study

1.3.1 General objective

The main objective of this study was to obtain new fixed point results for mappings satisfying rational type contractions in partially ordered b-metric spaces.

1.3.2 Specific objectives

This study has the following specific objectives

- To prove the existence of new fixed point results for rational type contractions in partially ordered b-metric spaces.
- To discuss additional conditions required to obtain a unique fixed point in partially ordered b-metric spaces.
- To provide examples in the support of the result of the study.

1.4 Significance of the study

The study may have the following importance

- The results obtained in this study may contribute to research activities in the area of fixed point.
- The researcher developed scientific research writing skills and scientific communication in Mathematics.
- The result may be applicable in solving differential equations, integral equations, partial differential and functional equations.

1.5 Delimitation of the Study

This study was delimited to prove new fixed point result for rational type contractions in partially ordered b-metric spaces.

Chapter 2

Review of Related Literatures

The Banach Contraction Principle is a very popular tool which is used to solve existence problems in many branches of Mathematical Analysis and its applications. It is no surprise that there is a great number of generalizations of this fundamental theorem. They go in several directions modifying the basic contractive condition or changing the ambient space.The Banach's contraction mapping principle is one of the cornerstones in the development of fixed point theory. In particular, this principle is used to demonstrate the existence and uniqueness of a solution of ordinary differential equations, integral equations, functional equations, partial differential equations and others. Stefan Banach (1922) stated his celebrated theorem on the existence and uniqueness of fixed point theorem for contraction of self maps defined on complete metric spaces for the first time, which is known as the Banach contraction mapping principle. Fixed point theory is one of the famous and traditional theories in mathematics and has a broad set of applications. In this theory, contraction is one of the main tools to prove the existence and uniqueness of a fixed point theorem. Banach's contraction principle, which gives an answer on the existence and uniqueness of a solution of an operator equation, $Tx = x$ is the most widely used fixed point theorem in all of analysis. This principle is constructive in nature and is one of the most useful tools in the study of nonlinear analysis.

If $(X, \preceq)$ is a partially ordered set and $T : X \to X$, we say that T is monotone non-decreasing if $x, y \in X, x \preceq y \Rightarrow Tx \preceq Ty$. This definition coincides with the notion of a non-decreasing function in the case where $X = \mathbb{R}$ and $\preceq$ represents the usual total order in $\mathbb{R}$. Geraghty (1973) proved a fixed point result, generalizing the Banach contraction principle. The extension of Banach contraction principle through rational expressions was also made by (Das and Gupta, 1975 and Harjani, 2010) proved a fixed point theorems for mappings satisfying contractive condition of rational type on a partially ordered metric space. Several authors proved later various results using Geraghty-type conditions. Fixed point results of this kind in b-metric spaces were obtained by (Dukic et al., 2011). Dukic *et al.* (2011) for a real number $s > 1$ (the case $s = 1$ is easily metric), let $F(s)$ denote the class of all function $\beta : [0, \infty) \to [0, \frac{1}{s})$ satisfying the following condition:

$$\lim_{n\to\infty} sup\beta t_n = \frac{1}{s} \;\; implies \;\; t_n \to \infty \;\; as \;\; n \to \infty.$$

Due to its importance, generalizations of Banachs contraction mapping principle have been investigated heavily by many authors (see Sintunavarat and Kumam, 2013; Arab and Zare, 2016; Amini-Harandi and Emami, 2014).

Theorem 2.0.1 *(Jaggi, 1977) Let (X,d) be a complete metric space and $T : X \to X$ be a mapping such that there exist $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$ satisfying the condition:*

$$d(Tx,Ty) \leq \alpha \tfrac{d(x,y)(1+d(x,Tx))}{1+d(x,y)} + \beta d(x,y)$$

for all $x, y \in X$.

Then T has a unique fixed point.
Das and Dey (2007) has developed fixed point theory in partially ordered metric spaces endowed with a partial ordering and the triplet $(X,d,\preccurlyeq)$is called partially ordered metric space if $(X,\preccurlyeq)$ is a partially ordered set and (X,d) is a metric space. The study of fixed points of mappings satisfying certain contractive conditions can be employed to establish existence of solutions of certain operator equations such as differential, functional and integral equations (Nieto and Lopez, 2005 and Amini-Harndi, 2014).
Arab and Zare (2016) established some fixed point results satisfying a generalized contraction mapping of rational type in *b*metric spaces endowed with partial order. Also, they established a result for existence and uniqueness of fixed point for such class of mappings.

Chapter 3

Methodology

This chapter contains study design, description of the research methodology, data collection procedures and data analysis process.

3.1 Study period and site

This study was conducted from September, 2017 G.C. to September, 2018 G.C. in Jimma University under Mathematics Department.

3.2 Study Design

In this research work we employed analytical design method.

3.3 Source of Information

The available sources of information for the study were books and published articles.

3.4 Mathematical Procedure of the Study

The mathematical procedure that the researcher followed for analysis was the standard technique used by (Reza Arab and Kolsoum Zare, 2016).
These procedures are:

- Establishing theorems.
- Constructing sequences.
- Showing whether the sequences are Cauchy or not.

- Showing the convergence of the sequences.
- Proving the existence of fixed points.
- Showing uniqueness of the fixed point.
- Giving examples in support of the main results.

Chapter 4

Preliminaries and Main Results

4.1 Preliminaries

We recall definitions of fixed point, contraction, monotone non-decreasing and continuity of mappings.

Definition 4.1.1 *Let X be a nonempty set and $T: X \to X$ be a self-map. A point $x \in X$ is said to be fixed point of T if $Tx = x$.*

Example 4.1.1 Let $X = \mathbb{R}$ and $T: X \to X$ be given by $Tx = x$. Then T has infinitely many fixed points.
Example 4.1.2 Let $X = \mathbb{R}$ and $T: X \to X$ be given by $Tx = x^3$, then T has three fixed points,namely $0, 1$ and -1.
Example 4.1.3 Let $X = \mathbb{R}$ and $T: X \to X$ be given by $Tx = x+1$, then T has no fixed points.

Definition 4.1.2 *Let (X,d) be a metric space. A mapping $T: X \to X$ is called contraction, if there exists $\lambda \in [0,1]$ such that*

$$d(Tx,Ty) \leq \lambda d(x,y)$$

for all $x, y \in X$.

Example 4.1.4 Let $X = [0,1]$ and $d : X \times X \to \mathbf{R}$ defined by $d(x,y) = \mid x-y \mid$.
Then $T : X \to X$ given by $Tx = \frac{1}{2}x$ is contraction since

$$d(Tx,Ty) = \mid \frac{1}{2}x - \frac{1}{2}y \mid = \frac{1}{2} \mid x-y \mid = \frac{1}{2}d(x,y).$$

which satisfies the property when $\lambda \geq \frac{1}{2}$.

Definition 4.1.3 *Let (X,d,s) be a b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$ be a given mapping. We say that T is continuous at $x_0 \in X$ if for every sequence x_n in X with $x_n \to x$ as $n \to \infty$, then $Tx_n \to Tx_0$ as $n \to \infty$. If T is continuous at each point $x_0 \in X$, then we say that T is continuous on X.*

Definition 4.1.4 *A mapping $T : X \to X$ in a partially ordered b-metric space $(X,d,\preceq)$ is said to be monotone non-decreasing mapping if for all $x,y \in X$,*

$$x \preceq y \Rightarrow T(x) \Rightarrow T(y).$$

Definition 4.1.5 *Let X be a non empty set.Then $(X,d,\preceq)$ is called partially ordered b-metric space if,*
i) (X,d) is a b-metric space
ii) $(X,\preceq)$ is a partially ordered set.

Definition 4.1.6 *Let $(X,\preceq)$ be a partially ordered set and $x,y \in X$ then x and y are said to be comparable elements of X if $x \preceq y$ or $y \preceq x$.*

4.2 Main Results

Theorem 4.2.1 Let $(X,\preccurlyeq,d)$ is a partially ordered complete b-metric space. Let $T : X \to X$ be a continuous and nondecreasing self mapping. Suppose there exist mappings $a_i\colon X\times X \to [0,1)$ such that for all $x,y\in X$ and $i=1,2,3,\ldots,8$,

$$a_i(Tx,Ty)\le a_i(x,y).$$

Also, for all $x,y\in X$ with $x\preccurlyeq y$,

$$\begin{aligned} d(Tx,Ty) \;\le\; & a_1(x,y)d(x,y)+a_2(x,y)[d(x,Tx)+d(y,Ty)]+a_3(x,y)\frac{d(y,Tx)+d(x,Ty)}{s}\\ & +a_4(x,y)d(y,Ty)\varphi(d(x,y),d(x,Tx))+a_5(x,y)d(y,Tx)\varphi(d(x,y),d(x,Ty))\\ & +a_6(x,y)d(x,y)\varphi(d(x,y),d(x,Tx))+d(y,Tx))+a_7(x,y)d(y,Tx),\\ & +a_8(x,y)\frac{d(x,Ty)(1+d(x,Tx))}{s(1+d(x,y))}, \end{aligned} \tag{4.1}$$

where $\varphi\colon \mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ is a function such that $\varphi(t,t)=1$ for all $t\in\mathbb{R}^+$ and $\sup\limits_{x,y\in X}\{\,a_1(x,y)+a_2(x,y)+a_3(x,y)+a_4(x,y)+a_6(x,y)+a_8(x,y)\}\le \dfrac{1}{s+1}$. If there exists $x_0\in X$ such that $x_0\preccurlyeq Tx_0$, then T has a fixed point.

Proof: If $x_0=Tx_0$, then we have the result. Suppose that $x_0\preccurlyeq Tx_0$.
Then we construct a sequence $\{x_n\}$ in X such that

$$x_{n+1}=Tx_n \;\; for\; every\; n=0,1,2,\cdots. \tag{4.2}$$

Since T is a nondecreasing mapping, we obtain by induction that

$$x_0\preccurlyeq Tx_0=x_1\preccurlyeq Tx_2\preccurlyeq\ldots\preccurlyeq Tx_{n-1}=x_n\preccurlyeq Tx_n=x_{n+1}\cdots. \tag{4.3}$$

If there exists some $k\in\mathbb{N}$ such that

$$x_k=x_{k+1},$$

then from (4.2)

$$x_{k+1}=Tx_k=x_k,$$

that is x_k is a fixed point of T and the proof is finished. So, we suppose that $x_n \neq x_{n+1}$, for all n in $\mathbb{N}$. Since, $x_n < x_{n+1}$ we set $x = x_n$ and $y = x_{n+1}$ in (4.1), we have

$$\begin{aligned}
d(Tx_n, Tx_{n+1}) \leq\ & a_1(x_n, x_{n+1})d(x_n, x_{n+1}) + a_2(x_n, x_{n+1})[d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})] \\
& + a_3(x_n, x_{n+1})\frac{d(x_{n+1}, Tx_n) + d(x_n, Tx_{n+1})}{s} \\
& + a_4(x_n, x_{n+1})d(x_{n+1}, Tx_{n+1})\varphi(d(x_n, x_{n+1}), d(x_n, Tx_n)) \\
& + a_5(x_n, x_{n+1})d(x_{n+1}, Tx_n)\varphi(d(x_n, x_{n+1}), d(x_n, Tx_{n+1})) \\
& + a_6(x_n, x_{n+1})d(x_n, x_{n+1})\varphi(d(x_n, x_{n+1}), d(x_n, Tx_n)) + d(x_{n+1}, Tx_n)) \\
& + a_7(x_n, x_{n+1})d(x_{n+1}, Tx_n) \\
& + a_8(x_n, x_{n+1})\frac{d(x_n, x_{n+2})(1 + d(x_n, x_{n+1}))}{s(1 + d(x_n, x_{n+1}))},
\end{aligned}$$

that is,

$$\begin{aligned}
d(x_{n+1}, x_{n+2}) \leq\ & a_1(x_n, x_{n+1})d(x_n, x_{n+1}) + a_2(x_n, x_{n+1})[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \\
& + a_3(x_n, x_{n+1})[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + a_4(x_n, x_{n+1})d(x_{n+1}, x_{n+2}) \\
& + a_6(x_n, x_{n+1})d(x_n, x_{n+1}) + a_8(x_n, x_{n+1})[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \\
=\ & [a_1(x_n, x_{n+1}) + a_2(x_n, x_{n+1}) + a_3(x_n, x_{n+1}) + a_6(x_n, x_{n+1}) + a_8(x_n, x_{n+1})]d(x_n, x_{n+1}) \\
& + [a_2(x_n, x_{n+1}) + a_3(x_n, x_{n+1}) + a_4(x_n, x_{n+1}) + a_8(x_n, x_{n+1})]d(x_{n+1}, x_{n+2}) \\
=\ & [a_1 + a_2 + a_3 + a_6 + a_8](Tx_{n-1}, Tx_n)d(x_n, x_{n+1}) \\
& + [a_2 + a_3 + a_4 + a_8](Tx_{n-1}, Tx_n)d(x_{n+1}, x_{n+2}) \\
\leq\ & [a_1 + a_2 + a_3 + a_6 + a_8](x_{n-1}, x_n)d(x_n, x_{n+1}) \\
& + [a_2 + a_3 + a_4 + a_8](x_{n-1}, x_n)d(x_{n+1}, x_{n+2}) \\
\vdots\ & \\
\leq\ & [a_1 + a_2 + a_3 + a_6 + a_8](x_0, x_1)d(x_n, x_{n+1}) \\
& + [a_2 + a_3 + a_4 + a_8](x_0, x_1)d(x_{n+1}, x_{n+2}),
\end{aligned}$$

which implies that

$$d(x_{n+1}, x_{n+2}) \leq \frac{(a_1 + a_2 + a_3 + a_6 + a_8)(x_0, x_1)}{1 - (a_2 + a_3 + a_4 + a_8)(x_0, x_1)} d(x_n, x_{n+1}).$$

Now,

$$\begin{array}{rcl} (a_1+2a_2+2a_3+a_4+a_6+2a_8)(x_0,x_1) & \leq & \{(s+1)[a_1+a_2+a_3+a_4+a_6+a_8](x_0,x_1)\} \\ & \leq & \sup\limits_{x,y\in X}\{(s+1)[a_1+a_2+a_3+a_4+a_6+a_8](x,y)\} \\ & < & 1. \end{array}$$

Thus we get

$$d(x_{n+1},x_{n+2}) \leq \lambda d(x_n,x_{n+1}). \tag{4.4}$$

Where,

$$\lambda = \frac{(a_1+a_2+a_3+a_6+a_8)(x_0,x_1)}{1-(a_2+a_3+a_4+a_8)(x_0,x_1)} < 1.$$

Obviously, $0 \leq \lambda < \frac{1}{s}$. Then by repeated application of (4.4), we have

$$d(x_{n+1},x_{n+2}) \leq \lambda d(x_n,x_{n+1}) \leq \lambda^2 d(x_{n-1},x_n) \leq \cdots \leq \lambda^{n+1} d(x_0,x_1). \tag{4.5}$$

Thus, setting any positive integers m and n where $(m > n)$, we have

$$\begin{array}{rcl} d(x_n,x_m) & \leq & sd(x_n,x_{n+1})+s^2d(x_{n+1},x_{n+2})+\cdots+s^{m-n}d(x_{m-1},x_m) \\ & \leq & [s\lambda^n+s^2\lambda^{n+1}+\cdots+s^{m-n}\lambda^{m-1}]d(x_0,x_1) \\ & = & s\lambda^n[1+s\lambda+(s\lambda)^2+\cdots+(s\lambda)^{m-n-1}]d(x_0,x_1) \\ & \leq & s\lambda^n[1+s\lambda+(s\lambda)^2+\cdots]d(x_0,x_1) \\ & = & \dfrac{s\lambda^n}{1-s\lambda}d(x_0,x_1). \end{array}$$

Since $0 \leq \lambda < \frac{1}{s}$, we notice that $\frac{s\lambda^n}{1-s\lambda} \to 0$ as $n \to \infty$ for any $m \in \mathbb{N}$. So $\{x_n\}$ is Cauchy in a complete b-metric space X, there exists $x \in X$ such that

$$\lim_{n\to\infty} x_{n+1} = x$$

Letting $n \to \infty$ in (4.2) and from the continuity of T, we get

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T(x_n) = T(\lim_{n\to\infty} x_n) = T(x).$$

This implies that x is a fixed point of T.
We shall prove the uniqueness of the fixed point. In addition to the hypotheses of Theorem 4.2.1, Suppose that for every $x, y \in X$, there exists $u \in X$ such that $u \preceq x$ and $u \preceq y$. Then T has a unique fixed point.
Proof: It follows from the Theorem 4.2.1 that the set of fixed points of T is non empty. We shall show that if x^* and y^* are two fixed points of T that is, if $x^* = Tx^*$ and $y^* = Ty^*$, then $x^* = y^*$.
By the assumption, there exists $u_0 \in X$ such that $u_0 \preceq x^*$ and $u_0 \preceq y^*$ we define the sequence $\{u_{n+1}\}$ Such that $u_{n+1} = Tu_n = T^{n+1}u_0, n = 0, 1, 2, ...$
Monotonicity of T implies that $T^n u_0 = u_n \preceq u_{n+1} \preceq x^* = T^n x^*$ and $T^n u_0 = u_n \preceq u_{n+1} \preceq y^* = T^n y^*$.
If there exists a positive integer m such that $u_m = x^*$ then, $x^* = Tx^* = Tu_n = u_{n+1}$, for all $n \geq m$. Then $u_{n+1} \to 0$ as $n \to \infty$. Now we suppose that $x^* \neq u_{n+1}$ for all $n \geq 0$.
Let $p_n = d(u_{n+1}, x^*)$, for all $n \geq 0$. Since $u_{n+1} \preceq x^*$ for all $n \geq 0$, By applying

Lemma 1.1 and Theorem 4.2.1 we have,

$$
\begin{aligned}
\frac{d(u_{n+1},x_*)}{s} &\leq \lim_{n\to\infty}\sup d(u_{n+2},Tx^*)\\
&\leq \lim_{n\to\infty}\sup a_1(u_{n+1}`,x^*)d(u_{n+1},x^*)\\
&\quad + a_2(u_{_{n+1}},x^*)[d(u_{n+1},u_{n+2})+d(x^*,Tx^*)]\\
&\quad + a_3(u_{n+1},x^*)\frac{d(x^*,u_{n+2})+d(u_{n+1},x^*)}{s}\\
&\quad + a_4(u_{n+1},x^*)d(x^*,Tx^*)\varphi(d(u_{n+1},x^*),d(u_{n+1},u_{n+2}))\\
&\quad + a_5(u_{n+1},x^*)d(x^*,u_{n+2})\varphi(d(u_{n+1},x^*),d(u_{n+1},Tx^*))\\
&\quad + a_6(u_{n+1},x^*)d(u_{n+1},x^*)\varphi(d(u_{n+1},x^*),d(u_{n+1},u_{n+2})\\
&\quad + d(x^*,u_{n+2})),\\
&\quad + a_7(u_{n+1},x^*)d(x^{*,un+2})\\
&\quad + a_8(u_{n+1},x^*)\frac{d(u_{n+1},Tx^*)(1+d(u_{n+1},u_{n+2}))}{s(1+d(u_{n+1}),x^*))}\\
&\leq [a_1(x^*,x^*)+a_4(x^*,x^*)+a_5(x^*,x^*)+a_6(x^*,x^*)]d(u_{n+1},x_*)\\
&\leq \frac{1}{s+1}d(u_{n+1},x_*)\\
&< \frac{1}{s}d(u_{n+1},x^*).
\end{aligned}
$$

Which contradicts the assumption. So, $p_{n+1} \leq p_n$ that is, $\{p_n\}$ is a decreasing sequence of positive real numbers and is bounded below. There exists $r \geq 0$ such that $p_n = d(u_{n+1},x^*) \to r$ as $n \to \infty$. Similarly, we can show that
$r = 0$, then, $p_n = d(u_{n+1},x^*) \to 0$, as $n \to \infty$, that is, $u_{n+1} \to x^*$ as $n \to \infty$.
Using similar argument, we can prove that $u_{n+1} \to y^*$ as $n \to \infty$. Finally, the uniqueness of the limit implies $x^* = y^*$. Hence T has a unique fixed point.
Now we give an example in support of Theorem 4.2.1.
Example 4.2.1: Let $X = [0,2]$ with $d : X \times X \to [0,\infty)$ given by $d(x,y) = (x-y)^2$ is a $b-$metric space with $s = 2$. We define the partial order "$\preceq$" on X by

$$\preceq = \{(x,y) : x,y \in [0,1), x = y\} \cup \{(x,y) : x,y \in [1,2], x \leq y\},\ \ then\ \ (X,\preceq)$$

is a partially ordered set.

$$T(x) = \begin{cases} \frac{x}{3} & \text{if } 0 \le x < 1, \\ \frac{1}{3} & \text{if } 1 \le x \le 2. \end{cases}$$

and

$$a_i(x,y) = \frac{i(x+y)}{20}.$$

We define $\varphi : [0,\infty) \times [0,\infty) \to [0,\infty)$ and also $\varphi(t,t) = 1$ and $\varphi(t,s) = \frac{1+s}{1+t}$ are given conditions.

Now we verify the condition of Theorem 4.2.1.

Case 1: Let $x,y \in [0,1)$, then $x = y$ so, $Tx = \frac{x}{3} = Ty$.

Then we have

$d(Tx,Ty) = d(\frac{x}{3} - \frac{x}{3}) = (\frac{x}{3} - \frac{x}{3})^2 = 0$, since $x = y \in [0,1)$.

$a_1(x,y)d(x,y) = \frac{(x+x)}{20}(x-y)^2 = 0.$

$$\begin{aligned} a_2(x,y)[d(x,Tx)+d(y,Ty)] &= a_2(x,y)[d(x,\frac{x}{3})+d(y,\frac{y}{3})] \\ &= \frac{8x^3}{45}. \end{aligned}$$

$$\begin{aligned} a_3(x,y)\frac{d(y,Tx)+d(x,Ty)}{s} &= a_3(x,y)d(y,\frac{x}{3}+d(x,\frac{y}{3}) \\ &= \frac{8x^3}{60}. \end{aligned}$$

$$a_4(x,y)d(y,\frac{y}{3})\varphi(d(x,y),d(x,\frac{x}{3}) = \frac{144x^3+64x^5}{90}.$$

$$\begin{aligned} a_5(x,y)d(y,Tx)\varphi(d(x,y),d(x,Ty)) &= a_5(x,y)d(x,\frac{x}{3})\varphi(d(x,x),d(x,\frac{x}{3})) \\ &= \frac{18x^3+8x^5}{81.} \end{aligned}$$

$$a_6(x,y)d(x,y)\varphi(d(x,y),d(x,Tx)+d(y,Tx))=0.$$

$$\begin{aligned} a_7(x,y)d(y,Tx) &= a_7(x,x)d(x,\frac{x}{3}) \\ &= \frac{7(2x)}{20}(x-\frac{x}{3})^2 \\ &= \frac{28x^3}{90}. \end{aligned}$$

$$a_8(x,y)\frac{d(x,Ty)(1+d(x,Tx))}{s(1+d(x,y))}=\frac{144x^3+64x^5}{405}.$$

This implies,

$$0\le 4.04x^3 1.28x^5.$$

Since the left hand side of the expression is 0 for any $x\in[0,1)$ it holds true.

Case 2: Let $x,y\in[1,2]$ such that $x\le y$, then we have the following.

$$Tx=\frac{1}{3},Ty=\frac{1}{3}$$

$$d(Tx,Ty)=d(\frac{1}{3}-\frac{1}{3})=(\frac{1}{3}-\frac{1}{3})^2=0.$$

$$a_1(x,y)d(x,y)=a_1(x,y)(x-y)^2=\frac{x+y}{20}(x-y^2)$$

$$a_2(x,y)[d(x,Tx)+d(y,Ty)]=\frac{(x+y)((3x-1)^2+(3y-1)^2)}{90}$$

$$a_3(x,y)\frac{d(y,Tx)+d(x,Ty)}{s}=\frac{(x+y)[(3y-1)^2+(3x-1)^2]}{120}$$

$$\begin{aligned} a_4(x,y)d(y,Ty)\varphi(d(x,y),d(x,Tx)) &= a_4(x,y)d(y,Ty)d(y,\frac{1}{3})\varphi(d(x,y),d(x,\frac{1}{3})) \\ &= \frac{(x+y)}{5}a_4(x,y)d(y,Ty)\varphi(d(x,y),d(x,Tx)) \\ &= \frac{(x+y)}{5}\frac{[(3y-1)^2(9+(3x-1)^2)]}{81(1+(x-y)^2)}. \end{aligned}$$

$$a_5(x,y)d(y,Tx)\varphi(d(x,y),d(x,Ty)) = \frac{(x+y)(3x-1)^2}{36}[\frac{9+(3x-1)^2}{9(1+(x-y)^2)}].$$

$$a_6(x,y)d(x,y)\varphi(d(x,y),d(x,Tx)+d(y,Tx)) = \frac{3(x+y)}{10}(x-y)^2[\frac{9+(3x-1)^2+(3y-1)^2}{9(1+(x-y)^2)}].$$

$$a_7(x,y)d(y,Tx) = \frac{7(x+y)(3y-1)^2}{180}.$$

$$a_8(x,y)\frac{d(x,Ty)(1+d(x,Tx))}{s(1+d(x,y))} = \frac{4(x+y)(3x-1)^2(9+(3x-1)^2)}{1620(1+(x-y)^2)}.$$

Since again from Case(2) the left hand side is 0 for $x,y \in [1,2]$ with $x \leq y$ it holds true.

From Case (1) and Case (2) considered above T satisfies the Inequality 4.2.1 and hence T satisfies all the hypothesis of Theorem 4.2.1 and T has a fixed point $x_0 = 0$.

Example 4.2.2: Let $X = [0,1]$ with the usual order $\leq$. Define $d(x,y) = \mid x-y \mid^2$. Then d is a b-metric space with $s = 2$. Also define $a_1(x,y) = \frac{x+y+1}{64}$ and $Tx = \frac{x^2}{32}$. We observe that

$$\begin{aligned}
a_1(Tx,Ty) &= \frac{1}{64}(\frac{x^2}{32}+\frac{y^2}{32}+1) \\
&= \frac{1}{64}(\frac{x.x}{32}+\frac{y.y}{32}+1) \\
&\leq \frac{x+y+1}{64} = a_1(x,y),
\end{aligned}$$

where $a_i(x,y) = 0$ for $i = 2,3,...,8$.
And for all comparable $x,y \in X$, we get

$$\begin{aligned}
d(Tx,Ty) &= \mid Tx - Ty \mid^2 \\
&= \mid \frac{x^2}{32} - \frac{y^2}{32} \mid^2 \\
&= \frac{1}{(32)^2} \mid x+y \mid\mid x+y \mid\mid x-y \mid^2 \\
&\leq \frac{1}{64 \times 16} \mid x+y \mid\mid x-y \mid^2 \\
&\leq \frac{1}{16}\frac{x+y+1}{64} \mid x-y \mid^2 = \frac{1}{16} a_1(x,y)d(x,y) \\
&\leq a_1(x,y)d(x,y).
\end{aligned}$$

Moreover, T is a non-decreasing continuous mapping with respect to the usual order $\leq$. Hence all conditions of Theorem 4.2.1 are satisfied. Therefore T has a fixed point $x = 0$.

Corollary 4.2.2 Suppose that $(X,d,\preceq)$ is a partially ordered complete b-metric space. Let T be a continuous and non-decreasing mapping such that the following conditions hold:
For all $x,y \in X$ with $x \preceq y$, where a_i are non negative coefficients for $i = 1,2,3,...,7$ with $a_1 + a_2 + a_3 + a_4 + a_6 \leq \frac{1}{s+1}$, and $\varphi : \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+$ is a function such that $\varphi(t,t) = 1$ for all $t \in \mathbf{R}^+$. If there exists $x \in X$ such that $x_0 \preceq Tx_0$,then T has a fixed point.

Proof: The result follows by taking $a_8(x,y) = 0$ in Theorem 4.2.1.

Theorem 4.2.2 Suppose that $(X,d,\preceq)$ is a partially ordered complete b-metric space. Assume that if x_n is a non- decreasing sequence in X such that $x_n \to x$, then $x_n \leq x$ for all $n \in N$. Let $T : X \to X$ be a non-decreasing mapping.Suppose there exist continuous mappings
$a_iX \times X \to [0,1)$ such that for all $x,y \in X$ and $i = 1,2,3,...,8$

$$a_i : (Tx,Ty) \leq a_i(x,y).$$

Also, for all $x, y \in X$ with $x \preceq y$,

$$\begin{aligned}
d(Tx,Ty) \;\leq\;& a_1(x,y)d(x,y) + a_2(x,y)[d(x,Tx)+d(y,Ty)] + a_3(x,y)\frac{d(y,Tx)+d(x,Ty)}{s} \\
&+ a_4(x,y)d(y,Ty)\varphi(d(x,y),d(x,Tx)) + a_5(x,y)d(y,Tx)\varphi(d(x,y),d(x,Ty)) \\
&+ a_6(x,y)d(x,y)\varphi(d(x,y),d(x,Tx))+d(y,Tx)) + a_7(x,y)d(y,Tx) \\
&+ a_8(x,y)\frac{d(x,Ty)(1+d(x,Tx))}{s(1+d(x,y))},
\end{aligned}$$

where $\varphi : \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+$ is a continuous function such that $\varphi(t,t) = 1$ for all $t \in \mathbf{R}^+$ and
$\sup\limits_{x,y\in X}\{\, a_1(x,y)+a_2(x,y)+a_3(x,y)+a_4(x,y)+a_6(x,y)+a_8(x,y)\} \leq \dfrac{1}{s+1}$.
If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point.

Proof: We take the same sequence x_n as in the Theorem 4.2.1.Then we have $x_0 \preceq x_1 \preceq x_2 \preceq \ldots \preceq x_n \preceq x_{n+1 \preceq \ldots}$ that is, x_n is a non-decreasing sequence. Also, this sequence converges to x. Suppose that $Tx \neq x$, that is, $d(x,Tx) > 0$. Since $x_n \preceq x$ for all n, applying condition of Theorem 4.2.1 and Lemma 1.1, we have

$$\begin{aligned}
\frac{d(x,Tx)}{s} \;&\leq\; \lim_{n\to\infty}\sup d(x_{n+1},Tx) \\
&=\; \lim_{n\to\infty}\sup d(Tx_n,Tx) \\
&\leq\; \lim_{n\to\infty}\sup\{a_1(x_n,x)d(x_n,x)+a_2(x_n,x)[d(x_n,x_{n+1})+d(x,Tx)] \\
&\quad + a_3(x_n,x)\frac{d(x_n,x_{n+1})+d(x_n,Tx)}{s} \\
&\quad + a_4(x_n,x)d(x,Tx)\varphi(d(x_n,x),d(x_n,x_{n+1})) \\
&\quad + a_5(x_n,x)d(x,x_{n+1})\varphi(d(x_n,x_{n+1}),d(x_n,Tx)) \\
&\quad + a_6(x_n,x)d(x_n,x)\varphi(d(x_n,x),d(x_n,x_{n+1})+d(x,x_{n+1})) \\
&\quad + a_7(x_n,x)d(x,x_{n+1}) + a_8(x_n,x)\frac{d(x_n,x_n)(1+d(x_n,x_{n+1}))}{s(1+d(x_n,x))}\} \\
&\leq\; [a_2(x,x)+a_3(x,x)+a_4(x,x)]d(x,Tx) \\
&\leq\; \frac{1}{s+1}d(x,Tx) \\
&<\; \frac{1}{s}d(x,Tx).
\end{aligned}$$

This is a contradiction. Hence, $Tx = x$, that is, x is a fixed point of T.

Chapter 5

Conclusion and Future Scope

5.1 Conclusion

In 2016 Arab and Zare established the existence of fixed point theorems for a self-maps satisfying rational type contractions in partially ordered complete b-metric space. In this research work, we have established existence and uniqueness of fixed points for a self-maps satisfying rational type contractions in partially ordered complete b-metric spaces. Also, we have given supporting examples to the main results of our work. Our study extend and generalize the main result of (Arab and Zare, 2016).

5.2 Future Scope

There are some published results related to the existence of fixed points of self-maps defined on b- metric spaces induced with partial order. The researcher believes, the search for the existence of fixed points of self-maps satisfying different conditions in partially ordered-metric spaces for rational type contractions is an active area of study. So, any interested researchers can use this opportunity to Conduct their research work in this area.

References

Abbas, M., Chemeda, Z. I., Razani, A. (2016).Existence of common fixed point for b-metric rational type contractions. *Published by Faculty of Sciences and mathematics,Univ.of Nis,Serbzia, 6,1413-1429.*

Amini-Harandi, A. and Emami, H. (2014).A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations. *Jour. of Nonlinear Anal. Theory Methods and Appl., 72, 22382242.*

Arab,R., Zare,K. (2016). New fixed point Results for rational type Contractions in Partially ordered b-metric space. *International Journal of analysis and Applications, 10(2), 64-70.*

Arshad,M. Karabiner, E. and Ahimad, J. (2013). Some unique fixed theorems for rational contractions in partially ordered metric spaces. *Jour. of Ineq. Appl.,* 248.

Aydi, H., Bota, M.F.,Karapinar, E. and Moradi, S. (2012). A common fixed point for weak on b-metric spaces. *Fixed point theory,* 1, (3), 337-346.

Banach, S. (1922). Surles operations dans les ensembles abstraitetleur application aux. equetion. *Journal of Integrals, fundam. Math.*, 3(2022), 133-186.

Bhaskar, T. G., Lakshmikantham (2006). Fixed point theory in partially ordered metric spaces and applications. *Nonlinear Anal.,* 65, 1379-1393.

Cabrera, I., Harjani, J., Sadarangani, K. (2013). A fixed point theorem for contractions of rational type in partially ordered metric spaces. *Ann. Univ. Ferrara,* 59, 251-258.

Chandok, S., T. D. Narang. M. Taoud, S. (2014). Fixed point theorem for generalized contractions satisfying rational type expressions in partially ordered metric spaces. *Gulf journal of mathematics,* 2, 87-93.

Chatterjea, S. K. (1972). Fixed point theorems. *C.R. Acada. Jour. Bulgar sci.,* 25, 72730.

Czerwik, S. (1993). Contraction mappings in *b*-metric spaces. *Acta. Math. Info. Univ. Ostrav.,* 1,5-11.

Das, B. K., Gupta, S. (1975). An extension of Banach contraction principle through rational expressions. *Indian. J. pure appl. math.,* 6(1975), 1455-1458.

Das, P. and L. K. Dey, (2007). A fixed point theorem in a generalized metric space. *Soochow Journal of Mathematics,* 33, (1).

Doric, (2009). Common fixed point for generalized weak contractions. *Appl. Math. Let.,* 22 , 18961900.

Dukic, D. Kadelburg, Z., Radenovic, S. (2011). Fixed points of Geraghty-type mappings in various generalized metric spaces. *Abstract Appl . Anal.,* 561245.

Geraghty, M. (1973). On contractive mappings, Proc. *Am. Math. Soc.,* 40, 604-608.

Glasnik, V. (2017). Contractive conditions in *b*-metric spaces. *Military technical courier,* 65, (4), 852.

Harjani, J., B. Lopez, and Sadarangan. K. (2010). A Fixed point theorem for mappings satisfying a contractive condition of rational type on a partially ordered metric spaces. *Abstract and Applied analysis,* 2010(2010), 8 pages, article ID 190171.

Jungck, G., Rhoades, B. E. (1998). Fixed points for set-valued functions with- out continuity. *Indian J.Pure. Appl. Math*., 29 (3), 227-238.

Kannan, T. K. (1968). Some results on fixed point. *Bull. Calcuta Math. Soc.*, 60(2), 71-76.

Kir, M., Kiziltunc, H. (2013). On some well-known fixed point Theorems in b-metric spaces. *Turkish Journal of Anal.and Number theory*, 1(1),13-16.

Nieto, J. J. and Lopez, R. R. (2005). Contractive mapping theorems in partially ordered sets and applications to ordinary differential equation. *Order,* 22 (3), 223239.

Mishra, K. P., Sachdeva, Sh., Banerjee, K. S. Some fixed point theorems in *b*-metric spaces. *Turkish Journal of Anal.and Number theory*, 2, (1), 19-22.

Rohades, B. E. (1977). A comparison of various definitions of contractive mappings, *Jour. Transfer Amer.* Soc., 226(32), 257-290.

Sintunavarat, W. and P. Kumam, (2013). PPF Dependent fixed point theorems for rational type Contraction mappings in Banach spaces. *Journal of Nonlinear Analysis and optimization*. 4, (2), 157-162.

Zamfirescu, T. (1972). Fixed point theorems in metric spaces. *Jour. of Arch. Math. (Basel), 23, 292-298.*