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Research Article

New Fixed Point Theorem in Dislocated Quasi Metric Space

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Abstract
The purpose of this paper is to establish new type of fixed point result for a single self-mapping in the setting of dislocated quasi-metric space. Our established result generalizes and modifies some existing fixed point theorems in the literature. Appropriate example for the usability of the established result is also given.

Keywords: Complete dislocated quasi metric space, Fixed point, Contraction mapping, Self-mapping, Cauchy sequence.

1 Introduction

Fixed point theory is one of the most dynamic research subjects in nonlinear analysis and can be used to many discipline branches such as; control theory, convex optimization, differential equation, integral equation ,economics etc. In this area, the first important and remarkable result was presented by Banach in 1922 for a contraction mapping in a complete metric space. Dass and Gupta [1] generalized the Banach contraction principle in a metric space for some rational type contractive conditions. Hitzler and Seda [3] investigated the useful applications of dislocated topology in the context of logic programming semantics. Furthermore, Zeyada et al. [4] generalized the results of Hitzler and Seda [3] and introduced the concept of complete dislocated quasi metric space. Aage and Salunke [2] derived some fixed point theorems in dislocated quasi metric spaces. In this manuscript, we established some fixed point result for a single continuous self-mapping in the context of dislocated quasi metric space which generalizes the result of Sarwar et al. [8].

2 Preliminaries

Definition 2.1 *[4] Let X be a non- empty set and $d : X \times X \to [0,\infty)$ be a function satisfying the following conditions:*

$$d_1)\ d(x,y) = d(y,x) = 0, \text{ implies } x = y,$$
$$d_2)\ d(x,y) \le d(x,z) + d(z,y), \text{ for all } x,y,z \in X.$$

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Then d *is called a dislocated quasi metric on* X*. A pair* (X,d) *is called dislocated quasi metric space. A dislocated quasi metric space is a function* $d: X\times X\to[0,\infty)$ *satisfying all the conditions of metric space with the exception of self-distance and symmetry. We note that the class of dislocated quasi metric space is effectively general than that of metric space. Here we note that every metric space is dislocated quasi metric space but the converse is not true.*

Definition 2.2 *[4] A sequence* $\{x_n\}$ *in a dislocated quasi metric space* (X,d) *is called Cauchy sequence if, for all* $\varepsilon>0$,$\exists\ n_0\in\mathbb{N}$ *such that for* $m,n\ge n_0$*, we have* $d(x_n,x_m)<\varepsilon$ *or* $d(x_m,x_n)<\varepsilon$.

Definition 2.3 *Let* (X,d) *be a dislocated quasi metric space. A self-mapping* $T:X\to X$ *is said to be a contraction map if there exists a constant* $k\in[0,1)$ *such that*

$$d(Tx,Ty)\le kd(x,y) \text{ for all } x,y \text{ in } (X,d).$$

Lemma 2.1 *[4] Limit of a convergent sequence in a dislocated quasi metric space is unique.*
In the following theorem, Zeyada et al. [4] generalized the Banach contraction principle in dislocated quasi metric space.

Theorem 2.1 *[4] Let* (X,d) *be a complete dislocated quasi metric space,* $T:X\to X$ *be a continuous contraction, then* T *has a unique fixed point in* X.
Isufati [5] derived the following two results, where the first one generalized the result of Dass and Gupta [1] in dislocated quasi metric spaces.

Theorem 2.2 *Let* (X,d) *be a complete dislocated quasi metric space and* $T:X\to X$ *be a continuous self-mapping satisfying the following condition:*

$$d(Tx,Ty)\le\alpha d(x,y)+\beta\frac{d(y,Ty)\left[1+d(x,Tx)\right]}{1+d(x,y)}$$

for all $x,y\in X$ *and* $\alpha,\beta\ge0$ *with* $\alpha+\beta<1$*. Then* T *has a unique fixed point.*

Theorem 2.3 *Let* (X,d) *be a complete dislocated quasi metric space and* $T:X\to X$ *be a continuous self-mapping satisfying the following condition:*

$$d(Tx,Ty)\le\alpha d(x,y)+\beta d(x,Ty)+\gamma d(y,Tx)$$

for all $x,y\in X$ *and* $\alpha,\beta,\gamma,\delta\ge0$ *with* $\sup\{\alpha+2\beta+2\gamma\}<1$.
Then T *has a unique fixed point.*
Kohli, Shrivastava and Sharma [6] proved the following theorem in the context of dislocated quasi metric space which generalized Theorem 2.2.

Theorem 2.4 *Let* (X,d) *be a complete dislocated quasi metric space and* $T:X\to X$ *be a continuous self-mapping satisfying the following condition:*

$$d(Tx,Ty)\le\alpha d(x,y)+\beta d(y,Ty)+\gamma\frac{d(y,Ty)\left[1+d(x,Tx)\right]}{1+d(x,y)}$$

for all $x,y\in X$ *and* $\alpha,\beta,\gamma\ge0$ *with* $\alpha+\beta+\gamma<1$*. Then* T *has a unique fixed point.*

For rational type contraction conditions Madhu Shrivastava et al. [7] proved the following theorem in a dislocated quasi metric space.

Theorem 2.5 *Let* (X,d) *be a complete dislocated quasi metric space and* $\mathrm{T}:\mathrm{X}\to\mathrm{X}$ *be a continuous self-mapping satisfying the following condition:*

$$d(Tx,Ty)\le \alpha d(x,y)+\beta\frac{d(y,Ty)\left[1+d(x,Tx)\right]}{1+d(x,y)}+\gamma\frac{d(y,Tx)+d(y,Ty)}{1+d(y,Tx)d(y,Ty)}$$

for all $x,y\in X$ *and* $\alpha,\beta,\gamma\ge 0$ *with* $\alpha+\beta+\gamma<1.$ *Then* T *has a unique fixed point. Sarwar et al. [8] proved the following theorem in the context of dislocated quasi metric space which generalized, modified and unified from Theorems 2.3, 2.4 and 2.5.*

Theorem 2.6 *Let* (X,d) *be a complete dislocated quasi metric space and* $\mathrm{T}:\mathrm{X}\to\mathrm{X}$ *be a continuous self-mapping satisfying the following condition:*

$$\begin{aligned} d(Tx,Ty)\le{}& a_1d(x,y)+a_2d(x,Ty)+a_3d(y,Tx)+a_4d(y,Ty)\\ &+a_5\frac{d(y,Ty)\left[1+d(x,Tx)\right]}{1+d(x,y)}+a_6\frac{d(y,Tx)+d(y,Ty)}{1+d(y,Tx)d(y,Ty)}\\ &+a_7\frac{d(x,Tx)[1+d(y,Tx)]}{1+d(x,y)+d(y,Ty)}\end{aligned}$$

for all $x,y\in X$ *and* $a_1,a_2,a_3,a_4,a_5,a_6,a_7\ge 0$ *with* $a_1+2(a_2+a_3)+a_4+a_5+3a_6+a_7<1.$
Then T *has a unique fixed point.*

3 Main Result

Theorem 3.1 *Let* X *be a non- empty set and* (X,d) *be a complete dislocated quasi metric space, and* $\mathrm{T}:\mathrm{X}\to\mathrm{X}$ *be a continuous self-mapping satisfying the following condition:*

$$\begin{aligned} d(Tx,Ty)\le{}& a_1d(x,y)+a_2d(x,Ty)+a_3d(y,Tx)+a_4d(y,Ty)\\ &+a_5\frac{d(y,Ty)\left[1+d(x,Tx)\right]}{1+d(x,y)}+a_6\frac{d(y,Tx)+d(y,Ty)}{1+d(y,Tx)d(y,Ty)}\\ &+a_7\frac{d(x,Tx)[1+d(y,Tx)]}{1+d(x,y)+d(y,Ty)}+a_8\frac{d(y,Ty)[1+d(x,Ty)]}{1+d(x,y)+d(y,Ty)}\end{aligned}\tag{3.1}$$

for all $x,y\in X$ *and* $a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8\ge 0$ *with* $a_1+2(a_2+a_3)+a_4+a_5+3a_6+a_7+a_8<1.$
Then T *has a unique fixed point.*

Proof.
Let x_0 be arbitrary in X, we define a sequence $\{x_n\}$ by
$x_0,\ x_1=Tx_0, x_2=Tx_1,..., x_{n+1}=Tx_n$ for $n=0,1,2,3,\ldots,$
Suppose

$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n).$

By using condition (3.1) we have

$$
\begin{aligned}
d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\
&\le a_1 d(x_{n-1}, x_n) + a_2 d(x_{n-1}, Tx_n) + a_3 d(x_n, Tx_{n-1}) + a_4 d(x_n, Tx_n) \\
&\quad + a_5 \frac{d(x_n, Tx_n)\left[1 + d(x_{n-1}, Tx_{n-1})\right]}{1 + d(x_{n-1}, x_n)} + a_6 \frac{d(x_n, Tx_{n-1}) + d(x_n, Tx_n)}{1 + d(x_n, Tx_{n-1}) d(x_n, Tx_n)} \\
&\quad + a_7 \frac{d(x_{n-1}, Tx_{n-1})[1 + d(x_n, Tx_{n-1})]}{1 + d(x_{n-1}, x_n) + d(x_n, Tx_n)} + a_8 \frac{d(x_n, Tx_n)[1 + d(x_{n-1}, Tx_n)]}{1 + d(x_{n-1}, x_n) + d(x_n, Tx_n)} \\
&\le a_1 d(x_{n-1}, x_n) + a_2 d(x_{n-1}, x_{n+1}) + a_3 d(x_n, x_n) + a_4 d(x_n, x_{n+1}) \\
&\quad + a_5 \frac{d(x_n, x_{n+1})\left[1 + d(x_{n-1}, x_n)\right]}{1 + d(x_{n-1}, x_n)} + a_6 \frac{d(x_n, x_n) + d(x_n, x_{n+1})}{1 + d(x_n, x_n) d(x_n, x_{n+1})} \\
&\quad + a_7 \frac{d(x_{n-1}, x_n)[1 + d(x_n, x_n)]}{1 + d(x_{n-1}, x_n) + d(x_n, x_{n+1})} + a_8 \frac{d(x_n, x_{n+1})[1 + d(x_{n-1}, x_{n+1})]}{1 + d(x_{n-1}, x_n) + d(x_n, x_{n+1})} \\
&\le a_1 d(x_{n-1}, x_n) + a_2 d(x_{n-1}, x_{n+1}) + a_3 [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + a_4 d(x_n, x_{n+1}) \\
&\quad + a_5 \frac{d(x_n, x_{n+1})\left[1 + d(x_{n-1}, x_n)\right]}{1 + d(x_{n-1}, x_n)} + a_6 \frac{[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + d(x_n, x_{n+1})}{1 + d(x_n, x_n) d(x_n, x_{n+1})} \\
&\quad + a_7 \frac{d(x_{n-1}, x_n)[1 + d(x_{n-1}, x_n) + d(x_n, x_{n+1})]}{1 + d(x_{n-1}, x_n) + d(x_n, x_{n+1})} + a_8 \frac{d(x_n, x_{n+1})[1 + d(x_{n-1}, x_{n+1})]}{1 + d(x_{n-1}, x_n) + d(x_n, x_{n+1})} \\
&\le a_1 d(x_{n-1}, x_n) + a_2 [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + a_3 [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\
&\quad + a_4 d(x_n, x_{n+1}) + a_5 d(x_n, x_{n+1}) + a_6 d(x_n, x_{n+1}) + a_6 [d(x_{n-1}, x_n) + 2d(x_n, x_{n+1})] \\
&\quad + a_7 d(x_{n-1}, x_n) + a_8 d(x_n, x_{n+1}),
\end{aligned}
$$

$$d(x_n, x_{n+1}) \le \frac{a_1 + a_2 + a_3 + a_6 + a_7}{1 - (a_2 + a_3 + a_4 + a_5 + 2a_6 + a_8)} d(x_{n-1}, x_n).$$

Let $\lambda = \dfrac{a_1 + a_2 + a_3 + a_6 + a_7}{1 - (a_2 + a_3 + a_4 + a_5 + 2a_6 + a_8)}.$

Clearly, $0 \le \lambda < 1$, since $a_1 + 2(a_2 + a_3) + a_4 + a_5 + 3a_6 + a_7 + a_8 < 1.$

So,

$$d(x_n, x_{n+1}) \le \lambda d(x_{n-1}, x_n).$$

Similarly,

$$d(x_{n-1}, x_n) \le \lambda d(x_{n-2}, x_{n-1}).$$

Thus

$$d(x_n, x_{n+1}) \le \lambda^2 d(x_{n-2}, x_{n-1}).$$

Continuing the same procedure, we get

$$d(x_n, x_{n+1}) \le \lambda^n d(x_0, x_1).$$

Now, for any $m, n \in \mathbb{N}$ with $m > n$, using the triangle inequality, we get

$$\begin{aligned} d(x_n, x_m) &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) \\ &= \lambda^n d(x_0, x_1) + \lambda^{n+1} d(x_0, x_1) + \ldots + \lambda^{m-1} d(x_0, x_1) \\ &\le (\lambda^n + \lambda^{n+1} + \lambda^{n+2} + \ldots + \lambda^{m-1}) d(x_0, x_1) \\ &\le \frac{\lambda^n}{1-\lambda} d(x_0, x_1). \end{aligned}$$

For any $\varepsilon > 0$, choose $N \ge 0$ such that, $\frac{\lambda^N}{1-\lambda} d(x_0, x_1) < \varepsilon.$

Then for any $m, n \ge N$, we have

$$d(x_n, x_m) \le \frac{\lambda^n}{1-\lambda} d(x_0, x_1) \le \frac{\lambda^N}{1-\lambda} d(x_0, x_1) < \varepsilon.$$

Similarly, we can show that $d(x_m, x_n) < \varepsilon.$

This shows that $\{x_n\}$ is a Cauchy sequence in a complete dislocated quasi metric space (X, d). So, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u.$ Since T is continuous, so we have

$$Tu = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} \mathrm{T}(x_n) = \lim_{n\to\infty} x_{n+1} = u.$$

Hence, u is a fixed point of T.

Uniqueness: Suppose that T has two distinct fixed points u and v. Consider

$$\begin{aligned} d(Tu, Tv) \le\; & a_1 d(u, v) + a_2 d(u, Tv) + a_3 d(v, Tu) + a_4 d(v, Tv) \\ & + a_5 \frac{d(v, Tv)\left[1 + d(u, Tu)\right]}{1 + d(u, v)} + a_6 \frac{d(v, Tu) + d(v, Tv)}{1 + d(v, Tu) d(v, Tv)} \\ & + a_7 \frac{d(u, Tu)[1 + d(v, Tu)]}{1 + d(u, v) d(v, Tv)} + a_8 \frac{d(v, Tv)[1 + d(u, Tv)]}{1 + d(u, v) + d(v, Tv)}. \end{aligned} \tag{3.2}$$

Since u and v are fixed points of T, condition (3.1) implies that

$d(u, u) = 0$ and $d(v, v) = 0.$

Finally, from (3.2) we get

$$d(u, v) \le (a_1 + a_2) d(u, v) + (a_3 + a_6) d(v, u). \tag{3.3}$$

Similarly, we have

$$d(v, u) \le (a_1 + a_2) d(v, u) + (a_3 + a_6) d(u, v). \tag{3.4}$$

Subtracting (3.4) from (3.3) we have

$$|d(u, v) - d(v, u)| \le |(a_1 + a_2) - (a_3 + a_6)| \, |d(u, v) - d(v, u)|. \tag{3.5}$$

Since $|(a_1 + a_2) - (a_3 + a_6)| < 1$, the above inequality (3.5) is possible if

$$d(u,v) - d(v,u) = 0. \quad (3.6)$$

Taking equations (3.3), (3.4) and (3.6) into account, we have $d(u,v) = 0$ and $d(v,u) = 0.$

Thus by (d_2), $u = v.$ Hence T has a unique fixed point in X.

Example 3.1 *Let* $X = [0,1]$ *with a complete dislocated quasi metric defined by* $d(x,y) = |x|$, *for all* $x, y \in X$ *and define the continuous self-mapping* T *by* $Tx = \frac{x}{4}$ *with* $a_1 = \frac{1}{6}, a_2 = \frac{1}{8}, a_3 = \frac{1}{12}, a_4 = \frac{1}{14}, a_5 = \frac{1}{16}, a_6 = \frac{1}{18}, a_7 = \frac{1}{20}$ *and* $a_8 = \frac{1}{24}$.
Then T satisfies all the conditions of Theorem 3.1 and $x = 0$ *is the unique fixed point of* T *in* X.

Remarks: In Theorem 3.1:

If $a_4 = a_5 = a_6 = a_7 = 0,$ then we get the result of Isufati [5].

If $a_2 = a_3 = a_4 = a_7 = 0,$ then we get the result of Madhu Shrivastava et al. [7].

If $a_2 = a_3 = a_4 = a_6 = a_7 = 0,$ then we get the result of Isufati [5].

If $a_2 = a_3 = a_6 = a_7 = 0,$ then we get the result of Manvi Kohli [6].

If $a_8 = 0,$ then we get the result of Sarwar et al. [8].

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