Non-Polynomial Cubic Spline Method for solving Singularly Perturbed Delay Convection-Diffusion Equations

A Thesis Submitted to the Department of Mathematics, Jimma University in Partial Fulfillment for the Requirements of the Degree of Masters of Science in Mathematics

(Numerical Analysis)

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DECLARATION

I undersigned declare that, this thesis entitled "non-polynomial cubic spline method for solving singularly perturbed delay convection diffusion equations." is my own original work and it has not been submitted for the award of any academic degree or the like in any other institution or university, and that all the sources I have used or quoted have been indicated and acknowledged.

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Abstract

In this thesis, non-polynomial cubic spline method is presented for solving singularly perturbed delay convection diffusion equations . First the second order singularly perturbed delay convection diffusion equation transformed in to an asymptotically equivalent singularly perturbed boundary value problem. Then non-polynomial cubic spline approximations are changed in to a three-term recurrence relation, which can be solved using Thomas Algorithm. The stability and convergence of the method have been established. The applicability of the proposed method is validated by implementing it by four model examples with different values of perturbation parameter ε *, delay parameter* δ *and mesh size h. The numerical results have been presented in tables and further to examine the effect of delay on the left and right boundary layer of the solution; graphs have been given for different values of* δ*. To show the accuracy of the method, the results are presented in terms of maximum absolute errors. Concisely, the present method gives more accuracy result than some existing numerical methods reported in the literature.*

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Chapter One

Introduction

1.1. Background of the Study

Numerical analysis is a branch of mathematics concerned with theoretical foundations of numerical algorithms for the solution of problems arising in scientific applications, (Wasow, 1942). In real life, we often encounter many problems which are described by parameter dependent differential equations. The behavior of the solutions of these types of differential equations depends on the magnitude of the parameters.

Singularly perturbed ordinary differential equation with a delay is ordinary differential equations in which the highest derivative is multiplied by a small parameter and involving at least one delay term. Such type of equations arises frequently from the mathematical modeling of various practical phenomena, for example, in the modeling of the human pupil-light reflex (Longtin and Milton, 1988), the study of bistable devices (Derstine, et al., 1982), and vibrational problems in control theory (Glizer, 2003), etc. When perturbation parameter ε is very small, most numerical methods for solving such problems may unstable and give inaccurate results. So, it is important to develop suitable numerical methods to solve singularly perturbed delay differential equations.

Hence, in the recent times, many researchers have been trying to develop numerical methods for solving singularly perturbed delay differential equations (SPDDE). For example, Soujanya and Reddy, (2016) proposed computational method of first order for solving singularly perturbed delay reaction-diffusion equations with layer or oscillatory behavior. Chakravarthy et al., (2017) presented exponentially fitted second order finite difference scheme for a class of singularly perturbed delay differential equations with large delay. Gemechis et al., (2017) presented fourth order finite difference scheme for singularly perturbed delay differential reaction diffusion equations. Gashu et al., (2018) proposed fourth order finite difference method for solving singularly perturbed delay differential equations.

Swamy et al., (2015) presented computational method of first order for singularly perturbed delay differential equation with twin layers or oscillatory behavior when negative shift is in the first derivative term. Phaneendra et al., (2014) presented fourth order finite difference scheme for second order singularly perturbed differential–difference equation with negative shift, but they

,mainly focus only on the constant coefficients. Sirisha and Reddy, (2017) proposed exponentially integrating method of first order for singularly perturbed delay convection-diffusion equations. But, still there is a lack of accuracy because of the treatment of singularly perturbed problems is not trivial and the solution depends on perturbation parameter ε and mesh size h ((Doolan et al, 1980) , (Kadalbajoo and Sharma, 2004) , (Roos et al, 1942)).

Thus, existing numerical methods produce good results only when we take step size $h \leq \varepsilon$. This shows that there is a challenge for singularly perturbed boundary value problems to get more accurate solution due to perturbation parameter is sufficiently small and no good result when

$\varepsilon < h$. Therefore, it is important to develop more accurate and convergent numerical method for solving singularly perturbed delay convection-diffusion equations.

Thus, the purpose of this study is to develop stable, convergent and more accurate numerical method for solving singularly perturbed delay convection-diffusion equations.

1.2. Objectives of the Study

1.2.1. General Objective

The general objective of this study is to develop non-polynomial cubic spline method for solving singularly perturbed delay convection diffusion equations.

1.2.2. Specific Objectives

The specific objectives of the present study are:

1. To describe the non-polynomial cubic spline method for solving singularly perturbed delay convection-diffusion equations.
2. To investigate the accuracy of the present scheme.
3. To establish the stability and convergence of the present scheme.

1.3. Significance of the Study

The outcomes of this study may have the following importance:.

- Provide some background information for other researchers who work on this area

- To introduce the application of numerical methods in solving problems arising in different field of studies.
- Help the graduate students to acquire research skills and scientific procedures.

1.4. Delimitation of the Study

The singularly perturbed problems are perhaps arises in variety mathematical and physical problems. However, this study is delimited to non-polynomial cubic spline method for solving singularly perturbed delay convection-diffusion equations of the form:

$$\varepsilon y''(x) + p(x)y'(x-\delta) + q(x)y(x) = f(x), \quad 0 \le x \le 1$$

with the interval and boundary conditions,

$$y(x) = \phi(x), \quad -\delta \le x \le 0 \quad \text{and} \quad y(1) = \varphi$$

where ε is perturbation parameter, $0 < \varepsilon << 1$ and δ is a small delay parameter of $o(\varepsilon), 0 < \delta \ll 1.$ Also $p(x),\ q(x),\ f(x)$ are assumed to be continuously differentiable on (0,1)

Chapter Two
Review of Related Literature

2.1. Singular Perturbation Theory

Singular perturbations were first described by (Prandtl, 1905) in a seven-page report presented at the Third International Congress of Mathematicians in Heidelberg in 1904. However, the term "Singular Perturbations" was first used by (Friedrichs and Wasow, 1946) in a paper presented at a seminar on nonlinear vibrations at New York University. The solution of singular perturbation problems typically contains layers. (Prandtl, 1905) originally introduced the term "boundary layer," but this term came into more general use following the work of (Wasow, 1942). (Pearson, 1968) was among the first to consider the finite-difference method to solve singular perturbation problems ε is not 0 but is small, the solution is expected, under certain conditions, to exhibit narrow regions of very fast variation (so called boundary or interior layers) which connect wider regions where it varies more slowly. In recent years researchers have used cubic splines and finite-element methods to solve singular perturbation problems, and a large number of papers and books have been published describing the various methods. For a detailed discussion on singular perturbation problems one may refer to the books and high level monographs: (O'Malley, 1974), (Nayfeh,1973), (Kevorkian and Cole, 1981), *"mathematics",* (Bender and Orszag, 1978), (Farrell et al., 2000) and (Roost et al., 1996).

The development of small parameter methods led to the efficient use of boundary layer theory in various fields of applied mathematics, for instance, modeling of water quality problems in river networks Baumer et al., (1981), financial modelling Black and Scholes, (1973) and mathematical models of turbulence, (Launder and Spalding, 1972).

The study of many theoretical and applied problems in science and technology leads to boundary value problems for singularly perturbed differential equations that have a multi-scale character. However, most of the problems cannot be completely solved by analytic techniques. Consequently, numerical simulations are of fundamental importance in gaining some useful insights on the solutions of the singularly perturbed differential equations.

2.2. Singularly Perturbed Delay Differential Equations

Singularly perturbed delay differential equation is an equation in which evolution of system at a certain time depends on the rate at an earlier time. The delay in process arises due to requirement of definite time to sense the instruction and react to it. The delay differential equation in which the highest derivative is multiplied by perturbation parameter is known as perturbed delay differential equation. The delay differential equation can be classified as retarded delay differential equation and neutral differential equation. A delay differential equation is said to be of retarded delay differential equation if the delay argument does not occur in the highest order derivative term, otherwise it is known as neutral delay differential equations. If we restrict it to a class in which the highest derivative term is multiplied by a small parameter, then we obtain singularly perturbed delay differential equations of the retarded type. Frequently, delay differential equations have been reduced to differential equations with coefficients that depend on the delay by means Taylor's series expansions of the terms that involve delay and the resulting differential equations have been solved either analytically when the coefficients of these equations are constant or numerically, when they are not. When the delay argument is sufficiently small, to tackle the delay term (Kadalbajoo and Sharma, 2004; 2008), used Taylor's series expansion and presented an asymptotic as well as numerical approach to solve such type boundary value problem. But the existing methods in the literature fail in the case when the delay argument is bigger than one because in this case, the use of Taylor's series expansion for the term containing delay may lead to a bad approximation.

A singularly perturbed problem is said to be of convection-diffusion type, if the order of differential equation is reduced by one, when the perturbation parameter is set equal zero. If the order is reduced by two, it is known as a reaction-diffusion type problem. Further if the order of the differential equation in singularly perturbed problem is greater than two, it is said to be a higher order SPP (Phaneendra et al., 2012). The theory and numerical solution of singularly perturbed delay differential equations are still at the initial stage. In the past, only very few people had worked in the area of numerical methods on singularly perturbed delay differential equations (SPDDEs). But in the recent years, there has been a growing interest in this area. In fact, Er-dogan,(2009) proposed an exponentially fitted operator method for singularly perturbed first order delay differential equation, Kadalbajoo and Sharma, (2007; 2008) proposed some numerical methods for SPDDEs with a small delay. Chakravarthy et al., (2015) proposed a fitted numerical scheme for second order singularly perturbed delay differential equations via cubic spline in compression.Gemechis and Reddy, (2013) presented Computational method for solving singularly perturbed delay differential equations with negative shift. Awoke and Reddy, (2013) presented second order parameter fitted scheme to solve singularly perturbed delay differential equations.Gülsu and Öztürk, (2011) presented an approximate solution of the singularly perturbed delay convection-diffusion equation by Chebyshev-gauss grid. The numerical treatment of singularly perturbed problems preserves some major computational difficulties and in recent years a large number of special purpose methods have been proposed to provide accurate numerical solutions. This type of problem has been intensively studied analytically and it is known that its solution generally has boundary layers where the solution varies rapidly. The outer solution corresponds to the reduced problem, *i.e.*, that obtained by setting the small perturbation parameter to zero. In recent years, the cubic-spline, B-spline, finite difference, stable central difference methods etc. has been used to find the approximate solutions of differential, difference, integral and integral-differential-difference equations. The main characteristic of this technique is that it reduces these problems to those of solving a system of algebraic equations, thus greatly simplifying the problem.

Accordingly, a numerical method named as initial value technique (IVT) is suggested to solve the second order ordinary differential equations of reaction-diffusion type with a negative shift in the differentiated term. The initial value method was also introduced by the authors (Gasparo and Macconi, 1990). Subburayan and Ramanujam, (2013) presented an initial value technique to solve the singularly perturbed boundary value problem for the second order ordinary differential equations of convection-diffusion type with delay. Tang and Geng, (2016) proposed Fitted reproducing kernel method for singularly perturbed delay initial value problems.

2.3. Spline Based Methods

In last 25 years remarkable progress has been made in the theory, methods, and applications for the singular perturbation in the mathematical circles and a lot of new results have appeared. To be more accessible for practicing engineers and applied mathematicians there is a need for methods, which are easy and ready for computer implementation. The spline technique appears to be an ideal tool to attain these goals. The spline based methods are easy and ready for computer implementation. Splines have many applications in the numerical solution of a variety of problems in applied mathematics and engineering. Some of them are data fitting, function approximation, Integral differential equations, optimal control problems, computer-aided geometric design (CAGD), and wavelets and so on. Programs based on spline functions have found their way in most of computer applications. It is commonly accepted that the first mathematical reference to splines was made in the year 1946 in an interesting paper by (Schoenberg, 1946), which is probably the first place that the word "spline" is used in connection with smooth, piece wise polynomial approximation. The use of cubic splines for the solution of linear two point boundary value problems was suggested by Bickley, (1968). Aziz and Khan, (2002) proposed a method based on cubic spline in compression for the linear second order singularly perturbed problems which have second and fourth order convergence.

Some of the earliest papers using spline functions for smooth approximate solution of ordinary and partial differential equations (PDEs) include (Siraj et al, 2007). These papers demonstrate the approximate methods of solving second, third and fourth-order linear boundary-value problems (BVPs) and solution of elliptic and parabolic equations by spline functions of various degrees. Today, there are hundreds of research papers on this subject, and it remains an active research area. Recently, non-polynomial spline method has turned out to be an effective tool for solving ordinary and partial differential equations. In many papers various techniques using quadratic, cubic, quartic, quintic, sextic, septic and higher degree non-polynomial splines have been discussed for the numerical solution of linear and nonlinear BVPs. Non-polynomial splines were used for numerical solution of system of second-order BVPs in (Brati and Rashidnia, 2011).

As introduced in the literature, most researchers have been tried to find approximate solution for singularly perturbed delay convection-diffusion equation, but mainly focuses on constant coefficients, and some others those who have done for variable coefficients did not get more accurate solutions. Owing this, we are going to find a more accurate and convergent numerical method for solving singularly perturbed delay convection-diffusion equation, by using non polynomial spline method of fourth order convergence.

Chapter Three

Methodology

3.1. Study Site

This study is conducted at Jimma University under the department of Mathematics from September 2017 to September 2018. Conceptually, the study focus on the area of non-polynomial cubic spline method for solving singularly perturbed delay convection-diffusion equations.

3.2. Study Design

The study employed mixed design (*i.e.*, document review and numerical experimentation designs).

3.3. Source of Information

The relevant sources of information for this study are books, published articles and the experimental result obtained by writing MATLAB code.

3.4. Mathematical Procedures

In order to achieve the stated objectives, the study followed the following procedures:

1. Defining the problem,
2. Discretizing the solution domain,
3. Formulate non-polynomial cubic spline approximation and obtained the scheme, which can be solved using Thomas Algorithm.
4. Establishing the stability and convergence of the proposed scheme,
5. Writing MATLAB code for the obtained scheme,
6. Validating the schemes by using numerical examples,
7. Presentation of the results in table and graphs .

Chapter Four

Description of the Method, Results and Discussion

4.1. Description of the Method

In this section, the description of non-polynomial cubic spline method: and its stability and convergence analysis is discussed. Consider singularly perturbed delay convection-diffusion equations (SPDCDEs) of the form:

$$\varepsilon y''(x) + p(x)y'(x-\delta) + q(x)y(x) = f(x), \qquad 0 \le x \le 1 \tag{4.1}$$

with the interval and boundary conditions,

$$y(x) = \phi(x) \ , -\delta \le x \le 0, \quad y(1) = \varphi \tag{4.2}$$

where ε is perturbation parameter, $0 < \varepsilon << 1$ and δ is a small delay parameter of $o(\varepsilon), 0 < \delta \ll 1; p(x), q(x), f(x), \phi(x)$ are assumed to be continuously differentiable in the interval (0, 1), φ is constant and assumed that $q(x) < 0$. Further, if $p(x) > 0, \forall x \in [0,1]$, then the boundary layer exists in the neighborhood of $x = 0$. If $p(x) < 0, \forall x \in [0,1]$, then the boundary layer exist in the neighborhood of $x = 1$. `Since the delay parameter is smaller than the perturbation parameter (*i.e.,* $\delta < \varepsilon$), an application of Taylor series expansion on the delay term yields,

$$y'(x-\delta) \approx y'(x) - \delta y''(x) + O(\delta^2) \tag{4.3}$$

Substituting Eq. (4.3) into Eq. (4.1), we obtain an asymptotically equivalent singularly perturbed boundary value problem (SPBVP) of the form:

$$\gamma y''(x) + p(x)y'(x) + q(x)y(x) = f(x) \tag{4.4}$$

where $\gamma = \varepsilon - \delta p(x), \quad 0 < \gamma << 1.$

with the boundary conditions,

$$y(0) = \phi_0 , \quad y(1) = \varphi \tag{4.5}$$

We consider a uniform mesh with nodal points on [0, 1] such that:

$0 = x_0 < x_1 < x_2, ..., x_{n-1} < x_n = 1,\ \ x_i = x_0 + ih,\ \ i = 0,1,...,\ n$ where $h = \dfrac{1}{n}, n$ is number of interval. For each segment $[x_i, x_{i+1}]$ $, i = 0,1,2,3,...,n-1$ the non-polynomial cubic spline $S_\Delta(x)$ has the form:

$$S_\Delta(x) = a_i \sin w(x - x_i) + b_i \cos w(x - x_i) + c_i \left(e^{w(x-x_i)} - e^{-w(x-x_i)}\right)$$

$$+ d_i \left(e^{w(x-x_i)} + e^{-w(x-x_i)}\right) \tag{4.6}$$

where, a_i, b_i, c_i and d_i are unknown coefficients, and $w \neq 0$ arbitrary parameter which will be used to increase the accuracy of the method.

To determine the unknown coefficients in Eq. (4.6) in terms of $y_i, y_{i+1}, M_i, M_{i+1}$ first we define:

$$S_\Delta(x_i) = y_i \ , \qquad S_\Delta(x_{i+1}) = y_{i+1},$$

$$S''_\Delta(x_i) = M_i \ , \qquad S''_\Delta(x_{i+1}) = M_{i+1} \tag{4.7}$$

Differentiating Eq. (4.6) successively, we get:

$$S'_\Delta(x) = a_i w \cos w(x - x_i) - b_i w \sin w(x - x_i) + c_i w(e^{w(x-x_i)} + e^{-w(x-x_i)})$$

$$+ d_i w(e^{w(x-x_i)} - e^{-w(x-x_i)}) \tag{4.8}$$

$$S''_\Delta(x) = -a_i w^2 \sin w(x - x_i) - b_i w^2 \cos w(x - x_i) + c_i w^2 \left(e^{w(x-x_i)} - e^{-w(x-x_i)}\right)$$

$$+ d_i w^2 \left(e^{w(x-x_i)} + e^{-w(x-x_i)}\right) \tag{4.9}$$

Substituting Eq. (4.7) in to Eq. (4.9), we have:

$$S''_\Delta(x_i) = M_i = -b_i w^2 + 2d_i w^2 \Rightarrow 2d_i = b_i + \frac{M_i}{w^2} \tag{4.10}$$

Substituting Eq. (4.7) into Eq. (4.6), we have:

$$S_\Delta(x_i) = y_i = b_i + 2d_i \Rightarrow y_i = b_i + 2d_i \tag{4.11}$$

Substituting Eq. (4.10) into Eq. (4.11), we obtain:

$$y_i = b_i + b_i + \frac{M_i}{w^2} \Rightarrow b_i = \frac{y_i w^2 - M_i}{2w^2} \tag{4.12}$$

Again substituting Eq. (4.7) into Eqs. (4.6) and (4.9) respectively and letting $\theta = wh$, we obtain:

$$S_\Delta(x_{i+1}) = y_{i+1} = a_i \sin wh + b_i \cos wh + c_i(e^{wh} - e^{-wh}) + d_i(e^{wh} + e^{-wh})$$

$$\Rightarrow y_{i+1} = a_i \sin(\theta) + \left(\frac{y_i w^2 - M_i}{2w^2} \right) \cos(\theta) + c_i(e^{\theta} - e^{-\theta}) + d_i(e^{\theta} + e^{-\theta}) \tag{4.13}$$

$$S''_\Delta(x_{i+1}) = M_{i+1} = -a_i w^2 \sin(wh) - b_i w^2 \cos(wh) + c_i w^2 (e^{wh} - e^{-wh}) + d_i w^2 (e^{wh} + e^{-wh})$$

$$\Rightarrow \frac{M_{i+1}}{w^2} = -a_i \sin(\theta) + \left(\frac{M_i - y_i w^2}{2w^2} \right) \cos(\theta) + c_i(e^{\theta} - e^{-\theta}) + d_i(e^{\theta} + e^{-\theta}) \tag{4.14}$$

Subtracting Eq. (4.14) from Eq.(4.13),we get:

$$y_{i+1} - \frac{M_{i+1}}{w^2} = 2a_i \sin(\theta) + \left(\frac{y_i w^2 - M_i}{w^2} \right) \cos(\theta)$$

$$\Rightarrow a_i = \frac{w^2 y_{i+1} - M_{i+1} + (M_i - y_i w^2)\cos(\theta)}{2w^2 \sin(\theta)} \tag{4.15}$$

Adding Eqs.(4.14) and (4.13), we get :

$$y_{i+1} + \frac{M_{i+1}}{w^2} = 2c_i(e^{\theta} - e^{-\theta}) + 2d_i(e^{\theta} + e^{-\theta}) \tag{4.16}$$

Substituting Eq. (4.12) into Eq. (4.10), we have:

$$2d_i = \frac{y_i w^2 - M_i}{2w^2} + \frac{M_i}{w^2}$$

$$\Rightarrow d_i = \frac{y_i w^2 + M_i}{4w^2} \tag{4.17}$$

Substitute Eq. (4.17) into Eq. (4.16), we get:

$$y_{i+1}+\frac{M_{i+1}}{w^2}=2c_i(e^{\theta}-e^{-\theta})+2\left(\frac{y_i w^2+M_i}{4w^2}\right)(e^{\theta}+e^{-\theta})$$

$$\Rightarrow c_i=\frac{2(y_{i+1}w^2+M_{i+1})-(y_i w^2+M_i)(e^{\theta}+e^{-\theta})}{4w^2(e^{\theta}-e^{-\theta})} \tag{4.18}$$

Using the continuity condition of the first derivative at x_i, $S'_{\Delta-1}(x_i)=S'_{\Delta}(x_i)$, we have:

$$a_{i-1}\cos(\theta)-b_{i-1}\sin(\theta)+c_{i-1}(e^{\theta}+e^{-\theta})+d_{i-1}(e^{\theta}-e^{-\theta})=a_i+2c_i \tag{4.19}$$

Reducing indices of Eqs. (4.12), (4.15), (4.17) and (4.18) by one and substituting into Eq. (4.19), we have`:

$$\left(\frac{w^2 y_i-M_i+(M_{i-1}-y_{i-1}w^2)\cos(\theta)}{2w^2\sin(\theta)}\right)\cos(\theta)-\left(\frac{y_{i-1}w^2-M_{i-1}}{2w^2}\right)\sin(\theta)$$
$$+\frac{2(y_i w^2+M_i)-(y_{i-1}w^2+M_{i-1})(e^{\theta}+e^{-\theta})}{4w^2(e^{\theta}-e^{-\theta})}(e^{\theta}+e^{-\theta})+\left(\frac{y_{i-1}w^2+M_{i-1}}{4w^2}\right)(e^{\theta}-e^{-\theta})$$
$$=\left(\frac{w^2 y_{i+1}-M_{i+1}+(M_i-y_i w^2)\cos(\theta)}{2w^2\sin(\theta)}\right)+\left(\frac{2(y_{i+1}w^2+M_{i+1})-(y_i w^2+M_i)(e^{\theta}+e^{-\theta})}{2w^2(e^{\theta}-e^{-\theta})}\right)$$

$$\Rightarrow\left(-\frac{\cos^2\theta}{2\sin\theta}-\frac{\sin\theta}{2}-\frac{(e^{\theta}+e^{-\theta})^2}{4(e^{\theta}-e^{-\theta})}+\frac{(e^{\theta}-e^{-\theta})}{4}\right)y_{i-1}$$

$$+\left(\frac{\cos\theta}{2\sin\theta}+\frac{\cos\theta}{2\sin\theta}+\frac{(e^{\theta}+e^{-\theta})}{2(e^{\theta}-e^{-\theta})}+\frac{(e^{\theta}+e^{-\theta})}{2(e^{\theta}-e^{-\theta})}\right)y_i+\left(-\frac{1}{2\sin\theta}-\frac{1}{e^{\theta}-e^{-\theta}}\right)y_{i+1}$$

$$=\left(-\frac{\cos^2\theta}{2w^2\sin\theta}-\frac{\sin\theta}{2w^2}+\frac{(e^{\theta}+e^{-\theta})^2}{4w^2(e^{\theta}-e^{-\theta})}-\frac{(e^{\theta}-e^{-\theta})}{4w^2}\right)M_{i-1}$$

$$\left(\frac{\cos\theta}{2w^2\sin\theta}+\frac{\cos\theta}{2w^2\sin\theta}-\frac{(e^{\theta}+e^{-\theta})}{2w^2(e^{\theta}-e^{-\theta})}-\frac{(e^{\theta}+e^{-\theta})}{2w^2(e^{\theta}-e^{-\theta})}\right)M_i+\left(-\frac{1}{2w^2\sin\theta}+\frac{1}{e^{\theta}-e^{-\theta}}\right)M_{i+1}$$

$$\Rightarrow\left(-\frac{1}{2\sin(\theta)}-\frac{1}{e^{\theta}-e^{-\theta}}\right)y_{i-1}+\left(\frac{\cos(\theta)}{\sin(\theta)}+\frac{e^{\theta}+e^{-\theta}}{e^{\theta}-e^{-\theta}}\right)y_i+\left(-\frac{1}{2\sin(\theta)}-\frac{1}{e^{\theta}-e^{-\theta}}\right)y_{i+1}$$

$$=\left(-\frac{1}{2w^2\sin(\theta)}+\frac{1}{w^2(e^{\theta}-e^{-\theta})}\right)M_{i-1}+\left(\frac{\cos(\theta)}{w^2\sin(\theta)}-\frac{(e^{\theta}+e^{-\theta}}{w^2(e^{\theta}-e^{-\theta})}\right)M_i$$

$$+\left(-\frac{1}{2w^2\sin(\theta)}+\frac{1}{w^2(e^{\theta}-e^{-\theta})}\right)M_{i+1}$$

$$\Rightarrow\left(-\frac{1}{2\sin(\theta)}-\frac{1}{e^{\theta}-e^{-\theta}}\right)y_{i-1}+\left(\frac{\cos(\theta)}{\sin(\theta)}+\frac{e^{\theta}+e^{-\theta}}{e^{\theta}-e^{-\theta}}\right)y_i+\left(-\frac{1}{2\sin(\theta)}-\frac{1}{e^{\theta}-e^{-\theta}}\right)y_{i+1}$$

$$=\left(-\frac{h^2}{2\theta^2\sin(\theta)}+\frac{h^2}{\theta^2(e^{\theta}-e^{-\theta})}\right)M_{i-1}+\left(\frac{h^2\cos(\theta)}{\theta^2\sin(\theta)}-\frac{h^2(e^{\theta}+e^{-\theta}}{\theta^2(e^{\theta}-e^{-\theta})}\right)M_i$$

$$+\left(-\frac{h^2}{2\theta^2\sin(\theta)}+\frac{h^2}{\theta^2(e^{\theta}-e^{-\theta})}\right)M_{i+1} \tag{4.20}$$

Multiplying both side of Eq. (4.20) by $\frac{-2\sin(\theta)(e^{\theta}-e^{-\theta})}{e^{\theta}-e^{-\theta}+2\sin(\theta)}$, we obtain:

$$y_{i-1}-2\left(\frac{e^{\theta}(\cos(\theta)+\sin(\theta))+e^{-\theta}(\sin(\theta)-\cos(\theta))}{e^{\theta}-e^{-\theta}+2\sin(\theta)}\right)y_i+y_{i+1}=\frac{h^2}{\theta^2}\left(\frac{e^{\theta}-e^{-\theta}-2\sin(\theta)}{e^{\theta}-e^{-\theta}+2\sin(\theta)}\right)M_{i-1}$$

$$-2h^2\left(\frac{e^{\theta}(\cos(\theta)-\sin(\theta))-e^{-\theta}(\sin(\theta)+\cos(\theta))}{\theta^2(e^{\theta}-e^{-\theta}+2\sin(\theta)}\right)M_i+\frac{h^2}{\theta^2}\left(\frac{e^{\theta}-e^{-\theta}-2\sin(\theta)}{e^{\theta}-e^{-\theta}+2\sin(\theta)}\right)M_{i+1}$$

$$\Rightarrow y_{i-1}+\rho y_i+y_{i+1}=h^2\left(\alpha M_{i-1}+\beta M_i+\alpha M_{i+1}\right) \tag{4.21}$$

where, $\rho=-2\left(\frac{e^{\theta}\left(\cos\theta+\sin\theta\right)+e^{-\theta}\left(\sin\theta-\cos\theta\right)}{2\sin\theta+e^{\theta}-e^{-\theta}}\right)$ $\quad\alpha=\left(\frac{e^{\theta}-e^{-\theta}-2\sin\theta}{\theta^2(2\sin\theta+e^{\theta}-e^{-\theta})}\right)$

$$\beta=-2\left(\frac{e^{\theta}\left(\cos\theta-\sin\theta\right)-e^{-\theta}\left(\sin\theta+\cos\theta\right)}{\theta^2(2\sin\theta+e^{\theta}-e^{-\theta})}\right)\quad M_j=y''(x_j),\ j=i-1,i,i+1 \text{ and } \theta=wh$$

$As,\theta\to 0,(\alpha,\beta,\rho)\to\left(\frac{1}{6},\frac{4}{6},-2\right)$. and then spline defined by (4.21) reduces to an ordinary cubic spline relation (Khan and Khandelwal,2017).

Using $S''_{\Delta}(x_j) = y''(x_j) = M_j,\ j = i-1, i, i+1$ into Eq. (4.4), we get:

$$\gamma M_i = f_i - p_i y_i' - q_i y_i \ ,\ \ \gamma M_{i-1} = f_{i-1} - p_{i-1} y'_{i-1} - q_{i-1} y_{i-1},\ \ \gamma M_{i+1} = f_{i+1} - p_{i+1} y'_{i+1} - q_{i+1} y_{i+1} \qquad (4.22)$$

substituting Eq. (4.22) into Eq. (4.21)

$$\frac{\gamma}{h^2}(y_{i-1} + \rho y_i + y_{i+1}) = \alpha\left(f_{i-1} - p_{i-1} y'_{i-1} - q_{i-1} y_{i-1}\right) + \beta\left(f_i - p_i y_i' - q_i y_i\right)$$

$$+\alpha\left(f_{i+1} - p_{i+1} y'_{i+1} - q_{i+1} y_{i+1}\right) \qquad (4.23)$$

using (4.23) and the following approximations for the first derivatives of y (Bawa, 2005).

$$y'_{i-1} \cong \frac{-3y_{i-1} + 4y_i - y_{i+1}}{2h} \qquad (4.24)$$

$$y'_{i+1} \cong \frac{y_{i-1} - 4y_i + 3y_{i+1}}{2h} \qquad (4.25)$$

$$y_i' \cong \left(\frac{1 - 2\psi h^2 q_{i+1} - \psi h(3p_{i+1} + p_{i-1})}{2h}\right) y_{i+1} + 2\psi\left(p_{i+1} + p_{i-1}\right) y_i$$

$$+\left(\frac{1 - 2\psi h^2 q_{i-1} - \psi h(p_{i+1} + 3p_{i-1})}{2h}\right) y_{i-1} + \psi h(f_{i+1} - f_{i-1}) \qquad (4.26)$$

where ψ is parameter used to raise the accuracy of the method .

we get:

$$\frac{\gamma}{h^2}(y_{i-1} + \rho y_i + y_{i+1}) + \frac{\alpha p_{i-1}}{2h}(-3y_{i-1} + 4y_i - y_{i+1}) + \alpha q_{i-1} y_{i-1}$$

$$+\beta p_i \left\{\left(\frac{1 - 2\psi h^2 q_{i+1} - \psi h(3p_{i+1} + p_{i-1})}{2h}\right) y_{i+1}\right.$$

$$\left. + 2\psi(p_{i+1} + p_{i-1}) y_i + \left(\frac{-1 + 2\psi h^2 q_{i-1} - \psi h(p_{i+1} + 3p_{i-1})}{2h}\right) y_{i-1} + q_i y_i\right\}$$

$$+\alpha p_{i+1}\left(\frac{(y_{i-1}-4y_i+3y_{i+1})}{2h}\right)+\alpha q_{i+1}y_{i+1}$$

$$=\left(\alpha+\beta p_i\psi h\right)f_{i-1}+\beta f_i+\left(\alpha-\beta p_i\psi h\right)f_{i+1}$$

which implies

$$\left\{\gamma-\frac{h}{2}u_i+\alpha h^2 q_{i-1}\right\}y_{i-1}-\left\{-\rho\gamma+\frac{h}{2}v_i-\beta h^2 q_i\right\}y_i+\left\{\gamma-\frac{h}{2}w_i+\alpha h^2 q_{i+1}\right\}y_{i+1} \qquad (4.27)$$

$$=h^2\left(\left(\alpha+\beta p_i\psi h\right)f_{i-1}+\beta f_i+\left(\alpha-\beta p_i\psi h\right)f_{i+1}\right) \quad i=1,2,\ldots,n-1$$

where,

$$u_i=3\alpha p_{i-1}+\beta p_i\psi h(p_{i+1}+3p_{i-1})-\beta p_i\psi h^2 q_{i-1}-\alpha p_{i+1}+\beta p_i$$

$$v_i=-4\alpha p_{i-1}-4\beta h p_i\psi(p_{i+1}+p_{i-1})+4\alpha p_{i+1}$$

$$w_i=\alpha p_{i-1}+\beta pi\psi h(3p_{i+1}+p_{i-1})-3\alpha p_{i+1}-\beta p_i+\beta p_i\psi h^2 q_{i+1}$$

Further, Eq. (4.28) can be rewritten as a three term recurrence relation of the form:

$$E_i y_{i-1}-F_i y_i+G_i y_{i+1}=H_i, \quad i=1,2\ldots,n-1 \qquad (4.28)$$

where,

$$E_i=\gamma-\frac{h}{2}u_i+\alpha h^2 q_{i-1}$$

$$F_i=-\rho\gamma+\frac{h}{2}v_i-\beta h^2 q_i$$

$$G_i=\gamma-\frac{h}{2}w_i+\alpha h^2 q_{i+1}$$

$$H_i=h^2\left(\left(\alpha+\beta p_i\psi h\right)f_{i-1}+\beta f_i+\left(\alpha-\beta p_i\psi h\right)f_{i+1}\right), \quad i=1,2,\ldots,n-1$$

The tri-diagonal system in Eq. (4.28) can be easily solved by the method of Thomas Algorithm

4.2. Truncation Error

From Eqs. (4.24-4.26), we have:

$$e'_{i+1} = Y'(x_{i+1}) - y'_{i+1} = \frac{h^2 y_i^{(3)}}{3} + \frac{h^3 y_i^{(4)}}{12} + \frac{h^4 y_i^{(5)}}{30} \tag{4.29}$$

$$e'_{i-1} = Y'(x_{i-1}) - y'_{i-1} = \frac{h^2 y_i^{(3)}}{3} - \frac{h^3 y_i^{(4)}}{12} + \frac{h^4 y_i^{(5)}}{30} \tag{4.30}$$

$$e'_i = Y'(x_i) - y'_i = -h^2(\frac{1}{6} + 2\psi\gamma) y_i^{(3)} - \frac{\psi\gamma h^4 y_i^{(5)}}{3} + \frac{h^4 y_i^{(5)}}{120} \tag{4.31}$$

Substituting Eq.(4.22) into Eq. (4.21), we get:

$$\gamma y_{i-1} + \gamma\rho y_i + \gamma y_{i+1} = h^2(\alpha(f_{i-1} - p_{i-1}y'_{i-1} - q_{i-1}y_{i-1}) + \beta(f_i - p_i y'_i - q_i y_i)$$

$$\alpha(f_{i+1} - p_{i+1}y'_{i+1} - q_{i+1}y_{i+1}) \tag{4.32}$$

Putting the exact solution in Eq. (4.32), we have:

$$\gamma Y(x_{i-1}) + \gamma\rho Y(x_i) + \gamma Y(x_{i+1}) = h^2 \left(\alpha(f_{i-1} - p_{i-1}Y'(x_{i-1}) - q_{i-1}Y(x_{i-1})\right. \tag{4.33}$$

$$+\beta(f_i - p_i Y'(x_i) - q_i Y'(x_i)) + \alpha(f_{i+1} - p_{i+1}Y'(x_{i+1}) - q_{i+1}Y(x_{i+1}) + T_0(h)$$

where,

$$T_0(h) = -(2+\rho)\gamma y_i + (2\alpha + \beta - 1)\gamma h^2 y''_i + (12\alpha - 1)\frac{\gamma h^4 y_i^{(4)}}{12} + (2\alpha + \beta - 1)\gamma h^6 y_i^{(6)} \tag{4.34}$$

Subtract Eq. (4.32) from Eq. (4.33) and putting:

$e_j = Y(x_j) - y_j$, $j = i \pm 1$, we get:

$$\gamma(Y(x_{i-1}) - y_{i-1}) + \gamma\rho(Y(x_i) - y_i) + \gamma(Y(x_{i+1}) - y_{i+1}) = -h^2 \left\{\alpha p_{i-1}(Y'(x_{i-1}) - y'_{i-1})\right.$$

$$+\alpha q_{i-1}(Y(x_{i-1}) - y_{i-1}) + \beta p_i(Y'(x_i) - y'_i) + \beta q_i(Y(x_i) - y_i) + \alpha p_{i+1}(Y'(x_{i+1}) - y'_{i+1})$$

$$+\alpha q_{i+1}(Y(x_{i+1})-y_{i+1})\}+T_0(h)$$

$$\Rightarrow\left(\gamma+h^2\alpha q_{i-1}\right)e_{i-1}+\left(\gamma\rho+\beta h^2 q_i\right)e_i+\left(\gamma+h^2\alpha q_{i+1}\right)e_{i+1}$$

$$=-h^2\left(\alpha p_{i-1}e'_{i-1}+\beta p_i e'_i+\alpha p_{i+1}e'_{i+1}\right)+T_0(h)$$

Using Eqs. (4.29) – (4.31), we have:

$$\left(\gamma+h^2\alpha q_{i-1}\right)e_{i-1}+\left(\gamma\rho+\beta h^2 q_i\right)e_i+\left(\gamma+h^2\alpha q_{i+1}\right)e_{i+1}$$

$$=-h^4\left((\frac{\alpha p_{i-1}}{3}-\beta p_i(\frac{1}{6}+2\psi\gamma)+\frac{\alpha p_{i+1}}{3})y_i^{(3)}\right)-h^5\left(\frac{-\alpha p_{i-1}}{12}+\frac{\alpha p_{i+1}}{12}\right)y^4$$

$$-h^6\left(\frac{1}{30}(\alpha p_{i-1}+\alpha p_{i+1})y^5+\beta p_i(-\frac{\psi\gamma}{3}+\frac{1}{120})y_i^{(5)}\right)+T_0(h) \qquad (4.35)$$

Let

$$p_{i+1}=p_i+hp'_i+\frac{h^2 p''_i}{2}$$

$$p_{i-1}=p_i-hp'_i+\frac{h^2 p''_i}{2} \qquad (4.36)$$

Substitute Eq. (4.36) into Eq. (4.35), we have:

$$\left(\gamma+h^2\alpha q_{i-1}\right)e_{i-1}+\left(\gamma\rho+\beta h^2 q_i\right)e_i+\left(\gamma+h^2\alpha q_{i+1}\right)e_{i+1}=T_i(h)$$

where,

$$T_i(h)=-(2+\rho)\gamma y_i+(2\alpha+\beta-1)\gamma h^2 y''_i+(\frac{-2\alpha}{3}+\beta(\frac{1}{6}+2\psi\gamma)h^4 y_i^{(3)}p_i$$

$$+(12\alpha-1)\frac{\gamma h^4 y_i^{(4)}}{12}+O(h^6) \qquad (4.37)$$

$T_i(h)$ is a local truncation error is associated with the scheme developed in Eq.(4.28). Thus, for different values of α, β, ψ in the scheme of Eq. (4.28), the following different order are presented.

(i) For any choice of arbitrary α and β with $\rho = -2,\ 2\alpha + \beta = 1$ and for any value of ψ the scheme of Eq. (4.28) gives the second-order method;

(ii) For $\rho = -2,\ \alpha = \frac{1}{12},\ \beta = \frac{10}{12},\ \psi = \frac{-1}{20\gamma}$, from Eq. (4.28) the fourth –order method is derived.

4.3. Stability and Convergence Analysis

In this section, we discuss the stability and convergence for the developed scheme.

To show this we considered the following two theorems, without proofs

Definition1: A matrix A is said to be L-matrix if and only if $a_{ii} > 0, i = 1, 2, ..., n-1$ and $a_{ij} \le 0, i \ne j\ , i = 1, 2, ..., n-1$

Theorem 4.1: For any partition $J \cup K$ of the index set $1, 2, ..., n$ of an $n \times n$ of matrix A, if there exists $j \in J$ and $k \in K$ such that a $a_{jk} \ne 0$, then A is an irreducible matrix, (Varga, 1962).

Theorem 4.2: If A is an L-matrix which is symmetric, irreducible and weak diagonal dominance, then A is a monotone matrix, (Young, 1971).

Writing the tri-diagonal system in Eq. (4.28) in matrix vector form, we have:

$$A\overline{Y} = C \tag{4.38}$$

where $A = (B_o + B_1 + h^2 B_2 Q)$ is a tridigonal matrix of order $n-1$.

Multiplying both sides of Eq. (4.28) by (-1), we get:

$$B_0=\begin{bmatrix} -\rho\gamma & -\gamma & & & \\ -\gamma & -\rho\gamma & -\gamma & & \\ & \ddots & \ddots & \ddots & \\ & & -\gamma & -\rho\gamma & -\gamma \\ & & & -\gamma & -\rho\gamma \end{bmatrix} \quad B_1=\begin{bmatrix} \frac{hv_1}{2} & \frac{hw_1}{2} & & & \\ \frac{hu_2}{2} & \frac{hv_2}{2} & \frac{hw_2}{2} & & \\ \ddots & & \ddots & \ddots & \\ & \frac{h}{2}u_{n-2} & \frac{hv_{n-2}}{2} & \frac{h}{2}w_{n-2} \\ & & \frac{h}{2}u_{n-1} & \frac{hv_{n-1}}{2} \end{bmatrix}$$

$$B_2=\begin{bmatrix} -\beta & -\alpha & & & \\ -\alpha & -\beta & -\alpha & & \\ & \ddots & \ddots & \ddots & \\ & & -\alpha & -\beta & -\alpha \\ & & & -\alpha & -\beta \end{bmatrix} \qquad Q=\begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ \vdots \\ \vdots \\ q_{n-2} \\ q_{n-1} \end{bmatrix}$$

and $C=(c_i)$ is the column vector,

where, $c_i=h^2\left\{(-\alpha-\beta\psi hp_1)f_0+\alpha\phi_0 q_0-\beta f_1-(\alpha-\beta\psi hp_1)f_2\right\}+\gamma\phi_0-\frac{h\phi_0}{2}u_1,$ for $i=1$

$$c_i=h^2\left\{(-\alpha-\beta\psi hp_i)f_{i-1}-\beta f_i-(\alpha-\beta\psi hp_i)f_{i+1}\right\}, \text{ for } 2\le i\le n-2$$

$$c_i=h^2\left\{(-\alpha-\beta\psi hp_{n-1})f_{n-2}-\beta f_{n-1}-(\alpha-\beta\psi hp_{n-1})f_n+\alpha h^2 q_n\right\}+\gamma\varphi-\frac{h\varphi}{2}w_n, \text{ for } i=n-1$$

and $\bar{Y}=\left[\bar{y}_1,\bar{y}_2,...,\bar{y}_{n-1}\right]^t$ is an approximation solution , $T(h)=O(h^6)$ for

$$\rho=-2,\ \ \alpha=\frac{1}{12},\ \beta=\frac{10}{12},\ \psi=\frac{-1}{20\gamma}$$

Now considering the above system with exact solution

$y=\left[y(x_1),y(x_2),...,y(x_{n-1})\right]^t$, we have:

$$AY-T(h)=C \tag{4.39}$$

and $T(h)=(T_1(h),T_2(h),...,T_{n-1})^t$ is the local truncation error.

From Eq. (4.38) and Eq. (4.39), we obtain:

$$A(Y-\bar{Y})=T(h) \tag{4.40}$$

This gives the error equation

$$AE=T(h) \tag{4.41}$$

where $E=Y-\bar{Y}=\left(e_0,e_1,e_2,...,\right)^T$

Let $|p_{i-1}|\le z_1, |p_i|\le z_2, |p_{i+1}|\le z_3, |q_{i-1}|\le k_1, |q_i|\le k_2, |q_{i+1}|\le k_3$ (4.42)

Let $r_{i,j}$, be the $(i,j)^{th}$ elements of the matrix $(B_1+h^2B_2Q)$ then :

$$|r_{i,i+1}|=\left|\frac{h}{2}w_i-\alpha h^2 q_{i+1}\right|=\left|\frac{h}{2}\left(\alpha p_{i-1}-\beta p_i(1+2\psi h^2 q_{i+1}+\psi h(3p_{i+1}+p_{i-1}))-3\alpha p_{i+1}-2\alpha h q_{i+1}\right)\right|$$

$$\le \frac{h}{2}\left(\alpha z_1+\beta z_2(1+2\psi h^2 k_3+\psi h(3z_3+z_1))+3\alpha z_3+2\alpha h k_3\right)$$

$$|r_{i,i-1}|=\left|\frac{h}{2}u_i-\alpha h^2 q_{i-1}\right|=\left|\frac{h}{2}(3\alpha p_{i-1}+\beta p_i(1+2\psi h^2 q_{i-1}-\psi h(p_{i+1}+3p_{i-1}))-\alpha p_{i+1}-2\alpha h q_{i-1})\right|$$

$$\le \frac{h}{2}\left(3\alpha z_1+\beta z_2(1+2h^2\psi k_1+\psi h(z_3+3z_1))+\alpha z_3+2\alpha h k_1\right)$$

Thus, for sufficiently small *h*, we have:

$$-\gamma+|r_{i,i+1}|\ne 0 \text{ for } i=1,2,...,n-2$$

$$-\gamma+|r_{i,i-1}|\ne 0 \text{ for } i=2,...,n-1$$

Hence, the matrix A is irreducible

Remark: From Eq. (4.28), we have:

$$E_i + G_i = -2\gamma + \frac{h}{2}\left(4\alpha p_{i-1} - 4\beta p_i \psi h(p_{i+1} + p_{i-1}) - 4\alpha p_{i+1}\right) - \alpha h^2 (q_{i-1} + q_{i+1}) + \beta\psi h^3 p_i (q_{i-1} + q_{i+1})$$
$$F_i = -2\rho + \frac{h}{2}\left(-4\alpha p_{i-1} + 4\beta p_i \psi h(p_{i+1} + p_{i-1}) + 4\alpha p_{i+1}\right)$$

Thus, for sufficiently small h, $\rho = -2$ and $\gamma = \varepsilon - \delta p(x), \quad 0 < \gamma << 1$, we get,

$$|E_i + G_i| < |F_i|$$

Hence, A is diagonally dominant.

Since the coefficient matrix A is diagonally dominant and irreducible the scheme is stable.

Let S_i be the sum of the elements of the i^{th} row of the matrix A then

$$S_1 = -\rho\gamma - \gamma + \frac{h}{2}\left(-3\alpha p_{i-1} - \beta p_i + \alpha p_{i+1}\right) + \frac{h^2}{2}\left(\beta\psi p_i (p_{i+1} + 3p_{i-1}) - 2(\beta q_i + \alpha q_{i+1})\right)$$

$$-h^3 \beta\psi p_i q_{i+1} \quad \text{for } i = 1$$

$$S_i = -\rho\gamma - 2\gamma + h^2 \left(\alpha(-q_{i-1}) + \beta(-q_i) + \alpha(-q_{i+1})\right) + O(h^3) \quad \text{for } 2 \le i \le n-2$$

$$S_{n-1} = -\rho\gamma - \gamma + \frac{h}{2}\left(-\alpha p_{i-1} + \beta p_i + 3\alpha p_{i+1}\right) + \frac{h^2}{2}\left(\beta\psi p_i (p_{i+1} + p_{i-1}) - 2\beta q_i - 2\alpha q_{i-1}\right)$$

$$+h^3 \beta\psi p_i q_{i-1} \quad \text{for } i = n-1$$

Let $d = \min_{1 \le i \le n-1} |q_i| = |-q_i|$

For sufficiently small h, A is monotone.

Hence, (A^{-1}) exist and $(A^{-1}) \ge 0$ .From Eq. (4.41),we have:

$$E = A^{-1} T(h) \tag{4.43}$$

$$\|E\| \le \|A^{-1}\| \|T(h)\| \tag{4.44}$$

For sufficiently small *h*, we have:

$$S_1 > h^2 dd_1 \quad \text{for } i = 1$$

$$S_i > h^2 dd_2 \quad \text{for } 2 \le i \le n-2$$

$$S_{n-1} > h^2 dd_1 \quad \text{for } i = n-1 \tag{4.45}$$

where, $d = \min_{1\le i\le n-1} |-q_i| = \min_{1\le i\le n-1} |q_i|$, $d_1 = \alpha + \beta$ and $d_2 = 2\alpha + \beta$

Let $(a_{i,k}^{-1}) \in (A^{-1})$, we define

$$\left\| a_{i,k}^{-1} \right\| = \max_{1\le i\le N-1} \sum_{k=1}^{N-1} \left| a_{i,k}^{-1} \right| \quad \text{and} \quad \left\| T(h) \right\| = \max_{1\le i\le N-1} \left| T_i(h) \right| \tag{4.46}$$

Since $(a_{i,k}^{-1}) \ge 0$ then from the theory of matrices, we have:

$$\sum_{k=1}^{N-1} (a_{i,k}^{-1}) S_k = 1 \quad \text{for } i = 1, 2, ..., n-1$$

Henc $(a_{i,1}^{-1}) \le \frac{1}{s_1} < \frac{1}{h^2 dd_1}$ for $k = 1$ (4.47)

$$(a_{i,n-1}^{-1}) \le \frac{1}{s_{n-1}} < \frac{1}{h^2 dd_1} \text{ for } k = n-1 \tag{4.48}$$

Further $\sum_{k=2}^{N-2} (a_{i,k}^{-1}) \le \frac{1}{\min_{2\le k\le n-2} s_k} < \frac{1}{h^2 dd_2}$ for $k = 2, ..., n-2$ (4.49)

From Eqs. (4.41-4.49)and (4.37),we get:

$$\|E\| \le \left(\frac{1}{h^2 dd_1} + \frac{1}{h^2 dd_1} + \frac{1}{h^2 dd_2} \right) \times O(h^6) = Dh^4$$

where, $D=\left(\frac{1}{dd_1}+\frac{1}{dd_1}+\frac{1}{dd_2}\right)$,which independent of mesh size h. This establishes that the present method is fourth order convergent for $\rho=-2,\ \alpha=\frac{1}{12},\ \beta=\frac{10}{12}$ and $\psi=\frac{-1}{20\gamma}$

4.4. Numerical Examples and Results

To demonstrate the applicability of the method, four model examples for SPDCD problems two example for right layer and Two example for left layer have been considered. The numerical results are presented for $\rho=-2,\ \alpha=\frac{1}{12},\ \beta=\frac{10}{12}$ and $\psi=\frac{-1}{20\gamma}$. Since the variable coefficient examples have no exact solution, the numerical solutions are computed using double mesh principle. The maximum absolute errors are computed using double-mesh principle given by:

$$Z_h=\max_i\left|y_i^h-y_i^{\frac{h}{2}}\right| \quad i=1,2,...,n-1$$

where y_i^h is the numerical solution at the nodal point x_i on the mesh $\{x_i\}^{n-1}$ and $x_i=x_0+ih, i=1,2,...,n-1,$ $y^{\frac{h}{2}}$ is the numerical solution at the nodal point x_i on the mesh $\{x_i\}^{2n-1}$ where $x_i=x_0+i\frac{h}{2}, i=1,2,...,2n-1$ (*i.e.*, the numerical solution on a mesh, obtained by bisecting the original mesh with n number of mesh intervals), Doolan et al (1980).

Example 4.1: Consider the singularly perturbed delay convection-diffusion equation,

$$\varepsilon y''(x)-y'(x-\delta)-y(x)=0$$

under the interval and boundary conditions

$$y(x)=1,\ -\delta\le x\le 0 \text{ and } y(1)=-1.$$

The analytical solution of this equation is given by:

$$y(x)=\frac{(1+e^{m_2})e^{m_1x}-(e^{m_1}+1)e^{m_2x}}{e^{m_2}-e^{m_1}}$$

Where,

$$m_1 = \frac{1-\sqrt{1+4(\varepsilon+\delta)}}{2(\varepsilon+\delta)} \quad \text{and} \quad m_1 = \frac{1+\sqrt{1+4(\varepsilon+\delta)}}{2(\varepsilon+\delta)}.$$

The maximum absolute errors are presented in Tables 4.1 and 4.5 for different values of ε and δ. The graph of the computed solution for $\varepsilon = 0.01$ and different values of δ is also given in Fig. 4.1.

Table 4.1: The maximum absolute errors of Example 4.1 for different values of δ with $\varepsilon = 0.01$.

$\delta \downarrow$	$N = 10^2$	$N = 10^3$	$N = 10^4$
Present Method			
0.1ε	5.2311e − 04	4.9987e − 08	3.7912e – 11
0.15ε	4.3505e − 04	4.1907e − 08	3.7991e – 11
0.25ε	3.0684e − 04	3.0145e − 08	4.0597e – 11
Sirisha and Reddy, 2017			
0.1ε	1.6595e − 01	2.2109e − 02	2.2850e – 03
0.15ε	1.5895e − 01	2.1173e − 02	2.1860e – 03
0.25ε	1.4603e − 01	1.9539e − 02	2.0130e – 03
Phaneendra et al., 2014			
0.07ε	1.8573e−01	8.4819e−05	8.4629e−09
0.15ε	8.0711e−02	1.7927e−05	1.8318e−09
0.25ε	3.2547e−02	4.7036e−06	4.9565e−10

Example 4.2: Consider the singularly perturbed delay convection-diffusion equation,

$$\varepsilon y''(x) - e^x y'(x-\delta) - xy(x) = 0$$

under the interval and boundary conditions

$$y(x) = 1, \ -\delta \le x \le 0 \ \text{ and } \ y(1) = 1.$$

The maximum absolute errors are presented in Tables 4.2 and 4.5 for different values of ε and δ. The graph of the computed solution for $\varepsilon = 0.01$ and different values of δ is also given in Fig. 4.2

Table 4.2: The maximum absolute errors of Example 4.2 for different values of δ with $\varepsilon = 0.1$.

$\delta \downarrow$	$N = 10^2$	$N = 10^3$	$N = 10^4$
Present Method			
0.1ε	2.6162e − 06	2.8352e − 08	2.9569e−10
0.3ε	4.3417e − 06	4.3980e − 08	4.8046e−10
0.6ε	4.4854e − 06	4.4995e − 08	5.2096e−10
0.8ε	4.2905e − 06	4.2980e − 08	4.4079e−10
Sirisha and Reddy, 2017			
0.1ε	7.7065e – 03	8.5743e − 04	8.6724e−05
0.3ε	5.5572e – 03	6.0006e − 04	6.0487e–05
0.6ε	3.8911e – 03	4.1085e − 04	4.1314e–05
0.8ε	3.2241e – 03	3.3750e − 04	3.3908e–05
Phaneendra et al., 2012			
0.1ε	5.7597e – 03	5.0842e − 04	5.02478e−05
0.3ε	3.9328e – 03	3.6132e − 04	3.58384e-05
0.6ε	2.7026e – 03	2.5507e − 04	2.53643e-05
0.8ε	2.2469e – 03	2.1413e − 04	2.13134e-05

Example 4.3: Consider the singularly perturbed delay convection-diffusion equation,

$$\varepsilon y''(x) + y'(x-\delta) - y(x) = 0$$

under the interval and boundary conditions

$$y(x) = 1, \ -\delta \le x \le 0 \ \text{ and } \ y(1) = 1 .$$

The analytical solution of this equation is given by:

$$y\left(x\right)=\frac{(1-e^{m_2})e^{m_1x}+(e^{m_1}-1)e^{m_2x}}{e^{m_1}-e^{m_2}}$$

Where,

$$m_1=\frac{-1-\sqrt{1+4(\varepsilon-\delta)}}{2(\varepsilon-\delta)} \quad \text{and} \quad m_1=\frac{-1+\sqrt{1+4(\varepsilon-\delta)}}{2(\varepsilon-\delta)}.$$

The maximum absolute errors are presented in Tables 4.3 and 4.5 for different values of ε and δ. The graph of the computed solution for $\varepsilon=0.01$ and different values of δ is also given in Fig. 4.3.

Table 4.3: The maximum absolute errors of Example 4.3 for different values of δ with $\varepsilon = 0.01$.

$\delta\downarrow$	$N=10^2$	$N=10^3$	$N=10^4$
Present Method			
0.1ε	5.4192e − 04	5.0759e − 08	3.1056e − 11
0.3ε	1.4385e − 03	1.3787e − 07	8.4393e − 11
0.6ε	9.9988e − 03	1.2836e − 06	8.3262e − 11
0.8ε	6.1971e − 02	2.0628e − 05	2.0325e − 09
Sirisha and Reddy, 2017			
0.1ε	9.0733e − 02	1.2286e − 02	1.2790e – 03
0.3ε	1.0803e − 01	1.5622e − 02	1.6440e – 03
0.6ε	1.2778e − 01	2.6309e − 02	2.8700e – 03
0.8ε	1.0040e − 01	4.8338e − 02	5.6880e – 03
Phaneendra et al., 2012			
0.1ε	9.0730e − 02	1.2280e − 02	1.2700e – 03
0.3ε	1.0803e − 01	1.5620e − 02	1.6400e – 03
0.6ε	1.2777e − 04	2.6300e − 02	2.8700e – 03
0.8ε	1.0040e − 04	4.8330e − 02	5.6800e – 03

Example 4.4: Consider the singularly perturbed delay convection-diffusion equation,

$$\varepsilon y''(x) + e^{-0.25x} y'(x-\delta) - y(x) = 0$$

under the interval and boundary conditions

$$y(x) = 1, \ -\delta \le x \le 0 \text{ and } y(1) = 1.$$

The maximum absolute errors are presented in Tables 4.4 and 4.5 for different values of ε and δ. The graph of the computed solution for $\varepsilon = 0.01$ and different values of δ is also given in Fig. 4.4.

Table 4.4: The maximum absolute errors of Example 4.4 for different values of δ with $\varepsilon = 0.1$.

$\delta \downarrow$	$N = 10^2$	$N = 10^3$	$N = 10^4$
Present Method			
0.1ε	5.0718e – 07	4.5393e – 09	5.0900e–11
0.3ε	2.4141e – 06	2.2786e – 08	2.4474e–10
0.6ε	1.5438e – 05	1.4239e – 07	1.8438e–09
0.8ε	9.5435e – 03	7.6953e – 07	8.6163e–09
Sirisha and Reddy, 2017			
0.1ε	6.2687e – 03	6.6646e − 04	6.7072e–05
0.3ε	8.0060e – 03	8.6458e − 04	8.7156e–05
0.6ε	1.3420e – 02	1.5282e − 03	1.5493e–04
0.8ε	2.3860e – 02	3.0459e − 03	3.1280e–04
Phaneendra et al., 2012			
0.1ε	6.3299e – 03	6.7427e − 04	6.7871e–05
0.3ε	8.1592e – 03	8.8256e − 04	8.8987e–05
0.6ε	1.3848e – 02	1.5797e − 03	1.6020e–04
0.8ε	2.4772e – 02	3.1732e − 03	3.2603e–04

Table 4.5: The point wise absolute errors for different values of ε, N and $\delta = 0.5\varepsilon$.

x	ε 2^{-8}		2^{-9}		2^{-10}	
	N=100	N = 200	N = 100	N = 200	N = 100	N = 200
Example 4.1						
0.2	1.8935e−09	1.1847e−10	3.8255e−09	2.3960e−10	7.6739e−09	4.8166e−10
0.4	3.1041e−09	1.9421e−10	6.2678e−09	3.9257e−10	1.2569e−08	7.8893e−10
0.6	3.8166e−09	2.3879e−10	7.7020e−09	4.8240e−10	1.5441e−08	9.6916e−10
0.8	4.1712e−09	2.6097e−10	8.4127e−09	5.2691e−10	1.6861e−08	1.0583e−09
Example 4.2						
0.2	3.6361e−07	8.8239e−08	3.7937e−07	8.9465e−08	4.0887e−07	9.1441e−08
0.4	5.7715e−07	1.4010e−07	6.0220e−07	1.4209e−07	6.4871e−07	1.4523e−07
0.6	6.5386e−07	1.5849e−07	6.8180e−07	1.6077e−07	7.3493e−07	1.6438e−07
0.8	6.2607e−07	1.5176e−07	6.5458e−07	1.5401e−07	7.0719e−07	1.5760e−07
Example 4.3						
0.2	1.2643e−08	7.9269e−10	2.5216e−08	1.5889e−09	5.1328e−06	3.1764e−09
0.4	1.1577e−08	7.2586e−10	2.3137e−08	1.4553e−09	4.5923e−08	2.9095e−09
0.6	9.4230e−09	5.9082e−10	1.8836e−08	1.1847e−09	3.7425e−08	2.3689e−09
0.8	5.7524e−09	3.6067e−10	1.1501e−08	7.2338e−10	2.2853e−08	1.4465e−09
Example 4.4						
0.2	6.2465e−07	1.5343e−07	6.4020e−07	1.5460e−07	2.2546e−06	1.5656e−07
0.4	5.8202e−07	1.4280e−07	5.9768e−07	1.4402e−07	6.2673e−07	1.4600e−07
0.6	4.8798e−07	1.1959e−07	5.0218e−07	1.2074e−07	5.2821e−07	1.2252e−07
0.8	3.1078e−07	7.6069e−08	3.2055e−07	7.6883e−08	3.3828e−07	7.8114e−08

The Effect of Delay Term on the Solution Profile

To analyze the effect of the delay term on the solution profile of the problem, the numerical solution of the problem for different values of the delay parameters have been given by the following graphs.

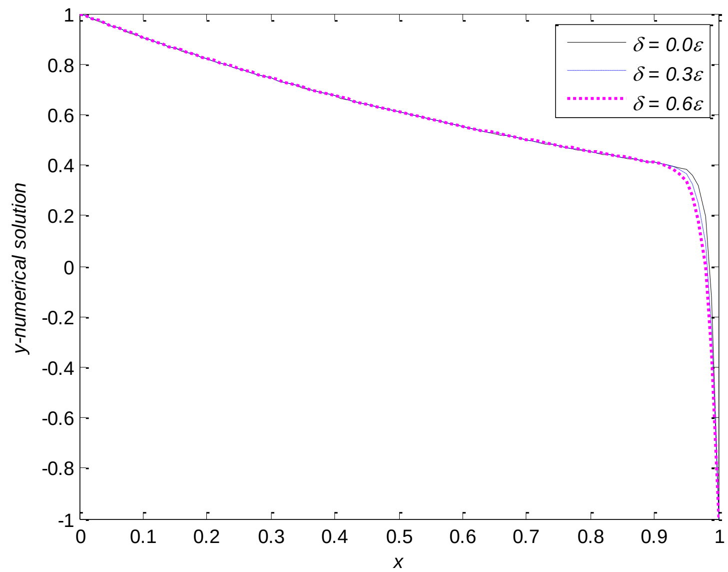

Figure 4.1: The numerical solution of Example 4.1 with $\varepsilon = 0.01$ and $N = 100$.

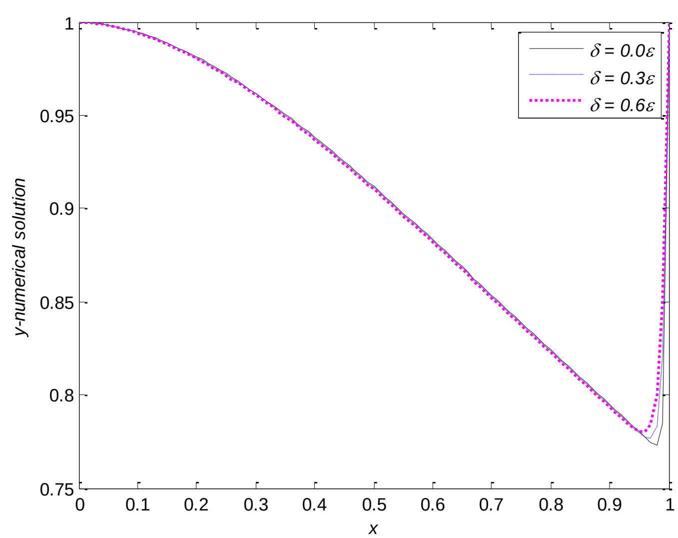

Figure 4.2: The numerical solution of Example 4.2 with $\varepsilon = 0.01$ and $N = 100$.

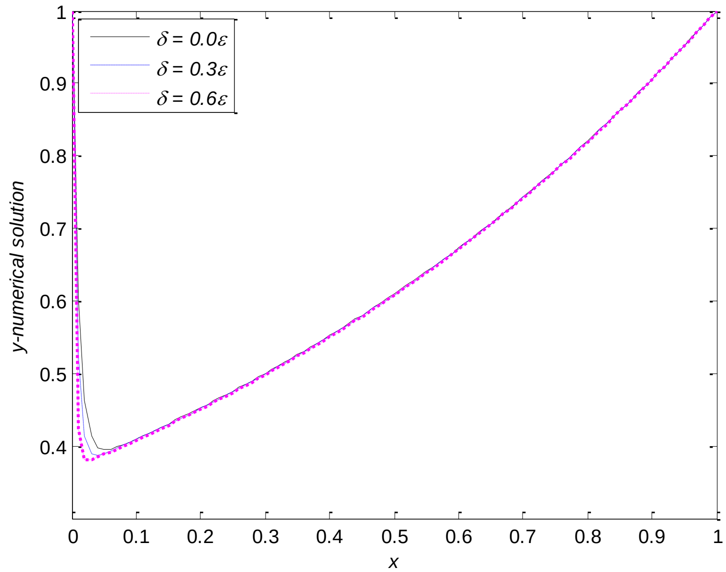

Figure 4.3: The numerical solution of Example 4.3 with $\varepsilon = 0.01$ and $N = 100$.

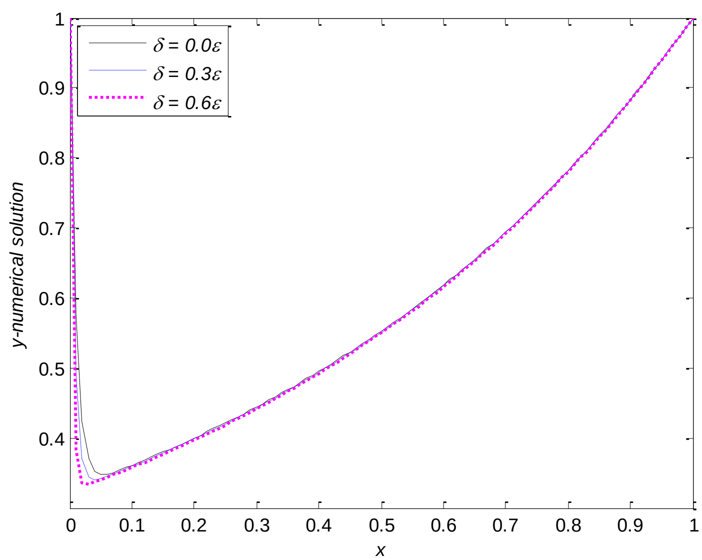

Figure 4.4: The numerical solution of Example 4.4 with $\varepsilon = 0.01$ and $N = 100$.

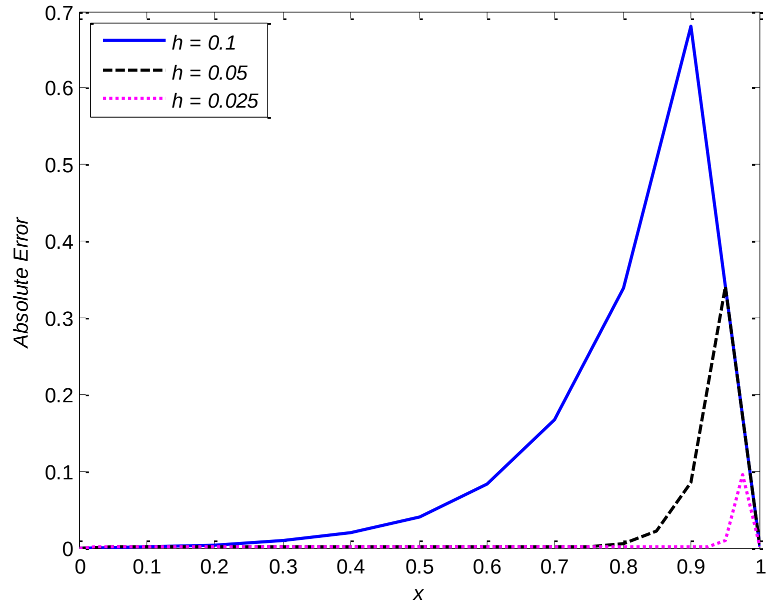

Figure 4.5: The pointwise absolute errors of Example 4.1 for different values of mesh size h, $\varepsilon = 2^{-8}$ and $\delta = 0.5\varepsilon$.

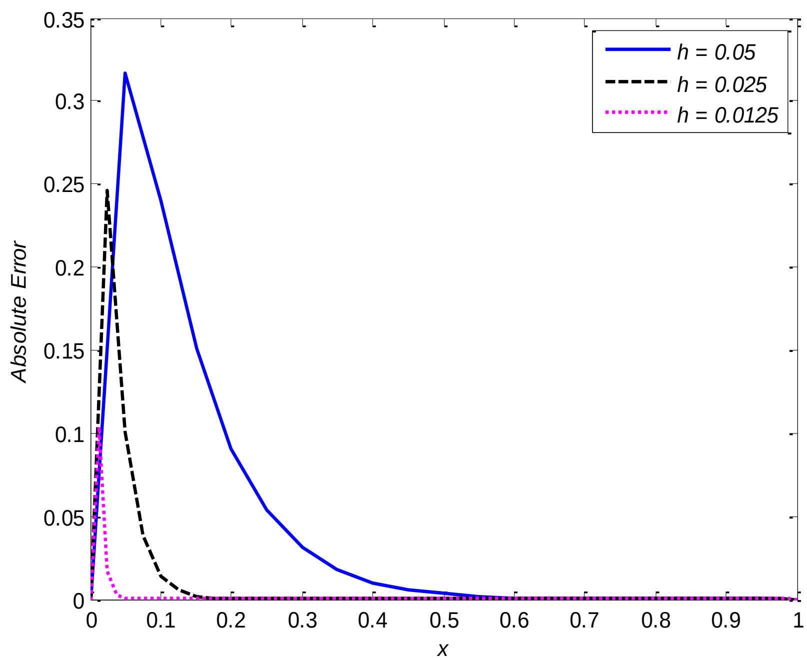

Figure 4.6: The pointwise absolute errors of Example 4.4 for different values of mesh size h, $\varepsilon = 2^{-8}$ and $\delta = 0.5\varepsilon$.

4.5. Discussion

In this thesis non polynomial cubic spline method is presented for solving singularly perturbed delay convection-diffusion equations. First the second order singularly perturbed delay convection diffusion equation transformed into an asymptotically equivalent singularly perturbed boundary value problem. Then using non-polynomial cubic spline approximation is changed in to a three-term recurrence relation, which can be solved using Thomas Algorithm. The stability and convergence of the method have been established. Two model examples of variable coefficient without exact solution and two model examples of constant coefficients with exact solution have been considered and solved for different values of perturbation parameter ε, delay parameter δ and mesh size h. The numerical result are organized in tables and further to examine the effect of delay on the left and right boundary layer of the solution graphs have been given for different values of δ. The maximum absolute errors are tabulated in Tables (4.1-4.4) for different values of ε and δ and the pointwise absolute errors for different values of ε and N and $\delta = 0.5\varepsilon$ tabulated in table (4.5).The results obtained by the present method improves the finding of Sirisha and Reddy (2017). Also, it is significant that all of the maximum absolute errors decrease rapidly as N increases. As perturbation parameter ε is sufficiently small, some researchers such as (Doolan et al, 1980), (Kadalbajoo and Sharma, 2004) and (Roos et al, 1942) state that there is a challenge to get more accurate solutions for singularly perturbed boundary value problems. And also they states that no good result for singularly perturbed boundary value problem when $\varepsilon < h$. But we get a good result for is sufficiently small ε and $\varepsilon < h$ Table (4.5).

To demonstrate the effect of delay term on the left and right boundary layer of the solution, the graphs for different values of delay parameter δ, and small size h are plotted in Figures. (4.1)-(4.6). Accordingly, depending on the coefficient of the delay term we observed that from Figures(4.1, and 4.2) as the delay increases, the thickness of the right boundary layer increases whereas from Figures(4.3and 4.4) as delay increases, the thickness of the left boundary layer decreases. Also it can see that as mesh size h decrease the absolute errors also decrease from Figures (4.5) and (4.6)

Chapter Five
Conclusion and Scope for Future Work

5.1 Conclusion

This study is implemented four model examples with exact and without exact solutions by taking different values of perturbation parameter ε and delay parameter δ, and the results are presented in the tables and figures. The result observed from the tables the present method approximate the solution very well. Furthermore the stability and convergence of the method is established well and the accuracy of the method presented in terms of maximum absolute errors .The effect of the delay on the solution of singularly perturbed delay convection-diffusion equation is showed by plotting graphs of four model examples. As the delay increases, the thickness of the right boundary layer increases whereas the left boundary layer decreases In general, the present method is stable, convergent and more accurate for solving singularly perturbed delay convection-diffusion equations.

5.2. Scope of the Future Work

In this thesis, the numerical method based on non-polynomial cubic spline method is introduced for solving singularly perturbed delay convection-diffusion equations. Hence, the scheme proposed in this thesis can also be extended to quartic, quantic, sextic, septic and higher order non polynomial spline methods for solving singularly perturbed delay convection-diffusion equation.

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